Generalizing Newman's Lemma for Left-Linear Rewrite Systems

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Abstract. Confluence criteria for non-terminating rewrite systems are known to be rare and notoriously difficult to obtain. Here we prove a new result in this direction. Our main result is a generalized version of Newman's Lemma for left-linear term rewriting systems that does not need a full termination assumption. We discuss its relationships to previous confluence criteria, its restrictions, examples of application as well as open problems. The whole approach is developed in the (more general) framework of context-sensitive rewriting which thus turns out to be useful also for ordinary (context-free) rewriting.

1 Introduction and Overview

Besides termination, confluence is the most fundamental property of virtually any kind of rewrite systems (cf. e.g. [1], [2]). Newman's Lemma [19] is well-known to be the major tool for checking confluence of rewrite systems. It states that local confluence implies confluence for terminating reduction relations. However, without termination Newman's Lemma is not applicable, i.e., local confluence may be insufficient for guaranteeing confluence. In general, confluence proofs without termination are much harder. For the case of not necessarily terminating term rewriting systems (TRSs), a couple of rather restrictive criteria - mostly via strong confluence properties – are known, both for abstract rewrite systems (cf. e.g. [11], [13], [2]) as well as for TRSs (cf. e.g. [26], [11], [27], [29], [8], [23], [24], [21]). Known related decidability results include [7], [6]. Also structural and modularity properties and considerations may help in certain cases to prove confluence of non-terminating systems (cf. e.g. [25], [28], [14], [20]). The latter type of criteria is based on a *divide-and-conquer* approach, where certain sub-TRSs are shown to be confluent which in turn implies, under certain combination conditions, confluence of the whole system.

In term rewriting it is well-known that local confluence of (finite) terminating TRSs is decidable since it amounts to joinability of all *critical pairs* (*Critical Pair Lemma*, [11]). Hence, for (finite) terminating TRSs **Newman's Lemma** combined with the **Critical Pair Lemma** yields a decision procedure for confluence.

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The approach for proving confluence of (non-terminating) TRSs that we are going to present here is novel and differs methodologically from virtually all of these previous approaches in the sense that we do not consider sub-TRSs but rather certain sub-relations of the rewrite relation that are not generated by sub-TRSs.

The basic idea of our approach is as follows: Given a (non-terminating) TRS $\mathcal{R}$ with induced rewrite relation $\to_\mathcal{R}$, we first try to identify an appropriate terminating sub-relation $\to' \subseteq \to_\mathcal{R}$ (that is not induced by a sub-TRS of $\mathcal{R}$), to prove its confluence via Newman's Lemma, and then to deduce confluence of the entire TRS $\mathcal{R}$, i.e., of $\to_\mathcal{R}$, under some additional conditions.

The setting we are working in is *context-sensitive term rewriting* (CSR), a framework that properly extends ordinary (context-free) term rewriting by introducing context-sensitivity restrictions in the rewrite relation (cf. e.g. [17]). The necessary technical background will be provided below. CSR has turned out to be very useful for obtaining better computational properties of equational and rewrite specifications, e.g., for increased efficiency, a better termination behaviour, and an effective handling of infinite data structures (cf. e.g. [15,16,17]). Given the fact that termination is sometimes very difficult to prove and that non-termination is in many cases inherently unavoidable, CSR often provides ways for such examples to get a (restricted) terminating context-sensitive rewrite relation, while still preserving the desired computational power (e.g., for computing normal forms). Our confluence criterion will be based on such a context-sensitive view of a given ordinary TRS $\mathcal{R}$.

Let us give two simple motivating examples illustrating the problem with proving confluence.

Example 1. Consider the following TRS $\mathcal{R}$, which is a slightly modified variant of [4, Ex. (27)], and the essential part of its reduction graph:

$$
\begin{array}{lrl}
(1) & g(a) & \to f(g(a)) \\
(2) & g(b) & \to c \\
(3) & a & \to b \\
(4) & f(x) & \to h(x,x) \\
(5) & h(x,y) & \to c
\end{array}
$$

$$
\begin{tikzcd}
g(a) \arrow[r] \arrow[dd] & f(g(a)) \arrow[d] \\
 & h(g(a),g(a)) \arrow[d] \\
g(b) \arrow[r] & c
\end{tikzcd}
$$

Example 2. This example involves the generation of (all) natural numbers (via the constant nats and using a recursive increment operation inc) with some list destructors h(ea)d, t(ai)l and : as (infix) list constructor. The TRS $\mathcal{R}$ and the essential part of its reduction graph are as follows:

$$
\begin{array}{lrl}
(1) & \mathsf{nats} & \to 0 : \mathsf{inc}(\mathsf{nats}) \\
(2) & \mathsf{inc}(x:y) & \to \mathsf{s}(x) : \mathsf{inc}(y) \\
(3) & \mathsf{hd}(x:y) & \to x \\
(4) & \mathsf{tl}(x:y) & \to y \\
(5) & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) & \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))
\end{array}
$$

$$\begin{tikzcd}
\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \arrow[r] \arrow[d] & \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \arrow[d] \\
\mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats}))) \arrow[d] & \mathsf{tl}(\mathsf{inc}(0 : \mathsf{inc}(\mathsf{nats}))) \arrow[d] \\
\mathsf{inc}(\mathsf{inc}(\mathsf{nats})) & \mathsf{tl}(\mathsf{s}(0) : \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \arrow[l]
\end{tikzcd}$$

In both examples the rewrite system $\mathcal{R}$ has the following properties: It is left-linear, non-terminating and locally confluent. But as far as we know there are no known confluence criteria in the literature that would allow us to directly infer confluence in these examples. In particular, since both systems are non-terminating, Newman's Lemma is not applicable, i.e., a test for joinability of critical pairs is not sufficient. The systems are not orthogonal, since there exist critical pairs (in Ex. 1 rule (3) overlaps into (1), and in Ex. 2 rule (1) overlaps into (5)). Even though these critical pairs are joinable (cf. the reduction graphs), none of the critical pair based confluence criteria for left-linear rewrite systems in [11,8,23,24,21] is applicable here. Also, decidability results of [7], [6] are not applicable.[1] Yet, both systems are indeed confluent as we shall prove later on with our new criterion.

2 Preliminaries

We assume familiarity with the basic theory, terminology and notations in term rewriting (cf. e.g. [1], [2]). For the sake of readability some important notions and notations are recalled here.

Given a set A, $\mathcal{P}(A)$ denotes the set of all subsets of A. Given a binary relation $\to$, on a set A, we denote the transitive closure of $\to$ by $\to^+$, and its reflexive and transitive closure by $\to^*$. The inverse $\to^{-1}$ of $\to$ defined by $\{(b,a) \mid (a,b) \in \to$ is also denoted by $\leftarrow$. An element $a \in A$ is an $\to$-normal form, if there exists no b such that $a \to b$; $\mathrm{NF}(\to)$ is the set of $\to$-normal forms. We say that b is a $\to$-normal form of a, if $a \to^* b \in \mathrm{NF}(\to)$. We say that $\to$ is *terminating* iff there is no infinite sequence $a_1 \to a_2 \to a_3 \cdots$. $\to$ is *locally confluent* iff $\leftarrow \cdot \to \subseteq \to^* \cdot {}^*\!\leftarrow$, and *confluent* (or *Church-Rosser*) iff $\leftarrow^* \cdot \to^* \subseteq \to^* \cdot \leftarrow^*$. *Terms* are constructed as usual over some countable set $\mathcal{V}$ of *variables* and a *signature* $\mathcal{F}$ of *functions symbols* equipped with a fixed arity given by $ar\colon \mathcal{F} \to \mathbb{N}$. The set of all terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. A term is *linear* if it has no multiple occurrences of a single variable. Terms are viewed as labelled trees in the usual way. *Positions* $p, q, \ldots$ in terms are

[1] With the *decreasing diagrams method* of [22], [2, Section 14.2], however, it is possible to prove confluence of the *linear* system of Example 2 above, by finding an appropriate well-founded labelling for rewrite steps. Yet, this powerful and general method does not directly yield easily applicable confluence criteria, but requires careful and smart design choices to become applicable. For Example 1, (practical) applicability of this method remains unclear, due to non-right-linearity.

represented by sequences of positive natural numbers. Given positions p, q, we denote their concatenation by $p.q$. Positions are ordered by the standard prefix ordering $\leq$. Two positions p and q are *parallel* (or *disjoint*), denoted $p \parallel q$, if neither $p \leq q$ nor $q \leq p$. The set of all *positions* of a term t is $Pos(t)$, the set of all its variable positions and of all its non-variable positions by $\mathcal{V}Pos(t)$ and $\mathcal{F}Pos(t)$, respectively. We denote the 'empty' root position by ϵ. The subterm of t at position p is denoted by $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s. We shall also make free use of (term) *contexts* as usual. The symbol labelling the root of t is denoted as $root(t)$. For the set of all variables occurring in a term s we write $Var(s)$.

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $l \notin \mathcal{V}$ and $Var(r) \subseteq Var(l)$. A *term rewriting system* (TRS) is a pair $\mathcal{R} = (\mathcal{F}, R)$ where $\mathcal{F}$ is a signature and R is a set of rewrite rules over $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We will often omit the signature when it is implicitly given by the set of rules, and identify $\mathcal{R}$ and R. A TRS $\mathcal{R}$ is left-linear if for all $l \to r \in \mathcal{R}$, l is linear. The *rewrite relation* induced by a TRS $\mathcal{R}$ is defined by $s \to_{\mathcal{R}} t$ if $s|_p = l\sigma$, $t = s[r\sigma]$ for some $l \to r \in \mathcal{R}$, some $p \in Pos(s)$ and some substitution σ. Instead of $s \to_{\mathcal{R}} t$ we also write $s \to t$ if $\mathcal{R}$ is clear from the context, and $s \to_{\mathcal{R},p} t$ or $s \to_p t$ to indicate the position of the *redex contraction*. *Critical pairs* and *critical peaks* of rewrite rules and systems are defined as usual. A TRS $\mathcal{R}$ is terminating, confluent, locally confluent, etc. if $\to$ has the respective property.

Next we need some additional notions and notations for context-sensitive rewriting. Given a signature $\mathcal{F}$, a mapping $\mu\colon \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if for all $f \in \mathcal{F}$, $\mu(f) \subseteq \{1, \ldots, ar(f)\}$ ([15]). The set $Pos^\mu(t)$ of *(μ-)replacing* or *active* positions of $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is given by $Pos^\mu(t) = \{\epsilon\}$, if $t \in \mathcal{V}$ or $t \in \mathcal{F}$ with $ar(f) = 0$, and $Pos^\mu(t) = \{\epsilon\} \cup \bigcup_{i \in \mu(root(t))} \{i.q \mid q \in Pos^\mu(t|_i)\}$, otherwise. The set $\overline{Pos^\mu(t)}$ of *non-(μ-)replacing* or *inactive* positions of $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is just the complement of the former, i.e., $\overline{Pos^\mu(t)} = Pos(t) \setminus Pos^\mu(t)$. Replacement maps are ordered by $\sqsubseteq$, with $\mu \sqsubseteq \mu'$ if for all $f \in \mathcal{F}$, $\mu(f) \subseteq \mu'(f)$. Thus, $\mu \sqsubseteq \mu'$ means that μ considers less positions than μ' (for reduction). If $\mu \sqsubseteq \mu'$, we also say that μ is *more restrictive than* μ'. A context-sensitive rewrite system (CSRS) is a pair $(\mathcal{R}, \mu)$ (also denoted by $\mathcal{R}_\mu$), where $\mathcal{R}$ is a TRS and μ is a replacement map (over the signature of $\mathcal{R}$). In *context-sensitive rewriting* (CSR [15]), only replacing redexes are contracted. s μ-rewrites to t, denoted by $s \to_{\mathcal{R},\mu} t$ or just $s \to_\mu t$, if $s \to_{\mathcal{R},p} t$ and $p \in Pos^\mu(t)$. Note that this means that $\to_{\mathcal{R},\mu} t$ is stable under substitutions, but in general not under contexts, i.e., the monotonicity property (of $\to_{\mathcal{R}}$) is lost. Slightly abusing notation, we denote rewriting at non-replacing positions by $\to_{\overline{\mu}}$, i.e., $s \to_{\overline{\mu}} t$ if $s \to_p t$ for some $p \in \overline{Pos^\mu(s)}$. Observe that in general $\to_\mu \cup \to_{\overline{\mu}}$ need not be a disjoint union. A simple example illustrating this is the TRS consisting of the two rules $f(x) \to f(b)$, $a \to b$ with $\mu(f) = \emptyset$. Here we have both $f(a) \to_\mu f(b)$ and $f(a) \to_{\overline{\mu}} f(b)$. A CSRS $\mathcal{R}_\mu$ is terminating, confluent, locally confluent etc. if $\to_\mu$ has the respective property. Finally, for a given CSRS $\mathcal{R}_\mu$, we will need certain replacement maps that are not very restrictive. More precisely, all positions of non-variable subterms in the left-hand sides of the rules should be replacing (this will guarantee in particular that rewrite steps that are involved in critical overlaps of $\mathcal{R}$ are also $\mathcal{R}_\mu$-steps). The *canonical* replacement map $\mu^{can}_{\mathcal{R}} : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is defined by $i \in \mu^{can}_{\mathcal{R}}(f) \iff \exists l \to r \in \mathcal{R}, p \in \mathcal{F}Pos(l) : root(l|_p) = f, p.i \in \mathcal{F}Pos(l)$. The set $CM_{\mathcal{R}}$ of replacement maps (for $\mathcal{R}$) that are at most as restrictive as $\mu^{can}_{\mathcal{R}}$ is given by $CM_{\mathcal{R}} = \{\mu \in M_{\mathcal{R}} \mid \mu^{can}_{\mathcal{R}} \sqsubseteq \mu\}$. The most liberal replacement map (for $\mathcal{R}$) $\mu_\top$ is the greatest element of $M_{\mathcal{R}}$, i.e., with $\mu_\top(f) = \{1, \ldots, ar(f)\}$ for all $f \in \mathcal{F}$.

3 Weakening the Termination Assumption in Newman's Lemma for Left-Linear Rewrite Systems

Suppose $\mathcal{R}$ is a locally confluent non-terminating TRS. In order to try to prove confluence of $\mathcal{R}$, we will impose context-sensitivity restrictions on $\mathcal{R}$, i.e., a replacement map μ such that $\to_{\mathcal{R},\mu}$ (hopefully) becomes terminating and such that confluence of $\to_{\mathcal{R},\mu}$ implies confluence of $\to_{\mathcal{R}}$ (hence of $\mathcal{R}$).

3.1 Confluence Via Context-Sensitive Confluence

For reasons that will become clear later on (cf. Lemmas 3, 4, 5) we need as general assumptions, besides termination of $\to_{\mathcal{R},\mu}$, that $\mathcal{R}$ is left-linear, and that μ is at most as restrictive as the canonical replacement map $\mu^{can}_{\mathcal{R}}$.

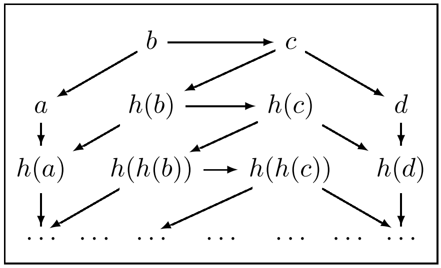
Remark 1. To see why requiring $\mu \in CM_{\mathcal{R}}$ makes sense, consider the rewrite system $\mathcal{R}$ consisting of the rules $c \to b$, $b \to c$ and $h(b) \to a$, together with $\mu(h) = \emptyset$. Here we have one critical overlap $h(c) \leftarrow h(b) \to a$ (which is joinable via $h(c) \to h(b) \to a$). However, we cannot deduce this using only $\to -\mu$-reduction (in $\mathcal{R}_m u$ this critical peak does not even exist, since $h(c) \leftarrow h(b)$ is not a $\to -\mu$-step; moreover, for joinability we need the $h(c) \to h(b)$ which is also not a $\to -\mu$-step).

- $\mathcal{R}$ is left-linear. (1)
- $\mu \in CM_{\mathcal{R}}$. (2)
- $\mathcal{R}_\mu$ is terminating. (3)

A first question is whether, for any such $\mathcal{R}_\mu$, its context-free version $\mathcal{R}$ is already (necessarily) confluent if $\mathcal{R}$ is locally confluent? Actually, when looking at examples in the literature, especially in papers on CSR, we have not found a single counterexample to this tempting conjecture. However, it turns out that conditions (1)–(3), together with local confluence of $\mathcal{R}$, are not yet sufficient for concluding confluence of $\mathcal{R}$. A simple counterexample is the following modified and extended version of the basic counterexample to the equivalence of local confluence and confluence (cf. [10]).

Example 3. Suppose the TRS $\mathcal{R}$ is given as follows, again with the relevant part of its reduction graph on the right.

(1) $b \to a$
(2) $b \to c$
(3) $c \to h(b)$
(4) $c \to d$
(5) $a \to h(a)$
(6) $d \to h(d)$



Clearly, $\mathcal{R}$ is not confluent, since for instance for $a \leftarrow b \to c \to d$ there is no common successor of a and d. However, $\mathcal{R}$ is obviously locally confluent. The two critical peaks $a \leftarrow b \to c$ and $h(b) \leftarrow c \to d$ are joinable via $a \to h(a) \leftarrow h(b) \leftarrow c$ and $h(b) \to h(c) \to h(d) \leftarrow c$, respectively. Moreover, choosing μ with $\mu(h) = \emptyset$ we have $\mu = \mu_{\mathcal{R}}^{can} \in CM_{\mathcal{R}}$. For this choice of μ, $\to_{\mathcal{R},\mu}$ is easily seen (and proved) to be terminating. However, what goes wrong in this example is the fact that $\to_\mu$ is not (locally) confluent. To see this, consider again the critical peaks. For $a \leftarrow_\mu b \to_\mu c$, reduction of a and c to a common successor is not possible by $\to_\mu$-steps only: $a \to_\mu h(a) \leftarrow_{\overline{\mu}} h(b) \leftarrow_\mu c$. Similarly, for $h(b) \leftarrow_\mu c \to_\mu d$ we only get $h(b) \to_{\overline{\mu}} h(c) \to_{\overline{\mu}} h(d) \leftarrow_\mu d$. In other words, although $\to$ (hence $\mathcal{R}$) is locally confluent, $\to_\mu$ is not. Thus we cannot argue using Newman's Lemma for $\to_\mu$.

Example 3 and Remark 1 suggest that in order to be able to use Newman's Lemma for the context-sensitive restriction $\to_\mu$ of $\to$ (in proofs of confluence of $\to$), we have to additionally require the following property of $\mathcal{R}_\mu$.

– Every critical peak $t_1 \leftarrow s \to t_2$ of $\mathcal{R}$ is joinable with $\to_\mu$-steps. (4)

For locally diverging μ-steps (i.e., of the form $t_1 \leftarrow_\mu s \to_\mu t_2$) that correspond to a variable overlap we also have to ensure $\to_\mu$-joinability.[2] Actually, for proof-technical reasons we need a stronger property. To describe this formally, we first need some additional terminology.

Definition 1 (level of subterms). *Given $\mathcal{R}_\mu$ and a term t, the level of a subterm $t|_p$ of t (and of p), denoted by $level(t,p)$ is the number of all non-replacing positions $q = q'.i$ on the path in t from ϵ to p with $i \notin \mu(root(t|_{q'}))$. Formally:*

$$level(t,p) = \begin{cases} 0, \mathit{if}\, p = \epsilon \\ level(t_i,p'), \mathit{if}\, p = i.p', t = f(t_1,\dots,t_n), i \in \mu(f) \\ 1 + level(t_i,p'), \mathit{if}\, p = i.p', t = f(t_1,\dots,t_n), i \notin \mu(f) \end{cases}$$

If $x \in Var(t)$ (for an arbitrary term t), we define – slightly abusing notation – $level(t,x) = \max\{level(t,p) \mid t|_p = x\}$ if $x \in Var(t)$, and $level(t,x) = 0$ if $x \notin Var(t)$.

[2] E.g., consider $f(x) \to g(x)$ and $a \to b$ with $\mu(f) = \{1\}$, $\mu(g) = \emptyset$. Then we have $f(b) \leftarrow_\mu f(a) \to_\mu g(a)$, which is $\to$-joinable via $f(b) \to g(b) \leftarrow g(a)$, but not $\to_\mu$-joinable, because the latter step is not a $\to_\mu$-step.

Intuitively, $level(t, p)$ describes 'the degree of how forbidden' it is to reduce the subterm $t|_p$ of t.

Definition 2 (level-decreasingness). *Given $\mathcal{R}_\mu$, a rule $l \to r \in \mathcal{R}$ is said to be* level-decreasing, *if for every variable $x \in Var(l)$ we have $level(l, x) \geq level(r, x)$. $\mathcal{R}_\mu$ is* level-decreasing *if every rule of $\mathcal{R}_\mu$ is level-decreasing.*

Note that ground TRSs are trivially level-decreasing.

Our last condition for the announced confluence criterion now reads as follows.

– $\mathcal{R}_\mu$ is level-decreasing. (5)

Definition 3 (level of reduction steps). *Given $\mathcal{R}_\mu$ we define binary relations $\to_{\mu,i}$ and $\to_{\mu,\leq i}$, for all $i \geq 0$ as follows:*

$$s \to_{\mu,i} t \iff s \to_{\mathcal{R},p} t, level(s, p) = i\ .$$

$$s \to_{\mu,\leq i} t \iff s \to_{\mu,k} t \text{ for some } k \leq i\ .$$

For the sake of readability, if μ is clear from the context, we also write $\to_i$ and $\to_{\leq i}$ instead of $\to_{\mu,i}$ and $\to_{\mu,\leq i}$, respectively.

From the definitions of $\to_{\overline{\mu}}$ and $\to_k$ it is obvious that $\to_{\overline{\mu}} = \bigcup_{k\geq 1} \to_k$ holds. Clearly, $s \to_k t$ ($s \to_{\leq k} t$) means that t can be obtained from s by contracting some redex at level k (at some level $\leq k$). And $s \to_{\overline{\mu}} t$ says that we can get t from s by contracting a redex $s|_p$ of s at some non-replacing position p of s (i.e., such that the level of $s|_p$ of s is equal to some $k \geq 1$).

Example 4 (Example 3 continued). Adding levels to reduction steps, we have here e.g. $a \leftarrow_0 b \to_0 c$ and $a \to_0 h(a) \leftarrow_1 h(b) \leftarrow_0 c$ as well as $h(b) \leftarrow_0 c \to_0 d$ and $h(b) \to_1 h(c) \to_1 h(d) \leftarrow_0 d$.

Proposition 1. *Given $\mathcal{R}_\mu$ the following properties hold:*

(a) $\to_k \subseteq \to_{\leq k} \subseteq \to_{\leq k+1}$ *for all* $k \geq 0$.
(b) $\to_\mu = \to_0$, $\to_{\overline{\mu}} = \bigcup_{k\geq 1} \to_k$.
(c) $\to = \bigcup_{k\geq 0} \to_{\leq k} = \bigcup_{k\geq 0} \to_k = \to_\mu \cup \to_{\overline{\mu}}$.

Proof. Straightforward by the respective definitions.

Lemma 1 (confluence criterion for $\to_\mu$, cf. [15]). *Let $\mathcal{R}_\mu$ be a CSRS satisfying (1), (3), (4) and (5). Then $\to_\mu$ is confluent.*

Proof. Local confluence of $\to_\mu$ can be easily directly shown by considering all cases of local divergences and exploiting properties (1), (4) and (5) which together with Newman's Lemma yields confluence because of (3). In particular, our conditions (1) and (5) imply the property that $\to_\mu$ *has left-homogeneous replacing variables* (cf. [15, Def. 5]) which guarantees that variable overlaps in $\mathcal{R}_\mu$ are uncritical.[3]

[3] Actually, condition (5) could still be weakened a bit here by requiring only that level-decreasingness need only hold for rules with variables of level 0 in left-hand sides. However, Lemma 1 is anyway a special case of [15, Theorem 5] and we will not use the above slight generalization later on.

Lemma 2 (extraction lemma). *Given $\mathcal{R}_\mu$, suppose $s \to_{\leq k+1} t$ at some level ≥ 1. Then s has the form $s = C[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$ where the p_i's are all minimal non-replacing positions in s, and $t = C[t_1, \ldots, t_n]_{p_1,\ldots,p_n}$, with $s_i \to_{\leq k} t_i$ for some $i \in \{1, \ldots, n\}$ and $s_j = t_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$.*

Proof. Straightforward by definition of $\to_{\leq i}$.

The next result gives conditions under which reduction sequences can be rearranged such that $\to_0$-steps are done first.[4]

Lemma 3 (exchange lemma). *Let $\mathcal{R}_\mu$ be given with (1), (2) and (5), i.e., such that $\mathcal{R}$ is left-linear, $\mu \in CM_\mathcal{R}$ and $\mathcal{R}_\mu$ is level-decreasing. Then the following rearrangement property holds for every $k \geq 0$:*[5] $\to_{\leq k} \cdot \to_0 \subseteq \to_0 \cdot \to^*_{\leq k}$.

Proof. First we observe that for $k = 0$ the inclusion holds trivially. Thus suppose $k \geq 1$. Furthermore let $s \to_{k'} t$ at position p with $0 \leq k' \leq k$ and $t \to_0 u$ at position q. Again, if $k' = 0$, the inclusion holds trivially, hence we may assume $k' \geq 1$. Now, p and q must be distinct (otherwise we have a contradiction, because a position cannot be both replacing and non-replacing). We distinguish the following cases.

(a) $p \parallel q$: Then the two steps commute and we get $s \to_0 s[u|_q]_q \to_{k'} s[u|_q][t|_p]_p = u$.
(b) $p < q$: This is impossible, since $s \to_{k'} t$ at p with $k' \geq 1$ implies $p \in \overline{Pos_\mu(s)}$ and $p \in \overline{Pos_\mu(t)}$, hence also $q \in \overline{Pos_\mu(t)}$ (because of $p < q$). But on the other hand, due to $t \to_0 u$ at position q, we have $q \in Pos_\mu(t)$, hence a contradiction.
(c) $p > q$: In this case we have $s = s[l\sigma]_p \to_{k'} s[r\sigma]_p = t = t[l'\tau]_q \to_0 t[r'\tau]_q = u$ (for some $l \to r, l' \to r' \in \mathcal{R}$ and substitutions σ, τ). If we can show that position p in t is below the pattern of l' in $t[l'\tau]_q$ we are done, because then we have – by left-linearity of $\mathcal{R}$ – $s = s[l'\tau']_q \to_{k'} t = t[l'\tau]_q \to_0 t[r'\tau]_q = u$ (for some rewrite rule $l' \to r' \in \mathcal{R}$ and some substitution τ') which commutes via $s = s[l'\tau']_q \to_0 s[r'\tau']_q \to^*_{\leq k'} s[r'\tau]_q = t[r'\tau]_q = u$. Note that for the reduction $s[r'\tau']_q \to^*_{\leq k'} s[r'\tau]_q$ in the variable parts of the right-hand side r' of $l' \to r'$, more precisely for the bound k on the level of the reduction steps) we need assumption (5). Now suppose p in t were in the pattern of l'. This would imply by (2) that $p \in Pos_\mu(t)$ and $p \in Pos_\mu(s)$, hence $k' = 0$. But this is a contradiction to our assumption $k' \geq 1$ from above. Hence we are done.

The next commutation result will be needed to prove a kind of backward preservation of $\to_\mu$-normal forms along non-replacing reduction steps (cf. Lemma 5).

[4] This is similar to *standardization* in left-linear TRSs (cf. e.g. [2]), except for the fact that we need additional information about the individual steps.

[5] Actually, from the proof it is clear that we even have the stronger statement $\to_{\leq k} \cdot \to_0 \subseteq \to_0 \cdot \mathrel{+\!\!\!\!\!\mapsto}_{\leq k}$ where $\mathrel{+\!\!\!\!\!\mapsto}_{\leq k}$ denotes a parallel reduction step with each contraction being at a level at most k. However, we don't need this stronger version later on.

Lemma 4 (commutation lemma). *Let $\mathcal{R}_\mu$ be given with (1), (2) and (5), i.e., such that $\mathcal{R}$ is left-linear, μ is at most as restrictive as $\mu_\mathcal{R}^{can}$ and $\mathcal{R}_\mu$ is level-decreasing. Then the following commutation property holds for every $k \geq 1$:*[6]

Proof. Suppose $t \leftarrow_k s \rightarrow_0 u$ at positions p and q, respectively. Hence, $t = s[t|_p]_p \leftarrow_k s[s|_p]_p = s = s[s|_q]_q \rightarrow_0 s[u|_q]_q = u$. We proceed by case analysis.

(a) $p \leq q$: Due to $k \geq 1$ we have $p \in \overline{Pos_\mu(s)}$ and thus also $q \in \overline{Pos_\mu(s)}$. However, $s \rightarrow_0 u$ at q implies $q \in Pos_\mu(s)$, hence a contradiction.

(b) $p \parallel q$: In this case the reductions commute as usual: $t = s[t|_p]_p \rightarrow_0 s[t|_p][u|_q]_q \leftarrow_k s[u|_q]_q = u$ where the left step is at position q and the right one at p.

(c) $p > q$: Because of assumption (2), $\mu \in CM_\mathcal{R}$, and since $k \geq 1$, this case corresponds to a variable overlap. Moreover, due to assumption (1), left-linearity of $\mathcal{R}$, we have $t = s[l\sigma']_q \leftarrow_k s = s[l\sigma]_q \rightarrow_0 s[r\sigma]_q = u$ for some $l \rightarrow r \in \mathcal{R}$ and substitutions σ, σ'. Hence, t and u are joinable via $t = s[l\sigma'] \rightarrow_0 s[r\sigma'] \leftarrow^*_{\leq k} s[r\sigma]_q = u$. Note that the (parallel) reduction $u = s[r\sigma]_q \rightarrow^* s[r\sigma']$ is of level at most k because of assumption (5), i.e., level-decreasingness of $\mathcal{R}$.

The next lemma states conditions under which reduction of some term to a $\rightarrow_\mu$-normal form implies that the original term is already a $\rightarrow_\mu$-normal form.

Lemma 5 (a condition for backward invariance of $\rightarrow_\mu$-normal forms). *Let $\mathcal{R}_\mu$ be given with (1), (2) and (5), i.e., such that $\mathcal{R}$ is left-linear and level-decreasing, and $\mu \in CM_\mathcal{R}$. Then $s \rightarrow^*_{\overline{\mu}} t \in NF(\rightarrow_\mu)$ implies $s \in NF(\rightarrow_\mu)$.*

Proof. We prove the statement for one step, i.e., $s \rightarrow_{\overline{\mu}} t \in \mathrm{NF}(\rightarrow_\mu)$ implies $s \in \mathrm{NF}(\rightarrow_\mu)$. The result then follows by transitivity (that is by induction on the number of steps in $s \rightarrow^*_{\overline{\mu}} t$). Suppose, for a proof by contradiction, that $s \notin \mathrm{NF}(\rightarrow_\mu)$. Hence, there exists some s' with $s \rightarrow_0 s'$. Moreover, $s \rightarrow_{\overline{\mu}} t$ means $s \rightarrow_{k_1} t$ for some $k_1 \geq 1$. By Commutation Lemma 4 this implies that there exists some t' with $s' \rightarrow^*_{\leq k_1} t' \leftarrow_0 t$. But this is a contradiction to $t \in \mathrm{NF}(\rightarrow_\mu)$.[7]

3.2 Main Results

Now we are ready to prove the main results of the paper. The first one is a 'level confluence' criterion for CSRSs.

Theorem 1 (level confluence criterion / technical key lemma). *Let $\mathcal{R}_\mu$ be given satisfying (1)-(5). Then $\rightarrow_{\leq k}$ is confluent for every $k \geq 0$.*

Proof. We prove confluence of $\rightarrow_{\leq k}$ for all k by induction on k.

(o) Base case $k = 0$: Confluence of $\rightarrow_{\leq 0} = \rightarrow_0 = \rightarrow_\mu$ follows from Lemma 1.

6 Again, from the proof it follows that even the stronger property $\leftarrow_{\leq k} \cdot \rightarrow_0 \subseteq \rightarrow_0 \cdot \mathrel{\leftarrow\!\!\!\!\!\Vert\!\!-}_{\leq k}$ holds.

7 Note that the assumptions (1), (2), (5) in the lemma are needed to enable applicability of Lemma 4.

(i) Induction step "$k \implies k+1$": Consider an arbitrary $\to_{\leq k+1}$-divergence $t_1' \leftarrow^*_{\leq k+1} s \to^*_{\leq k+1} t_2'$. Due to assumption (3) we may $\to_\mu$-normalize t_1' and t_2' yielding $D_1 : s \to^*_{\leq k+1} t_1' \to^*_0 t_1 \in \mathrm{NF}(\to_\mu)$ and $D_2 : s \to^*_{\leq k+1} t_2' \to^*_0 t_2 \in \mathrm{NF}(\to_\mu)$. Now, by repeated application of Exchange Lemma 3 and termination of $\to_\mu$ we can rearrange these derivations into $D_1' : s \to^*_0 s_1 \to^*_{\leq k+1} t_1 \in \mathrm{NF}(\to_\mu)$ and $D_2' : s \to^*_0 s_2 \to^*_{\leq k+1} t_2 \in \mathrm{NF}(\to_\mu)$, where the $\to_{\leq k+1}$-reduction steps are all non-replacing, i.e., of level ≥ 1. From Lemma 5 we infer that $s_1, s_2 \in \mathrm{NF}(\to_\mu)$. Together with confluence of $\to_0$ (see base case) this implies $s_1 = s_2$. Hence the divergence diagram collapses to $t_1 \leftarrow^*_{\leq k+1} s_1 = s_2 \to_{\leq k+1} t_2$. Now, let $s' = s_1 = s_2$. Repeated applications of the Extraction Lemma 2 yield $s' = C[u_1, \ldots, u_m] \to^*_{\leq k+1} C[w_1, \ldots, w_m] = t_2$ and $s' = C[u_1, \ldots, u_m] \to^*_{\leq k+1} C[v_1, \ldots, v_m] = t_1$ for some context $C[\ldots]$ such that $v_i \leftarrow^*_{\leq k} u_i \to^*_{\leq k} w_i$ for $1 \leq i \leq m$. Applying the induction hypothesis (for k) to all i, $1 \leq i \leq m$, we conclude that there exist $\overline{u_i}$ for all i with $v_1 \to^*_{\leq k} \overline{u_i} \leftarrow^*_{\leq k} w_i$. Putting back these reductions in the non-extracted version, we get $t_1 = C[v_1, \ldots, v_m] \to^*_{\leq k+1} C[\overline{u_1}, \ldots \overline{u_m}] \leftarrow_{\leq k+1} C[w_1, \ldots, w_m] = t_2$. Hence, $\overline{s'} = C[\overline{u_1}, \ldots \overline{u_m}]$ is a common $\to_{\leq k+1}$-reduct of both t_1 and t_2 as desired, and we are done.

As a consequence of this level confluence criterion we thus obtain our main result, a generalized version of Newman's Lemma for left-linear TRSs.[8]

Theorem 2 (main result). *Let $\mathcal{R}$ be a TRS and μ be a replacement map on the signature of $\mathcal{R}$ such that (1)-(5) are satisfied, i.e., such that $\mathcal{R}$ is left-linear, $\mu \in CM_{\mathcal{R}}$, $\mathcal{R}_\mu$ is terminating and level-decreasing and all critical pairs of $\mathcal{R}$ are $\mathcal{R}_\mu$-joinable. Then $\mathcal{R}$ is confluent.*

Proof. Due to Proposition 1(c) this is an immediate corollary of Theorem 1.

Note that Newman's Lemma (for left-linear TRSs) is obtained from Theorem 2 as a special case, namely by taking – for some given $\mathcal{R}$ – μ to be the most liberal replacement map $\mu = \mu_\top$. This choice of μ clearly implies (2) and (5), and also that $\to = \to_\mu$, hence termination of $\to$ is equivalent to termination of $\to_\mu$. Actually, Theorem 2 properly generalizes Newman's Lemma (for left-linear TRSs) since there are cases (cf. e.g. Examples 1, 2) where the former is applicable, but not the latter because the system (as a TRS) is not terminating.

3.3 Examples and Comparison

Let us first reconsider our Examples 3, 1 and 2. In Example 3, $\mathcal{R}$ with μ as specified satisfies all preconditions of Theorem 2 except (4). Hence the latter is not (and should not be) applicable. In Example 1, choosing $\mu^{can}_{\mathcal{R}}$, i.e., with $\mu(f) = \mu(h) = \emptyset$, conditions (1)-(5) are all satisfied as is easily verified. In particular, termination of $\mathcal{R}_\mu$ is not difficult to prove by some of the methods proposed in the literature (cf. e.g. [3], [9], [5], [18]). Observe that, when choosing some $\mu \in CM_{\mathcal{R}}$, in order to ensure termination of $\mathcal{R}_\mu$ we must obviously have

[8] more precisely, for the abstract reduction systems induced by left-linear TRSs.

$\mu(f) = \emptyset$ (because otherwise this entails non-termination of $\mathcal{R}_\mu$). Hence, for h the only choice is $\mu(h) = \emptyset$. Otherwise, (5) would be violated.

In Example 2, choosing $\mu = \mu^{can}_{\mathcal{R}}$ (hence $\mu(:) = \mu(s) = \mu(hd) = \emptyset$, $\mu(\mathsf{inc}) = \mu(\mathsf{tl}) = \{1\}$) it is easy to verify that (1)-(5) do indeed hold. Hence, confluence of the TRS follows by Theorem 2.

When applying Theorem 2, there is a certain flexibility in the sense that the parameter μ may be chosen differently. We require $\mu \in CM_{\mathcal{R}}$, but not necessarily $\mu = \mu^{can}_{\mathcal{R}}$ as in the above examples. In fact, in certain cases the canonical $\mu^{can}_{\mathcal{R}}$ need not be a good choice, whereas a more liberal μ can work, cf. conditions (3)-(5).

Example 5 (Example 1 modified). Consider $\mathcal{R}$ consisting of the rules (1) $\{g(a) \to f(g(a))$, (2′) $g(b) \to c(a)$, (3) $a \to b$, (4) $f(x) \to h(x)$ and (5′) $h(x) \to c(b)$. Here, choosing $\mu^{can}_{\mathcal{R}}$ we cannot apply Theorem 2 to infer confluence, since property (4) is violated. However, choosing $\mu \in CM_{\mathcal{R}}$ with $\mu(g) = \mu(c) = \{1\}$ and $\mu(i) = \emptyset$ for all other function symbols i, Theorem 2 is applicable and shows indeed confluence of $\mathcal{R}$.

Comparing our new confluence criterion of Theorem 2 with other known criteria (or decision procedures, respectively) for (possibly non-terminating) TRSs, cf. in particular those of [26], [11], [8], [23], [24], [21], [7], [6] it turns out to be incomparable w.r.t. all of them. This is easy to show by exhibiting examples where our criterion is applicable whereas the other ones are not, and vice versa. This incomparability is not really surprising, because all other confluence criteria above do not rely on a (partial) termination assumption, whereas our criterion crucially does.

3.4 Discussion

Let us first discuss the preconditions for applying Theorem 2, namely, (1)-(5), the effectiveness of using it for confluence proofs, and the inherent limitations of this confluence criterion. Then we will see how these latter limitations naturally lead to some interesting open problems.

Recall that applicability of Theorem 2 requires the following properties:

- $\mathcal{R}$ is left-linear. (1)
- $\mu \in CM_{\mathcal{R}}$. (2)
- $\mathcal{R}_\mu$ is terminating. (3)
- Every critical peak $t_1 \leftarrow s \to t_2$ of $\mathcal{R}$ is joinable with $\to_\mu$-steps. (4)
- $\mathcal{R}_\mu$ is level-decreasing. (5)

Note that checking (1), (2) (and (4) provided (3) holds) is easy. Furthermore, in (2), there are only finitely many possibilities for choosing some $\mu \in CM_{\mathcal{R}}$ (for finite $\mathcal{R}$), hence the search for an appropriate $\mu \in CM_{\mathcal{R}}$ can also be automated. Proving termination of $\mathcal{R}_\mu$, i.e., (3), is of course undecidable in general, but nowadays numerous powerful methods and tools exist for such context-sensitive termination proofs, cf. e.g. [5], [18]. Thus, the applicability of the confluence criterion of Theorem 2 is effectively decidable, provided that (3) holds, for some $\mu \in CM_{\mathcal{R}}$.

Having a closer look at the preconditions, termination of $\rightarrow_\mu$ (3) is crucial to get a Newman style confluence criterion. The conditions (1) left-linearity of $\mathcal{R}$, and (2) $\mu \in CM_{\mathcal{R}}$, are essential for several important lemmas (especially Lemmas 3, 4 and 5) used in the proof of the main Theorem 2. Condition (4), at least in combination with (2), seems to be unavoidable to infer confluence of $\rightarrow_\mu$ using (1). The only condition which appears to be less clear and intuitive is level-decreasingness of $\rightarrow_\mu$ (5). Besides termination of $\rightarrow_\mu$, this condition is the most restrictive application condition in practical examples. It would be nice if it could be dropped or weakened. Currently we do not know any counterexample to the modified (generalized) statement of Theorem 2 where condition (5) is dropped. On the other hand, the proof of Theorem 2 (via the "level confluence" criterion of Theorem 1) as well as Lemmas 3 and 4 heavily rely on this condition. Hence we have the following

Open Problem 1 (necessity of level-decreasingness?). *Does the statement of Theorem 2 also hold if precondition (5) is omitted? In other words, is any TRS $\mathcal{R}$ s.t*

- *$\mathcal{R}$ is left-linear,*
- *$\mu \in CM_{\mathcal{R}}$,*
- *$\mathcal{R}_\mu$ is terminating, and*
- *every critical pair of $\mathcal{R}$ is $\mathcal{R}_\mu$-joinable*

necessarily confluent?

A positive solution to this open problem would be particularly nice, since there are numerous examples (cf. e.g. the literature on CSR) where level-decreasingness is not satisfied. A basic one is the following (cf. e.g.[17]).

Example 6. Consider the $\mathcal{R}$ given by

$$\begin{aligned} \mathsf{from}(x) &\rightarrow x : \mathsf{from}(\mathsf{s}(x)) \\ \mathsf{sel}(0, y : z) &\rightarrow y \\ \mathsf{sel}(\mathsf{s}(x), y : z) &\rightarrow \mathsf{sel}(x, z) \end{aligned}$$

where from models a kind of parameterized version of generating infinite lists of natural numbers (cf. Example 2 for a non-parameterized version), and sel serves for extracting elements from a list. This system is clearly non-terminating as a TRS, but becomes terminating as $\mathcal{R}_\mu$ with e.g. $\mu = \mu_{\mathcal{R}}^{can}$ (hence, with $\mu(:) = \{1\}$). Conditions (1)-(4) of Theorem 2 are easily verified, but (5) is violated, since the first rule is not level-decreasing. Hence Theorem 2 cannot be applied, although $\mathcal{R}$ is indeed confluent, simply because it is orthogonal ([26]).

Another issue that is related to (the preconditions and the statement of) Theorem 2 is the following which we will only touch (cf. e.g. [2] for more details and background). Let us reconsider the introductory counterexample 3 that we used to motivate the requirement that all critical pairs should be $\to_\mu$-joinable. In the example this was not the case, and $\mathcal{R}$ was not confluent, because a and d with $a \leftarrow b \to c \to d$ did not have a common reduct. But, interestingly, it turns out that, when switching from *finitary* rewriting and confluence to *infinitary* rewriting and confluence (cf. e.g. [4], [12], [16], [2]), then Example 3 behaves nicely, in the sense that $\mathcal{R}$ is *infinitary confluent* (ω-confluent). Intuitively this is easy to see since the *infinitary normal forms* of both a and d are h^ω, hence the system is indeed ω-confluent. A tempting conjecture in this direction which we state as open problem is the following.

Open Problem 2 (criterion for ω-confluence?). *Is any left-linear, non-collapsing, locally confluent TRS $\mathcal{R}$, with $\mu \in CM_\mathcal{R}$ and $\to_{\mathcal{R},\mu}$ terminating, necessarily ω-confluent?*

Orthogonal systems are known to be ω-confluent (for *strongly converging reductions*) provided they are non-collapsing (cf. [12], [2]). The typical counterexample showing that the non-collapsingness in this result cannot be dropped is as follows ([12]): Let $\mathcal{R}$ consist of the rules $a(x) \to x$, $b(x) \to x$ and $c \to a(b(c))$. Then we get the reductions

$$c \to a(b(c)) \to a(c) \to a(a(b(c))) \to a(a(c)) \to^\omega a^\omega$$

$$c \to a(b(c)) \to b(c) \to b(a(b(c))) \to b(b(c)) \to^\omega b^\omega ,$$

hence $a^\omega \leftarrow^\omega c \to^\omega b^\omega$, but there is no term t with $a^\omega \to^{\leq\omega} t \leftarrow^{\leq\omega} b^\omega$. Now, $\mathcal{R}$ is clearly non-terminating, but $\mathcal{R}_\mu$ is also non-terminating for any μ here. This phenomenon also applies to other collapsing counterexamples (to ω-confluence) in [12]. If instead of the above system we consider $\mathcal{R}$ consisting of $a(x) \to x$, $b(x) \to x$ and $c \to d(a(b(c)))$, then $\mathcal{R}_\mu$ becomes obviously terminating e.g. for $\mu \in CM_\mathcal{R}$ with $\mu(a) = \mu(b) = \{1\}$, $\mu(d) = \emptyset$. However, for $(da)^\omega \leftarrow^\omega c \to^\omega (db)^\omega$ we can now find a common reduct (in infinitary rewriting): $(da)^\omega \to^\omega d^\omega \leftarrow^\omega (db)^\omega$. Hence, in the above Open Problem 2 it could even make sense to generalize the statement by omitting the non-collapsing requirement.

If the answer to the above open problem were "yes", then this would be a nice way to prove ω-confluence. To the best of our knowledge it would also be the first confluence criterion for non-orthogonal infinitary rewrite systems.

4 Conclusion

To conclude, we have presented a new confluence criterion for (possibly non-terminating) left-linear TRSs which properly generalizes Newman's Lemma (for left-linear TRSs). The criterion is We think that not only the result itself is interesting, but also the proof technique employed that uses the more general framework of context-sensitive rewriting to finally derive a result about ordinary (context-free) rewriting. Methodologically, the approach strongly differs from related confluence criteria. It is neither based on critical pair criteria nor on modularity properties, but rather on a combination of Newman's Lemma (for a terminating sub-relation of the rewrite relation, that is not induced by a sub-TRS) with a level-based approach that exploits rearrangement and commutation properties.

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