Extending First-Order Unification by Tractable Second-Order Features

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Abstract. We present a new approach for solving certain infinite sets of first order unification problems represented by term schemes. Within the framework of second-order equational logic solving such scheme unification problems amounts exactly to solving (variable-)restricted unification problems. Finally, we show how this approach yields a generic solution technique for infinitely many ordinary first-order unification problems.

1 Introduction and Overview

Solving infinitely many first-order unification problems (of certain types) is a typical infinitary problem that occurs in quite different contexts in computer science, and in particular in automated reasoning and first-order theorem proving. Technically, such kinds of infinitely many *repeated unifiability problems* are e.g. at the heart of the *divergence phenomenon* in Knuth-Bendix-like completion procedures for equational theories (or, more generally, for first-order equational clauses), of *termination analysis* and representation of *infinite solution sets* in logic programming, and of *failure phenomena* (i.e., non-termination) in first-order (deductive or inductive) theorem proving. In general such problems occur quite often in various areas where infinitely many objects (e.g., terms or equations) are successively created by some more or less complex generation mechanism like *deduce*-inference rules in automated first-order theorem proving.

Let us give a simple example to illustrate some of the problems typically encountered.

Example 1. For $A : f(h(u), v) = f(u, h(v))$, $B_0 : f(w, a) = w$ and $C_0 : f(w, g(w)) = w$ Knuth-Bendix completion with input $\{A, B_0\}$ and $\{A, C_0\}$, respectively, and an appropriate reduction ordering produces the following infinite sequence(s) of rewrite rules:

$$\begin{array}{lr}
A & f(h(u),v) \to f(u,h(v)) \\
B_0 & f(w,a) \to w \\
B_1 & f(w,h(a)) \to h(w) \\
B_2 & f(w,h(h(a))) \to h(h(w)) \\
 & \vdots \\
B_n & f(w,h^n(a)) \to h^n(w) \\
 & \vdots
\end{array}$$

$$\begin{array}{lr}
A & f(h(u),v) \to f(u,h(v)) \\
C_0 & f(w,g(w)) \to w \\
C_1 & f(w,h(g(h(w)))) \to h(w) \\
C_2 & f(w,h(h(g(h(h(w)))))) \to h(h(w)) \\
 & \vdots \\
C_n & f(w,h^n(g(h^n(w)))) \to h^n(w) \\
 & \vdots
\end{array}$$

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The reason for divergence (i.e., non-termination) obviously comes from the fact that any left-hand side l_n of B_n and C_n, respectively, is unifiable with the left-hand side l of A yielding via critical pair construction $l_{n+1} \to r_{n+1}$. Now, why are all those (infinitely many) l_n indeed unifiable with l? One might conjecture that this *repeated unifiability* of l with every l_n is due to the fact that l is unifiable with any *instance* of the *term scheme* $f(w, X)$ and $f(w, Y(w))$, respectively.

In order to give a formal definition of this kind of unification problems and to provide solutions for them we have to make precise the notions *term scheme* and *unification of term schemes*. This will be done in the framework of second order equational logic.[1]

In section 2 we provide basic terminology and facts about second-order equational logic. The main results about the mentioned unification problems as well as solution methods for them are presented in section 3. We conclude by briefly discussing (some) related work and further perspectives. The present paper is to a considerable extent based on an old technical report[2] where some proofs and more examples missing here (due to lack of space) are given in detail.

2 Second Order Term Languages

The basic definitions, notions and lemmas about second order term languages are essentially taken from [12] and [24]. Moreover we assume familiarity with the basic notions and results of λ-calculus (cf. [12], [2]).

Definition 1 (second order terms). *For all $i \geq 0$ let $\mathcal{V}_i$ be a denumerable set of distinct (function) variables of arity i with $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for $i \neq j$. The set of all variables is defined as $\mathcal{V} = \bigcup_{i \geq 0} \mathcal{V}_i$. Let $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{V} = \emptyset$ be a finite or denumerable set of distinct (function) constants together with their respective arity (denoted by $ar(\cdots)$). We use X, Y, Z, ... for denoting variables, x, y, z, ... for variables of arity 0, f, g, h, ... for function constants and A, B, C, ... for denoting variables or constants. The set of* restricted second order terms *$\mathcal{T} = \mathcal{T}(C, V)$ is the smallest set satisfying*

(i) $A \in \mathcal{V} \cup \mathcal{C}, ar(A) = 0 \Longrightarrow A \in \mathcal{T}$, *and*
(ii) $A \in \mathcal{V} \cup \mathcal{C}, ar(A) > 0, t_1, \ldots, t_{ar(A)} \in \mathcal{T} \Longrightarrow A(t_1, \ldots, A_{ar(A)}) \in \mathcal{T}$.

$\mathcal{T}_0 := \{t \in \mathcal{T} \mid V(t) \subseteq \mathcal{V}_0\}$ is the subset of first order terms of $\mathcal{T}$, where the set $V(t)$ of variables of a a term t is defined by $V(A(t_1, \ldots, t_{ar(A)})) := \bigcup_{i=1}^{ar(A)} V(t_i) \cup (\{A\} \cap \mathcal{V})$. In order to get general second order terms (i.e., functional objects, too) we complete $\mathcal{T}$ into $\underline{\mathcal{T}}$ by introducing bound variables as follows. $\underline{\mathcal{T}}$ is the smallest set containing $\mathcal{T}$ that satisfies the closure property $\forall t \in \underline{\mathcal{T}}\ \forall x \in \mathcal{V}_0 \colon \lambda x.t \in \underline{\mathcal{T}}$. As usual we abbreviate a term of the form $\lambda x_1 \cdots \lambda x_n.t$ as $\lambda x_1 \ldots x_n.t$. In fact, as pointed out in [12], $\underline{\mathcal{T}}$ may be regarded as the restriction of the language of extensional normal forms of typed λ-calculus (with one base type) to second order terms. The definition of $V(t)$ is recursively extended to $\underline{\mathcal{T}}$ by defining $V(\lambda x.t) := V(t) \setminus \{x\}$, which describes the set of free variables *of t. By $V(M)$ with $M \subseteq \mathcal{T}$ we mean $\bigcup_{t \in M} V(t)$. In the following terms are always compared modulo α-conversion (i.e., renaming of bound*

[1] Note that we will not deal here explicitly with the (important) questions related to guessing / deducing the general form of the infinitely many objects (e.g., terms, rules) created.
[2] See: B. Gramlich: *Unification of Term Schemes – Theory and Applications*, SEKI-Report SR-88-18, Fachbereich Informatik, Universität Kaiserslautern, Germany, 1988.

variables) which greatly simplifies the presentation. Furthermore w.l.o.g. we assume that for any term under consideration the set of free variables is distinct from the set of all bound variables. By $\eta[A] := \lambda x_1 \cdots x_{ar(A)}.A(x_1, \ldots, x_{ar(A)})$ *we denote the extensional normal form of* $\mathcal{V} \cup \mathcal{C}$ *(i.e., the normal form of* A *under the converse of the* η*-reduction rule in* λ*-calculus).*

Definition 2 (second order substitutions, cf. [12], [24]). *A* second order substitution σ *is a finite set of (type respecting) variable assignments, i.e., pairs* (X,t) *with* $t \in \mathcal{T}$ *also denoted by* $X \leftarrow t$. *It may be considered to be a total function* $\sigma \colon \mathcal{V} \to \underline{t}$ *by defining* $\sigma(X) := \eta[X]$, *if there is no pair* $X \leftarrow t$ *in* σ. *The* domain *of a substitution* σ *is* $D(\sigma) := \{X \in \mathcal{V} \mid \sigma(X) \neq \eta[X]\}$, *the set of* introduced variables *is* $I(\sigma) := \bigcup_{X \in D(\sigma)} I(\sigma, X)$, *where* $I(\sigma, X) := V(t)$ *for* $(X \leftarrow t) \in \sigma$, $X \in D(\sigma)$. *The application of substitutions to terms yielding again terms is recursively defined by (denoting also the extension by* σ*)*

(i) $\sigma(A(t_1, \ldots, t_{ar(A)}) := A(\sigma(t_1), \ldots, \sigma(t_{ar(A)}))$ *for* $A \in (\mathcal{C} \cup \mathcal{V}) \setminus D(\sigma)$,

(ii) $\sigma(X(t_1, \ldots, t_{ar(X)})) := \phi(t)$ *for* $X \in \mathcal{V} \cap D(\sigma)$, $X \leftarrow \lambda x_1 \ldots x_{ar(X)}.t \in \sigma$ *and the substitution*
$\phi := \{x_i \leftarrow \sigma(t_i) \mid 1 \leq i \leq ar(X)\}$, *and*

(iii) $\sigma(\lambda x.t) := \lambda x.\sigma(t)$, *where the bound variable* x *is supposed not to be in* $D(\sigma)$, *which is implicitly satisfied since we work modulo renaming of bound variables (note that* $\sigma(X) = \sigma(\eta[X]) \iff X \notin D(\sigma)$*).*

By $SUB := SUB(\underline{\mathcal{T}})$ *we denote the set of all (second order) substitutions, by* $SUB(\mathcal{T}_0)$ *the subset of all first order substitutions, i.e., substitutions* σ *with* $D(\sigma), I(\sigma) \subseteq \mathcal{V}_0$. Equality for substitutions *is defined by* $\sigma = \sigma' \iff D(\sigma) = D(\sigma')$ *and* $\forall X \in D(\sigma) \colon \sigma(X) = \sigma'(X)$. *By* id *we denote the* identity substitution *(*$D(id) = \emptyset$*). The* composition of substitutions *is defined by* $\phi\sigma := \{X \leftarrow \phi(\sigma(X)) \mid X \in D(\sigma), \phi(\sigma(X)) \neq \eta[X]\} \cup \{X \leftarrow \phi(X) \mid X \in D(\phi) \setminus D(\sigma)\}$. *The* restriction of a substitution σ to a (finite) set V of variables *is defined as* $\sigma|_V := \{(X \leftarrow t) \in \sigma \mid X \in V \cap D(\sigma)\}$. Equality of substitutions restricted to a (finite) set of variables V *is defined by* $\sigma =_V \phi \iff \sigma|_V = \phi|_V$.

A substitution is idempotent *iff* $\sigma\sigma = \sigma$. *It is a* renaming substitution away from $V \subseteq \mathcal{V}$ *iff (1)* $\forall X \in D(\sigma) \colon \sigma(X) = \eta[Y]$ *for some* $Y \in \mathcal{V} \setminus V$ *and (2)* $\forall X, Y \in D(\sigma) \colon \sigma(X) = \sigma(Y) \Longrightarrow X = Y$. *It is a* W-renaming *for* $W \subseteq \mathcal{V}$ *iff (1')* $\forall X \in W : \sigma(X) = \eta[Y]$ *for some* $Y \in \mathcal{V}$ *and (2')* $\forall X, Y \in W : \sigma(X) = \sigma(Y) \Longrightarrow X = Y$. *And* σ *is said to be* strict *iff* $\forall X \in D(\sigma), \sigma(X) = \lambda u_1, \ldots, u_{ar(X)}.t \colon \{u_1, \ldots, u_{ar(X)}\} \subseteq V(t)$. *For* $M \subseteq \underline{\mathcal{T}}$ *we denote the set* $\{\phi(t) \mid t \in M\}$ *by* $\phi(M)$.

Definition 3 (preordering on terms and substitutions). *The preordering* $\leq$ *on terms is defined by* $s \leq t$ *iff* $\exists \sigma \in SUB \colon t = \sigma(s)$. *For* $s \leq t$ *we also say that* t *is an* instance of s *or* s *is* more general than t. *Analogously (and abusing notation)* $\leq$ *for substitutions is defined by* $\sigma \leq \phi \iff \exists \tau \in SUB \colon \phi = \tau\sigma$. *We say that* σ *is* more general than ϕ *on* $V \subseteq \mathcal{V}$ *(written* $\sigma \leq_V \phi$*) iff* $\exists \tau \in SUB \colon \phi =_V \tau\sigma$. *The preordering* $\leq_V$ *induces an equivalence relation* $\equiv_V$ *on SUB via* $\sigma \equiv_V \phi$ *iff* $\sigma \leq_V \phi$ *and* $\phi \leq_V \sigma$.

Lemma 1 (basic properties properties of substitutions). *For all* $\sigma, \phi \in SUB$ *we have:*

(a) $I(\sigma) \cap D(\sigma) = \emptyset \Longrightarrow \sigma$ *is idempotent.*
(b) $D(\sigma) \cap D(\phi) = I(\phi) \cap (D(\sigma) \cup D(\phi)) = \emptyset \Longrightarrow \phi\sigma = (\phi\sigma)\phi$.
(c) $D(\sigma \cap D(\phi) = D(\sigma) \cap I(\sigma) = D(\phi) \cap I(\sigma) = \emptyset \Longrightarrow \phi\sigma = \sigma\phi$.
(d) $D(\sigma \cap D(\phi) = D(\sigma) \cap I(\phi) = \emptyset \Longrightarrow \phi\sigma = (\phi\sigma)|_{D(\sigma)}\phi$.

3 Restricted Unification and Unification of Term Schemes

We now come back to our original problem and make precise the notions of term schemes and scheme unification. In order to solve scheme unification problems we then generalize the notion of (variable-)restricted unification (cf. [3]) to the second order case, develop solution methods for them and finally show that scheme unification problems may be solved as a special case in this more general framework.

Definition 4 (term scheme, scheme unification). *A* term scheme *is any term $s \in \mathcal{T}$ with its variables partitioned into a set of ordinary first order variables and a set of (possibly second order) scheme variables. A* scheme unification problem (SUP) *is represented by a finite set E of (unordered) pairs of term schemes where the role of any variable occurring in E is fixed, either as an ordinary variable or as a scheme variable. Thus, we may denote the SUP by $\langle E/W \rangle$ where W is the set of ordinary variables of E and $W^c := V(E) \setminus W$ the set of scheme variables of E. Solving the SUP $\langle E/W \rangle$ consists in deciding whether every first order unification problem $\langle \psi(E) \rangle$ with $\psi \in SUB$, $D(\psi) \subseteq W^c$, $I(\psi) \cap W = \emptyset$ is solvable, and if so, in computing the unifiers (see Definition 5) for all these problems.*

In order to be able to handle scheme unification problems, we first deal now with so-called (variable-)restricted unification problems. Let us start with a generalized definition of (second order) unification problems following [24] but forbidding certain variables to be instantiated.

Definition 5 (variable-restricted unification). *A* (variable-)restricted unification problem (RUP) *is a finite set E of (unordered) term pairs (s_i, t_i) together with a set of variables $W \subseteq V(E) := \bigcup_i (V(s_i) \cup V(t_i))$, also denoted by $\langle E/W \rangle = \langle s_1 \doteq t_1, \ldots, s_n \doteq t_n / W \rangle$. A* solution *or* (variable-)restricted unifier *of $\langle E/W \rangle$ is any substitution σ with $D(\sigma) \cap V(E) \subseteq W$ and $\sigma(s_i) = \sigma(t_i)$ for $i = 1, \ldots, n$. The set of all solutions of $\langle E/W \rangle$ is denoted by $U(E/W)$. Since $\sigma \in U(E/W)$ iff $\sigma|_W \in U(E/W)$ we are in most cases interested in $U_W(E/W) := \{\sigma \in U(E/W) \mid D(\sigma) \subseteq W\}$, the set of all solutions of $U(E/W)$ with their domain restricted to the relevant variables. We say $\langle E/W \rangle$ is* solvable *iff $U(E/W) \neq \emptyset$. Two term s and t are said to be* W-unifiable *for $W \subseteq V(s) \cup V(t)$ iff $\langle s \doteq t/W \rangle$ is solvable.*

Choosing $W = V(E)$ we get the ordinary unification problem (UP) $\langle E/V(E) \rangle$ for (second order) terms, simply denoted $\langle E \rangle$. Intuitively W describes the set of variables substitution is restricted to, whereas $W^c := V(E) \setminus W$ contains the "protected" variables where substitution is not allowed.

Second order unification is in general undecidable as shown in [8][3] , but of course the set of unifiers for a given problem is recursively enumerable. Next we give some basic definitions that are important for finite descriptions of unifiers sets. These definitions are a straightforward generalization for RUP's of the ones given in [24] for ordinary UP's.

Definition 6 (complete / minimal set of unifiers, most general unifier). *Given a RUP $\langle E/W \rangle$, a set S of substitutions and a finite set of additional "protected" variables V with $V \cap W^c = \emptyset$, we say that S is a* complete *set of unifiers (csu) for $\langle E/W \rangle$ away from V iff*

(i) $\forall \sigma \in S$: $D(\sigma) \subseteq W$ and $I(\sigma) \cap (D(\sigma) \cup V) = \emptyset$,

[3] but see, for instance, [14], [16] for decidable subcases

(ii) $S \subseteq U(E/W)$, *and*
(iii) $\forall \phi \in U(E/W) \exists \sigma \in S \colon \sigma \leq_{V(E)} \phi$.

S is a complete minimal *of unifiers (cmsu) for* $\langle E/W \rangle$ *away from V if additionally*

(iv) $\forall \sigma, \sigma' \in S, \sigma \neq \sigma' \colon \sigma \not\leq_{V(E)} \sigma'$ *holds.*

σ *is a* most general *unifier (mgu) of* $\langle E/W \rangle$ *away from V iff* $\{\sigma\}$ *is a csu of* $\langle E/W \rangle$ *away from V.*

The set V of "protected" variables may be used to separate new variables which are introduced by a csu of a given problem from the variables of the context where the problem was extracted from. Indeed, in contrast to first order (syntactic) unification, it is in general necessary for higher order unification to introduce new (free) variables in order to describe csu's.[4] When V is not significant we drop it.

For any solvable RUP $\langle E/W \rangle$ a csu clearly exists but may be not finite. If a cmsu exists it is unique up to an isomorphism (see [12]).

Condition (iv) above for csu's is of technical nature only. This is shown by the following

Lemma 2 (cf. [24]). *Let a RUP* $\langle E/W \rangle$, $\phi \in SUB$ *and a set V of "protected" variables with* $V \cap W^c = \emptyset$ *be given. If* $\phi \in U(E/W)$ *then there exists* $\sigma \in SUB$ *such that*

(i) $D(\sigma) \subseteq W$ *and* $I(\sigma) \cap (D(\sigma) \cup V) = \emptyset$,
(ii) $\sigma \in U(E/W)$, *and*
(iii) $\sigma \equiv_{V(E)} \phi$.

This lemma shows that w.l.o.g. we can restrict ourselves to idempotent unifiers σ with $I(\sigma) \cap D(\sigma) = \emptyset$ when regarding CSU's for a given RUP $\langle E/W \rangle$. The following monotonicity lemma is straightforward from the definition of solution sets.

Lemma 3. *For any set E of equations and sets* W, W' *of variables with* $W \subseteq W' \subset V(E)$ *the inclusion* $U(E/W) \subseteq U(E/W')$ *holds.*

Definition 7. *A RUP* $\langle E/W \rangle$ *is* in solved form *iff for all* $s \doteq t$ *in E the term s is of the form* $\eta[X]$ *with* $X \in W \subseteq \mathcal{V}$ *and X does not occur anywhere else in E. For any RUP* $\langle E/W \rangle$ *in solved form we define* $\sigma_E := \{X \leftarrow t \mid \eta[X] \doteq t \in E\}$ *and conversely for* $\sigma \in SUB$ *let* E_σ *be defined by* $E_\sigma := \{\eta[X] \doteq t \mid (X \leftarrow t) \in \sigma\}$.

The relevance of solved forms is explained by

Lemma 4. *If* $\langle E/W \rangle$ *is a RUP in solved form the the substitution* σ_E *is a mgu for* $\langle E/W \rangle$ *away from V for any* $V \subseteq V(E) = \emptyset$.

We will show now that a RUP $\langle E/W \rangle$ may be transformed into an ordinary (unrestricted UP $\langle E' \rangle$ such that the solutions of $\langle E/W \rangle$ can be obtained from the solutions of $\langle E' \rangle$ by an inverse transfromation. The transformation function involved interprets the forbidden variables in W^c as distinct new constants of corresponding arity. To be precise, for $W^c = \{X_1, \ldots, X_n\}$ let C_i be a new constant with $ar(X_i) = ar(C_i)$ for every i, $1 \leq i \leq n$. Defining $\mathcal{C}^* := \mathcal{C} \uplus \{C_i \mid 1 \leq i \leq n\}$, $\mathcal{V}^* := V \setminus W^c$,

[4] This phenomenon also appears in general in first-order unification modulo non-empty equational theories.

the set of transformed terms becomes $\mathcal{T}^* := \mathcal{T}(\mathcal{C}^*, \mathcal{V}^*)$ and $\underline{\mathcal{T}^*} := \underline{\mathcal{T}(\mathcal{C}^*, \mathcal{V}*)}$ respectively. The transformation function $\Phi: \underline{\mathcal{T}} \to \underline{\mathcal{T}^*}$ is defined by homeomorphic extension of $\Phi: \mathcal{V} \to \underline{\mathcal{T}^*}$, $\Phi(X_i) := \eta[C_i]$ for $X_i \in W^c$ and $\Phi(X) := \eta[X]$ else. It may also be extended to substitutions that leave the variables of W^c unchanged. With $\mathrm{SUB}^*(\underline{\mathcal{T}}) := \{\sigma \in \mathrm{SUB}(\underline{\mathcal{T}}) \mid D(\sigma) \cap W^c = \emptyset\}$ we get $\Phi_S: \mathrm{SUB}^*(\underline{\mathcal{T}}) \to \mathrm{SUB}(\underline{\mathcal{T}})$ defined by $\Phi_S(\sigma) := \{X \leftarrow \Phi(\sigma(X)) \mid X \in D(\sigma)\} = (\Phi\sigma)|_{D(\sigma)}$. It is easily verified that Φ, Φ_S are bijections with Φ^{-1}, Φ_S^{-1} defined by $\Phi^{-1}(\lambda u_1 \ldots u_m.A(t_1, \ldots, t_n)) := \lambda u_1 \ldots u_m.A'(\Phi^{-1}(t_1), \ldots, \Phi^{-1}(t_n))$ with $A' := X_i$ if $A = C_i$ and $A' := A$ else, and $\Phi_S(\sigma^*) := \{X \leftarrow \Phi^{-1}(\sigma^*(X)) \mid X \in D(\sigma^*)\}$ (note that Φ^{-1} is not a substitution!). Moreover we have the following simple properties:

Lemma 5. *For all $\sigma, \sigma_1, \sigma_2 \in SUB^*(\underline{\mathcal{T}}), \sigma^*, \sigma_1^*, \sigma_2^* \in SUB(\underline{\mathcal{T}^*}), t \in \mathcal{T}$ and $t^* \in \underline{\mathcal{T}^*}$ the following properties hold:*

(a) $\Phi(\sigma(y)) = \Phi_S(\sigma)(\Phi(t))$.
(b) $\Phi^{-1}(\sigma^*(t^*)) = \Phi_S^{-1}(\sigma^*)(\Phi^{-1}(t^*))$.
(c) $\Phi_S(\sigma_2\sigma_1) = \Phi_S(\sigma_2)\Phi_S(\sigma_1)$.
(d) $\Phi_S^{-1}(\sigma_2^*\sigma_1^*) = \Phi_S^{-1}(\sigma_2^*)\Phi_S^{-1}(\sigma_1^*)$.

With $\Phi(E) := \{\Phi(s) \doteq \Phi(t) \mid s \doteq t \in E\}$ the connection between the solution sets of the RUP $\langle E/W \rangle$ and the transformed (unrestricted) UP $\langle \Phi(E) \rangle$ is captured by

Lemma 6 (correspondence between solution sets). *The properties of substitutions to be a (most general) solution and of solution sets to be complete (and minimal) are preserved under the transformation Φ, i.e., for all $\sigma \in SUB^*(\underline{\mathcal{T}})$, $Q \subseteq SUB^*(\underline{\mathcal{T}})$ the following properties hold:*

(a) $\sigma \in U(E/W) \iff \Phi_S(\sigma) \in U(\Phi(E))$.
(b) Q is a csu for $\langle E/W \rangle \iff \Phi_S(Q)$ is a csu for $\langle \Phi(E) \rangle$.
(c) Q is a cmsu for $\langle E/W \rangle \iff \Phi_S(Q)$ is a cmsu for $\langle \Phi(E) \rangle$.
(d) σ is a mgu for $\langle E/W \rangle \iff \Phi_S(\sigma)$ is a mgu for $\langle \Phi(E) \rangle$.

In the following we are mainly interested in RUP's, where only the forbidden variables may be of second order. In this case the transformed problem clearly is a first order one (since all second-order variables are forbidden and hence have been transformed into fresh function symbols of corresponding arity). Solving an unrestricted UP $\langle E \rangle$ with $E \subseteq \mathcal{T}^2$, $V(E) \subseteq \mathcal{V}_0$ over $\underline{\mathcal{T}}$, i.e., as an ordinary first order UP over first order terms. This is obvious considering the following solution preserving and terminating set $\mathcal{R}$ of transformation rules for such UP's over $\mathcal{T}$, where the symbol FAIL is to denote unsolvability (cf. [6], [24], [17], [1]):

(1) **Deletion of trivial pairs**
$\{s \doteq s\} \cup E \to E$
(2) **Term decomposition**
$\{f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n\} \cup E \to \{s_i \doteq t_i \mid 1 \leq i \leq n\} \cup E$
(3) **Variable elimination**
$\{x \doteq t\} \cup E \to \{x \doteq t\} \cup \sigma(E)$, if $x \in V(E), x \notin V(t), \sigma = \{x \leftarrow t\}$
(4) **Clash**
$\{f(s_1, \ldots, s_m) \doteq g(t_1, \ldots, t_n)\} \cup E \to$ FAIL, if $f \neq g$
(5) **Occur check**
$\{x \doteq t\} \cup E \to$ FAIL, if $x \neq t, x \in V(t)$.

Using the rules (1)-(3) any solvable UP $\langle E\rangle$ can be transformed into an equivalent UP $\langle E'\rangle$ which is in solved form and thus describes an mgu for $\langle E'\rangle$ as well as for $\langle E\rangle$ (see [6]). Rules (4) and (5) are needed to detect unsolvability. The rules system $\mathcal{R}$ may be looked upon as an abstract formulation of a big class of unification algorithms since it is terminating but does not fix the control structure for rule application (note that in the above presentation there is an additional source of nondeterminism because pairs $s = t$ are considered as unordered multisets). Moreover it provides a very useful computational proof technique as we will see later on.[5]

Using Lemma 6 (d) we can deduce that any RUP $\langle E/W\rangle$, $E \subseteq \mathcal{T}^2$, with $W \subseteq \mathcal{V}_0$ is either unsolvable or possesses an mgu (which is unique up to $\equiv_{V(E)}$-equivalence and may be computed by any of the well-known algorithms for first order unification (see figure).

$$\begin{array}{ccc}
\langle E/W\rangle & \xrightarrow{\;\Phi\;} & \langle \Phi(E)\rangle \\
\big\downarrow & & \big\downarrow \text{first order unification} \\
\Phi^{-1}(E') & \xleftarrow{\;\Phi^{-1}\;} & E' \text{ in solved form}
\end{array}$$

The explicit translation using Φ, Φ^{-1} may be avoided by freezing the forbidden variables from W^c within the first order unification process.

We have seen that, if an RUP $\langle E/W\rangle$ with $W \subseteq \mathcal{V}_0$ is solvable, then it has an mgu which is unique (up to $\equiv$-equivalence). It is possible that $\langle E\rangle$ is solvable, but $\langle E/W\rangle$ is not. The next result shows that if $\langle E/W\rangle$ with $V(E) \subseteq \mathcal{V}_0$ is solvable the resulting mgu is also most general for the unrestricted UP $\langle E\rangle$ whereas this property cannot be generalized to arbitrary second order RUP's.

Lemma 7. *Let $\langle E/W\rangle$, $E \subseteq \mathcal{T}^2$ be given.*

(a) If $V(E) \subseteq \mathcal{V}_0$ and σ is a mgu for $\langle E/W\rangle$, then σ is also a mgu for $\langle E\rangle$.
(b) If $V(E) \not\subseteq \mathcal{V}_0$ and S is a csu for $\langle E/W\rangle$, then S is not necessarily a csu for $\langle E\rangle$.

Proof. **(a)** Let σ be a mgu for $\langle E/W\rangle$ obtained by transformation of $\langle \Phi(E)\rangle$ into solved form $\langle E'\rangle$ using rules (1)-(3) of $\mathcal{R}$: $\Phi(E) \rightarrow^*_{\mathcal{R}} E'$, $\sigma = \Phi_S^{-1}(\sigma_E)$. Any step $E_1 \rightarrow_{(i)} E_2$ using rule (i), $1 \le i \le 3$, in this derivation may be translated into a step $\Phi^{-1}(E_1) \rightarrow_{(i)} \Phi^{-1}(E_2)$. For $i = 1, 2$ this is obvious. For the variable elimination rule

$$\{x \doteq t^*\} \cup E^* \rightarrow_{(3)} \{x \doteq t^*\} \cup \sigma^*(E^*), \text{ if } x \in V(E^*), x \notin V(t^*), \sigma^* = \{x \leftarrow t^*\}$$

we have

$$\Phi^{-1}(x) = x \notin V(\Phi^{-1}(t^*)) \subseteq V(t^*), x \in V(\Phi^{-1}(E^*))$$

implying

$$\{x \doteq \Phi^{-1}(t^*)\} \cup \Phi^{-1}(E^*) \rightarrow_{(3)} \{x \doteq \Phi^{-1}(t^*)\} \cup \sigma(\Phi^{-1}(E^*))$$

with $\sigma := \{x \leftarrow \Phi^{-1}(t^*)\}$. Using $\sigma = \Phi_S^{-1}(\sigma^*)$, $t^* = \Phi(t)$ and Lemma 5 (b) we get

$$\sigma(\Phi^{-1}(E^*)) = \Phi_S^{-1}(\sigma^*)(\Phi^{-1}(E^*)) = \Phi^{-1}(\sigma^*(E^*))$$

[5] Actually, this proof-technical usage is the main reason for explicitly including this well-known inference system here.

and thus

$$\Phi^{-1}(\{x \doteq t^*\} \cup E^*) \rightarrow_{(3)} \Phi^{-1}(\{x \doteq t^*\} \cup \sigma^*(E^*))$$

as desired. From $\langle \Phi(E) \rangle \rightarrow^*_{\mathcal{R}} \langle E' \rangle$ and $\langle \Phi^{-1}(\Phi(E)) \rangle$ we finally deduce that $\Phi^{-1}(E')\rangle$ is also in solved form yielding the mgu $\Phi_S^{-1}(\sigma_{E'} = \sigma$ for the unrestricted UP $\langle E \rangle$ (note that this result could also be proved without the rule system $\mathcal{R}$ using only reasoning about first order substitutions and renamings).
(b) Consider the following counterexample: For $E : X(y) \doteq X(a), X \in \mathcal{V}_1$ it may easily be verified that $\{\sigma_1, \sigma_2\}$ with $\sigma_1 := \{y \leftarrow a\}$, $\sigma_2 := \{X \leftarrow \lambda u.z\}$ is a csu for $\langle E \rangle$ such that σ_1 and σ_2 are incomparable w.r.t. $\leq_{V(E)}$. For $W = \{y\}$, $W^c = \{X\}$, σ_1 is a mgu for $\langle E/W \rangle$. For $W = \{X\}$, $W^c = \{y\}$, σ_2 is again a mgu for $\langle E/W \rangle$, but neither $\{\sigma_1\}$ nor $\{\sigma_2\}$ is a csu for the unrestricted UP $\langle E \rangle$. □

If the RUP $\langle E/W \rangle$ with $E \subseteq \mathcal{T}^2$, $W \subseteq \mathcal{V}_0$ is solvable, say with mgu σ, then the instantiated problem $\langle \Psi(E)/W \rangle$ is solvable, too, for every substitution Ψ that leaves the variables from W untouched and does not introduce any such variable. Moreover, in the case of a strict (cf. Def. 2) Ψ the resulting mgu may be computed from σ and Ψ without explicitly unifying again. This is detailed by

Theorem 1 (solutions for instantiated RUP's). *Let $\langle E/W \rangle$ with $E \subseteq \mathcal{T}^*$, $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$ be a solvable RUP and σ be a mgu for it (w.l.o.g. let $I(\sigma) \subseteq V(E)$). Assume further that Ψ is a substitution with $D(\Psi) \subseteq V(E) \setminus W$, $I(\Psi) \cap W = \emptyset$. Then $(\Psi\sigma)|_{D(\sigma)}$ is a solution for the RUP $\langle \Psi(E)/W \rangle$ which is most general for strict Ψ.*

Proof. Let $\langle E/W \rangle$, σ and Ψ satisfy the assumptions. By $\rightarrow_{\mathcal{R},V}$, $V \subseteq \mathcal{V}$ we denote the reduction relation defined by the transformation system $\mathcal{R}$, where the variables from V are *frozen*, i.e., considered as distinct new constants. Analogously $\rightarrow_{(i),V}$ denotes one $(\mathcal{R}, V)$-step using rule (i), $1 \leq i \leq 3$. W.l.o.g. we assume further that the mgu σ for $\langle E/W \rangle$ has been constructed by transformation of $\langle E/W \rangle$ into solved form E' using $\mathcal{R}$: $E \rightarrow^*_{\mathcal{R},P} E'$, $P := V(E) \setminus W$, $\sigma = \sigma_E$. We will show now that for strict ψ this derivation can be translated into a derivation $\Psi(E) \rightarrow^*_{\mathcal{R},Q} \Psi(E')$ with $Q := V(\Psi(E)) \setminus W$. The conditions $D(\Psi) \subseteq V(E) \setminus W$ and $I(\Psi) \cap W = \emptyset$ imply then that $\langle \Psi(E') \rangle$ is in solved form, too. Thus, $\sigma_{\Psi(E')} = (\Psi\sigma)|_{D(\sigma)}$ is a mgu for $\langle \Psi(E)/W \rangle$ as required. So, let us consider an arbitrary step $E_1 \rightarrow_{\mathcal{R},P} E_2$ within the derivation $E \rightarrow_{\mathcal{R},P} E'$. Translation will yield $\Psi(E_1) \rightarrow^+_{\mathcal{R},Q} \Psi(E_2)$ according to the following cases.

(1) $\{s \doteq s\} \cup E_1 \rightarrow_{(1),P} E_1$ obviously implies $\{\Psi(s) \doteq \Psi(s)\} \cup \Psi(E_1) \rightarrow_{(1),Q} \Psi(E_1)$.
(2) $\{f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n)\} \cup E_1 \rightarrow_{(2),P} \{s_i \doteq t_i \mid 1 \leq i \leq n\} \cup E_1$:
If f is not among the frozen variables from $P = V(E) \setminus W$ interpreted as new constants then we have $\Psi(\{f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n)\} \cup E_1) \rightarrow_{(2),Q} \Psi(\{s_i \doteq t_i \mid 1 \leq i \leq n\} \cup E_1)$. Otherwise $\Psi(f(s_1, \ldots, s_1)) \doteq \Psi(f(t_1, \ldots, t_n))$ can be simplified using $\rightarrow_{\mathcal{R},Q}$ by repeated application of decomposition (2) and deletion of trivial problems (1) until only the pairs $\Psi(s_1) \doteq \Psi(t_1)$, $1 \leq i \leq n$ are left. Note that the s_i, t_i do not disappear because Ψ is strict. Thus we get

$$\Psi(\{f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n)\} \cup E_1) \rightarrow^+_{\mathcal{R},Q} \Psi(\{s_i \doteq t_i \mid 1 \leq i \leq n\} \cup E_1) .$$

(3) $\{x \doteq t\} \cup E_1 \rightarrow_{(3),P} \{x \doteq t\} \cup \phi(E_1)$ with $x \in V(E_1) \setminus P$, $x \notin V(t)$, $\phi = \{x \leftarrow t\}$:
From the fact that no new variables are introduced by rule application and the

assumptions about W and Ψ we deduce $x \in V(E_1) \setminus P \subseteq V(E) \setminus P = W$, $x \in V(\Psi(E_1)) \setminus Q$, $x = \psi(x) \notin V(\psi(t))$. This implies $\psi(\{x \doteq t\} \cup E_1) \rightarrow_{(3),Q} \Psi(\{x \doteq t\}) \cup \phi'(\psi(E_1))$ with $x \in V(\psi(E_1)) \setminus Q$, $x \notin V(\psi(t))$, $\phi' := \{x \leftarrow \psi(t)\}$. Since $\phi'(\psi(E_1)) = (\psi\phi)|_{D(\phi)}(\psi(E))$ using Lemma 1 (d) we can conclude $\psi(\{x \doteq t\} \cup E_1) \rightarrow_{(3),Q} \psi(\{x \doteq t\} \cup \phi_{(1)})$ as desired.

Note that $(\psi\sigma)|_{D(\sigma)} \in U(\psi(E)/W)$ also holds for every (not necessarily strict) ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$ because of Lemma 1 (d). □

The strictness assumption for ψ in Theorem 1 is necessary to ensure that the solution $(\psi\sigma)|_{D(\sigma)}$ of $\langle\psi(E)/W\rangle$ is most general. Take for example $E\colon f(u, X(u)) \doteq f(v, X(a))$, $W = \{u, v\}$ and $\psi = \{X \leftarrow \lambda z.w\}$. Here $\sigma := \{u \leftarrow a, v \leftarrow a\}$ is a mgu for $\langle E/W\rangle$ but $(\psi\sigma)|_{D(\sigma)} = \sigma \in U(\psi(E)/W)$ is not most general. Indeed we get $\psi(E) : f(u, w) \doteq f(v, w)$, which has as mgu $\tau := \{u \leftarrow v\}$ such that $\tau < \sigma$, i.e., $\tau \leq \sigma$, but $\sigma \not\leq \tau$.

Theorem 1 essentially provides a characterization of a whole class of RUP's as described by

Corollary 1. *For $E \subseteq \mathcal{T}^2$, $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$, the RUP $\langle E/W\rangle$ is solvable iff $\langle\psi(E)/W\rangle$ is solvable for every ψ with $D(\psi) \subseteq V(E) \setminus W$ and $I(\psi) \cap W = \emptyset$, and if so, for any such ψ which is strict the mgu for $\langle\psi(E)/W\rangle$ may be computed from the mgu for $\langle E/W\rangle$ as described in Theorem 1.*

Proof. The only-if direction is provided by Theorem 1 and for the if-direction we simply choose the identity substitution for ψ. □

We now come back to our original problem of solving scheme unification problems (SUP's, see Definition 4), i.e., to decide for given $\langle E/W\rangle$ with $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$, whether all first order (unrestricted) UP's $\langle\psi(E)/W\rangle$ with $D(\psi) \subseteq (V(E) \setminus W) \cap \mathcal{V}_0$ and $I(\psi) \cap W = \emptyset$ are solvable, and if so, to compute the corresponding mgu's. The following result shows how to do that. Essentially it is a stronger version of Corollary 1 stating that under slightly stronger preconditions concerning the underlying signature the "if-direction" remains true, when ψ is restricted to first order substitutions (i.e., $I(\psi) \subseteq \mathcal{V}_0$).

Theorem 2 (solving scheme unification problems). *Let $E \subseteq \mathcal{T}(\mathcal{C}, \mathcal{V})^2$, $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$ be given. Assume further that the underlying signature contains at least one function constant of arity ≥ 1 and another one of arity ≥ 2. Then the RUP $\langle E/W\rangle$ is solvable iff the SUP $\langle E/W\rangle$ is solvable, i.e., $\langle\psi(E)\rangle$ is solvable for every ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$, and if so, for any such ψ which is strict the mgu for $\langle\psi(E)\rangle$ may be computed from the mgu σ for $\langle E/W\rangle$ as $(\psi\sigma)|_{D(\sigma)}$.*

Proof. The interesting if-direction is again proved by considering $\mathcal{R}$-derivations to construct mgu s. Assume $\langle E/W\rangle$ is unsolvable but $\langle\psi(E)\rangle$ is solvable for every ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$. Using the same notations as in the proof of Theorem 1 we construct from a derivation $E \rightarrow^*_{\mathcal{R},P} E' \rightarrow_{\mathcal{R},P}\rightarrow$ FAIL with $P := V(E) \setminus W$ another derivation $\psi(E) \rightarrow^*_{\mathcal{R},Q} \psi(E') \rightarrow_{\mathcal{R},Q}$ FAIL for some strict ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$ contradicting the assumption that $\langle\psi(E)\rangle$ is solvable for every such ψ. Obviously the last step of the first derivation must be an application of clash (4) or occur check (5) rule. Thus we have the following two cases:

(a) $E \rightarrow^*_{\mathcal{R},P} E' \rightarrow_{(4),P}$ **FAIL with** $E' = \{f(s_1,\dots,s_m) \doteq g(t_1,\dots,t_n)\} \cup E''$**,** $f \neq g$**:**
Now, depending on whether $f, g \in \mathcal{C}$, we choose any strict ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$ and construct a derivation $\psi(E) \rightarrow^*_{\mathcal{R},Q} \psi(E') \rightarrow_{(4),Q}$ FAIL with $\psi(E') = \{f(\psi(s_1),\dots,\psi(s_m)) \doteq g(\psi(t_1),\dots,\psi(t_n))\} \cup \psi)E'')$, $f \neq g$, $Q := V(\psi(E)) \setminus W$ as in Theorem 1. Again we have four subcases:
(i) For $f, g \in \mathcal{C}$ we can choose any such ψ.
(ii) For $f \in \mathcal{C}, g \notin \mathcal{C}$, g a frozen variable corresponding to $X \in Q$, we choose a strict ψ with $\psi(X) = \lambda u_1,\dots,u_{ar(X)}.t$, $f \neq top(t) \in \mathcal{C}$.
(iii) The case $g \in \mathcal{C}, f \notin \mathcal{C}$ is symmetric to (ii).
(iv) For $f \notin \mathcal{C}, g \notin \mathcal{C}$, f and g corresponding to frozen variables $X \in Q$ and $Y \in Q$ respectively, we choose a strict ψ with $\psi(X) = \lambda u_1,\dots,u_{ar(X}.s$, $\psi(Y) = \lambda u_1,\dots,u_{ar(Y)}.t$, $top(s) \neq top(t)$, $top(s), top(t) \in \mathcal{C}$.
Note that for the cases (ii)-(iv) the assumption that there exist at least a function constant of arity ≥ 1 and another one of arity ≥ 2 is essential for constructing a strict ψ with the required properties.

(b) $E \rightarrow^*_{\mathcal{R},P} E' \rightarrow_{(4),P}$ **FAIL with** $E' = \{x \doteq t\} \cup E$**,** $x \neq t$**,** $x \in V(t)$**:**
In this case any strict ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$ is sufficient to construct a derivation $\psi(E) \rightarrow^*_{\mathcal{R},Q} \psi(E') \rightarrow_{(4),Q}$ FAIL, since $x \neq t$, $x \in V(t)$ implies $x \neq \psi(t)$, $x \in V(\psi(t))$ for strict ψ.

□

In order to illustrate this result we consider the second part of the introductory Example 1. Here we have the SUP $\langle E/W \rangle$, $E := \{f(h(u), v) \doteq f(w, Y(w))\}$, $W := \{u, v, w\}$ with only one (unary) scheme variable, namely Y. Solving this problem with the techniques described amounts to solving the RUP $\langle E/W \rangle$ which yields a mgu $\sigma := \{w \leftarrow h(u), v \leftarrow Y(h(u))\}$. Now, take for instance the strict substitution $\psi := \{Y \leftarrow \lambda x.f(y, h(x))\}$. Then $(\psi\sigma)|_{D(\sigma)} = \{w \leftarrow h(u), v \leftarrow f(y, h(h(u)))\}$ is a mgu for $\langle \psi(E) \rangle = \langle f(h(u), v) \doteq f(w, f(y, h(w))) \rangle$.

The preconditions concerning the signature in the above theorem are really necessary in general for the validity of the "if-direction". Consider the following counterexamples:

(a) $\mathcal{C}$ **contains only function constants of arity** 0 **or** 1**:**
For $E: X(X(v, w), X(w, v)) \doteq X(u, u)$, $W := \{u\}$, the RUP $\langle E/W \rangle$ is unsolvable, because we have $\{\Phi(E)\} \rightarrow^*_{\mathcal{R},P} \{u \doteq X(v, w), v \doteq w\} \rightarrow_{(4),P}$ FAIL. But for every ψ with $D(\psi) \subseteq V(E) \setminus W$, $I(\psi) \cap W = \emptyset$, $I(\psi) \subseteq \mathcal{V}_0$ the UP $\langle \psi(E) \rangle$ is solvable. For $\psi(X) = \lambda yz.t$ we know that t cannot contain both y and z because of the restricted signature. By an easy case analysis for t we can prove now that $\langle \psi(E) \rangle$ is always solvable. For example, if $y \in V(t)$, then y is the only variable occurring in t (moreover it occurs only once). Thus, with the notation $t = t[y]$ we get $\psi(E): t[t[\psi(v)]] \doteq t[u]$, which, transformed into solved form, yields $E': u \doteq t[\psi(v)]$.

(b) $\mathcal{C}$ **contains one function constant** f **of arity** ≥ 2 **and all others have arity** 0**:**
For X with $ar(X) = ar(f)$ and
$E: X(X(y,\dots,y),\dots,X(y,\dots,y)) \doteq X(f(z_1,\dots,z_{ar(f)}),\dots,f(z_1,\dots,z_{ar(f)}))$, $W := \{y, z_1,\dots,z_{ar(f)}\}$ one may prove by case analysis analogous to (a) that again the RUP $\langle E/W \rangle$ is unsolvable, but for every ψ satisfying the corresponding conditions $\langle \psi(E) \rangle$ is solvable.

Let us conclude this section with some historical remarks. The idea of freezing certain variables in unification problems, i.e., considering them as distinct new constants, has already been used in the pioneering work of Huet ([12]) in order to compute complete independent sets of preunifiers for semi-unification problems in typed λ-calculus. From a logical point of view restricted unification problems may be represented by formulae of the form

$$\forall \underline{x}\, \exists \underline{y} \colon s_1 = t_1 \wedge \ldots \wedge s_n = t_n, \tag{1}$$

where the variables of the s_i, t_i are partitioned into the disjoint variable sets $\underline{x}$ and $\underline{y}$. Solving (1) as a restricted unification problem amounts to looking for values for the existentially quantified variables $\underline{y}$ that make the formula valid, whereas the universally quantified variables have to be left untouched (cf. [3]). Within this logical framework it becomes obvious that considering the universally quantified variables as distinct new constants essentially is an application of the "Theorem on Constants" (cf. [23]). This fundamental result, roughly speaking, states that universally quantified variables in logical formulae may be equivalently replaced by distinct new constants. In our context we do not only have the equivalence between (1) and

$$\exists \underline{y} \colon \Phi(s_1) = \Phi(t_1) \wedge \ldots \wedge \Phi(s_n) = \Phi(t_n)\ , \tag{2}$$

i.e., equivalence concerning solvability, but also the correspondence between solutions as described above.

4 Discussion and Related Work

The approach presented provides an interesting means for solving *a priori* certain infinite sets of unification problems in generic (parameterized) form. Many questions, however, arise quite naturally from what has been presented on one side, and – on the other side – from what is known about schematization of infinite sets of terms using other well-known approaches. To mention a few of them: [9] deals with a syntactical analysis of sufficient conditions for *repeated unifiability* (in the context of completion procedures), [13] uses terms with *meta variables* to describe infinite sets of terms (rules and equations), [21, 22], [4], [5], [10], [11] treat unification (and related) problems of infinite sets of terms described by *primal grammars* and less general *recurrent domains*. Constraint solving on terms with integer exponents (for the purpose of improved *model building tools* is dealt with in [18]), and *loop-detection* (in *hyper-tableaux*) using related techniques and computer algebra systems for handling terms with exponents is the subject of [25]. Other approaches for tackling infinity are *regular tree grammars*, cf. e.g. [7], [19], and so-called *tree (tuple) synchronized grammars* as developed in e.g. [15], [20].

Due to lack of space we cannot go into any details concerning the relationships between our approach and the latter ones mentioned above. This is the subject of ongoing work. However, observe that the focus in this paper was on deciding solvability – and, in the positive case, on how to compute parameterized mgu's – of certain infinite sets of (ordinary) unification problems, and not on an explicit generation and finite schematization of infinitely many symbolic objects (e.g. terms, or equations to be solved).

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[6] Cf. `http://www.univ-orleans.fr/SCIENCES/LIFO/Members/limet/amadeus.html`.

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