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Abstract: We investigate the question whether semi-completeness, i.e., weak termination plus confluence, of term rewriting systems is preserved under the normalization of right-hand sides. We give a simple counterexample showing that in general this transformation neither preserves weak termination nor (local) confluence. Moreover we present two conditions which are sufficient for the preservation of semi-completeness. In particular, we show that (almost) orthogonal systems enjoy this preservation property.

Key-words: Theory of Computation, term rewriting, confluence, termination, semi-completeness.

(Résumé : tsvp)

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Sur la semi-complétude de systèmes de réécriture

Résumé : Nous étudions le problème de savoir si la semi-complétude, c'est-à-dire, la terminaison faible et la confluence, de systèmes de réécriture est préservée par la normalisation des membres droits. Nous donnons un contre-exemple simple qui montre qu'en général cette transformation ne préserve ni la terminaison faible, ni la confluence (locale). Puis nous proposons deux conditions suffisantes permettant de préserver la semi-complétude. En particulier, nous démontrons que les systèmes (presque) orthogonaux possèdent cette propriété de préservation.

Mots-clé : Théorie du calcul, réécriture, confluence, terminaison, semi-complétude.

1 Introduction

We assume familiarity with the basic no(ta)tions, terminology and theory of term rewriting (cf. e.g. [DJ90], [Klo92]) but recall some no(ta)tions for the sake of readability. The set of terms over some given signature $\mathcal{F}$ and some (disjoint) countably infinite set $\mathcal{V}$ of variables is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. A term rewriting system (TRS) is a pair $(\mathcal{F},\mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules over $\mathcal{F}$, i.e., pairs (l,r) — also denoted by $l \to r$ — with $l, r \in \mathcal{T}(\mathcal{F},\mathcal{V})$. Here we require that l is not a variable, and that all variables of r occur in l. Instead of $(\mathcal{F},\mathcal{R})$ we also write $\mathcal{R}$ if $\mathcal{F}$ is clear from the context or irrelevant. The rewrite relation $\to$ induced by $\mathcal{R}$ (and $\mathcal{R}$ itself) is said to be *confluent*, if ${}^*\!\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\!\leftarrow$, *locally confluent* if $\leftarrow \circ \to \subseteq \to^* \circ {}^*\!\!\leftarrow$, *strongly confluent* if $\leftarrow \circ \to \subseteq \to^= \circ {}^*\!\!\leftarrow$ ($\to^=$ denotes the reflexive closure of $\to$). $\to$ (and $\mathcal{R}$) is *(strongly) terminating* or *strongly normalizing* if there is no infinite reduction sequence $s_1 \to s_2 \to s_3 \to \ldots$. It is *weakly terminating* (or *weakly normalizing*) if every term s has a normal form, i.e., if $s \to^* s'$ with s' irreducible. A term s is said to be weakly ($\mathcal{R}$-)terminating if it has an ($\mathcal{R}$-) normal form, and ($\mathcal{R}$-)terminating if all reduction sequences issuing from s are finite. A TRS is *complete* or *convergent* if it is confluent and terminating, and *semi-complete* if it is confluent and weakly terminating. If a term s has a unique $\mathcal{R}$-normal form the latter is denoted by $s{\downarrow_\mathcal{R}}$. A term is *root-reducible* if it is a *redex*, i.e., an instance $l\sigma$ of the left-hand side of some rule $l \to r$.[1]

It is well-known that any complete TRS $\mathcal{R}$ can be simplified into an irreducible, complete TRS $\mathcal{R}_{irr}$ which induces the same equational theory ([Mét83]). For instance, this can be done by first normalizing all right-hand sides, and then omitting all rules rules with a left-hand side that is reducible by the remaining rules (cf. e.g. [Klo92]):

(a) $\mathcal{R}{\downarrow} := \{l \to r{\downarrow_\mathcal{R}} \mid l \to r \in \mathcal{R}\}$ (**right-normalization**).

(b) $\mathcal{R}_{irr} := \{l \to r \in \mathcal{R}{\downarrow} \mid l \text{ is } (\mathcal{R}{\downarrow}\backslash\{l \to r\})\text{-irreducible}\}$ (**deletion of left-reducible rules**).[2]

If $\mathcal{R}$ is complete then $\mathcal{R}{\downarrow}$ as well as $\mathcal{R}_{irr}$ are also complete, and their induced equational theories coincide: $\leftrightarrow^*_\mathcal{R} = \leftrightarrow^*_{\mathcal{R}\downarrow} = \leftrightarrow^*_{\mathcal{R}_{irr}}$. These properties are usually exploited in typical Knuth-Bendix like completion procedures for equational specifications ([KB70]).

Right-normalization, i.e., step (a) above, is also well-defined for semi-complete TRSs, because for such systems every term has a unique normal form. Hence, it is quite natural to ask the following basic questions due to Kahrs ([Kah95]):

(1) Does right-normalization preserve semi-completeness?

(2) Does right-normalization of semi-complete TRSs preserve the equational theory?

As mentioned by Kahrs in [Kah95] it is easy to see that the preservation of weak termination would be sufficient to settle both questions in the positive (see Lemma 2.1 below). Moreover,

[1] More formally, a redex (in some term s) can be defined by the pair $(p, l \to r)$ indicating that the subterm $s/p = l\sigma$ (for some σ) in s is to be rewritten or *contracted* into $r\sigma$ yielding the result $s[p \leftarrow r\sigma]$.

[2] Note that as usual equality of rewrite rules is interpreted as equality modulo variable renamings!

if the answer to question (2) is negative, then one could enforce preservation of the equational theory by 'adding the potentially missing part', i.e., by defining

$$\mathcal{R}{\downarrow}_{ext} \;:=\; \mathcal{R}{\downarrow} \cup \{ r \to r{\downarrow}_{\mathcal{R}} \mid l \to r \in \mathcal{R}, r \neq r{\downarrow}_{\mathcal{R}} \} \,.$$

If $\mathcal{R}$ is semi-complete, the equational theories of $\mathcal{R}$ and $\mathcal{R}{\downarrow}_{ext}$ clearly coincide and both systems have the same sets of normal forms, but the question

(1a) Does semi-completeness of $\mathcal{R}$ imply semi-completeness of $\mathcal{R}{\downarrow}_{ext}$?

remains non-trivial ([Kah95]). In view of the above discussion about complete systems we add another question:

(3) Does deletion of left-reducible rules preserve semi-completeness and the equational theory of right-normalized semi-complete TRSs?

Question (1) is certainly the most interesting one from a computational point of view. It is closely related to the general question which reduction strategies are normalizing for a given TRS. In many examples right-normalization of a semi-complete, but non-terminating TRS yields again a semi-complete or even complete system. Somehow surprisingly, we will show in Section 2 that in general both weak termination and (even local) confluence may get lost when right-normalizing semi-complete systems. Yet, in Section 3 we shall identify two classes of TRSs for which semi-completeness is preserved under right-normalization. Questions (1a) and (3) will also be settled in the negative. However, (2) remains open. We shall exhibit necessary properties of potentially existing counterexamples for (2).

2 A Simple Counterexample

Subsequently, we will use the notations introduced above. In particular, $\mathcal{R}{\downarrow}$ will always denote the right-normalized version of a semi-complete TRS $\mathcal{R}$. Let us start with some simple facts about right-normalization.

Lemma 2.1 *Let $\mathcal{R}$ be semi-complete. Then the following properties hold:*

(1) A term is $\mathcal{R}$-reducible iff it is $\mathcal{R}{\downarrow}$-reducible.

(2) If $\mathcal{R}{\downarrow}$ is weakly terminating, then $\mathcal{R}{\downarrow}$ is confluent (hence semi-complete) and $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}\downarrow}$.

Proof: Straightforward. ■

The following answers question (1) in the negative.

Theorem 2.2 *There exists a semi-complete TRS $\mathcal{R}$ such that $\mathcal{R}{\downarrow}$ is neither weakly terminating nor locally confluent.*

$$\begin{tikzcd} h(f(a)) \arrow[r] \arrow[d] & h(g(a)) \arrow[r] \arrow[d] & c \\ h(f(b)) \arrow[r] & h(g(b)) \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} h(f(a)) \arrow[r] \arrow[d] & h(g(a)) \arrow[r] \arrow[d] & c \\ h(f(b)) \arrow[r, shift left] & h(g(b)) \arrow[l, shift left] & \end{tikzcd}$$

Figure 1: reductions in $\mathcal{R}$ (left) and in $\mathcal{R}\!\downarrow$ (right)

Proof: Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} a \to b \\ g(b) \to f(a) \\ h(f(x)) \to h(g(x)) \\ h(g(a)) \to c \end{array} \right\}$$

Obviously, right-normalization yields

$$\mathcal{R}\!\downarrow = \left\{ \begin{array}{c} a \to b \\ g(b) \to f(b) \\ h(f(x)) \to h(g(x)) \\ h(g(a)) \to c \end{array} \right\}$$

We have to show

(1) $\mathcal{R}$ is confluent.

(2) $\mathcal{R}$ is weakly terminating.

(3) $\mathcal{R}\!\downarrow$ is not locally confluent.

(4) $\mathcal{R}\!\downarrow$ is not weakly terminating.

The relevant parts of the reduction graphs of $\mathcal{R}$ and $\mathcal{R}\!\downarrow$ are displayed in Figure 1.
For (1), it suffices to show confluence of $\to_{\widehat{\mathcal{R}}}$ for any $\widehat{\mathcal{R}}$ with $(*)$ $\to_{\mathcal{R}} \subseteq \to_{\widehat{\mathcal{R}}} \subseteq \to_{\mathcal{R}}^*$. Taking for instance $\widehat{\mathcal{R}} := \mathcal{R} \cup \{h(g(b)) \to c, h(f(a)) \to c, h(f(b)) \to c\}$, we have $l \to_{\mathcal{R}}^{+} r$ for all $l \to r \in \widehat{\mathcal{R}} \setminus \mathcal{R}$ as is easily verified. Hence, $(*)$ holds for $\widehat{\mathcal{R}}$. Moreover, $\widehat{\mathcal{R}}$ is strongly closed ([Hue80]), i.e., for every critical pair $\langle s, t \rangle$ there exist terms u, v such that $s \to^* u \ {}^{=}\!\!\leftarrow t$ and $s \to^{=} v \ {}^{*}\!\!\leftarrow t$. Furthermore, $\widehat{\mathcal{R}}$ is left-linear and right-linear, hence by [Hue80, Lemma 3.2] it is (strongly) confluent. This implies confluence of $\mathcal{R}$ as desired.
For (2) we show that all terms t are weakly ($\mathcal{R}$-)terminating, by structural induction and case analysis. In the base case, if t is a variable or a constant (from $\{a, b, c\}$), it obviously has a normal form. In the induction step we have the following cases. If $t = f(t')$ then

by induction hypothesis t' has some normal form $\widehat{t'}$. Since there is no rule in $\mathcal{R}$ with f as left-hand side root symbol, $f(\widehat{t'})$ is a normal form of $f(t)$. If $t = g(t')$ then by induction hypothesis t' has some normal form $\widehat{t'}$. If $\widehat{t'} = b$ then $t = g(t')$ reduces to the normal form $f(b)$. Otherwise, i.e., if $\widehat{t'} \neq b$, then $g(\widehat{t'})$ is a normal form of t. Finally, if $t = h(t')$ then by induction hypothesis t' has some normal form $\widehat{t'}$. If $\widehat{t'}$ is of the form $\widehat{t'} = f(\widetilde{t'})$ then, if $\widetilde{t'} = b$ then t reduces to the normal form c otherwise t reduces to the normal form $h(g(\widetilde{t'}))$. Otherwise, i.e., if $\widehat{t'} \neq f(\widetilde{t'})$ then t reduces to $h(\widehat{t'})$ which is already in normal form (note that $\widehat{t'} = g(a)$ is impossible since $\widehat{t'}$ is a normal form).
For (3) we observe that in $\mathcal{R}{\downarrow}$ we have $h(g(b)) \leftarrow h(g(a)) \rightarrow c$. Here, c is irreducible and the only reductions from $h(g(b))$ are of the form $h(g(b)) \rightarrow h(f(b)) \rightarrow h(g(b)) \rightarrow \ldots$. Hence, $\mathcal{R}{\downarrow}$ is not locally confluent. Moreover, (4) holds since for instance $h(g(b))$ does not have a normal form in $\mathcal{R}{\downarrow}$. ■

A slight modification of the above counterexample shows that also for the special case of *string rewriting systems* (cf. e.g. [BO93]) semi-completeness is not preserved under right-normalization. To this end one may simply turn the constants a, b, c in $\mathcal{R}$ above into unary function symbols and add some variable x in the corresponding rules. This yields the system $\{a(x) \rightarrow b(x), g(b(x)) \rightarrow f(a(x)), h(f(x)) \rightarrow h(g(x)), h(g(a(x))) \rightarrow c(x)\}$ which corresponds (in the usual way) to the string rewriting system $\{a \rightarrow b, gb \rightarrow fa, hf \rightarrow hg, hga \rightarrow c\}$.

Notice, that the above counterexample also constitutes a counterexample for (1a), since

$$\mathcal{R}{\downarrow}_{ext} = \mathcal{R}{\downarrow} \cup \{r \rightarrow r{\downarrow}_{\mathcal{R}} \mid l \rightarrow r \in \mathcal{R}, r \neq r{\downarrow}_{\mathcal{R}}\} = \left\{ \begin{array}{c} a \rightarrow b \\ g(b) \rightarrow f(b) \\ h(f(x)) \rightarrow h(g(x)) \\ h(g(a)) \rightarrow c \\ f(a) \rightarrow f(b) \end{array} \right\}$$

is also neither weakly terminating nor (locally) confluent (for the same reasons as $\mathcal{R}{\downarrow}$).

3 Two Sufficient Conditions

It is obvious that right-normalization disables certain reduction steps which are possible in the original system. One case where this is harmless is the following.

Theorem 3.1 *Let $\mathcal{R}$ be a semi-complete TRS such that $\mathcal{R}{\downarrow}$ satisfies: $(*)$ no non-variable subterm of a right-hand side in $\mathcal{R}{\downarrow}$ is unifiable with a left-hand side in $\mathcal{R}{\downarrow}$. Then $\mathcal{R}{\downarrow}$ is semi-complete.*

Proof: Let $\mathcal{R}$ satisfy the above assumptions. By Lemma 2.1(2) it suffices to show weak termination of $\mathcal{R}{\downarrow}$. We show by structural induction that every term t has a normal form (in $\mathcal{R}{\downarrow}$). First we consider the base case: If t is a variable or irreducible constant then we already have a normal form. If t is a reducible constant, then t is a left-hand side of $\mathcal{R}$ and of $\mathcal{R}{\downarrow}$. Consequently, t reduces in one $\mathcal{R}{\downarrow}$-step to a normal form. In the induction step t is of the form $t = f(t_1, \ldots, t_n)$. By induction hypothesis, every t_i $(1 \leq i \leq n)$ has some normal form $\widehat{t_i}$ in $\mathcal{R}\!\downarrow$, hence $t = f(t_1, \ldots, t_n) \to^*_{\mathcal{R}\downarrow} f(\widehat{t_1}, \ldots, \widehat{t_n}) =: \widehat{t}$. If $\widehat{t}$ is irreducible, we are done. Otherwise, we must have $\widehat{t} = l\sigma$ for some rule $l \to r \in \mathcal{R}$ and some irreducible substitution σ, i.e., with $\sigma(x)$ irreducible for every $x \neq \sigma(x)$ (note that σ may be assumed to be irreducible because all proper subterms of $\widehat{t}$ are irreducible). Hence, $\widehat{t} = l\sigma \to_{\mathcal{R}\downarrow} (r\!\downarrow_{\mathcal{R}})\sigma$. The latter term is irreducible by the assumption $(*)$ and irreduciblity of σ. Hence, $t = f(t_1, \ldots, t_n)$ reduces in $\mathcal{R}\!\downarrow$ to the normal form $(r\!\downarrow_{\mathcal{R}})\sigma$ as desired. Hence we are done. ∎

Note that the counterexample in the proof of Theorem 2.2 violates the condition $(*)$, since the non-variable subterm $g(x)$ of the right-hand side $h(g(x))$ of the $\mathcal{R}\!\downarrow$-rule $h(f(x)) \to h(g(x))$ is unifiable with the left-hand side $g(b)$.

Remark 3.2 *We observe that for the preservation of weak termination in Theorem 3.1 confluence of $\mathcal{R}$ is not really necessary. More precisely, if $\mathcal{R}$ is weakly terminating then we may consider any system $\mathcal{R}\!\downarrow$, which is obtained by replacing in $\mathcal{R}$ every right-hand side by one of its $\mathcal{R}$-normal forms. If $\mathcal{R}\!\downarrow$ satisfies $(*)$ above, then it is innermost terminating, hence in particular weakly terminating (this is easily proved by structural induction or minimal counterexample). Moreover, in such an $\mathcal{R}\!\downarrow$ the contractum of an innermost redex in some term is irreducible. This means that the length of innermost derivations in $\mathcal{R}\!\downarrow$ is bounded by the size of the initial term.*

In some sense the condition $(*)$ in Theorem 3.1 above is not very satisfactory because it is formulated in terms of the resulting system and not of the original one. Moreover the class of TRSs to which it applies seems to be rather restricted (cf. Remark 3.2 above).

Another more interesting sufficient condition relies on a known result of O'Donnell about normalization of reduction strategies. More precisely, we shall exploit the fact that parallel-outermost reduction and — more generally — outermost-fair reduction are normalizing for almost orthogonal TRSs ([O'D77]). The latter result has recently been extended to the higher-order case by van Raamsdonk ([Raa96]).[3] First let us explain some notions needed here and subsequently. Recall that a TRS $\mathcal{R}$ is *non-overlapping* if it has no critical pairs (note that an overlap of a rule with itself at the root position is not considered to be critical). A critical pair $\langle s, t\rangle$ is *trivial* if $s = t$, and an *overlay* if it is obtained by overlapping two rules at root position. A left-linear TRS $\mathcal{R}$ is called *orthogonal* if it is non-overlapping, *almost orthogonal* if all its critical pairs are trivial overlays, and *weakly orthogonal* if all its critical pairs are trivial. A reduction strategy is said to be *normalizing* if for any term t possessing a normal form reduction (of t) according to the strategy eventually ends in a normal form. A (possibly infinite) reduction sequence (where for every step the redex contracted is specified) is said to be *outermost-fair* if every outermost redex occurrence is eventually eliminated. In almost orthogonal TRSs outermost redex occurrences can only be eliminated and created by contracting redex occurrences which are outermost themselves (this property does not hold for weakly orthogonal systems).

[3]Note that *outermost-fair* reduction ([Klo92]) is called *eventually outermost* in [O'D77].

Theorem 3.3 *Let $\mathcal{R}$ be an almost orthogonal TRS. If $\mathcal{R}$ is semi-complete then $\mathcal{R}{\downarrow}$ is semi-complete, too.*

Proof: Let $\mathcal{R}$ be semi-complete and almost orthogonal. To show semi-completeness of $\mathcal{R}{\downarrow}$ it suffices by Lemma 2.1 to prove that $\mathcal{R}{\downarrow}$ is weakly terminating. For an arbitrary term s we construct the $\mathcal{R}{\downarrow}$-reduction (parallel reduction by contraction of some outermost redexes is denoted by $\underset{\circ}{\Vert\!\!\!\rightarrow}$)

$$D' : s =: s_1 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}\downarrow} s_2 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}\downarrow} s_3 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}\downarrow} \cdots$$

where s_{k+1} is obtained from s_k by the parallel contraction of all (parallel) outermost $\mathcal{R}{\downarrow}$-redexes in s_k. If some outermost root-reducible subterm s_k/p of s_k is root-reducible by several distinct rules then we choose an arbitrary applicable one. In any case, the result of contracting s_k/p is unique, since $\mathcal{R}$ is almost orthogonal. By definition of $\mathcal{R}{\downarrow}$, we have $l \rightarrow_{\mathcal{R}} r \rightarrow^*_{\mathcal{R}} r'$ for every $l \rightarrow r \in \mathcal{R}$, $l \rightarrow r' \in \mathcal{R}{\downarrow}$. Thus we may refine D' into an $\mathcal{R}$-derivation

$$D : s =: s_1 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}} s_2' \rightarrow^*_{\mathcal{R}} s_2 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}} s_3' \rightarrow^*_{\mathcal{R}} s_3 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \cdots$$

where s_{k+1} is obtained from s_k by the parallel contraction of all (parallel) outermost $\mathcal{R}$-redexes in s_k (yielding s'_{k+1}) followed by the $\mathcal{R}$-normalization of the right-hand side patterns of the applied $\mathcal{R}$-rules,[4] i.e., if $s_k = C[l_1\sigma_1, \ldots, l_n\sigma_n]$ (with $l_i \rightarrow r_i \in \mathcal{R}$) where all outermost $\mathcal{R}$-redexes are displayed then

$$s_k \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}} C[r_1\sigma_1, \ldots, r_n\sigma_n] = s'_{k+1} \rightarrow^*_{\mathcal{R}} C[(r_1{\downarrow_{\mathcal{R}}})\sigma_1, \ldots, (r_n{\downarrow_{\mathcal{R}}})\sigma_n] = s_{k+1}\,.$$

Now, for every k every outermost $\mathcal{R}$-redex of s_k is eliminated in the step $s_k \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}} s'_{k+1}$.[5] Hence, D is in particular an outermost-fair $\mathcal{R}$-derivation. Since outermost-fair reduction is normalizing for almost orthogonal TRSs and $\mathcal{R}$ is semi-complete and almost orthogonal, D must end with an $\mathcal{R}$-normal form s_n (for some n). By definition of $\mathcal{R}{\downarrow}$ we know that s_n is also a normal form w.r.t. $\mathcal{R}{\downarrow}$. Consequently, D' must also end in the ($\mathcal{R}$- and $\mathcal{R}{\downarrow}$-) normal form s_n. Hence,

$$D' : s =: s_1 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}\downarrow} s_2 \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}\downarrow} \cdots \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}\downarrow} s_n$$

is a normalizing $\mathcal{R}{\downarrow}$-derivation for s as desired. ■

We observe that the right-normalized version of a semi-complete orthogonal TRS is not only semi-complete but by construction also orthogonal. This preservation of orthogonality does not extend to almost orthogonality, i.e., the right-normalized version of a semi-complete, almost orthogonal TRS is semi-complete, but need not be almost orthogonal (however, it is still an overlay system). For instance, right-normalization of the complete, almost orthogonal TRS

$$\mathcal{R} = \left\{ \begin{array}{c} f(x, g(a)) \rightarrow h(x) \\ f(g(a), g(x)) \rightarrow h(g(x)) \\ h(g(x)) \rightarrow b \end{array} \right\}$$

[4] For unicity one may assume for every rule $l \rightarrow r \in \mathcal{R}$ one fixed normalizing $\mathcal{R}$-derivation $r \rightarrow^*_{\mathcal{R}} r{\downarrow_{\mathcal{R}}}$.

[5] Note that, for $k \geq 2$, all outermost redexes of all intermediate terms occurring in $s'_k \rightarrow^*_{\mathcal{R}} s_k$ are eliminated at the latest in the step $s_k \underset{\circ}{\Vert\!\!\!\rightarrow}_{\mathcal{R}} s'_{k+1}$.

yields

$$\mathcal{R}{\downarrow} = \left\{ \begin{array}{c} f(x, g(a)) \to h(x) \\ f(g(a), g(x)) \to b \\ h(g(x)) \to b \end{array} \right\}$$

which is complete, but obviously not almost orthogonal.

A careful inspection of the proof of Theorem 3.3 reveals that the only property which is really essential for the construction to go through is normalization of outermost-fair reduction. However, we are not aware of any further results (besides the one mentioned above) about classes of TRSs for which outermost-fair reduction is normalizing. For instance, it seems to be open whether outermost-fair reduction is also normalizing for weakly orthogonal TRSs. Note that weak orthogonality is only slightly more general than almost orthogonality.

Actually, considering the proof of Theorem 3.3 it is obvious that instead of normalization of outermost-fair reduction *hyper normalization*[6] of parallel-outermost reduction may also be used as essential property (this is exactly what is needed in the construction). And it seems plausible ([Mid97b]) that hyper normalization of parallel-outermost reduction for weakly orthogonal TRSs can be proved by using ideas of [SR93], [OR94], [Mid97a]. Yet, this remains to be checked in detail.

4 Discussion

Right-normalization of semi-complete TRSs is a transformation which is incompatible with the usual (position selection) reduction strategies like (leftmost, parallel) innermost or outermost, namely in the following sense. Applying some rule $l \to r{\downarrow_{\mathcal{R}}} \in \mathcal{R}{\downarrow}$ (for some $l \to r \in \mathcal{R}$) to a term $t = C[l\sigma]_p$ means to apply $l \to r$ to t/p yielding $t' = C[r\sigma]_p$ followed by some derivation below p where the pattern of $r\sigma$, i.e., r is normalized. If $t/p = l\sigma$ is outermost in t, then a redex which is outermost in r (let's say at position q) need not correspond to an outermost redex in $t' = C[r\sigma]_p$ because some new outermost redex in t' above pq may have been created, via the surrounding context $C[.]$ or via instantiation with σ. Similarly, if $t/p = l\sigma$ is innermost in t, then a redex which is innermost in r (let's say at position q) need not correspond to an innermost redex in $t' = C[r\sigma]_p$ because some new innermost redex in t' (strictly) below pq may have been created by instantiation. Hence, in general it seems difficult to predict the effect of right-normalization of some semi-complete TRS on a previously existing normalizing reduction strategy (for the original system).

We have isolated two conditions under which the right-normalized version of a semi-complete TRS is again semi-complete (and has the same equational theory). Furthermore we have shown that in general both weak termination and (local) confluence may be lost under right-normalizing semi-complete TRSs. However, in the counterexample presented at least the equational theory is preserved as is easily verified. In fact, we did not succeed in finding a counterexample for question (2), i.e., a semi-complete TRS for which right-normalization

[6] We say that a strategy $\to_s$ (i.e., satisfying $\to_s \subseteq \to^*$) whose normal forms are also $\to$-normal forms is *hyper normalizing* if every term that is weakly terminating under $\to$ terminates under $\to^* \circ \to_s \circ \to^*$.

does not preserve the equational theory. Let us only mention that any (potentially existing) counterexample $\mathcal{R}$ would have to satisfy the following property: There exists a right-hand side r in $\mathcal{R}$ which is

(a) weakly $\mathcal{R}$-terminating (since $\mathcal{R}$ is semi-complete),

(b) not $\mathcal{R}$-terminating,

(c) not weakly $\mathcal{R}{\downarrow}$-terminating, and such that

(d) $r \not\leftrightarrow^*_{\mathcal{R}\downarrow} r{\downarrow}_{\mathcal{R}}$.

By definition of $\mathcal{R}{\downarrow}$ we have $\leftrightarrow^*_{\mathcal{R}\downarrow} \subseteq \leftrightarrow^*_{\mathcal{R}}$. Hence, if $\leftrightarrow^*_{\mathcal{R}\downarrow} \neq \leftrightarrow^*_{\mathcal{R}}$ then there exists some rule $l \to r \in \mathcal{R}$ with (d) $r \not\leftrightarrow^*_{\mathcal{R}\downarrow} r{\downarrow}_{\mathcal{R}}$ (otherwise, for any $l \to r \in \mathcal{R}$ we would have $l \to_{\mathcal{R}\downarrow} r{\downarrow}_{\mathcal{R}} \leftrightarrow^*_{\mathcal{R}\downarrow} r$, hence $l \leftrightarrow^*_{\mathcal{R}\downarrow} r$, and consequently $\leftrightarrow^*_{\mathcal{R}\downarrow} = \leftrightarrow^*_{\mathcal{R}}$). Moreover, (b) holds, because otherwise r would be $\mathcal{R}{\downarrow}$-terminating, hence $r \to^*_{\mathcal{R}} r{\downarrow}_{\mathcal{R}}$, $r \to^*_{\mathcal{R}\downarrow} r{\downarrow}_{\mathcal{R}\downarrow}$ and $r{\downarrow}_{\mathcal{R}} = r{\downarrow}_{\mathcal{R}\downarrow}$ (by confluence of $\mathcal{R}$), and thus $r \leftrightarrow^*_{\mathcal{R}\downarrow} r{\downarrow}_{\mathcal{R}}$ which contradicts (d). Finally, (c) holds, because otherwise there would exist some $\mathcal{R}{\downarrow}$- (and $\mathcal{R}$-) irreducible $\hat{r}$ with $r \to^*_{\mathcal{R}\downarrow} \hat{r}$, $r \to^*_{\mathcal{R}} r{\downarrow}_{\mathcal{R}}$ and $\hat{r} = r{\downarrow}_{\mathcal{R}}$ (by $\to_{\mathcal{R}\downarrow} \subseteq \to^*_{\mathcal{R}}$ and confluence of $\mathcal{R}$), hence $r \leftrightarrow^*_{\mathcal{R}\downarrow} r{\downarrow}_{\mathcal{R}}$ again contradicting (d).

To conclude, we present a simple counterexample to question (3) of the introduction, i.e., a semi-complete, right-normalized TRS where deletion of left-reducible rules destroys both semi-completeness as well as the equational theory. Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} g(a) \to f(a) \\ h(f(x)) \to h(g(x)) \\ h(g(a)) \to b \end{array} \right\}$$

which is easily shown to be semi-complete and which is obviously right-normalized. The only rule with a reducible left-hand side is the last one, hence we obtain

$$\mathcal{R}{\downarrow}_{ext} = \left\{ \begin{array}{c} g(a) \to f(a) \\ h(f(x)) \to h(g(x)) \end{array} \right\}$$

which is neither weakly terminating (for instance, $h(f(a))$ has no normal form) nor does it satisfy $\leftrightarrow^*_{\mathcal{R}\downarrow_{ext}} = \leftrightarrow^*_{\mathcal{R}}$ (for instance, $h(g(a)) \to_{\mathcal{R}} b$ but $h(g(a)) \not\leftrightarrow^*_{\mathcal{R}\downarrow_{ext}} b$).

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