On Termination and Confluence of Conditional Rewrite Systems

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Abstract. We consider the problem of verifying confluence and termination of conditional term rewriting systems. Recently we have obtained some interesting results for unconditional term rewriting systems (TRSs) which are non-overlapping or, more generally, locally confluent overlay systems. These results provide sufficient criteria for termination plus confluence in terms of restricted termination and confluence properties (Gramlich 1994a). Here we generalize our approach to the conditional case and show how to solve the additional complications due to the presence of conditions in the rules. Our main result can be stated as follows: Any conditional TRS (CTRS) which is an innermost terminating overlay system such that all (conditional) critical pairs are joinable is complete, i.e., terminating and confluent.

1 Introduction and Overview

Due to the fact that termination is a fundamental property of TRSs but undecidable in general (Huet & Lankford 1978), many sufficient criteria, techniques and methods for proving termination have been developed, see (Dershowitz 1987) for a survey. Most practically applicable approaches are based on reduction orderings, i.e. well-founded term orderings which are stable w.r.t. substitutions and monotonic w.r.t. the term structure.

Here we follow another line of reasoning and investigate under which conditions one can infer termination (plus confluence) from restricted termination (and confluence) properties. More precisely, we are interested in cases for which termination of rewriting under some strategy implies termination under arbitrary strategies. Particularly interesting from a practical point of view is the innermost reduction strategy since it corresponds closely to the evaluation mechanism employed in many rewriting based computation models, e.g. in functional programming languages or formalisms for recursive function definitions. Note moreover that for the latter case sophisticated techniques for verifying termination by means of recursion analysis are available, cf. e.g. (Boyer & Moore 1979; Walther 1988). The notion of termination in this functional framework

* This research was supported by the 'Deutsche Forschungsgemeinschaft, SFB 314 (D4-Projekt)'.

corresponds to innermost termination from a more abstract rewriting point of view. Since innermost termination may be much easier to verify than general termination, our results bear considerable practical relevance.

The first results about termination of rewriting under such restrictions and its relation to termination (of unrestricted rewriting) have been obtained for orthogonal, i.e. left-linear and non-overlapping (unconditional) TRSs (cf. e.g. (Klop 1992) for a survey). Recently we have shown (Gramlich 1994a) that most of these results can be generalized to non-overlapping TRSs. The main result of (Gramlich1994a) states that even for locally confluent (unconditional) overlay systems innermost termination suffices for guaranteeing general termination and hence also completeness.

Since it is well-known that conditional rewriting is much more difficult to handle in theory and practice, it was not clear a priori that our results carry over to the conditional case. But this is indeed possible (for the main results) as can be shown by a careful analysis and inspection of the proofs for the unconditional case and by taking into account the additional complications arising with CTRSs. From a technical point of view one of the main problems in generalizing our results is the well-known fact that *variable overlaps* may be critical for CTRSs and need to be explicitly handled. Another related problem with CTRSs is the fact that – in contrast to the unconditional case – joinability of critical pairs is not sufficient anymore for guaranteeing local confluence.

The rest of the paper is structured as follows. After introducing some basic definitions and notions we present and discuss in section 3 our main results on restricted termination and confluence properties of CTRSs. Missing details can be found in (Gramlich 1993, 1994a). Finally we briefly discuss some related work.

2 Preliminaries

We assume familiarity with the basic terminology needed for dealing with (C)TRSs, cf. e.g. (Dershowitz et al. 1990; Klop 1992; Middeldorp 1993).

A TRS $\mathcal{R}$ is *(strongly) terminating* or *strongly normalizing (SN)* if $\rightarrow$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \cdots$. It is said to be *weakly terminating* or *weakly normalizing (WN)* if every term can be reduced to some normal form. If $s \rightarrow t$ then, in order to make explicit the position p of the reduced subterm, the applied substitution σ and the applied rule $l \rightarrow r$, we shall sometimes use the notation $s \rightarrow_{p,\sigma,l\rightarrow r} t$ or $s \rightarrow_p t$. The topmost (or *root*) position in a term is denoted by λ. By $s \rightarrow^k t$ we mean that s reduces to t in k steps. A reduction step $s \rightarrow t$ by applying some rule of $\mathcal{R}$ at position p in s is *innermost* if every proper subterm of s/p is irreducible. In that case we also write $s \underset{i}{\rightarrow} t$. $\mathcal{R}$ is *(strongly) innermost terminating* or *(strongly) innermost normalizing (SIN)* if every sequence of innermost reduction steps terminates. It is *weakly innermost terminating* or *weakly innermost normalizing (WIN)* if every term can be reduced to some normal form by innermost reduction steps.

A TRS is *confluent* or has the *Church-Rosser property (CR)* if ${}^* \leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^* \leftarrow$ and *weakly Church-Rosser (WCR)* or *locally confluent* if $\leftarrow \circ \rightarrow$

$\subseteq \rightarrow^* \circ {}^* \leftarrow$.[2] $\mathcal{R}$ has the property WCR^1 if $t \leftarrow s \rightarrow u$ and $t \neq u$ imply that there exists some v with $t \rightarrow v \leftarrow u$. A confluent and terminating TRS is said to be *convergent* or *complete* . If $l_1 \rightarrow r_1$, $l_2 \rightarrow r_2$ are two rules of $\mathcal{R}$ (w.l.o.g. we assume that they do not share common variables), with p some non-variable position of $l_2 \rightarrow r_2$ such that l_1 and l_2/p are unifiable with most general unifier σ then $\langle \sigma(l_2[p \leftarrow r_1]), \sigma(r_2) \rangle$ is said to be a *critical pair (CP)* of $\mathcal{R}$ (obtained by overlapping $l_1 \rightarrow r_1$ into $l_2 \rightarrow r_2$ at position p), provided that p is not the top position if the two rules are renamed versions of the same rule. Joinability of all critical pairs is abbreviated by JCP. A critical pair of the form $\langle s, s \rangle$ is said to be *trivial*.

A TRS $\mathcal{R}$ is said to be *non-overlapping (NO)* if there is no critical pair between rules of $\mathcal{R}$. *Weakly non-overlapping* TRSs have only trivial critical pairs. $\mathcal{R}$ is *left-linear (LL)* if every variable occurs at most once in every left hand side of an $\mathcal{R}$-rule. $\mathcal{R}$ is *orthogonal / weakly orthogonal* if it is left-linear and non-overlapping / weakly non-overlapping. It is *non-erasing (NE)* if r and l have the same set of variables for every rule $l \rightarrow r \in \mathcal{R}$. If every critical pair of a TRS $\mathcal{R}$ is obtained by an *overlay*, i.e. by overlapping left hand sides of rules at top position then $\mathcal{R}$ is said to be an *overlay system (OS)*.

By $P(\mathcal{R})$ we mean that the TRSs $\mathcal{R}$ has property P. Moreover we also ambiguously use the notation $P(t)$ for terms t provided there is a sensible local interpretation for $P(t)$. For instance, $CR(t)$ is to denote the property that whenever we have $t \rightarrow^* v$ and $t \rightarrow^* w$ then there exists a term s with $v \rightarrow^* s$ and $w \rightarrow^* s$.

Some basic properties of TRSs (or, more generally, *abstract reduction systems*, cf. (Newman 1942; Klop1992)) which are explicitly or implicitly used subsequently, are the following:

- WCR and SN imply CR (*Newman's Lemma*).
- WCR^1 and WN imply SN.
- WCR^1 implies CR.

For TRSs the following well-known results are fundamental.

- WCR is equivalent to JCP (*Critical Pair Lemma*, (Huet 1980)).
- JCP and SN imply CR (Knuth & Bendix 1970).

Moreover, we need some basic terminology about conditional term rewriting systems (CTRSs). A CTRS is set of *conditional rewrite rules* of the form

$$s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \rightarrow r$$

. Here we require $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$, i.e. no variable left hand sides and no extra variables on the right hand side. Extra variables in conditions are allowed if not stated otherwise. Depending on the interpretation of the equality sign in the conditions of rewrite rules, different reduction relations may be associated with a given CTRS as usual. In a *join CTRS* $\mathcal{R}$ the equality sign in the conditions of rewrite rules is interpreted as joinability, i.e. as $\downarrow$. In a *normal CTRS* $\mathcal{R}$

[2] Here, '$\circ$' denotes relation composition.

a condition $s = t$ is satisfied if $s \to^* t$ and t is a ground normal form (w.r.t. the unconditional version of $\mathcal{R}$). *Semi-equational* CTRSs are obtained by interpreting the equality sign in the conditions as convertibility, i.e. as $\overset{*}{\leftrightarrow}$. A *generalized* CTRS has rules of the form $P_1 \wedge \ldots, \wedge P_n \Longrightarrow l \to r$ where the conditions P_i, $1 \leq i \leq n$, are formulated in a general mathematical framework, e.g. in some first order language.

For the sake of readability we shall use in the following some compact notations for conditional rules and conjunctions of conditions. When writing $P \Longrightarrow l \to r$ for some conditional rewrite rule then P stands for the conjunction of all conditions. Similarly, $P\downarrow$ means joinability of all conditions in P, and σP means that all conditions in P are instantiated by σ.

Definition1 (reduction relation, depth). The reduction relation corresponding to a given (join, semi-equational or normal) CTRS $\mathcal{R}$ is inductively defined as follows ($\Box$ denotes $\downarrow$ or $\overset{*}{\leftrightarrow}$, respectively):

$$\begin{aligned} \mathcal{R}_0 &= \emptyset\ , \\ \mathcal{R}_{i+1} &= \{\sigma l \to \sigma r | P \Longrightarrow l \to r \in \mathcal{R}, \sigma u \Box_{\mathcal{R}_i} \sigma v \text{ for all } u \Box v \text{ in } P\}\ ,^{3} \\ s \to_{\mathcal{R}} t :\Longleftrightarrow\ & s \to_{\mathcal{R}_i} t \text{ for some } i \geq 0, \text{ i.e. } \to_{\mathcal{R}} = \bigcup_{i \geq 0} \to_{\mathcal{R}_i}\ . \end{aligned}$$

If $s \to_{\mathcal{R}} t$ then the *depth* of $s \to_{\mathcal{R}} t$ is defined to be the minimal n with $s \to_{\mathcal{R}_n} t$. For $s \to^*_{\mathcal{R}} t$ and $s \downarrow_{\mathcal{R}} t$ depths are defined analogously. More precisely, if $s \to^*_{\mathcal{R}} t$ then the *depth* of $s \to^*_{\mathcal{R}} t$ is defined to be the minimal n with $s \to^*_{\mathcal{R}_n} t$. The *depth* of $s \downarrow_{\mathcal{R}} t$ is the minimal n with $s \downarrow_{\mathcal{R}_n} t$. If the depth of $s \to^*_{\mathcal{R}} t$ is at most n we denote this by $s \overset{n}{\longrightarrow}^*_{\mathcal{R}} t$.

Remark. Note that instead of a CTRS $\mathcal{R}$ one may somehow equivalently consider the extended system $\mathcal{R}' := \mathcal{R} \uplus \{eq(x,x) \to true\}$. More precisely, taking – within a many-sorted framework – $\mathcal{R}' := \mathcal{R} \uplus \{eq(x,x) \to true\}$, with eq a fresh binary function symbol and $true$ a fresh constant of a new sort (with x a variable of the 'old' sort), it is easily shown that $\mathcal{R}'$ is a conservative extension of $\mathcal{R}$ in the following sense: for all 'old' terms s, t we have:[4] $s \overset{n}{\longrightarrow}_{\mathcal{R}} t \iff s \overset{n}{\longrightarrow}_{\mathcal{R}'} t$, $s \to_{\mathcal{R}} t \iff s \to_{\mathcal{R}'} t$, $s \Box^n_{\mathcal{R}} t \iff eq(s,t) \Box^n_{\mathcal{R}'} true$ (for $n \geq 1$),[5] $eq(s,t) \overset{n}{\longrightarrow}^*_{\mathcal{R}'} eq(u,v) \iff s \overset{n}{\longrightarrow}^*_{\mathcal{R}} u \wedge t \overset{n}{\longrightarrow}^*_{\mathcal{R}} v$, $eq(s,t) \overset{n}{\longrightarrow}^*_{\mathcal{R}'} true \iff \exists w : eq(s,t) \overset{n}{\longrightarrow}^*_{\mathcal{R}} eq(w,w) \to_{\mathcal{R}'} true$ (again for $n \geq 1$), for $\Box$ denoting $\downarrow$ or $\overset{*}{\leftrightarrow}$, respectively. From these properties it is straightforward to infer that properties like termination, confluence, local confluence and joinability of critical pairs are

[3] Note in particular that all unconditional rules of $\mathcal{R}$ are contained in $\mathcal{R}_1$ (because the empty conditions are vacuously satisfied) as well as all conditional rules with trivial conditions only, i.e. conditions of the form $s \Box s$. In fact, rules of the latter class can be considered to be essentially unconditional.

[4] Note that $s \Box^n_{\mathcal{R}} t$ is to denote that the depth of $s \Box_{\mathcal{R}} t$ is at most n.

[5] In order to obtain the equivalence $s \Box^n_{\mathcal{R}} t \iff eq(s,t) \Box^n_{\mathcal{R}'} true$ for $n = 0$, too, one would have to include the rule $eq(x,x) \to true$ into $\mathcal{R}_0$ instead of $\mathcal{R}_1$ in Definition 1 as it is sometimes done in the literature.

not affected by considering $\mathcal{R}'$ instead of $\mathcal{R}$, or vice versa. Note in particular, that for join CTRSs the equivalence $s \downarrow^n_{\mathcal{R}} t \iff eq(s,t) \downarrow^n_{\mathcal{R}'} true$ (for $n \geq 1$) means: $s \downarrow^n_{\mathcal{R}} t \iff eq(s,t) \xrightarrow{n}{}^*_{\mathcal{R}'} true$, since $true$ is irreducible. This (depth preserving!) encoding of joinability into reducibility by means of an equality predicate is particularly useful for proof-technical reasons as we shall see later on (in the proof of Theorem 7).

Definition 2 (*conditional critical pairs*)**.** Let $\mathcal{R}$ be a join CTRS, and let $P_1 \Longrightarrow l_1 \to r_1$ and $P_2 \Longrightarrow l_2 \to r_2$ be two rewrite rules of $\mathcal{R}$ which have no variables in common. Suppose[6] $l_1 = C[t]_p$ with $t \notin \mathcal{V}$ for some (possibly empty) context $C[]_p$ such that t and l_2 are unifiable with most general unifier σ, i.e. $\sigma(t) = \sigma(l_1/p) = \sigma(l_2)$. Then $\sigma(P_1) \wedge \sigma(P_2) \Longrightarrow \sigma(C[r_2]) = \sigma(r_1)$ is said to be a *(conditional) critical pair* of $\mathcal{R}$. If the two rules are renamed versions of the same rule of $\mathcal{R}$, we do not consider an overlap at top position (i.e. with $p = \lambda$). A (conditional) critical pair $P \Longrightarrow s = t$ is said to be *joinable* if $\sigma(s) \downarrow_{\mathcal{R}} \sigma(t)$ for every substitution σ with $\sigma(P) \downarrow$. A substitution σ which satisfies the conditions, i.e. for which $\sigma(P) \downarrow$ holds, is said to be *feasible*. Otherwise σ is *unfeasible*. Analogously, a (conditional) critical pair is said to be *feasible (unfeasible)* if there exists some (no) feasible substitution for it.

Note that testing joinability of conditional critical pairs is in general much more difficult than in the unconditional case since one has to consider all substitutions which satisfy the correspondingly instantiated conditions. Moreover, the Critical Pair Lemma does not hold for CTRSs in general as shown e.g. by the following example.

Example 1. (Bergstra & Klop 1986) Consider the join CTRS

$$\mathcal{R} = \begin{cases} x \downarrow f(x) \Longrightarrow f(x) \to a \\ \qquad\qquad\qquad\quad b \to f(b)\,. \end{cases}$$

Here we get $f(b) \to a$ due to $b \downarrow f(b)$ and hence $f(f(b)) \to f(a)$. We also have $f(f(b)) \to a$ because of $f(b) \downarrow f(f(b))$. But a and $f(a)$ do not have a common reduct which is easily shown. Thus $\mathcal{R}$ is not locally confluent despite the lack of critical pairs. Note moreover that $\mathcal{R}$ is even orthogonal when considered as unconditional TRS, i.e. when omitting the condition in the first rule.

Definition 3. (Bergstra & Klop 1986; Klop 1992) Let $\mathcal{R}$ be a CTRS and let $\mathcal{R}_u$ be its unconditional version, i.e. $\mathcal{R}_u := \{l \to r \,|\, P \Longrightarrow l \to r \in \mathcal{R}\}$. Then $\mathcal{R}$ is said to be *left-linear / non-overlapping / weakly non-overlapping / orthogonal / weakly orthogonal)* if $\mathcal{R}_u$ is left-linear / non-overlapping / weakly non-overlapping / orthogonal / weakly orthogonal.

[6] Here and subsequently we shall freely make use of *contexts* like $C[]$, $C[]_p$, $C[]_\Pi$, $C[,\ldots,]$ which denote terms with one or more 'holes' at some position p or, more generally, at some set Π of mutually disjoint positions. This is easily formalized by treating the 'holes' as fresh constant symbols or as fresh variables.

According to this definition Example 1 above shows that orthogonal CTRSs need not be confluent. But note that the CTRS $\mathcal{R}$ defined in Example 1 is not innermost terminating.

Definition 4. A CTRS $\mathcal{R}$ is said to be a *(conditional) overlay system (OS)* if $\mathcal{R}_u$ is an unconditional overlay system.

The careful reader may have observed that the definition of being non-overlapping above is somehow rather restrictive. Namely, the case that there exist conditional critical pairs all of which are infeasible (and hence should not be 'properly critical') is not covered. Analogously, it may be the case that for some CTRS $\mathcal{R}$ all feasible (conditional) critical pairs are overlays. This observation may be exploited to slightly generalize the notions of being an overlay system and being non-overlapping as it is done in (Gramlich 1993). In fact, all results presented here for non-overlapping and overlay CTRSs do also hold for these slightly more general 'semantic' versions of being an overlay system and of being non-overlapping.

The other basic notions for unconditional TRSs introduced above generalize in a straightforward manner to CTRSs.

In the following we shall tacitly assume that all CTRSs considered are join CTRSs (which is the most important case in practice), except for cases where another kind of CTRSs is explicitly mentioned.

3 Restricted Termination and Confluence Properties of Conditional Term Rewriting Systems

We shall study now under which conditions various restricted kinds of termination imply (strong) termination (and also confluence under some additional assumptions) of CTRSs. Firstly we summarize the most important known results on confluence and termination of unconditional orthogonal TRSs.

Theorem 5. *(Rosen 1973; O'Donnell 1977; Klop 1992) Any orthogonal TRS $\mathcal{R}$ is confluent and satisfies the following properties:*

(1a) $\forall t : [WIN(t) \iff SIN(t) \iff SN(t)]$.
(1b) $WIN(\mathcal{R}) \iff SIN(\mathcal{R}) \iff SN(\mathcal{R})$.
(2) If $s \xrightarrow{}_i t$ *and* $SN(t)$ *then* $SN(s)$.
(3a) $NE(\mathcal{R}) \implies [\forall t : [WN(t) \iff WIN(t) \iff SIN(t) \iff SN(t)]]$.
(3b) $NE(\mathcal{R}) \implies [WN(\mathcal{R}) \iff WIN(\mathcal{R}) \iff SIN(\mathcal{R}) \iff SN(\mathcal{R})]$.

Our recently obtained results on confluence and termination of TRSs show that all normalization properties of Theorem 5 above (i.e. (1)-(3)) still hold for non-overlapping but not necessarily left-linear TRSs (Gramlich 1994a). Moreover we have shown there that any innermost terminating (unconditional) overlay system with joinable critical pairs is terminating and hence confluent and complete.

For CTRSs much less is known concerning similar criteria for termination and confluence (see Example 1 above which shows that orthogonal CTRSs need

not be confluent). Next we summarize the most important known criteria for confluence of (possibly non-terminating) CTRSs.

Let $\mathcal{R}$ be a CTRS with rewrite relation $\rightarrow$ and let P be an n-ary predicate on the set of terms of $\mathcal{R}$. Then P is said to be *closed with respect to* $\rightarrow$ (Klop 1992) if for all terms t_i, t_i' such that $t_i \rightarrow^* t_i'$ $(i = 1, \ldots, n)$: $P(t_1, \ldots, t_n) \implies P(t_1', \ldots, t_n')$. $\mathcal{R}$ is said to be *closed* if all conditions (appearing in some conditional rewrite rule of $\mathcal{R}$), viewed as predicates with the variables ranging over $\mathcal{R}$-terms, are closed with respect to $\rightarrow$.

Theorem 6.

(1) Any generalized, weakly orthogonal, closed CTRS is confluent (O'Donnell 1977; Klop1992).
(2) Any weakly orthogonal, semi-equational CTRS is confluent.[7]
(3) Any weakly orthogonal, normal CTRS is confluent (Bergstra & Klop 1986; Klop 1992).

Under the assumption of termination and joinability of all critical pairs some further confluence criteria for CTRSs are possible if additional requirements are fulfilled, cf. (Dershowitz et al. 1988) for details.

In the following we shall show that our results on restricted termination and confluence properties of non-overlapping and even of overlay systems can be generalized to the conditional case. This generalization has to take into account the additional complications arising with CTRSs. In particular, we need a kind of 'local completeness' property implying that variable overlaps are not critical for certain conditional overlay systems. More precisely, we have the following result which is a generalized local version of Lemma 2 in (Dershowitz et al. 1988)[8] which in turn is the main technical result for inferring confluence of terminating CTRSs, provided that all conditional critical pairs are joinable overlays (cf. Theorem 4 in (Dershowitz et al. 1988), cf. also Theorem 6.2 in (Wirth & Gramlich 1994) which handles the more general case of positive / negative conditional rewrite systems). Note that extra variables (in conditions) are allowed here.

Theorem 7. *Let $\mathcal{R}$ be a CTRS with $OS(\mathcal{R})$ and $JCP(\mathcal{R})$ and let s be a term with $SN(s)$. Furthermore let $C[]_\Pi$ be a context (with 'holes' at some set Π of mutually disjoint positions), and t, u, v be terms. Then we have the following implication:*

$$u = C[s]_\Pi \rightarrow^* v \wedge s \rightarrow^* t \quad \Longrightarrow \quad C[t]_\Pi \downarrow v\,.$$

Proof. A detailed proof is provided in the Appendix. ■

[7] This is a corollary of (1).

[8] In Lemma 2 of (Dershowitz et al. 1988) the proof (i.e. the induction ordering) makes use of the general termination assumption $SN(\mathcal{R})$ for the considered CTRS $\mathcal{R}$. Our proof of Theorem 7 has a similar structure but is based on a slightly different notion of *depth* and – more importantly – the induction ordering only needs the local termination assumption $SN(s)$.

Note that Theorem 7 is interesting in itself because it entails a non-trivial structural confluence property for (conditional) overlay systems with joinable critical pairs without a full termination assumption (it is applicable even in situations where the whole system need not terminate and may have other potential applications than the ones mentioned below), namely in the following sense.

Lemma 8. *Let $\mathcal{R}$ be a CTRS with $OS(\mathcal{R})$ and $JCP(\mathcal{R})$ and let u, v, w be terms with $v^* \leftarrow u \rightarrow^* w$. Then $v \downarrow w$ holds provided that v is obtained from u by performing only reductions in strongly normalizing (parallel) subterms of u, formally: $u = C[s_1, \ldots, s_n]$, $v = C[t_1, \ldots, t_n]$ for some context $C[,\ldots,]$, and $SN(s_i)$, $s_i \rightarrow^* t_i$ for $i = 1, \ldots, n$.*

Proof. Straightforward by repeated application of Theorem 7. ■

One may ask now whether the assumption $OS(\mathcal{R})$ is really crucial in the above results. This is indeed the case, clearly, for CTRSs, since there exist terminating CTRSs with joinable critical pairs that are non-confluent, hence necessarily not even locally confluent (cf. (Dershowitz et al. 1988) for some illustrative counterexamples). A very simple counterexample (due to Aart Middeldorp) is the following:

Example 2.

$$\mathcal{R} = \begin{cases} g(x) \downarrow f(x) \Longrightarrow g(x) \rightarrow f(b) \\ \qquad\qquad\qquad\quad f(a) \rightarrow g(a) \\ \qquad\qquad\qquad\quad\ \ a \rightarrow b \end{cases}$$

This system is easily shown to be terminating and the only critical peak (between the last two rules) $f(b) \leftarrow f(a) \rightarrow g(a)$ is joinable since $g(a) \rightarrow f(b)$ by the first rule (due to $g(a) \downarrow f(a)$). But we have $g(b) \leftarrow g(a) \rightarrow f(b)$ (again due to $g(a) \downarrow f(a)$) with both $g(b)$ and $f(b)$ irreducible.

For unconditional TRSs such a counterexample involving a terminating TRS with joinable critical pairs cannot exist since for TRSs – in contrast to CTRSs – the Critical Pair Lemma holds (Huet 1980), i.e. the property $WCR(\mathcal{R})$ is equivalent to $JCP(\mathcal{R})$. By Newman's Lemma (Newman 1942) any such TRS must then be confluent. Hence, in the unconditional case the above question may be rephrased as follows: Does there exist a locally confluent (non-terminating) TRS which is not an overlay system violating the confluence property in Lemma 8 (and Theorem 7) above? This is indeed the case as shown by the following simple example.

Example 3.

$$\mathcal{R} = \begin{cases} f(a) \rightarrow f(b) \\ f(b) \rightarrow f(a) \\ \quad a \rightarrow c \\ \quad b \rightarrow d \end{cases}$$

This system which is not an overlay system is clearly locally confluent, but non-confluent (and hence necessarily non-terminating). We have for instance $f(c) \leftarrow f(a) \rightarrow^* f(d)$ with both $f(c)$ and $f(d)$ irreducible, and moreover, in the step $f(a) \rightarrow f(c)$ the proper subterm a of $f(a)$ which is contracted clearly satisfies $SN(a)$.

Thus, even for unconditional TRSs, Theorem 7 and Lemma 8 capture indeed a non-trivial confluence property of overlay systems!

But let us return now to the conditional case. Similar to Lemma 8 above we obtain from Theorem 7 in particular the following sufficient criterion for a variable overlap in CTRSs to be non-critical.

Lemma 9. *Let $\mathcal{R}$ be a CTRS with $OS(\mathcal{R})$ and $JCP(\mathcal{R})$, and let s, t be terms with $s \rightarrow_{p,\sigma,P \Longrightarrow l \rightarrow r} t$. Furthermore let σ' be given with $\sigma \rightarrow^* \sigma'$, i.e. $\sigma(x) \rightarrow^* \sigma'(x)$ for all $x \in dom(\sigma)$, such that $SN(\sigma(x))$ holds for all $x \in dom(\sigma)$. Then we have: $s = C[\sigma(l)]_p \rightarrow^* C[\sigma'(l)]_p \rightarrow_{p,\sigma',P \Longrightarrow l \rightarrow r} C[\sigma'(r)]_p$ (due to $\sigma'(P) \downarrow$), and $t = C[\sigma(r)]_p \rightarrow^* C[\sigma'(r)]_p$, for some context $C[]_p$.*

Proof. Straightforward by verifying $\sigma'(P) \downarrow$ using the encoding of $s \downarrow_{\mathcal{R}} t$ into $eq(s,t) \rightarrow^*_{\mathcal{R}'} true$[9] and applying Lemma 8 together with the fact that *true* is irreducible. ■

Choosing $C[]_\Pi$ to be the empty context (and accordingly $\Pi = \{\lambda\}$) in Theorem 7 we obtain as corollary the following local version of a confluence criterion.

Corollary 10. *Let $\mathcal{R}$ be a CTRS with $OS(\mathcal{R})$ and $JCP(\mathcal{R})$, and let s be a term with $SN(s)$. Then we have $CR(s)$. In other words, for a conditional overlay system with joinable critical pairs, a term is strongly normalizing if and only if it is complete.*

The termination assumption concerning s in this result is crucial as demonstrated by the following example.

Example 4 (example 1 continued). Here

$$\mathcal{R} = \begin{cases} x \downarrow f(x) \Longrightarrow f(x) \rightarrow a \\ \qquad\qquad\qquad\quad b \rightarrow f(b)\,. \end{cases}$$

is clearly an overlay system with joinable critical pairs (it is even non-overlapping). Moreover we have $f(f(b)) \rightarrow a$ and $f(f(b)) \rightarrow f(a)$ but not $a \downarrow f(a)$. Obviouly, $SN(f(f(b)))$ does not hold due to the presence of the rule $b \rightarrow f(b)$ in $\mathcal{R}$ (note that we even do not have $SIN(\,f(f(b)))\,)$.

Under the stronger assumption of global termination we get from Corollary 10 the following known critical pair criterion for confluence of conditional overlay systems.

Theorem 11. *(Dershowitz et al. 1988) A terminating CTRS which is an overlay system such that all its conditional critical pairs are joinable is confluent, hence complete.*

[9] Cf. the Remark after Definition 1.

3.1 Non-Overlapping CTRS

Now let us consider non-overlapping CTRSs. We shall show that all normalization properties of Theorem 5 also hold for non-overlapping CTRSs. Since the proofs are very similar to those of the corresponding results for unconditional non-overlapping TRSs in (Gramlich 1994a) we shall only mention additional problems arising with CTRSs. Throughout this subsection we assume that $\mathcal{R}$ is a non-overlapping CTRS, i.e. $NO(\mathcal{R})$ holds.

Let us start with an easy result about innermost reduction.

Lemma 12. *Innermost reduction in $\mathcal{R}$ is WCR^1, i.e. $WCR^1(\underset{i}{\rightarrow})$ holds.*

Proof. Cf. (Gramlich 1994a). ■

Corollary 13. *Innermost reduction in $\mathcal{R}$ is confluent, i.e. $CR(\underset{i}{\rightarrow})$ holds.*

The following result shows that for non-overlapping systems the existence of a terminating innermost derivation for some term t implies that any innermost derivation initiated by t is finite.

Lemma 14. $\forall t : [WIN(t) \iff SIN(t)]$.

Proof. Cf. (Gramlich 1994a). ■

Furthermore, strong innermost normalization is equivalent to strong normalization.

Theorem 15. $\forall t : [SIN(t) \iff SN(t)]$.

Proof. For a proof we refer to the more general Theorem 21. ■

Combined with Lemma 14 this yields the following

Corollary 16. $\forall t : [WIN(t) \iff SIN(t) \iff SN(t)]$.

The next result says that innermost reduction steps in non-overlapping CTRSs cannot be critical in the sense that they may destroy the possibility of infinite derivations.

Lemma 17. *If $s \underset{i}{\rightarrow} t$ and $SN(t)$ then $SN(s)$.*

Proof. This is an immediate consequence of Corollary 16. ■

Furthermore, as in the unconditional case, the non-erasing property is crucial for the equivalence of weak and strong termination of non-overlapping CTRSs.

Lemma 18. *Suppose $NE(\mathcal{R})$. If $s \rightarrow t$ and $SN(t)$ then $SN(s)$.*

Proof. As in (Gramlich 1994a) by using Lemma 17 and Corollary 16. Here we additionally need the fact that variable overlaps are not critical provided the substitution part is strongly normalizing, i.e. Lemma 9. ■

Lemma 19. *If $NE(\mathcal{R})$ then : $\forall t : [WN(t) \iff WIN(t) \iff SIN(t) \iff SN(t)]$.*

Proof. As in (Gramlich 1994a) using Lemma 18. ■

Finally let us summarize the results obtained for non-overlapping, but not necessarily left-linear CTRSs.

Theorem 20. *Any non-overlapping CTRS $\mathcal{R}$ satisfies the following properties:*

(1a) $\forall t : [WIN(t) \iff SIN(t) \iff SN(t)]$.
(1b) $WIN(\mathcal{R}) \iff SIN(\mathcal{R}) \iff SN(\mathcal{R})$.
(2) If $s \underset{i}{\rightarrow} t$ *and* $SN(t)$ *then* $SN(s)$.
(3a) $NE(\mathcal{R}) \implies [\forall t : [WN(t) \iff WIN(t) \iff SIN(t) \iff SN(t)]]$.
(3b) $NE(\mathcal{R}) \implies [WN(\mathcal{R}) \iff WIN(\mathcal{R}) \iff SIN(\mathcal{R}) \iff SN(\mathcal{R})]$.
(4) $WIN(\mathcal{R}) \implies CR(\mathcal{R})$.
(5) $NE(\mathcal{R}) \wedge WN(\mathcal{R}) \implies CR(\mathcal{R})$.

Proof. (1)-(3) have been shown above. To verify the confluence criteria (4) and (5) one has to combine (1) and (3), respectively, with Theorem 21 (below), observing that a non-overlapping CTRS is in particular an overlay system with joinable critical pairs. ■

In fact, all properties of Theorem 20 do also hold for the slightly more general class of weakly non-overlapping CTRSs (where trivial critical pairs are allowed), as can be verified by a careful inspection of the proofs for the non-overlapping (and the overlay) case.

3.2 Conditional Overlay Systems with Joinable Critical Pairs

Furthermore we can show that Theorem 15 above may be generalized by requiring only $OS(\mathcal{R}) \wedge JCP(\mathcal{R})$ instead of the more restrictive $NO(\mathcal{R})$.

Theorem 21. *For any CTRS $\mathcal{R}$ we have:*

(a) $OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \wedge SIN(\mathcal{R}) \implies SN(\mathcal{R}) \wedge CR(\mathcal{R})$, *and*
(b) $OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \implies [\forall s : [SIN(s) \implies SN(s) \wedge CR(s)]]$.

Proof. The proof (by minimal counterexample) is essentially the same as in (Gramlich 1994a) for the unconditional case, but taking into account that we have to ensure two additional properties which one obtains for free in the unconditional case: Namely, in an unconditional overlay system with joinable critical pairs a term is complete if and only if it strongly normalizing, and secondly, variable overlaps are not critical (if the substitution part is strongly normalizing).[10]

[10] More precisely, this property is needed in the following form: if $\sigma l \rightarrow \sigma r$ then $(\sigma\downarrow)l \rightarrow (\sigma\downarrow)r$ using the same rule $P \implies l \rightarrow r$ where σ is assumed to be strongly normalizing, and $(\sigma\downarrow)$ denotes the corresponding normalized substitution which is obtained by replacing σx by its (unique) normal form, for all $x \in dom(\sigma)$.

Now, for conditional overlay systems with joinable critical pairs the first property is ensured by Corollary 10, and the second one by Lemma 9. Furthermore, the minimal counterexample considered must be an infinite derivation

$$s = s_0 \to s_1 \to s_2 \to \ldots$$

such that in each step the redex contracted is either terminating (hence complete) or, if not, all its proper subterms are terminating (hence complete). The latter property for those steps $s_i = C[\sigma l]_p \to_{\sigma, P \Longrightarrow l \to r} C[\sigma r]_p = s_{i+1}$ which contract a non-terminating redex $\sigma l = s_i/p$ ensures[11] that the substitution part is terminating, hence that Lemma 9 is applicable and yields $C[(\sigma\downarrow)l]_p \to_{\sigma\downarrow, P \Longrightarrow l \to r} C[(\sigma\downarrow)r]_p$. ■

Theorem 21 states that any (strongly) innermost terminating (conditional) overlay system with joinable critical pairs is (strongly) terminating and confluent (part (a)), hence complete, which even holds in the stronger local version (part (b)). In other words, for (conditional) overlay systems it suffices to verify innermost termination and joinability of all critical pairs in order to infer general termination and confluence, i.e. completeness. The non-triviality of this result is obvious taking into account the fact that for CTRSs the Critical Pair Lemma does not hold in general and almost all known sufficient criteria for confluence presume even stronger properties than termination plus joinability of (conditional) critical pairs.

Finally let us mention that it is possible to apply the above results to the analysis of properties of (disjoint and restricted forms of non-disjoint) combinations of CTRSs, as it is done in (Gramlich 1994a) for the case of unconditional TRSs. But this generalization requires a careful analysis and is not straightforward, since for instance the properties weak termination, weak innermost termination and (strong) innermost termination are not preserved in general under disjoint unions of CTRSs (Middeldorp 1993; Gramlich 1994b). Some very recently obtained positive and negative results in this direction are presented in (Gramlich 1994b). Related results for combinations of constructor systems can be found in (Middeldorp & Toyama 1993) for the unconditional case and (Middeldorp 1994) for the conditional case.

Acknowledgement: I am indebted to Aart Middeldorp for suggesting a couple of improvements in (Gramlich 1994a) on which the present paper is partially based.

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[11] This stronger minimality property is not necessary for the proof of the unconditional version of Theorem 21 since for unconditional TRSs $\sigma l \to_{\lambda, \sigma, l \to r} \sigma r$ and $\sigma \to^* \sigma'$ always imply $\sigma' l \to_{\lambda, \sigma', l \to r} \sigma' r$.

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Appendix (proof of Theorem 7)

Proof. Let $\mathcal{R}$ be given as above. W.l.o.g. let us assume that $\mathcal{R}$ contains the rule $eq(x,x) \to true$ (with eq a fresh binary function symbol and $true$ a fresh constant of new sort, and x some 'old' variable).[12] For a context $C[]_\Pi$ and terms s, t, u, v we define the predicate $P(s,t,u,v,C[]_\Pi)$ by the following implication:

$$SN(s) \;\wedge\; u = C[s]_\Pi \to^* v \;\wedge\; s \to^* t \quad\Longrightarrow\quad C[t]_\Pi \downarrow v\,.$$

We have to show $P(s,t,u,v,C[]_\Pi)$ for all s, t, u, v, $C[]_\Pi$. To this end it is sufficient that $Q(s,n,k)$ defined by

$$\forall\, C[]_\Pi, t, u, v, : SN(s) \;\wedge\; u = C[s]_\Pi \xrightarrow{n}{}^{k} v \;\wedge\; s \to^* t \quad\Longrightarrow\quad C[t]_\Pi \downarrow v$$

holds for all s and for all n, k. We will show this by contradiction as follows. Assume that there exists a counterexample, i.e. $\langle s,n,k\rangle$ with $Q(s,n,k)$ not holding for. That means we have

$$(*)\quad SN(s) \quad\wedge\quad u = C[s]_\Pi \xrightarrow{n}{}^{k} v \quad\wedge\quad s \to^* t$$

for some $C[]_\Pi$, t, u, v, but

$$(**)\quad \neg\,(\,C[t]_\Pi \downarrow v\,)\,.$$

Now we define a complexity measure for $Q(\bar{s},\bar{n},\bar{k})$ by the triple $\langle \bar{s},\bar{n},\bar{k}\rangle$ using the lexicographic combination $\succ := lex(>_1,>_2,>_3)$ with $>_1 := (\to \cup >_{st})^+|_{below s}$, $>_2 := >_3 := > := >_{\mathbb{N}}$, where – for some binary (ordering) relation R – $R|_{below s}$ is defined by $R \cap (\{t | s R^* t\}^2)$, for comparing these triples. Now, $>_1$ is well-founded — due to $SN(s)$ — and $>_2 = >_3 = >_{\mathbb{N}}$ is obviously well-founded, too. Hence, their lexicographic combination $\succ$ is also well-founded. Thus, we may assume w.l.o.g. that

$$(***)\quad \langle s,n,k\rangle \text{ is a minimal counterexample w.r.t. } \succ\,.$$

In order to obtain a contradiction we proceed by case analysis and induction[13] showing that $\langle s,n,k\rangle$ cannot be a (minimal) counterexample.

If $u = v$ (i.e. $n = k = 0$) or $s = t$ we are done since $(**)$ is violated. Otherwise, let $s \to s' \to^* t$. If we can show that $C[s']_\Pi \downarrow v$ holds then by induction (on the first component) we get $C[t]_\Pi \downarrow v$ because we have $s \to s'$, hence $s >_1 s'$. But this is a contradiction to $(***)$. We shall distinguish the following cases:

[12] Cf. the Remark after Definition 1.

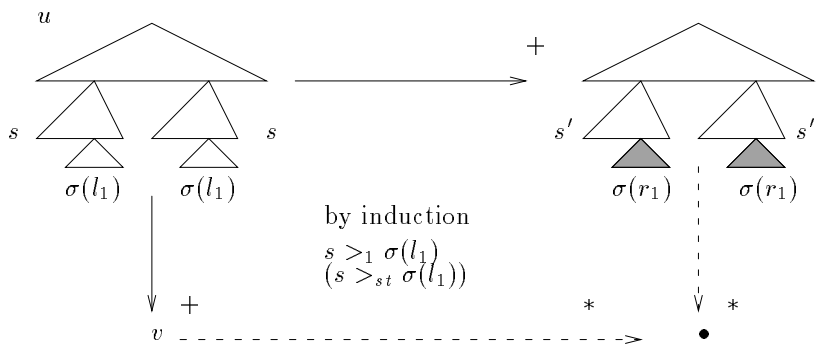
[13] This means that we shall exploit the minimality assumption of the counterexample $\langle s,n,k\rangle$.

(1) Proper subterm case (see Figure 1): If the first step $s \to s'$ reduces a proper subterm of s, i.e. $s \to_p s'$ for some $p > \lambda$, then we have

$$C[s]_\Pi = C[C'[s/p]_p]_\Pi = (C[C'[]_p]_\Pi)[s/p]_{\Pi p} \to^+ C[s']_\Pi = (C[C'[]_p]_\Pi)[s'/p]_{\Pi p}$$

with $s = C'[s/p]_p$ for some context $C'[]_p$, hence $C[s']_\Pi \downarrow v$ as desired by induction on the first component because $s >_{st} s/p = \sigma(l_1)$ implies $s >_1 s/p$.

Figure 1 : Proper subterm case



(2) Otherwise, we may suppose

$$s \to_{\lambda,\sigma,P_1 \Longrightarrow l_1 \to r_1} s', \text{ i.e. } s = \sigma(l_1) \ , \ s' = \sigma(r_1) \text{ and } \sigma(P_1) \downarrow$$

for some rule $P_1 \Longrightarrow l_1 \to r_1 \in \mathcal{R}$ and some substitution σ. Moreover assume

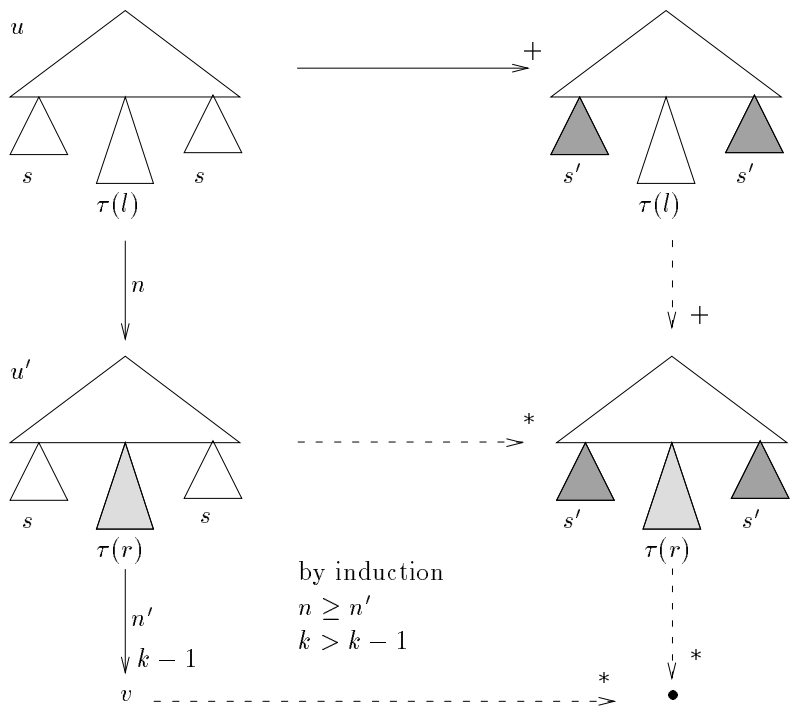
$$u = C[s]_\Pi \xrightarrow{n}_{q,\tau,P \Longrightarrow l \to r} u' \xrightarrow{n}{}^{k-1} v \ ,$$

i.e. $C[s]_\Pi/q = \tau(l)$, $u'/q = \tau(r)$ and $\tau(P) \downarrow$, for n minimal with $u \xrightarrow{n}{}^{*} v$ and $k \geq 1$ minimal with $u \xrightarrow{n}{}^{k} v$. Then we have to distinguish the following four subcases according to the relative positions of q and Π:

(2.1) $q \mid \Pi$ (disjoint peak, see Figure 2.1): Then we have $u = C[s]_\Pi \xrightarrow{n}_q C'[s]_\Pi = u' \xrightarrow{n}{}^{k-1} v$, $C'[s]_\Pi \to^* C'[s']_\Pi$ and $C[s]_\Pi \to^* C[s']_\Pi \to_q C'[s']_\Pi$ for some context $C'[,\ldots,]$, hence by induction on the second or third component ($n \geq n'$, $k > k-1$) $C'[s']_\Pi \downarrow v$ and thus $C[s']_\Pi \to C'[s']_\Pi \downarrow v$ as desired.

(2.2) $q \in \Pi$ (critical peak, see Figure 2.2): In this case we have a critical peak which is an instance of a critical overlay of $\mathcal{R}$, i.e. $s = \sigma(l_1) = \tau(l)$. Since all conditional critical pairs are joinable (overlays) we know that there exists some term w with $s = \sigma(l_1) \to \sigma(r_1) = s' \to^* w$ and $s = \tau(l) \to \tau(r) \to^* w$. Obviously, we have $u = C[s]_\Pi \to_{P \Longrightarrow l \to r} (C[\tau(l)]_\Pi)[q \leftarrow \tau(r)] =$

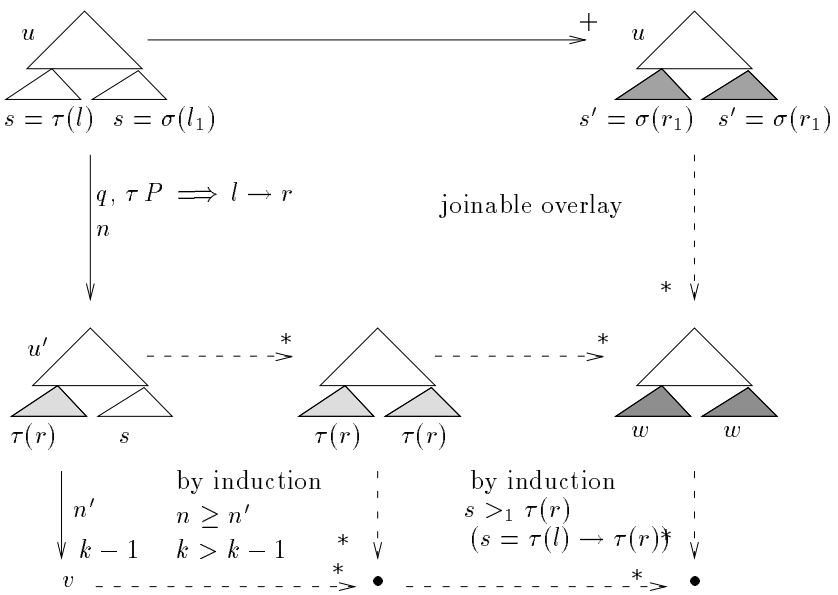
Figure 2.1 : Disjoint peak case



$u' \to^*_{P,\Longrightarrow_{l\to r}} C[\tau(r)]_\Pi$. For $|\Pi| = 1$ we obtain $\Pi = \{q\}$ and $(C[\tau(l)]_\Pi)[q \leftarrow \tau(r)] = u' = C[\tau(r)]_\Pi \to^* v$. Otherwise, we have $C[\tau(l)]_\Pi[q \leftarrow \tau(r)] \xrightarrow{n'}_{k-1} v$ with $n' \leq n$. Hence, by induction on the second or third component we obtain $C[\tau(r)]_\Pi \downarrow v$ (due to $n' \leq n$, $k-1 < k$). Moreover, $\tau(r) \to^* w$ yields $C[\tau(r)]_\Pi \to^* C[w]_\Pi$ which by induction on the first component implies $C[w]_\Pi \downarrow v$ (due to $s = \tau(l) \to \tau(r)$, hence $s >_1 \tau(r)$). Thus, $C[s']_\Pi = C[\sigma(r_1)]_\Pi \to^* C[w]_\Pi \downarrow v$ because of $\sigma(r_1) \to^* w$. Hence we get $C[s']_\Pi \downarrow v$ as desired. The remaining case is that of a *variable overlap*, either above or in some subterm $C[s]_\Pi/\pi = s$ ($\pi \in \Pi$) of $C[s]_\Pi$. Note that a critical peak which is not an overlay cannot occur due to $OS(\mathcal{R})$.

(2.3) $q < \pi$ for some $\pi \in \Pi$ (variable overlap above, see Figure 2.3): Let Π' be the set of positions of those subterms $s = \sigma(l_1)$ of $u/q = \tau(l)$ which correspond to some $u/\pi = s$, $\pi \in \Pi$. Formally, $\Pi' := \{\pi' \mid q\pi' \in \Pi\}$. Moreover, for every $x \in dom(\tau)$, let $\Delta(x)$ be the set of positions of those subterms s in $\tau(x)$ which are rewritten into s' in the derivation $u = C[s]_\Pi \to^+ C[s']_\Pi$, i.e. $\Delta(x) := \{\rho' \mid \exists\, \rho : l/\rho = x \wedge \rho\rho' \in \Pi'\}$. Then τ' is defined by $\tau'(x) := \tau(x)[\rho' \leftarrow$

Figure 2.2 : Critical peak (overlay)



$s' \mid \rho' \in \Delta(x)]$ for all $x \in dom(\tau)$. Obviously, we have $\tau(x) \to^*_{P_1 \Longrightarrow l_1 \to r_1} \tau'(x)$ for all $x \in dom(\tau)$. Thus we get

$$u = C[s]_\Pi = C'[\tau(l)]_q \xrightarrow{n}_{q,\tau,P \Longrightarrow l \to r} C'[\tau(r)]_q = u' \xrightarrow{n'}{}^{k-1} v$$

for some context $C'[]_q$ and some $n' \leq n$,

$$u = C[s]_\Pi \to^*_{P_1 \Longrightarrow l_1 \to r_1} C[s']_\Pi \to^*_{P_1 \Longrightarrow l_1 \to r_1} C'''[\tau'(l)]_q$$

for some context $C'''[]_q$, and

$$u' = C'[\tau(r)]_q \to^*_{P_1 \Longrightarrow l_1 \to r_1} C'''[\tau'(r)]_q \,.$$

Moreover we have

$$C'''[\tau'(l)]_q \to_{q,\tau',P \Longrightarrow l \to r} C'''[\tau'(r)]_q$$

by induction (due to $\tau'(P) \downarrow$ as shown below) and finally

$$C[s']_\Pi \to^* C'''[\tau'(r)]_q \downarrow v$$

as desired by induction (on the second or third component) due to $n \geq n'$, $k > k - 1$.

It remains to prove the claim $\tau'(P)\downarrow$. This means that we have to show $\tau'(z_1)\downarrow\tau'(z_2)$ for all $z_1\downarrow z_2\in P$. If P is empty or trivially satisfied (i.e. $n\leq 1$) we are done. Otherwise, we know by assumption that $\tau(z_1)\downarrow\tau(z_2)$ for all $z_1\downarrow z_2\in P$ in depth $n-1$. This means $eq(\tau(z_1),\tau(z_2))\stackrel{n-1}{\longrightarrow}{}^{*}\ true$. By construction of τ' we know $\tau(z_1)\rightarrow^{*}_{P_1\Rightarrow l_1\rightarrow r_1}\tau'(z_1)$, $\tau(z_2)\rightarrow^{*}_{P_1\Rightarrow l_1\rightarrow r_1}\tau'(z_2)$. Moreover, $eq(\tau(z_1),\tau(z_2))$ is of the form $E[s]_Q$, for some context $E[]_Q$, such that $E[s]_Q\rightarrow^{*}E[s']_Q=eq(\tau'(z_1),\tau'(z_2))$. By induction on the second component (due to $n>n-1$) we obtain $E[s']_Q=eq(\tau'(z_1),\tau'(z_2))\downarrow true$, hence $eq(\tau'(z_1),\tau'(z_2))\rightarrow^{*}true$ (since $true$ is irreducible) which means that $\tau'(z_1)$ and $\tau'(z_2)$ are joinable (without using the rule $eq(x,x)\rightarrow true$). This finishes the proof of the claim $\tau'(P)\downarrow$. Summarizing we have shown $C[s']_\Pi\downarrow v$ as desired.

(2.4) $\pi<q$ for some $\pi\in\Pi$ (variable overlap below, see Figure 2.4): Remember that we have $u/\pi=\sigma(l_1)=s$ and $u/q=\tau(l)$. Now let q', q'', q''', Π', Π'' and contexts $C'[]_q$, $D[]_{q'''}$, $D'[]_{\Pi'}$, $D''[]_{\Pi''}$ be (uniquely) defined by $u=C'[\tau(l)]_q$, $q=\pi q'$, $q'=q''q'''$, $l_1/q''=x\in\mathcal{V}$, $\sigma(x)=D[\tau(l)]_{q'''}$, $\Pi'=\{\pi'\,|\,l_1/\pi'=x\}$, $\Pi''=\{\pi''\,|\,r_1/\pi''=x\}$, $\sigma(l_1)=D'[D[\tau(l)]_{q'''}]_{\Pi'}$, $\sigma(r_1)=D''[D[\tau(l)]_{q'''}]_{\Pi''}$. Moreover let σ' be the substitution on $V(l_1)$ defined by

$$\sigma'(y)=\begin{cases}\sigma(y)\,,\ y\neq x\\ D[\tau(r)]_{q'''}\,,\ y=x,\sigma(x)=D[\tau(l)]_{q'''}\ .\end{cases}$$

Then we get

$$\begin{aligned}C[s]_\Pi&=C[\sigma(l_1)]_\Pi\\&=C[D'[D[\tau(l)]_{q'''}]_{\Pi'}]_\Pi\rightarrow^{+}_{\sigma,P_1\Rightarrow l_1\rightarrow r_1}C[s']_\Pi=C[\sigma(r_1)]_\Pi\\&=C[D''[D[\tau(l)]_{q'''}]_{\Pi''}]_\Pi\rightarrow^{*}_{\tau,R\Rightarrow l\rightarrow r}C[D''[D[\tau(r)]_{q'''}]_{\Pi''}]_\Pi\\&=C[\sigma'(r_1)]_\Pi\,,\end{aligned}$$

and

$$\begin{aligned}C[s]_\Pi&=C[\sigma(l_1)]_\Pi=C[D'[D[\tau(l)]_{q'''}]_{\Pi'}]_\Pi\\&=C'[\tau(l)]_q\rightarrow_{q,\tau,P\Rightarrow l\rightarrow r}u'\\&=C'[\tau(r)]_q\rightarrow^{*}_{\tau,P\Rightarrow l\rightarrow r}C[D'[D[\tau(r)]_{q'''}]_{\Pi'}]_\Pi=C[\sigma'(l_1)]_\Pi\,.\end{aligned}$$

By induction on the first component we obtain

$$C[s]_\Pi=C[\sigma(l_1)]_\Pi\rightarrow^{+}C[\sigma'(l_1)]_\Pi\downarrow v$$

(due to $s>_{st}\tau(l)$, hence $s>_1\tau(l)$). Moreover, we get

$$C[\sigma'(l_1)]_\Pi\rightarrow^{*}_{\sigma',P_1\Rightarrow l_1\rightarrow r_1}C[\sigma'(r_1)]_\Pi$$

since $\sigma'(P_1)\downarrow$ is satisfied by induction on the first component ($s>_{st}\tau(l)$, hence $s>_1\tau(l)$). [14] Furthermore we have $C[\sigma'(r_1)]_\Pi\downarrow v$ by induction on the

[14] Note that – in contrast to case 2.3 – a 'proper eq-reasoning' can be avoided here (by applying the induction hypothesis twice), since the inductive argument needed does not involve the *depth* of rewriting.

first component (due to $s = \sigma(l) \rightarrow^+ \sigma'(l_1)$, hence $s >_1 \sigma'(l_1)$). Summarizing we have shown

$$C[s']_\Pi \rightarrow^* C[\sigma'(r_1)]_\Pi \downarrow v$$

as desired.

Thus, for all cases we have shown $C[s']_\Pi \downarrow v$ yielding a contradiction to $(***)$, hence we are done. ■

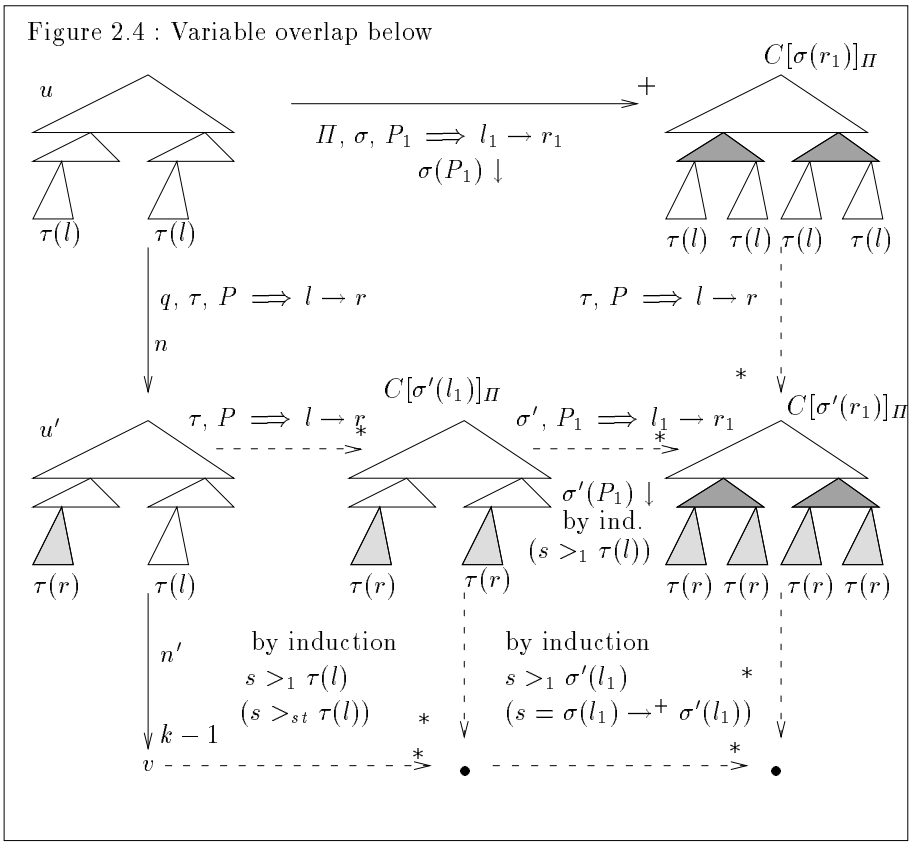
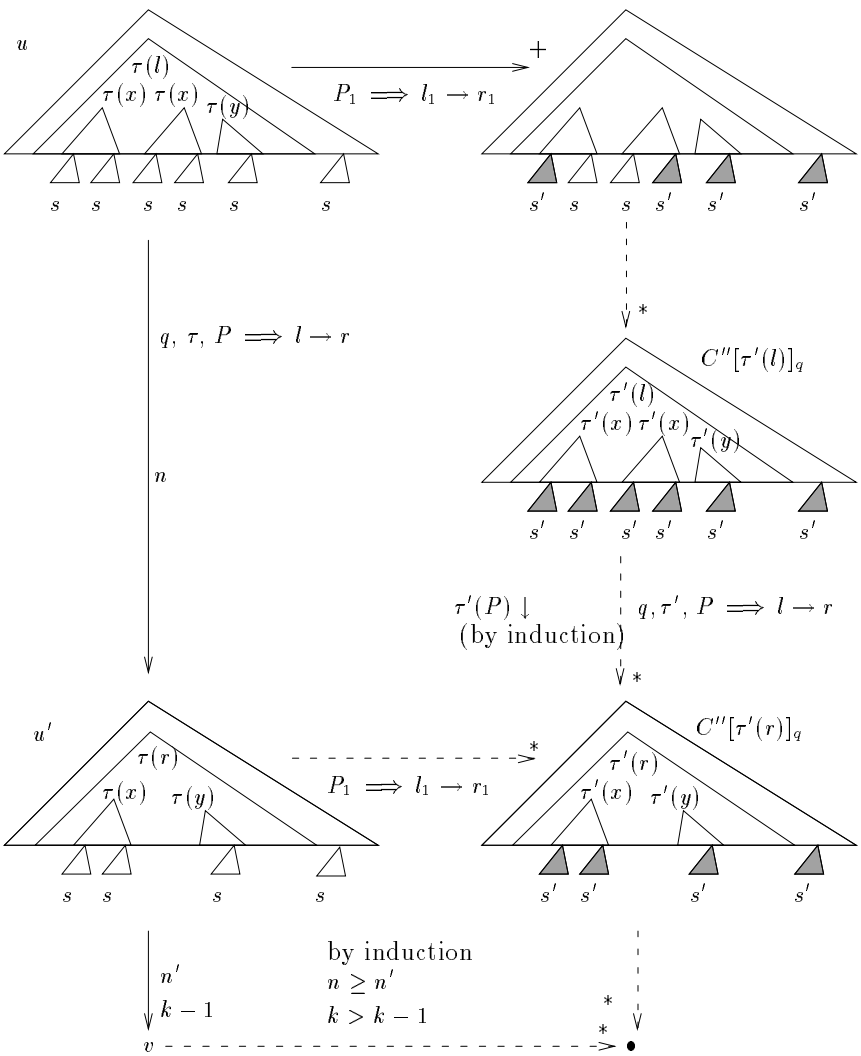



Figure 2.3 : Variable overlap above