Abstract Relations between Restricted Termination and Confluence Properties of Rewrite Systems*

Bernhard Gramlich**
Fachbereich Informatik, Universität Kaiserslautern
Postfach 3049, 67653 Kaiserslautern, Germany
gramlich@informatik.uni-kl.de

Abstract. We investigate restricted termination and confluence properties of term rewriting systems, in particular weak termination, weak innermost termination, (strong) innermost termination, (strong) termination, and their interrelations. New criteria are provided which are sufficient for the equivalence of these properties. These criteria provide interesting possibilities to infer completeness, i.e. termination plus confluence, from restricted termination and confluence properties. Our main result states that any (strongly) innermost terminating, locally confluent overlay system is terminating, and hence confluent and complete.
Using these basic results we are also able to prove some new results about modular termination of rewriting. In particular, we show that termination is modular for some classes of innermost terminating and locally confluent term rewriting systems, namely for non-overlapping and even for overlay systems. As an easy consequence this latter result also entails a simplified proof of the fact that completeness is a decomposable property of constructor systems. Similarly, a combined overlay system with shared constructors is complete if and only if its component sytems are complete overlay systems. Interestingly, these modularity results are obtained by means of a proof technique which itself constitutes a *modular* approach.

Key Words: Term rewriting systems, confluence, termination, weak termination, innermost termination, modularity, disjoint union, constructor systems, combination of constructor systems, combined systems with shared constructors.

1 Introduction

Term rewriting systems (TRSs for short) play an important role in various areas, e.g. in abstract data type specifications, for automated theorem proving and as a basic computation model for functional programming languages. In theory and practice, one of the most important properties of term rewriting systems is the strong normalization or (strong, finite or uniform) termination property. For ensuring this property which

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is undecidable in general (see [15]), many sufficient criteria, techniques and methods have been developed (see [4] for a survey). Most practically applicable approaches are based on reduction orderings, i.e. well-founded term orderings which are stable w.r.t. substitutions and monotonic w.r.t. the term structure.

On the other hand, in many rewriting based computation models, e.g. in functional programming languages, the indeterminism of general rewriting is often restricted by imposing some fixed rewriting strategy. For instance, a frequent restriction is innermost reduction, i.e. to require that every reduction step takes place at an innermost position of the term to be reduced. Innermost reduction corresponds closely to the functional evaluation mechanism employed in functional programming languages like LISP or ML. Of course, it may be the case that correspondingly restricted computations, i.e. innermost reduction sequences, always terminate but arbitrary computations (reduction sequences) do not necessarily terminate. A very simple example illustrating this gap is the following:

Example 1.1 *Let* $R = \{f(a) \to f(a), a \to b\}$. *Then we have e.g. the infinite reduction sequence* $f(a) \to f(a) \to f(a) \to \ldots$, *which uses only non-innermost reduction steps. But of course, every innermost derivation in* R *(e.g.* $f(a) \to f(b)$*) is terminating.*

Other kinds of restrictions imposed on rewriting steps might also be conceivable according to the intended purpose, e.g. leftmost outermost, top-down, bottom-up or other context-dependent strategies. Unfortunately, very little is known about termination of rewriting under such restrictions and its relation to (uniform) termination. In fact, there is one major exception, namely concerning the important and thoroughly investigated class of so-called *orthogonal* TRSs, i.e. TRSs which are left-linear and non-overlapping (see [16] for a survey of basic ideas, concepts and results about the theory of orthogonal TRSs). It is well-known that any orthogonal TRSs is confluent notwithstanding the fact that it may be non-terminating. For arbitrary TRSs one can conclude in general nothing about confluence or the existence of (unique) normal forms if termination is not guaranteed since these properties are undecidable in the general case.

In the following we shall study in particular under what conditions strong innermost normalization implies strong normalization of rewriting. More generally, we shall investigate and develop some extensions and generalizations of known results about orthogonal TRSs. This is done by weakening both the no-overlap and the left-linearity requirement but still guaranteeing local confluence. Moreover, in a second part of the paper we shall apply these abstract results in a modular fashion in order to obtain new sufficient criteria for the modularity of strong normalization of TRSs and corresponding invariance properties of certain classes of non-disjoint combinations of TRSs. Before going into details let us give a summary of our main results:

- If a TRS $\mathcal{R}$ is non-overlapping then weak innermost normalization, strong innermost normalization and strong normalization of $\mathcal{R}$ are equivalent properties (Theorem 3.13(1)).
- If a TRS $\mathcal{R}$ is non-overlapping, weakly normalizing and non-erasing then it is strongly normalizing (and hence confluent and complete) (Theorem 3.13(3)).
- If a TRS $\mathcal{R}$ is a strongly innermost normalizing, locally confluent overlay system then it is strongly normalizing (and hence confluent and complete) (Theorem 3.23).

- Strong innermost normalization is a modular property of TRSs (see Lemma 4.1).
- Strong normalization (and hence completeness) is modular for locally confluent overlay systems (Theorem 4.2).
- Strong normalization (and hence completeness) is modular for non-overlapping TRSs (Corollary 4.3).
- The union of two constructor systems with disjoint sets of defined symbols is complete if and only if both systems are complete (Theorem 4.7; cf. also [27]).
- A combined system with shared constructors is a complete overlay system if both component TRSs are complete overlay systems (Theorem 4.10).

The rest of the paper which is based on [11][1] is structured as follows. Firstly, we introduce the basic definitions and notions needed later on. In section 3 we review known results on orthogonal TRSs, and present new results for non-overlapping systems and for locally confluent overlay systems. Then, in section 4, we consider modular properties of TRSs, applying previously developed ideas and results. And finally, related work as well as some open problems are discussed.

2 Preliminaries

2.1 Abstract Reduction Systems

An *abstract reduction system* (ARS for short) is a pair $\mathcal{A} = \langle A, \to \rangle$ consisting of a set A and a *reduction relation*, i.e., a binary relation $\to \subseteq A \times A$ for which we use infix notation. A *reduction sequence* or *derivation* (in $\mathcal{A}$) is a (finite or infinite) sequence $a_1 \to a_2 \to a_3 \to \ldots$. For $b \to a$ we also write $a \leftarrow b$. The symmetric, transitive, transitive-reflexive and symmetric-transitive-reflexive closures of $\to$ are denoted by $\leftrightarrow$, $\to^+$, $\to^*$ and $\overset{*}{\leftrightarrow}$, respectively. If $a \to^* b$ we say that a *reduces* or *rewrites* to b and we call b a *reduct* of a. By $a \to^m b$ we mean that a reduces to b in m steps. Accordingly $a \to^{\leq n} b$ means $a \to^m b$ for some $m \leq n$. Two elements $a, b \in A$ are *joinable*, denoted by $a \downarrow b$, if there exists an element $c \in A$ with $a \to^* c {}^*\!\leftarrow b$. An element $a \in A$ is *irreducible* or *a normal form* if there is no $b \in A$ with $a \to b$. An element $a \in A$ *has a normal form* if there exists a normal form $b \in A$ with $a \to^* b$. In that case b is called a *normal form of* a.

If an ARS $\mathcal{A} = \langle A, \to \rangle$ has a certain property P (denoted by $P(\mathcal{A})$), we also say that $\to$ has the property P (and write $P(\to)$). This notation is extended in a straightforward manner to local versions, i.e. corresponding properties of single elements of $\mathcal{A}$, provided there is a sensible local interpretation of P. In order to make clear the underlying reduction relation $\to$ we shall also write $P(a, \to)$ instead of $P(a)$ (for $a \in A$).

$\mathcal{A} = \langle A, \to \rangle$ is said to be *weakly normalizing* (or *weakly terminating*) if every element of A has a normal form. $\mathcal{A}$ is *strongly normalizing (SN)* (or *(strongly) terminating* or *noetherian*) if there is no infinite reduction sequence $a_1 \to a_2 \to a_3 \to \ldots$, i.e. every reduction sequence eventually ends in some normal form. $\mathcal{A}$ is *confluent* or *Church-Rosser (CR)* if for all $a, b, c \in A$ with $b {}^*\!\leftarrow a \to^* c$ we have $b \downarrow c$. $\mathcal{A}$ is *locally*

[1] In [11] some additional results on the preservation of normalization properties under hierarchical combinations of TRSs are proved.

confluent or *weakly Church-Rosser (WCR)* if for all $a, b, c \in A$ with $b \leftarrow a \rightarrow c$ we have $b \downarrow c$. A confluent and strongly normalizing system is said to be *convergent* or *complete (COMP)*.

The following is a fundamental result due to Newman [28] which will be needed later on (in particular its strengthened localized version (b)).

Lemma 2.1 *(Newman's Lemma) Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS.*

(a) $WCR(\mathcal{A}) \wedge SN(\mathcal{A}) \implies CR(\mathcal{A})$.

(b) $\forall a \in A\,[\, WCR(\mathcal{A}) \wedge SN(a) \implies CR(a)\,]$.

(c) $\forall a \in A\,[\,[\,\forall b \in A\,[\, a \rightarrow^* b \implies WCR(b)\,]\,] \wedge SN(a) \implies CR(a)\,]$.

Note that the localized versions (b) and (c) are easily obtained from (a) by considering the 'sub-ARS' of $\mathcal{A}$ determined by the element a.

Moreover we shall need another result due to Newman [28] which allows to infer strong normalization from restricted normalization and confluence properties. Let WCR^1 be the following property of ARSs $\langle A, \rightarrow \rangle$: For all $a, b, c \in A$ with $b \neq c$ and $b \leftarrow a \rightarrow c$ there exists $d \in A$ with $b \rightarrow d \leftarrow c$.

Lemma 2.2 *([28]) For any ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ with $WCR^1(\mathcal{A})$ we have:*

(a) $WN(\mathcal{A}) \implies SN(\mathcal{A})$, *and*

(b) $\forall a \in A\,[\, WN(a) \implies SN(a)\,]$.

Note again that the stronger localized version (b) is easily obtained from (a) by considering the 'sub-ARS' of $\mathcal{A}$ determined by the element a (cf. e.g. [16]).

Furthermore let $\mathrm{WCR}^{\leq 1}$ be defined by: For all $a, b, c \in A$ with $b \leftarrow a \rightarrow c$ there exists $d \in A$ with $b \rightarrow^{\leq 1} d \leftarrow^{\leq 1} c$.

Then we obtain the following confluence criterion.

Lemma 2.3 *([28]) Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS.*

(a) $WCR^{\leq 1}(\mathcal{A}) \implies CR(\mathcal{A})$.

(b) $WCR^1(\mathcal{A}) \implies WCR^{\leq 1}(\mathcal{A})$, *hence also* $WCR^1(\mathcal{A}) \implies CR(\mathcal{A})$.

2.2 Term Rewriting Systems

We briefly recall the basic terminology needed for dealing with TRSs which are special ARSs (see e.g. [16], [6]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated to every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an

obvious way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$. In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. The set of variables (function symbols) occurring in a term t is denoted by $V(t)$ ($F(t)$). The top symbol of a term t is denoted by $root(t)$.

A *context* $C[,\ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$[2] where $\Box$ is a new special constant symbol. If $C[,\ldots,]$ is a context with n occurrences of $\Box$ and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ is the term obtained from $C[,\ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1,\ldots,t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. A *non-empty* context is a term from $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ with $t = C[s]$. If in addition $C[] \neq \Box$ then s is a *proper* subterm of t. A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma) = \{x \in \mathcal{V} \mid \sigma x \neq x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r) denoted by $l \to r$ with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$.[3] Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant.

Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\to_{\mathcal{R}^{\mathcal{F}}}$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s = C[\sigma l]$ and $t = C[\sigma r]$. The subterm σl of s which is rewritten (or *reduced*, *contracted*) is called a *redex* of s, the resulting term t a *reduct* of s. We also write $\to_{\mathcal{R}}$ or simply $\to$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$ is clear from the context, respectively. *Positions* or *occurrences* of subterms of a term consist of sequences of natural numbers and are compared by the usual lexicographic ordering (which we shall ambiguously denote by $\leq$). The set of all positions of a term s is denoted by $O(s)$. The topmost position of a term is denoted by λ, the 'empty' string. Two uncomparable positions p and q are said to be *parallel* or *disjoint* which is denoted by $p|q$. If $p \leq q$ we say that p is above q or q is below p. The result of replacing in s the subterm at position p by t is denoted by $s[p \leftarrow t]$. If $s \to t$, then, in order to make explicit the position p of the reduced subterm and the applied rule $l \to r$, we shall sometimes use the notation $s \to_{p,l\to r} t$ or $s \to_p t$. A step of the form $s \to_\lambda t$ is said to be a *root reduction (step)*. A reduction step $s \to t$ by applying some rule of $\mathcal{R}$ at position p in s is *innermost* if every proper subterm of s/p is irreducible. In that case we also write $s \xrightarrow[i]{} t$. $\mathcal{R}$ is *(strongly) innermost normalizing (SIN)* or *(strongly) innermost terminating* if $\xrightarrow[i]{}$ is strongly normalizing. $\mathcal{R}$ is *weakly innermost normalizing (WIN)* or *weakly innermost terminating* if $\xrightarrow[i]{}$ is weakly normalizing.

If $l_1 \to r_1$, $l_2 \to r_2$ are two rules (w.l.o.g. we assume that they do not have common variables) of $\mathcal{R}$ and p some non-variable position of l_2 such that l_1 and l_2/p are unifiable with most general unifier σ then $(\sigma(l_2[p \leftarrow r_1]), \sigma(r_2))$ is said to be a *critical pair (CP)* of $\mathcal{R}$ (obtained by overlapping $l_1 \to r_1$ into $l_2 \to r_2$ at position p). If the two rules are renamed copies of the same rule we do not consider the case $p = \lambda$. It is well-known that by means of critical pairs the test for local confluence of TRSs can be localized.

Lemma 2.4 *([14] Critical Pair Lemma) A TRS is locally confluent if and only if all*

[2] The symbol $\uplus$ is to denote disjoint union.

[3] This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

its critical pairs are joinable.

Combining Newman's Lemma and the Critical Pair Lemma one obtains the following result which is the basis for *completion techniques* for TRSs.

Theorem 2.5 *([17]) A strongly normalizing TRS is confluent if and only if all its critical pairs are joinable.*

A TRS $\mathcal{R}$ is said to be *non-overlapping (NO)* if there is no critical pair between rules of $\mathcal{R}$. It is *left-linear (LL)* if every variable occurs at most once in every left-hand side of an $\mathcal{R}$-rule. $\mathcal{R}$ is *orthogonal* if it is left-linear and non-overlapping. It is *non-erasing (NE)* if $V(r) = V(l)$ for every rule $l \to r \in \mathcal{R}$. If every critical pair of a TRS $\mathcal{R}$ is obtained by an *overlay*, i.e. by overlapping left-hand sides of rules at top positions, then $\mathcal{R}$ is said to be an *overlay system (OS)*.

For properties P and Q of TRSs we write $P \,\&\, Q$ for denoting the conjunction of P and Q. By $P(\mathcal{R})$ we mean that the TRS $\mathcal{R}$ (considered as specialized ARS) has property P. Moreover we also use the notation $P(t)$ for terms t provided there is a (sensible) local interpretation for $P(t)$. For instance, $\mathrm{CR}(t)$ is to denote the property that whenever we have $t \to^* v$ and $t \to^* w$ then there exists a term s with $v \to^* s$ and $w \to^* s$.

3 Restricted Termination and Confluence Properties of Term Rewriting Systems

In the following we shall study under which conditions various restricted kinds of normalization imply strong normalization (and possibly also confluence under some additional assumptions). First we consider orthogonal TRSs, i.e. systems which are left-linear and have no critical pairs. Then, admitting non-left-linearity, the no-overlap case is considered. And finally, we generalize the analysis by admitting overlaps, but only at top positions. Moreover, where possible, we give examples showing that no precondition of results obtained can be dropped.

According to Theorem 2.5 for strongly normalizing TRSs the confluence property can be easily tested by simply investigating joinability of all critical pairs (for finite systems this can can be done effectively). If strong normalization is not guaranteed it is in general much more difficult to establish confluence. The problem is that for non-terminating systems joinability of critical pairs is only equivalent to local confluence but not equivalent to confluence any more. There are some sufficient criteria for confluence which do not depend on the strong normalization property but instead on rather strong conditions concerning the syntactical form of the rules and on how joinability of critical pairs has to be possible (see e.g. [14], [16]). Some of them are based on corresponding abstract confluence criteria for ARSs (e.g. Lemma 2.3). Due to the strong preconditions the practical applicability of these criteria is rather limited.

Our overall goal in the following will be to establish strong normalization (and confluence) by looking for sufficient conditions involving restricted normalization (and confluence) properties.

3.1 Orthogonal Systems

Perhaps the most important and fundamental result concerning confluence of (possibly) non-terminating TRSs is the following: Any orthogonal TRS is confluent (cf. e.g. [33]). This fundamental property is crucial for instance within the field of designing and implementing equational programming languages (cf. e.g. [29], [30]) and has initiated a couple of investigations about the class of orthogonal TRSs.

In particular, for orthogonal TRSs one also knows some sufficient criteria for strong normalization which are formulated in terms of restricted normalization properties (cf. e.g. [29], [16]). But as soon as the orthogonality requirement is weakened, either by allowing critical overlaps or by admitting non-left-linear rules, the main results (at least concerning confluence) do not hold any more, in particular the so-called 'parallel moves lemma' (cf. [14], [33]) which is the technical key lemma for inferring confluence for orthogonal TRSs.

The most important known results about confluence and termination properties of orthogonal TRSs can be summarized as follows.

Theorem 3.1 *(cf. e.g. [33], [29], [16]) Any orthogonal TRS $\mathcal{R}$ is confluent and satisfies the following properties:*

(1a) $\forall t : [\, WIN(t) \iff SIN(t) \iff SN(t)\,]$.

(1b) $WIN(\mathcal{R}) \iff SIN(\mathcal{R}) \iff SN(\mathcal{R})$.

(2) If $s \xrightarrow{}_{i} t$ *and* $SN(t)$ *then* $SN(s)$.

(3a) $NE(\mathcal{R}) \implies [\, \forall t : [\, WN(t) \iff WIN(t) \iff SIN(t) \iff SN(t)\,]\,]$.

(3b) $NE(\mathcal{R}) \implies [\, WN(\mathcal{R}) \iff WIN(\mathcal{R}) \iff SIN(\mathcal{R}) \iff SN(\mathcal{R})\,]$.

The following example shows that if the left-linearity condition in Theorem 3.1 is dropped then confluence can get lost.

Example 3.2 *([14]) Let $\mathcal{R}$ be the TRS given by the rules*

$$f(x,x) \to a\,, \quad f(x,g(x)) \to b\,, \quad c \to g(c)\,.$$

Clearly, $\mathcal{R}$ is non-overlapping but neither left-linear nor confluent. We have e.g. the derivations

$$f(c,c) \to a \quad \textit{and} \quad f(c,c) \to f(c,g(c)) \to b\,.$$

Here a and b are in normal from. Note that $\mathcal{R}$ is obviously not strongly normalizing and moreover neither weakly innermost normalizing nor weakly normalizing (since for instance c does not possess a normal form).

Weak normalization is not crucial for the existence of such counterexamples as shown by the following example.

Example 3.3 *(Sivakumar'86 [6]) Let $\mathcal{R}$ be the TRS given by the rules*

$$f(x,x) \to g(x)\,, \quad f(x,g(x)) \to b\,, \quad h(c,y) \to f(h(y,c),h(y,y))\,.$$

Clearly, $\mathcal{R}$ is non-overlapping but not confluent. We have e.g. the derivations

$$h(c,c) \to f(h(c,c),h(c,c)) \to f(h(c,c),f(h(c,c),h(c,c))) \to f(h(c,c),g(h(c,c))) \to b$$

and

$$h(c,c) \to f(h(c,c),h(c,c)) \to g(h(c,c)) \to^+ g(b)\,.$$

Here b and $g(b)$ are in normal from. Note that $\mathcal{R}$ is obviously not strongly normalizing and even not weakly innermost normalizing (consider e.g. the term $h(c,c)$) but weakly normalizing.

3.2 Non-Overlapping Systems

In the following we shall consider TRSs which are still non-overlapping but not necessarily left-linear. In fact, we will show that all normalization properties of Theorem 3.1 still hold for this more general class of TRSs. Throughout the following we assume that $\mathcal{R}$ is a non-overlapping TRS, i.e. NO($\mathcal{R}$) holds.

Let us start with an easy result about innermost reduction.

Lemma 3.4 *Innermost reduction in $\mathcal{R}$ is WCR^1, i.e. $WCR^1(\ \underset{i}{\to})$ holds.*

Proof: Let $s\ \underset{i}{\to}_p\ t$ and $s\ \underset{i}{\to}_q\ u$. If the innermost redex positions p, q of s are the same then the applied rule is unique due to $NO(\mathcal{R})$ which implies $t = u$. Otherwise p and q are disjoint and v is uniquely defined by $s\ \underset{i}{\to}_p\ t\ \underset{i}{\to}_q\ v$ and $s\ \underset{i}{\to}_q\ u\ \underset{i}{\to}_p\ v$. ■

Combined with Lemma 2.3(b) this yields the following.

Corollary 3.5 *Innermost reduction in $\mathcal{R}$ is confluent, i.e. $CR(\ \underset{i}{\to})$ holds.*

Next we show that the existence of an innermost normal form for some term t implies that any innermost derivation initiated by t is finite.

Lemma 3.6 $\forall t : [\, WIN(t) \iff SIN(t)\,]$.

Proof: By Lemma 3.4 we know that innermost reduction is WCR^1. Hence, observing that WIN (SIN) is equivalent to "innermost reduction is WN (SN)", we can simply apply Lemma 2.2(b) which yields the desired result. ■

Note that for the proof of Lemma 3.4 it suffices that $\mathcal{R}$ lacks non-trivial critical pairs. Hence also Corollary 3.5 and Lemma 3.6 hold for this larger class of TRSs.

Furthermore, strong innermost normalization is equivalent to strong normalization.

Theorem 3.7 $\forall t : [\, SIN(t) \iff SN(t)\,]$.

Proof: For a proof we refer to the more general Theorem 3.23. ■

Combined with Lemma 3.6 this yields the following

Corollary 3.8 $\forall t : [\, \mathit{WIN}(t) \iff \mathit{SIN}(t) \iff \mathit{SN}(t)\,]$.

The next result says that innermost reduction steps in non-overlapping TRSs cannot be critical in the sense that they may destroy the possibility of infinite derivations.

Lemma 3.9 *If* $s \xrightarrow[i]{} t$ *and* $\mathit{SN}(t)$ *then* $\mathit{SN}(s)$.

Proof: This is an immediate consequence of Corollary 3.8. ■

The properties mentioned in Corollary 3.8 are in general not equivalent to $\mathrm{WN}(t)$. To see this, consider the following

Example 3.10 $\mathcal{R} := \{a \to f(a), f(x) \to b\}$ *is clearly a non-overlapping TRS. We have* $\mathit{WN}(f(a))$ *since for instance* $f(a) \to b$ *with* b *irreducible, but obviously not* $\mathit{SN}(f(a))$.

Note that in this example the innermost redex a which is not strongly normalizing disappears by applying the *erasing* rule $f(x) \to b$. In fact, the absence of erasing rules turns out to be sufficient for the equivalence of all mentioned normalization properties as shown next.

Lemma 3.11 *Suppose* $\mathit{NE}(\mathcal{R})$. *If* $s \to t$ *and* $\mathit{SN}(t)$ *then* $\mathit{SN}(s)$.

Proof: Let $s \to t$ and $\mathrm{SN}(t)$. By Lemma 3.9 it is sufficient to consider the case that $s \to t$ is a non-innermost reduction step, let's say with rule $l \to r$ and matching substitution σ: $s = C[\sigma l] \to C[\sigma r] = t$. $\mathrm{NO}(\mathcal{R})$ implies that σx is reducible for some $x \in V(l)$. Now $\mathrm{SN}(t)$ implies in particular $\mathrm{SN}(\sigma x)$ for all $x \in V(r)$. Hence, due to $\mathrm{NE}(\mathcal{R})$ we know $\mathrm{SN}(\sigma x)$ for all $x \in V(l)$. By innermost normalizing σ we thus obtain $s = C[\sigma l] \xrightarrow[i]{}{}^{+} C[\sigma' l] \to C[\sigma' r]$ and $t = C[\sigma r] \xrightarrow[i]{}{}^{*} C[\sigma' r]$ with $\sigma' x$ a normal form, for all $x \in V(l) = V(r)$. Furthermore, the step $C[\sigma' l] \to C[\sigma' r]$ is innermost because of $\mathrm{NO}(\mathcal{R})$. Hence, from $s \xrightarrow[i]{}{}^{*} C[\sigma' r]$, $t \xrightarrow[i]{}{}^{*} C[\sigma' r]$ and $\mathrm{SN}(t)$ we obtain $\mathrm{WIN}(s)$ which, by Corollary 3.8, yields $\mathrm{SN}(s)$. ■

Lemma 3.12 *If* $\mathit{NE}(\mathcal{R})$ *then :* $\forall t : [\, \mathit{WN}(t) \iff \mathit{WIN}(t) \iff \mathit{SIN}(t) \iff \mathit{SN}(t)\,]$.

Proof: Let t be a term with $\mathrm{WN}(t)$. This means there exists some normalizing derivation from t:

$$D : t =: t_0 \to t_1 \to t_2 \ldots \to t_n$$

with t_n in normal form, hence in particular $\mathrm{SN}(t_n)$. By a straightforward induction on the length of the derivation D we obtain $\mathit{SN}(t)$ from $\mathrm{SN}(t_n)$ using Lemma 3.11. Hence we are done. ■

Finally let us summarize the results obtained for non-overlapping, but not necessarily left-linear TRSs.

Theorem 3.13 *Any non-overlapping TRS $\mathcal{R}$ satisfies the following properties:*

(1a) $\forall t : [\, \mathit{WIN}(t) \iff \mathit{SIN}(t) \iff \mathit{SN}(t)\,]$.

(1b) $\mathit{WIN}(\mathcal{R}) \iff \mathit{SIN}(\mathcal{R}) \iff \mathit{SN}(\mathcal{R})$.

(2) If $s \xrightarrow{}_{\mathrm{i}} t$ *and* $\mathit{SN}(t)$ *then* $\mathit{SN}(s)$.

(3a) $\mathit{NE}(\mathcal{R}) \Longrightarrow [\forall t : [\, \mathit{WN}(t) \iff \mathit{WIN}(t) \iff \mathit{SIN}(t) \iff \mathit{SN}(t)\,]]$.

(3b) $\mathit{NE}(\mathcal{R}) \Longrightarrow [\, \mathit{WN}(\mathcal{R}) \iff \mathit{WIN}(\mathcal{R}) \iff \mathit{SIN}(\mathcal{R}) \iff \mathit{SN}(\mathcal{R})\,]$.

(4) $\mathit{WIN}(\mathcal{R}) \Longrightarrow \mathit{CR}(\mathcal{R})$.

(5) $\mathit{NE}(\mathcal{R}) \wedge \mathit{WN}(\mathcal{R}) \Longrightarrow \mathit{CR}(\mathcal{R})$.

3.3 Locally Confluent Overlay Systems

Considering Theorem 3.13(1) one may ask now whether instead of requiring NO($\mathcal{R}$) and WIN($\mathcal{R}$) it could also be sufficient to require WCR($\mathcal{R}$) and WIN($\mathcal{R}$) for guaranteeing SN($\mathcal{R}$) (and CR($\mathcal{R}$)). But this is not sufficient as witnessed by

Example 3.14 *For the TRS* $\mathcal{R} = \{b \to a, b \to c, c \to b, c \to d\}$ *we clearly have* $\mathit{WCR}(\mathcal{R})$ *and* $\mathit{WIN}(\mathcal{R})$ *but neither* $\mathit{SN}(\mathcal{R})$ *nor* $\mathit{CR}(\mathcal{R})$.

Even the stronger requirement WCR($\mathcal{R}$) and SIN($\mathcal{R}$) is not sufficient for ensuring SN($\mathcal{R}$) (and CR($\mathcal{R}$)) as can be seen from

Example 3.15 *Consider the TRS* $\mathcal{R} = \{f(b) \to a, f(b) \to f(c), f(c) \to f(b), f(c) \to d, b \to e, c \to e', f(e) \to a, f(e') \to d\}$. *Here it is easily verified that* $\mathit{WCR}(\mathcal{R})$ *and* $\mathit{SIN}(\mathcal{R})$ *hold but neither* $\mathit{SN}(\mathcal{R})$ *nor* $\mathit{CR}(\mathcal{R})$.

Another possibly tempting conjecture might be to insist on SIN($\mathcal{R}$) and even require CR($\mathcal{R}$) instead of the weaker WCR($\mathcal{R}$) in order to infer SN($\mathcal{R}$). But this is also not true in general.

Example 3.16 *Consider the TRS* $\mathcal{R}$ *given by* $f(a) \to f(a)$, $a \to b$ *for which we have* $\mathit{CR}(\mathcal{R})$ *and* $\mathit{SIN}(\mathcal{R})$ *but not* $\mathit{SN}(\mathcal{R})$.

But a common feature of the latter two counterexamples consists in the fact that for both systems critical pairs were constructed by overlaps strictly below the root. This is crucial as will be shown. More precisely, we will prove that any strongly innermost normalizing, locally confluent overlay system is strongly normalizing (hence also confluent and complete). To this end, let us assume in the following that we are dealing with a TRS $\mathcal{R}$ which is locally confluent, i.e., WCR($\mathcal{R}$) holds.

To simplify the presentation we introduce some easy definitions. Every term t can be (uniquely) written as $C[t_1, \ldots, t_n]$ where $t_1, \ldots, t_n$ are the maximal complete subterms of t. We define $\Phi(t) = C[t_1{\downarrow}, \ldots, t_n{\downarrow}]$. Here $t_i{\downarrow}$ denotes the unique normal form of t_i. Clearly we have $t \to^* \Phi(t)$. Because of the global assumption WCR($\mathcal{R}$) and Lemma 2.1(b) a term is complete if and only if it is strongly normalizing. Hence a term is complete if and only if all its subterms are complete. For an arbitrary reduction step

$s \to t$ we write $s \to_c t$ if the contracted redex is complete, and $s \to_{nc} t$ if the contracted redex is not complete. Clearly every reduction step can be written as either $s \to_c t$ or $s \to_{nc} t$. Note that whenever $s \to_c t$ is a root reduction step then this implies $\mathrm{SN}(s, \to_c)$ (which in this case is equivalent to $\mathrm{SN}(s, \to)$. Furthermore, if $C[s] \to_c C[t]$ (by contracting some redex in s) then $s \to_c t$, and vice versa.

Lemma 3.17 *The relation $\to_c$ is strongly normalizing, i.e., for all terms s we have $SN(s, \to_c)$.*

Proof: We proceed by induction on the term structure. If s is a variable we are obviously done. If s is a constant then it is either irreducible or else any reduction step $s \to_c t$ is a root reduction step. In both cases s is complete implying $\mathrm{SN}(s, \to_c)$. For the induction step consider the case $s = f(t_1, \ldots, t_n)$ $(n \geq 1)$. Assume $D : s =: s_0 \to_c s_1 \to_c s_2 \to_c \ldots$ is an infinite $\to_c$-derivation. Since we have $\mathrm{SN}(t_i, \to_c)$ for all i by induction hypothesis, some (first) step in D must be a root reduction step $s_k \to_c s_{k+1}$. But then s_k is complete which contradicts the infinity of D. ■

Lemma 3.18 *Suppose $\neg SN(\mathcal{R})$. Then every infinite derivation contains infinitely many $\to_{nc}$-steps.*

Proof: This is an immediate consequence of Lemma 3.17. ■

Lemma 3.19 *If $s \to_c t$ then $\Phi(s) \to^* \Phi(t)$.*

Proof: Clearly we have $t \to^* \Phi(s)$ by performing only reductions in complete subterms of t. Hence we get $\Phi(s) \to^* \Phi(t)$. ■

Observe that in general we do not have $\Phi(s) = \Phi(t)$ in Lemma 3.19 since the step $s \to_c t$ within some maximal complete subterm s/p of s may generate a new maximal complete subterm t/q in t with $q < p$. Take for instance $\mathcal{R}_1 = \{a \to b, f(a) \to f(a), f(b) \to c\}$ and consider the step $s = f(a) \to_c f(b) = t$ for which we have $\Phi(s) = f(b) \to c = \Phi(t)$. The analogous statement (of Lemma 3.19) for $\to_{nc}$ does not hold. Consider for instance the TRS $\mathcal{R}_2 = \{a \to b, f(a) \to g(a), g(x) \to f(x)\}$ and the step $s = f(a) \to_{nc} g(a) = t$. Here we have $\Phi(s) = f(b)$ and $\Phi(t) = g(b)$, but not $\Phi(s) \to^* \Phi(t)$. Note that $\mathcal{R}_2$ is not an overlay system. This is essential as will be shown next.

To this end we need the following crucial property of overlay systems.

Lemma 3.20 *Suppose $OS(\mathcal{R})$. If l is a left-hand side of a rule from $\mathcal{R}$ and σ a substitution such that σl is not complete then $\Phi(\sigma l) = (\Phi \circ \sigma)l$. Here $\Phi \circ \sigma$ denotes the composition of the substitution σ with the mapping Φ, i.e., the substitution defined by $(\Phi \circ \sigma)x = \Phi(\sigma x)$.*

Proof: We have to show that normalization of all maximal complete subterms in σl can be achieved by normalizing all maximal complete subterms in the "substitution part σ of σl". If no subterm of σl is complete we clearly obtain $\Phi(\sigma l) = (\Phi \circ \sigma)l$ by definition of Φ. Hence, we may assume that some subterm of σl is complete. Let

$\sigma l = C[t_1, \ldots, t_n]$, $n \geq 1$, where the t_i's are the maximal complete subterms of σl, let's say with $\sigma l/p_i = t_i$. Note that, due to the assumption that σl is not complete, we have $\lambda < p_i$ for all p_i. Now, if p_i is below the position p of some variable x in l then we get $t_i\downarrow = (\Phi \circ \sigma)(l)/p_i$ since t_i is also a maximal complete subterm of σx. If p_i is a non-variable position of l then we have $t_i = \sigma(l/p_i)$. Since t_i is complete, for every variable x which occurs in l (strictly) below p_i, σx is also complete. Let $\sigma' x = (\Phi \circ \sigma)(x) = (\sigma x)\downarrow$ for these variables. By definition of Φ we get $t_i\downarrow = \sigma(l/p_i)\downarrow = \sigma'(l/p_i)\downarrow$. We still have to show $\sigma'(l/p_i)\downarrow = \sigma'(l/p_i)$. From OS($\mathcal{R}$), irreducibility of σ' and $\lambda < p_i$ we conclude that $\sigma'(l/p_i)$ must be irreducible (because otherwise there would exist a critical pair in $\mathcal{R}$ which is not an overlay). Hence we are done. ■

Lemma 3.21 *Suppose OS($\mathcal{R}$). If $s \to_{nc} t$ then $\Phi(s) \to^+ \Phi(t)$*

Proof: Suppose $s \to_{nc} t$ by applying some rule $l \to r$ at position p with substitution σ, so $s/p = \sigma l$ and $t = s[p \leftarrow \sigma r]$. Because s/p is not complete, p is a position in $\Phi(s)$. Now, due to OS($\mathcal{R}$) we can apply Lemma 3.20 which yields $\Phi(s)/p = \tau l$ with substitution τ defined by $\tau x = \Phi(\sigma x)$ for all variables x. Hence we obtain $\Phi(s) \to \Phi(s)[p \leftarrow \tau r]$ again by the rule $l \to r$. Clearly we have $t \to^* \Phi(s)[p \leftarrow \tau r]$ by performing only reductions in complete subterms of t, and thus $\Phi(s)[p \leftarrow \tau r] \to^* \Phi(t)$. This implies $\Phi(s) \to \Phi(s)[p \leftarrow \tau r] \to^* \Phi(t)$, and hence $\Phi(s) \to^+ \Phi(t)$. ■

Lemma 3.22 *Suppose OS($\mathcal{R}$). If SN($\Phi(t)$) then SN(t)*

Proof: Assume $\neg$SN(t), i.e. there exists an infinite derivation

$$D : t = t_0 \to t_1 \to t_2 \to \ldots$$

where every step is either an $\to_c$-step or a $\to_{nc}$-step. By Lemma 3.18 there are infinitely many $\to_{nc}$-steps in D. Hence, by applying Φ to D and using Lemmas 3.21 and 3.19 we obtain the infinite derivation

$$\Phi(D) : \Phi(t) = \Phi(t_0) \to^* \Phi(t_1) \to^* \Phi(t_2) \to \ldots$$

which implies $\neg$SN($\Phi(t)$). ■

Now we are prepared to prove the main result.

Theorem 3.23 *Suppose OS($\mathcal{R}$). If SIN(t) then SN(t). (Taking into account the general assumption WCR($\mathcal{R}$), the global version of this result may be rephrased as follows: Any strongly innermost normalizing, locally confluent overlay system is strongly normalizing.)*

Proof: For a proof by contradiction, suppose that t admits an infinite derivation. Every infinite derivation starting from t must contain a non-innermost step, due to SIN(t). Now consider an infinite derivation D starting from t with the property that the first non-innermost step is essential: Selecting any innermost redex at that point would result in a term with the property SN. Let

$$D : t = t_0 \to t_1 \to \ldots \to t_n \to t_{n+1} \to \ldots$$

where $t_n \to t_{n+1}$ is the first non-innermost step. By assumption, contracting an innermost redex in t_n yields a strongly normalizing term. This implies that every innermost redex in t_n is complete. Since there is at least one innermost redex in t_n (the step $t_n \to t_{n+1}$ is non-innermost!), we conclude that $\mathrm{SN}(\Phi(t_n))$ holds. Since we also have $\neg\mathrm{SN}(t)$, this contradicts Lemma 3.22. ■

Note that we cannot weaken the precondition of Theorem 3.23 by omitting the global assumption $\mathrm{WCR}(\mathcal{R})$. To wit, consider

Example 3.24 *Let $\mathcal{R}$ be the TRS consisting of the rules*

$$f(a,b,x) \to f(x,x,x)\,, \quad G(x,y) \to x\,, \quad G(x,y) \to y\,.$$

Obviously, $\mathcal{R}$ is an strongly innermost normalizing overlay system but it is not strongly normalizing as can be seen from the infinite (cyclic) derivation

$$\begin{array}{rcl} f(a,b,G(a,b)) & \to & f(G(a,b),G(a,b),G(a,b)) \\ & \to & f(a,G(a,b),G(a,b)) \\ & \to & f(a,b,G(a,b)) \\ & \to & \cdots\,. \end{array}$$

Moreover, omitting the condition $\mathrm{OS}(\mathcal{R})$ in Theorem 3.23 is not possible, either, as demonstrated by Example 3.15.

According to Theorem 3.1(1a) (NO & LL & SIN) implies SN. Theorem 3.13(1) says that even (NO & SIN) implies SN. Moreover, from Theorem 3.23 we know that the weaker property (OS & WCR & SIN) implies SN, too. In view of these results another interesting conjecture would be the following:

$$(\mathrm{CR}\,\&\,\mathrm{LL}\,\&\,\mathrm{SIN}) \quad \Longrightarrow \quad \mathrm{SN}\,.$$

But again this is not true in general.

Example 3.25 *(Example 3.16 continued) Let $\mathcal{R}$ be given by the rules $f(a) \to f(a)$ and $a \to b$. Obviously, this system is confluent, left-linear and strongly innermost normalizing but is clearly not strongly normalizing.*

Considering Theorems 3.13 and 3.23 one might be tempted to state the following conjectures by weakening the requirement $\mathrm{NO}(\mathcal{R})$ to $\mathrm{OS}(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R})$.

$(C1)$ $\quad \mathrm{OS}(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R}) \implies [\,\mathrm{WIN}(\mathcal{R}) \implies \mathrm{SIN}(\mathcal{R})\,]\,,$

$(C2)$ $\quad \mathrm{OS}(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R}) \implies [\,s \underset{i}{\to} t \wedge \mathrm{SN}(t) \implies \mathrm{SN}(s)\,]$, and

$(C3)$ $\quad OS(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R}) \wedge NE(\mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}) \implies \mathrm{SN}(\mathcal{R})\,]$.

But $(C1)$, $(C2)$ and $(C3)$ are all refuted by the following very simple counterexample.

Example 3.26 *Let $\mathcal{R}$ consist of the two rules $a \to a$ and $a \to b$. Clearly, $\mathcal{R}$ is a non-erasing overlay system where the only critical pair is a joinable overlay. Moreover, every term has a normal form that can be computed by innermost reduction, but obviously $\mathcal{R}$ is not strongly innermost normalizing and hence not strongly normalizing, too. Furthermore we have $a \underset{i}{\to} b$ with SN(b), but not SN(a).*

4 Applications to Disjoint and Non-Disjoint Unions of TRSs

We shall now consider modular properties of TRSs and apply our previous results in order to derive some interesting new results and provide simplified proofs of known ones. Let us first give a brief (and incomplete) overview of known results in this field. Modular properties of term rewriting systems, i.e. properties which are preserved under disjoint unions, have attracted an increasing attention within the last few years. From a theoretical point of view and also for efficiency reasons it is very useful to know whether a combined TRS has some property whenever this property already holds for the single 'modules'. A simple and natural way of such 'modular' constructions is given by the concept of 'direct sum' ([36]) or 'disjoint union'. Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over disjoint signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $\mathcal{R}_1 \oplus \mathcal{R}_2$. We shall also speak of the disjoint union of $\mathcal{R}_1$ and $\mathcal{R}_2$ using the implicit convention that $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be disjoint TRSs. A property P of TRSs is said to be *modular* if the following holds for all disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property P iff both $\mathcal{R}_1$ and $\mathcal{R}_2$ have property P. Later on we shall also consider invariance properties of combination mechanisms which are more general than 'direct sum modularity'.

Toyama [36] has shown that confluence is modular. The strong normalization property, however, is in general not modular as witnessed by Example 3.24, partitioning the TRS there into $\mathcal{R}_1 = \{f(a,b,x) \to f(x,x,x)\}$ and $\mathcal{R}_2 = \{G(x,y) \to x, G(x,y) \to y\}$.

Note, that in this example $\mathcal{R}_2$ is not confluent. Other, more complicated examples by Klop & Barendregt as well as by Toyama gathered in [35] show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ may be not strongly normalizing even if $\mathcal{R}_1$ and $\mathcal{R}_2$ are both strongly normalizing, confluent and interreduced.

All these counterexamples have some common feature. Namely, one of the systems contains a duplicating rule, i.e. a rule $l \to r$ where some variable occurs strictly more often in r than in l, and the other system contains a collapsing rule $l' \to r'$, i.e. r' is a variable. As proved in [34] strong normalization is modular for the class of collapse-free TRSs as well as for the class of non-duplicating TRSs. In [23] it is shown that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two strongly normalizing TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is again strongly normalizing whenever one of the systems is non-duplicating and collapse-free. Moreover, as shown in [21], simple termination is a modular property of (finite) TRSs, i.e. whenever two (finite) TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ can be shown to be terminating by means of simplification orderings then this holds for their disjoint union, too. A unified approach to modular termination of rewriting is provided in [10], [32] where still more general sufficient conditions for modularity of termination are presented. Some extensions and generalizations of these results on modular termination to the case of conditional TRSs as well as to some restricted classes of non-disjoint unions of TRSs can be found in [24], [22], [26], [12], [10] and [32]. Further interesting preservation results for (disjoint and) non-disjoint unions of TRSs are presented in [27], [25], [18], [19], [5], [8], [31]. An interesting result not subsumed by other ones is due to [37] where it is shown that completeness is modular for left-linear TRSs.

The main link between our abstract results proved in the previous section and the problem of modular termination of rewriting is provided by the following easy but fundamental result.

Lemma 4.1 *Strong innermost normalization (SIN) is a modular property of TRSs.*

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint strongly innermost normalizing TRSs. Then we show by structural induction that SIN(t) holds for all $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$. If t is a variable then SIN(t) is trivially satisfied. If t is a constant of $\mathcal{F}_1 \uplus \mathcal{F}_2$ then SIN(t) is satisfied by assumption. If $t = f(t_1, \ldots, t_n)$ then we have by induction hypothesis SIN(t_1)$\wedge \ldots \wedge$SIN(t_n). W.l.o.g. we may further assume $f \in \mathcal{F}_1$. Now, if t is irreducible (w.r.t. $\mathcal{R}_1 \uplus \mathcal{R}_2$) we are done. Otherwise, we know by induction hypothesis that for every $i \in \{1, \ldots, n\}$ every innermost derivation of t_i is eventually terminating. This means that every innermost derivation starting with t is either terminating or has the form $t = f(t_1, \ldots, t_n) \ {}_i\!\!\rightarrow^* f(t'_1, \ldots, t'_n) \rightarrow_\lambda \ldots$ where $t'_1, \ldots, t'_n$ are all irreducible (w.r.t. $\mathcal{R}_1 \uplus \mathcal{R}_2$). If in the latter case $t' := f(t'_1, \ldots, t'_n)$ were not strongly innermost normalizing, we could conclude that $\mathcal{R}_1$ is not strongly innermost normalizing (this is easily seen by replacing every maximal subterm $s_1, \ldots, s_m$ of t with $root(s_i) \in \mathcal{F}_2$ by fresh variables x_i such that $x_j = x_k$ whenever $s_j = s_k$).[4] ■

Note that weak normalization and weak innermost normalization are modular properties of TRSs, too, as shown in [1], [7] and [20].

Combining Lemma 4.1 with Theorem 3.23 we obtain

Theorem 4.2 *Termination (and hence completeness) is modular for the class of locally confluent overlay systems.*

Proof: Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint, locally confluent and strongly normalizing overlay systems. Hence we have (OS & WCR & SN)($\mathcal{R}_1$) and (OS & WCR & SN)($\mathcal{R}_2$) implying (OS & WCR & SIN)($\mathcal{R}_1$) and (OS & WCR & SIN)($\mathcal{R}_2$). Since both OS and WCR are obviously modular, and SIN, too, by Lemma 4.1 we conclude (OS & WCR & SIN)($\mathcal{R}_1 \oplus \mathcal{R}_2$). By applying Theorem 3.23 we finally obtain (OS & WCR & SN)($\mathcal{R}_1 \oplus \mathcal{R}_2$) and hence COMP($\mathcal{R}_1 \oplus \mathcal{R}_2$) as desired. The other direction is straightforward. ■

Observing that NO is obviously modular, too, we obtain the following consequence.

Corollary 4.3 *Termination (and hence completeness) is modular for the class of non-overlapping TRSs.*

Note that Theorem 4.2 is in a sense a variation of the main result of [37] which says that (LL & COMP) is modular. Theorem 4.2 states that the left-linearity restriction can be dropped if a stronger kind of confluence property is satisfied, namely, every critical pair must be a joinable overlay. In other words, (OS & COMP) is modular.

Moreover, Example 3.24 shows that dropping WCR is not possible because in general (OS & SN) is not modular.

Furthermore let us show now that Theorem 3.23 can be used to give a substantially simplified proof of the main result from [27]. For a precise formulation we need some terminology from [27].

[4] Note that *identifying abstraction*, i.e., taking the *same* fresh variable for all maximal aliens, instead of *injective abstraction* as here, does not preserve innermost redexes in general!

Definition 4.4 *([27]) Let $\mathcal{R}^{\mathcal{F}}$ be a TRS with $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ such that $\mathcal{D} = \{root(l) \mid l \to r \in \mathcal{R}^{\mathcal{F}}\}$ and $\mathcal{C} = \mathcal{F} \backslash \mathcal{D}$. The symbols in $\mathcal{D}$ are said to be* defined symbols *and those in $\mathcal{C}$* constructors. *Then $\mathcal{R}^{\mathcal{F}}$ is said to be a* constructor system (CS) *if $\mathcal{C} \supseteq \bigcup_{f(t_1,\ldots,t_{n_i}) \to r_i \in \mathcal{R}^{\mathcal{F}}} \bigcup_{j=1}^{n_i} F(t_j)$,[5] i.e. if no left-hand side argument contains a defined symbol.[6] To emphasize the partitioning of $\mathcal{F}$ into $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ we also write $(\mathcal{R}^{\mathcal{F}}, \mathcal{C}, \mathcal{D})$ or $(\mathcal{R}, \mathcal{C}, \mathcal{D})$ instead of $\mathcal{R}^{\mathcal{F}}$. If $(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{C}_1, \mathcal{D}_1)$, $(\mathcal{R}_2^{\mathcal{F}_2}, \mathcal{C}_2, \mathcal{D}_2)$ are constructor systems with $\mathcal{D}_1 \cap \mathcal{F}_2 = \mathcal{D}_1 \cap (\mathcal{C}_2 \uplus \mathcal{D}_2) = \emptyset = \mathcal{D}_2 \cap (\mathcal{C}_1 \uplus \mathcal{D}_1) = \mathcal{D}_2 \cap \mathcal{F}_1$ then $(\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2} = (\mathcal{R}_1 \cup \mathcal{R}_2, \mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D}_1 \uplus \mathcal{D}_2)$ is said to be a* combination of constructor systems (with disjoint sets of defined symbols and common constructors).

Middeldorp and Toyama have shown in [27] that completeness is preserved for combinations of constructor systems. In fact, a slightly more general result is proved in [27].

Since any constructor system is by definition a special overlay system we easily obtain from Theorem 3.23 the following

Corollary 4.5 *For any TRS $\mathcal{R}$ we have $CS(\mathcal{R}) \wedge SIN(\mathcal{R}) \wedge WCR(\mathcal{R}) \Longrightarrow SN(\mathcal{R}) \wedge CR(\mathcal{R})$, i.e. any strongly innermost normalizing, locally confluent constructor system is strongly normalizing and confluent, hence complete.*

Moreover we need the following result.

Lemma 4.6 *The combination of strongly innermost normalizing constructor systems (with disjoint sets of defined symbols) is again a strongly innermost normalizing constructor system.*

Proof: The proof is very similar to that of Lemma 4.1 and uses similar arguments as given in [27] in the proof of the fact that weak normalization is preserved under the union of constructor systems with disjoint sets of defined symbols.

Let $(\mathcal{R}, \mathcal{C}, \mathcal{D}) = ((\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}, \mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D}_1 \uplus \mathcal{D}_2)$ be the combination of two (strongly) innermost normalizing constructor systems $(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{C}_1, \mathcal{D}_1)$ and $(\mathcal{R}_2^{\mathcal{F}_2}, \mathcal{C}_2, \mathcal{D}_2)$. We will show by structural induction that for every term $t \in \mathcal{T}(\mathcal{C} \uplus \mathcal{D}, \mathcal{V})$ we have SIN(t). The case $t \in \mathcal{C} \uplus \mathcal{D} \uplus \mathcal{V}$, i.e. if t is a constant or a variable, is easy. Now suppose $t = f(t_1, \ldots, t_n)$ and assume by induction hypothesis $\mathrm{SIN}(t_1) \wedge \ldots \wedge \mathrm{SIN}(t_n)$. If f is a constructor, i.e. $f \in \mathcal{C}$ then SIN(t) follows from $\mathrm{SIN}(t_1) \wedge \ldots \wedge \mathrm{SIN}(t_n)$. Else we may assume w.l.o.g. $f \in \mathcal{D}_1$. Clearly, every innermost derivation issued by t is terminating or has the form

$$t = f(t_1, \ldots, t_n) \underset{i}{\to}^* f(t'_1, \ldots, t'_n) \underset{i}{\to}_\lambda \cdots$$

with t'_j irreducible (w.r.t. $\mathcal{R}_1 \cup \mathcal{R}_2$), $1 \leq j \leq n$. If in the latter case $t' := f(t'_1, \ldots, t'_n)$ were not strongly innermost normalizing, we could conclude that $\mathcal{R}_1$ is not strongly innermost normalizing. This is easily seen by replacing every maximal subterm $s_1, \ldots, s_m$ of t with $root(s_i) \in \mathcal{D}_2$ by fresh variables x_i such that $x_j = x_k$ whenever $s_j = s_k$. ■

Now it is easy to give a substantially simplified and 'modular' proof of

[5] Note that function symbols from $\mathcal{F}$ which do not occur in rules of $\mathcal{R}^{\mathcal{F}}$ are (by definition) considered to be constructors.

[6] This definition of constructor system corresponds to what is usually meant when one speaks of a constructor discipline (for specifying functions).

Theorem 4.7 *([27]) The combination of complete constructor systems (with disjoint sets of defined function symbols) is again complete.*[7]

Proof: Straightforward by combining Corollary 4.5 and Lemma 4.6. ■

Moreover it is straightforward to cover the more general case of combined systems with shared constructors, too.

Definition 4.8 *(cf. [22]) Let* $\mathcal{R}_1^{\mathcal{F}_1} = (\mathcal{R}_1, \mathcal{C}_1, \mathcal{D}_1)$, $\mathcal{R}_2^{\mathcal{F}_2} = (\mathcal{R}_2, \mathcal{C}_2, \mathcal{D}_2)$ *be TRSs with constructors such that* $\mathcal{D}_1 \cap (\mathcal{C}_2 \uplus \mathcal{D}_2) = \mathcal{D}_2 \cap (\mathcal{C}_1 \uplus \mathcal{D}_1) = \emptyset$. *Then the TRS* $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}, \mathcal{C}, \mathcal{D}) = (\mathcal{R}_1 \cup \mathcal{R}_2, \mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D}_1 \uplus \mathcal{D}_2)$ *is said to be a* combined system with shared constructors ($\mathcal{C}$ and disjoint sets $\mathcal{D}_1$, $\mathcal{D}_2$ of defined symbols).

Clearly, every combination of constructor systems (with disjoint sets of defined symbols) is a combined system with shared constructors but not vice versa in general, since for the latter defined symbols may occur in left-hand side arguments which is not allowed for constructor systems.

It is easy to see that if $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2} = (\mathcal{R}_1 \cup \mathcal{R}_2, \mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D}_1 \uplus \mathcal{D}_2)$ is a combined system with shared constructors such that $\mathcal{R}_1$, $\mathcal{R}_2$ are overlay systems then the combined system $\mathcal{R}^{\mathcal{F}}$ is an overlay system, too. Moreover we also have $\mathrm{CP}(\mathcal{R}) = \mathrm{CP}(\mathcal{R}_1) \cup \mathrm{CP}(\mathcal{R}_2)$, hence local confluence of $\mathcal{R}$ is inherited from local confluence of $\mathcal{R}_1$, $\mathcal{R}_2$. Furthermore strong innermost normalization of $\mathcal{R}$ is also inherited from strong innermost normalization of $\mathcal{R}_1$, $\mathcal{R}_2$.

Lemma 4.9 *Let* $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ *be a combined system with shared constructors. Then we have:*

$$SIN(\mathcal{R}_1^{\mathcal{F}_1}) \wedge SIN(\mathcal{R}_2^{\mathcal{F}_2}) \iff SIN(\mathcal{R}^{\mathcal{F}}).$$

Proof: The proof is essentially the same as for Lemma 4.6. ■

With the above observation, the preceding lemma and Theorem 3.23 we obtain the following result as corollary.

Theorem 4.10 *Let* $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ *be a combined system with shared constructors. Then* $\mathcal{R}^{\mathcal{F}}$ *is a complete overlay system if and only if both* $\mathcal{R}_1^{\mathcal{F}_1}$ *and* $\mathcal{R}_2^{\mathcal{F}_2}$ *are complete overlay systems.*

Theorem 3.23 provides us with a rather general approach for ensuring the invariance of completeness under certain combinations of TRSs. What is necessary to apply this result is to define appropriate classes of (not necessarily disjoint) TRSs and/or combination mechanisms which ensure that the properties OS, WCR and SIN are preserved. We have seen that this is possible for combinations of constructor systems and combined systems with shared constructors. More general cases of non-disjoint combinations of TRSs, in particular hierarchical combinations, are treated in [11] using this approach (cf. also [18], [5]).

[7] One may allow here common defined function symbols under the additional assumption that for such symbols the corresponding definition rules have to coincide in both systems. See [27] for more details and a precise definition of 'composable' constructor systems and 'decomposable' properties of constructor systems.

5 Discussion, Related Work and Open Problems

As noticed by Dershowitz (cf. [5]) (one of) the first to consider modularity issues in rewriting was Bidoit ([2]) with his 'gracious' conditions. He proved in [2] that completeness is modular for orthogonal, fully defined[8] constructor systems. Other interesting results concerning termination of rewriting are investigated by Dershowitz [5] and Geupel [9] by means of overlap (and forward) closures. In particular, it is shown in [9] that — as conjectured in [3] — a non-overlapping TRS is terminating if and only if no-right-hand side of a forward closure initiates an infinite derivation. This implies as an easy consequence that completeness is modular for non-overlapping TRS (cf. Theorem 4.3 above) as noticed in [5]. Moreover, the following generalization of the above result about forward closures is stated in [5](cf. Proposition 17):[9] A uniquely normalizing overlay system is terminating if and only if no right-hand side of a forward closure initiates an infinite derivation.[10] This result can be used alternatively to provide a proof of our Theorem 4.2 above as sketched in [5].

Before concluding now let us finally discuss the general idea underlying the approach presented and some open problems. In fact, what has been done in section 3, is on the one hand side an abstract analysis of the interrelation between restricted and uniform termination under some additional assumptions. On the other hand the goal of the analysis and the results obtained may be considered to be a kind of 'modular' approach to modular properties of TRSs in the following sense. We wanted to find properties P such that e.g.

$$\mathrm{SIN} \,\&\, \mathrm{P} \quad \Longrightarrow \quad \mathrm{SN}\,(\,\&\,\mathrm{P})$$

holds. The additional knowledge available was that SIN is modular. Generalizing this situation we get the following abstract 'modular' approach for obtaining sufficient conditions for the modularity of some property Q. Assume that Q is not modular. Then look for other properties $\mathrm{P}_1, \ldots, \mathrm{P}_n$ which are modular – or, slightly weaker, for which $(\mathrm{P}_1 \,\&\, \cdots \,\&\, \mathrm{P}_n)$ is modular – with

$$(\mathrm{P}_1 \,\&\, \cdots \,\&\, \mathrm{P}_n) \quad \Longrightarrow \quad \mathrm{Q}\,.$$

Then the property $(\mathrm{P}_1 \,\&\, \cdots \,\&\, \mathrm{P}_n \,\&\, \mathrm{Q})$ is obviously modular, hence Q is modular for the class of TRSs satisfying $(\mathrm{P}_1 \& \cdots \,\&\, \mathrm{P}_n)$.

Unfortunately, this kind of modular approach is not easily applicable to the main result of [37] which says that $(\mathrm{LL} \,\&\, \mathrm{CR} \,\&\, \mathrm{SN})$ is a modular property of TRSs.

For obtaining a modular proof here we would have to find some property P (probably a kind of restricted termination property) satisfying

$$(1) \quad (\mathrm{LL} \,\&\, \mathrm{CR} \,\&\, \mathrm{P}) \text{ is modular , and}$$

$$(2) \quad (\mathrm{LL} \,\&\, \mathrm{CR} \,\&\, \mathrm{P}) \quad \Longrightarrow \quad \mathrm{SN}\,.$$

Since P cannot be SIN, this is an interesting open problem. But note that even for the case that such a (hopefully easy to understand) property P exists it is not clear a priori that the proofs of (1) and (2) would be simple.

[8] This means that every ground term can be reduced to a constructor ground term.

[9] In [5] no formal proof for this result is given but some proof idea which roughly spoken says that the proof of [9] can be adapted to the more general case of uniquely normalizing overlay systems.

[10] Here, a TRS is said to be *uniquely normalizing* if no term has more than one normal form.

We suppose that it should be possible to extend our approach and proof ideas to the more general case of conditional TRSs. This might in particular also yield a relatively simple proof of the fact that completeness is a decomposable property of conditional constructor systems, a result which has recently been obtained in [25].[11] Moreover our abstract results also provide a theoretical basis for investigating applications like (termination and uniqueness properties of) different function definition formalisms and inductive theorem proving problems.

6 Conclusion

We have provided an abstract analysis of how various kinds of restricted normalization (and confluence) properties are related to strong normalization (and confluence). In particular, we have proved some new results about sufficient criteria for strong normalization which can be considered as generalizations of known results about orthogonal TRSs. Moreover we have shown how these results can be applied in a modular fashion in order to derive simple modular proofs of known and new results concerning properties of combined term rewriting systems.

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