Sufficient Conditions for Modular Termination of Conditional Term Rewriting Systems

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Abstract. Recently we have shown the following abstract result for unconditional term rewriting systems (TRSs). Whenever the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two (finite) terminating TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is non-terminating, then one of the systems, say $\mathcal{R}_1$, enjoys an interesting property, namely it is not termination preserving under non-deterministic collapses, i.e. $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is collapsing, i.e. contains a rule with a variable right hand side. This result generalizes known sufficient syntactical criteria for modular termination of rewriting. Here we extend this result and derived sufficient criteria for modularity of termination to the case of conditional term rewriting systems (CTRSs). Moreover we relate various definitions of notions related to termination of CTRSs to each other and discuss some subtleties and problems concerning extra variables in the rules.

1 Introduction

From a theoretical point of view and also for efficiency reasons it is very useful to know whether a combined TRS has some property whenever this property already holds for the single 'modules'. A simple and natural way of such 'modular' constructions is given by the concept of 'direct sum' ([17]) or 'disjoint union'. Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case the rule sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $R_1 \oplus R_2$. We shall also speak of the disjoint union of $\mathcal{R}_1$ and $\mathcal{R}_2$ using the implicit convention that $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be disjoint TRSs. A property P of TRSs is said to be *modular* if the following holds for all disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property P iff both $\mathcal{R}_1$ and $\mathcal{R}_2$ have property P. Toyama [17] has shown that confluence is modular. The termination property, however, is in general not modular as witnessed by the following counterexample of [17]:

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Example 1. $\mathcal{R}_1$: $f(a,b,x) \to f(x,x,x)$ $\quad$ $\mathcal{R}_2$: $G(x,y) \to x$, $G(x,y) \to y$

Clearly, both R_1 and R_2 are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite derivation:

$$\begin{aligned} f(a,b,G(a,b)) &\to_{\mathcal{R}_1} f(G(a,b),G(a,b),G(a,b)) \\ &\to_{\mathcal{R}_2} f(a,G(a,b),G(a,b)) \\ &\to_{\mathcal{R}_2} f(a,b,G(a,b)) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

In the last years a couple of sufficient conditions for modularity of termination of (unconditional) TRSs have been found. Some of these criteria are formulated syntactically (in terms of collapsing and duplicating rules, see Rusinowith [16], Middeldorp [11]). Other criteria involve more semantical requirements. For instance, it is shown by Toyama, Klop & Barendregt [18] that for left-linear and confluent TRSs termination (and hence completeness, i.e. confluence plus termination) is modular. In Gramlich [6] we have shown that termination (and hence completeness) is modular for locally confluent overlay systems (a TRS is an overlay system if all its critical pairs are obtained by 'root overlaps'). Kurihara & Ohuchi [10] proved that simple termination, i.e. termination by means of some simplification ordering, is a modular property of (finite) TRSs. A structural analysis of potential counterexamples and a couple of derived sufficient conditions generalizing results from [16], [11] and [10] have been presented in Gramlich [4], [5]. Below we shall show how the main ideas and results of this approach can be extended to CTRSs.

For CTRSs results from [16] and [11] concerning collapse-free and non-duplicating systems for the unconditional case have been adapted and extended in Middeldorp [12], [13]. Recently Middeldorp has shown in [14] that the main result of Middeldorp & Toyama [15] stating that completeness is invariant under the (non-disjoint) combination of constructor systems (which may share common constructors) does also hold for CTRSs without extra variables in the conditions of the rewrite rules.

Before going into details now, let us motivate and sketch our approach for analyzing modularity of termination. Having again a closer look on example 1 above and the non-terminating derivation indicated there, it is obvious that the collapsing $\mathcal{R}_2$-steps using the rules $G(x,y) \to x, G(x,y) \to y$ play an essential role for enabling the derivation to be infinite. In fact, this observation can be generalized to arbitrary situations where we have (finite) terminating (C)TRSs $\mathcal{R}_1, \mathcal{R}_2$ over disjoint signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, such that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. In any infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation $s_0 \to s_1 \to s_2 \to s_3 \to \ldots$, all the s_i's must be 'mixed' terms, i.e. involve function symbols from both signatures $\mathcal{F}_1$ and $\mathcal{F}_2$. We shall show that for any counterexample satisfying a certain minimality property concerning the 'layer structure' of its terms, one can construct from this counterexample an infinite derivation in $R_i \oplus \{G(x,y) \to x, G(x,y) \to y\}$ for $i = 1$ or $i = 2$, let's say for $i = 1$. This is achieved by an appropiate transformation from terms over $\mathcal{F}_1 \uplus \mathcal{F}_2$ into

terms over $\mathcal{F}_1 \uplus \{A, G\}$ (here A is a new constant symbol and G is a new binary function symbol) which abstracts from the concrete form of $\mathcal{F}_2$-layers but retains enough relevant information for the translation of the reduction steps. This characteristic property of minimal counterexamples provides the basis for a couple of modularity results derived subsequently. It also corresponds nicely to the intuition that the existence of counterexamples crucially depends on 'non-deterministic collapsing' reduction steps. Hence, example 1 above is in a sense the simplest conceivable counterexample.

In the next section we briefly recall some basic notions, definitions and facts for TRSs, CTRSs and for disjoint unions needed later on. In section 3 the main results and their applications for CTRSs will be presented. Finally some open problems and perspectives will be discussed. For a more detailed presentation and discussion of the whole approach see [4]).

2 Preliminaries

2.1 Basic Notations and Definitions for TRSs

We briefly recall the basic terminology needed for dealing with TRSs (e.g. [9]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated to every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an obvious way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$. In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $V(t)$.

A *context* $C[, \ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ where $\Box$ is a new special constant symbol. If $C[, \ldots,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the term obtained from $C[, \ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. For the set $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ we also write $\mathcal{CON}(\mathcal{F}, \mathcal{V})$. A *non-empty* context is a term from $\mathcal{CON}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ with $t \equiv C[s]$. If in addition $C[] \not\equiv \Box$ then s is a *proper* subterm of t. A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma)$ $\{x \in \mathcal{V} | \sigma x \not\equiv x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r) denoted by $l \to r$ with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$. [2]

[2] This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

Instead of $(\mathcal{R},\mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\rightarrow_{\mathcal{R}^{\mathcal{F}}}$ for terms $s,t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ is defined as follows: $s \rightarrow_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \rightarrow r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$ and $t \equiv C[\sigma r]$. We also write $\rightarrow_{\mathcal{R}}$ or simply $\rightarrow$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$ is clear from the context, respectively. The symmetric, transitive and transitive-reflexive closures of $\rightarrow$ are denoted by $\leftrightarrow$, $\rightarrow^+$ and $\rightarrow^*$, respectively. Two terms s,t are *joinable in* $\mathcal{R}$, denoted by $s \downarrow_{\mathcal{R}} t$, if there exists a term u with $s \stackrel{*}{_{\mathcal{R}}}\leftarrow u \rightarrow^*_{\mathcal{R}} t$. A term s is *irreducible* or a *normal form* if there is no term t with $s \rightarrow t$. A TRS $\mathcal{R}$ is *terminating* or *strongly normalizing* if $\rightarrow$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \cdots$.

A *partial ordering* $>$ on a set D is a transitive and irreflexive binary relation on D. A partial ordering $>$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ is said to be *monotonic (w.r.t. the term structure)* if it possesses the *replacement property*

$$s > t \quad \Longrightarrow \quad C[s] > C[t]$$

for all $s,t,C[]$. It is *stable (w.r.t. substitutions)* if

$$s > t \quad \Longrightarrow \quad \sigma s > \sigma t$$

for all s,t,σ. A *term ordering on* $\mathcal{T}(\mathcal{F},\mathcal{V})$ is a monotonic and stable partial ordering on $\mathcal{T}(\mathcal{F},\mathcal{V})$. A *reduction ordering* is a well-founded term ordering. A term ordering $>$ is said to be a simplification ordering if it additionally enjoys the subterm property

$$C[s] > s$$

for any s and any non-empty context $C[]$.[3]

A TRS is confluent if $^* \leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^* \leftarrow$ and locally confluent if $\leftarrow \circ \rightarrow \subseteq \rightarrow^* \circ {}^* \leftarrow$.[4] A confluent and terminating TRS is said to be *convergent* or *complete*.

2.2 Conditional Term Rewriting Systems

Moreover, we need some basic terminology about CTRSs.

Definition 1. A *CTRS* is a pair $(\mathcal{R},\mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set of *conditional rewrite rules* of the form

$$s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \rightarrow r$$

with $s_1,\ldots,s_n,t_1,\ldots,t_n,l,r \in \mathcal{T}(\mathcal{F},\mathcal{V})$. Moreover, we require $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$ as for unconditional TRSs, i.e. no variable left hand sides and no extra variables on the right hand side. For our purposes it will be useful to exclude extra variables in the conditions, too. This means to require additionally

[3] For the case that varyadic function symbols are allowed one additionally requires here the so-called 'deletion'-property (cf. Dershowitz [1]).

[4] Here, '$\circ$' denotes relation composition.

$\bigcup_{i=1}^{n} \{V(s_i), V(t_i)\} \subseteq V(l)$.[5] If the condition is empty, i.e. $n = 0$, we simply write $l \to r$. Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant.

Depending on the interpretation of the equality sign in the conditions of rewrite rules, different reduction relations may be associated with a given CTRS.

Definition 2.

(1) In a *join CTRS* $\mathcal{R}$ the equality sign in the conditions of rewrite rules is interpreted as joinability. Formally this means: $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$, $t \equiv C[\sigma r]$ and $\sigma s_i \downarrow_{\mathcal{R}} \sigma t_i$ for all $i \in \{1, \ldots, n\}$. For rewrite rules of a join CTRS we shall use the notation $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \to r$.

(2) Semi-equational CTRSs are obtained by interpreting the equality sign in the conditions as convertibility, i.e. as $\overset{*}{\leftrightarrow}$.

Definition 3. The reduction relation corresponding to a given CTRS $\mathcal{R}$ is inductively defined as follows ($\square$ denotes $\downarrow$ or $\overset{*}{\leftrightarrow}$, respectively):

$$\begin{aligned}
\mathcal{R}_0 &= \{l \to r | l \to r \in \mathcal{R}\} , \\
\mathcal{R}_{i+1} &= \{\sigma l \to \sigma r | \; s_1 \square t_1 \wedge \ldots \wedge s_n \square t_n \Longrightarrow l \to r \in \mathcal{R}, \\
&\quad \sigma s_j \square_{\mathcal{R}_i} \sigma t_j \text{ for } j = 1, \ldots, n\} , \\
s \to_{\mathcal{R}} t : \Longleftrightarrow\ & s \to_{\mathcal{R}_i} t \text{ for some } i \geq 0, \text{i.e.} \to_{\mathcal{R}} = \bigcup_{i \geq 0} \to_{\mathcal{R}_i} .
\end{aligned}$$

The *depth* of a rewrite step $s \to_{\mathcal{R}} t$ is defined to be the minimal i with $s \to_{\mathcal{R}_i} t$.

The other basic notions for unconditional TRSs generalize in a straightforward manner to CTRSs.

In general, conditional rewriting is much more complicated than unconditional rewriting. For instance, the rewrite relation may be undecidable even for complete CTRSs without extra variables in the conditions (cf. [8]).

Definition 4. ([3]) A CTRS $\mathcal{R}$ is *decreasing* if there exists an extension $>$ of the reduction relation induced by $\mathcal{R}$ which satisfies the following properties:

(1) $>$ is noetherian.
(2) $>$ has the subterm property, i.e. $C[s] > s$ for every term s and every non-empty context $C[]$.
(3) If $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies l \to r$ is a rule in $\mathcal{R}$ and σ is a substitution then $\sigma l > \sigma s_i$ and $\sigma l > \sigma t_i$ for $i = 1, \ldots, n$.

[5] Extra variables in the conditions may be quite natural in many situations, in particular from a specification or programming point of view. Later on we will discuss the reason for excluding them here.

A CTRS $\mathcal{R}$ is *reductive* (cf. [7]) if there exists a well-founded monotonic extension $>$ of the reduction relation induced by $\mathcal{R}$ satisfying (3).

Clearly, every reductive system is decreasing and any decreasing system is terminating. Decreasingness exactly captures the finiteness of recursive evaluation of terms (cf. [2]). For decreasing (join) CTRSs all the basic notions are decidable, e.g. reducibility and joinability. Moreover, fundamental results like the critical pair lemma hold for decreasing (join) CTRSs which is not the case in general for arbitrary (terminating join) CTRSs.

In the following we shall tacitly assume that all CTRSs considered are join CTRSs (which is the most important case in practice), except for cases where another kind of CTRSs is explicitly mentioned.

2.3 Basic Notations for Disjoint Unions

The following notations and definitions for dealing with disjoint unions of (C)TRSs mainly follow [17] and [13]. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be (C)TRSs with disjoint signatures $\mathcal{F}_1, \mathcal{F}_2$. Their *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ is the (C)TRS $(\mathcal{R}_1 \uplus \mathcal{R}_2, \mathcal{F}_1 \uplus \mathcal{F}_2)$.[6]

Let $t \equiv C[t_1, \ldots, t_n]$ with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C[, \ldots,] \in CON(\mathcal{F}_a, \mathcal{V})$ and $root(t_1), \ldots, root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. In this case the t_i's are the *principal subterms* or *principal aliens* of t. Note that every $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V}) \setminus (\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}))$ has a unique representation of the form $t \equiv C[\![t_1, \ldots, t_n]\!]$. The set of all *special subterms* or *aliens* of t is recursively defined in a straightforward manner.

The *rank* of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ is defined by

$$rank(t) = \begin{cases} 1 & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}) \\ 1 + max\{rank(t_i) | 1 \leq i \leq n\} & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}$$

An important basic fact about the rank of terms occurring in a $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation is the following ([17]): If $s \to^* t$ then $rank(s) \geq rank(t)$. Moreover, if $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ with $rank(s) = n$ then there exists a ground instance σs of s with $rank(\sigma s) = n$, too. [7] A (finite or infinite) derivation $D: \; s_1 \to s_2 \to s_3 \ldots$ is said to have rank n ($rank(D) = n$) if n is the minimal rank of all the s_i's, i.e. $n = \min\{rank(s_i) | 1 \leq i\}$.

The topmost homogeneous part of a term $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V})$, denoted by $top(t)$, is obtained from t by replacing all principal subterms by $\Box$, i.e.

$$top(t) = \begin{cases} t & \text{if} \quad rank(t) = 1 \\ C[, \ldots,] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}$$

Furthermore we shall use the abbreviations $\mathcal{GT}_\oplus$ for $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$, $\mathcal{GT}_\oplus^n$ for $\{t \in \mathcal{GT}_\oplus | \; rank(t) = n\}$, $\mathcal{GT}_\oplus^{<n}$ for $\{t \in \mathcal{GT}_\oplus | rank(t) < n\}$ and $\mathcal{GT}_\oplus^{\leq n}$ for

[6] The symbol '$\uplus$' denotes union of disjoint sets.

[7] This is easily verified by substituting appropriately $\mathcal{F}_1$- or $\mathcal{F}_2$-ground terms for those variables which occur in the 'deepest layer' of s.

$\{t \in \mathcal{GT}_\oplus | rank(t) \leq n\}$. For the sake of better readability the function symbols from $\mathcal{F}_1$ are considered to be black and those of $\mathcal{F}_2$ to be white. Variables have no colour. A *top black (white)* term has a black (white) root symbol.

For $s, t \in \mathcal{GT}_\oplus$ the one-step reduction $s \to t$ is said to be *inner* – denoted by $s \stackrel{i}{\to} t$ – if the reduction takes place in one of the principal subterms of s. Otherwise, we speak of an *outer* reduction step and write $s \stackrel{o}{\to} t$. A rewrite step $s \to t$ is *destructive at level 1* if the root symbols of s and t have different colours. The step $s \to t$ is *destructive at level* $n+1$ (for $n \geq 1$) if $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \stackrel{i}{\to} C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level n. Clearly, if a rewrite step is destructive (at some level) then the applied rewrite rule is collapsing, i.e. has a variable right-hand side. This is a basic fact which should be kept in mind subsequently.

3 Modularity of Termination for Conditional Term Rewriting Systems

For generalizing results concerning modular properties of TRSs to the conditional case a careful analysis is necessary. As mentioned by Middeldorp (cf. [13]), the additional complications mainly arise from the fact that the fundamental property

$$s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t \quad \Longrightarrow \quad s \to_{\mathcal{R}_1} t \quad \vee \quad s \to_{\mathcal{R}_2} t \quad (*)$$

only holds for unconditional TRSs but not for CTRSs in general. This is due to the fact that for verifying the applicability of an $\mathcal{R}_1$-rule, i.e. for proving the corresponding instantiated conditions, rules from $\mathcal{R}_2$ may be crucial. Consider for instance

Example 2.

$\mathcal{R}_1 = \{\ x \downarrow b \ \wedge\ x \downarrow c \implies a \to a\ \}$ over $\mathcal{F}_1 = \{a, b, c\}$,
$\mathcal{R}_2 = \{G(x, y) \to x, G(x, y) \to y\}$ over $\mathcal{F}_2 = \{G, A\}$.

Here, we have $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ by applying the $\mathcal{R}_1$-rule (x is substituted by $G(b, c)$), but neither $a \to_{\mathcal{R}_1} a$ nor $a \to_{\mathcal{R}_2} a$. Hence, this is also a very simple counterexample to modularity of termination for CTRSs, because both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (the reduction relation of $\mathcal{R}_1$ is empty). Moreover, the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation $a \to a \to a \to \ldots$ has rank one, a phenomenon which cannot occur in the unconditional case. Note that the system $\mathcal{R}_1$ above has the extra variable x in the condition part of its rule.[8] Therefore it cannot be decreasing. When we forbid extra variables in the conditions then the minimal rank of potential counterexamples is at least 3 as we shall see. But the fundamental property $(*)$ above may still be violated as shown by

Example 3. (see [13] for similar counterexamples)

[8] Hence, strictly spoken $\mathcal{R}_1$ is no CTRS in the sense of definition 1.

$\mathcal{R}_1 = \{\ x \downarrow a\ \wedge\ x \downarrow b\ \Longrightarrow\ f(x) \to f(x)\ \}$ over $\mathcal{F}_1 = \{a, b, f\}$,
$\mathcal{R}_2 = \{G(x,y) \to x, G(x,y) \to y\}$ over $\mathcal{F}_2 = \{G, A\}$.

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly decreasing (and even reductive), hence terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating since we have e.g. the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation

$$f(G(a,b)) \to f(G(a,b)) \to f(G(a,b)) \to \ldots$$

where the $\mathcal{R}_1$-rule is enabled because the instantiated conditions can be verified using $\mathcal{R}_2$.

For unconditional TRSs it is known that each of the conditions

(a) neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a duplicating[9] rule ([16]),
(b) neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a collapsing rule, and
(c) one of the system $\mathcal{R}_1$, $\mathcal{R}_2$ contains neither collapsing nor duplicating rules ([11])

is sufficient for ensuring modularity of termination. The above example shows that (a) and (c) do not generalize to the conditional case. In [12] it is shown that (b) is still sufficient in the conditional case, and that (a) and (c) are sufficient under the additional assumption that both CTRSs are confluent. Moreover, confluence turns out to be a modular property of CTRSs as shown by Middeldorp in [13].

In the following we shall show that the essential features and results of our structural analysis of modular termination for the unconditional case presented in [5] can be generalized to CTRSs in a rather straightforward manner.

It is easy to see that conditional reduction steps are rank decreasing, i.e. $s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ implies $rank(s) \geq rank(t)$.

Next some structural properties of minimal counterexamples are exhibited.

Lemma 5. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two terminating disjoint CTRSs such that*

$$D : s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ (involving only ground terms) of minimal rank, i.e. any derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of smaller rank is finite. Then we have:

(a) $rank(D) \geq 3$.
(b) Infinitely many steps in D are outer steps.

Proof.

(b) Assume for a proof by contradiction that only finitely many steps in D are outer ones. W.l.o.g. we may further assume that no step in D is an outer one. Hence, for $s_1 \equiv C[\![t_1, \ldots, t_n]\!]$ all reductions in D are inner ones and take place below one of the positions of the t_i's. Since D is infinite we conclude by the pigeon hole principle that at least one of the t_i's initiates an infinite derivation whose rank is smaller than $rank(D)$. But this is a contradiction to the minimality assumption concerning $rank(D)$.

[9] A rule is said to be duplicating if some variable occurs in the right side strictly more often than in the left side.

(a) For proving (a) we first remark that $rank(D) = 1$ is impossible.[10] For showing by contradiction that $rank(D) = 2$ is impossible, too, let us assume w.l.o.g. that all s_j's are top black and have $rank\,2$. Then it is easy to see that for any step $s_j \xrightarrow{i}_{\mathcal{R}_1 \oplus \mathcal{R}_2} s_{j+1}$ we have $top(s_j) \equiv top(s_{j+1})$. Moreover, for $s_j \xrightarrow{o}_{\mathcal{R}_1 \oplus \mathcal{R}_2} s_{j+1}$ using an $\mathcal{R}_1$-rule we can show by induction over the depth of rewriting that $top(s_j) \xrightarrow{o}_{\mathcal{R}_1} top(s_{j+1})$ holds using the same $\mathcal{R}_1$-rule. Hence, the $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation D translates into the $\mathcal{R}_1$-derivation

$$D' : top(s_1) \to^*_{\mathcal{R}_1} top(s_2) \to^*_{\mathcal{R}_1} top(s_3) \to^*_{\mathcal{R}_1} \cdots .$$

which according to (b) contains infinitely many proper $\mathcal{R}_1$-steps. But this contradicts termination of $\mathcal{R}_1$. ■

Note, that for unconditional TRSs in any minimal counterexample there are infinitely many inner reduction steps which are destructive at level 2. This property does not hold for CTRSs in general. To wit, consider the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation

$$f(G(a,b)) \to f(G(a,b)) \to f(G(a,b)) \to \ldots$$

in example 3 above where all reductions are outer steps.

In order to be able to state our main result we need the following definitions.

Definition 6. A CTRS $\mathcal{R}$ is said to be *termination preserving under non-deterministic collapses* if termination of $\mathcal{R}$ implies termination of $\mathcal{R} \oplus \{G(x,y) \to x, G(x,y) \to y\}$.

Definition 7. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two (finite) terminating disjoint CTRSs, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ and $n \in N$ such that for every $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ there is no infinite $\mathcal{R}$-derivation starting with s. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$. Then the $\mathcal{F}_2$- (or *white*) *abstraction* is defined to be the mapping

$$\Phi : \mathcal{GT}^{\leq n}_{\oplus} \uplus \{t \in \mathcal{GT}^{n+1}_{\oplus} | root(t) \in \mathcal{F}_1\} \longrightarrow \mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$$

given by

$$\Phi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1) \\ A & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_2) \\ C[\![\Phi(t_1), \ldots, \Phi(t_m)]\!] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{F}_1 \\ CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{F}_2 \end{cases}$$

with

$$\begin{aligned} SUCC^{\mathcal{F}_1}(t) &:= \{t' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | t \to^*_{\mathcal{R}} t', root(t') \in \mathcal{F}_1\}, \\ \Phi^*(M) &:= \{\Phi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Phi), \\ CONS(\langle\rangle) &:= A, \\ CONS(\langle s_1, \ldots, s_{k+1}\rangle) &:= G(s_1, CONS(\langle s_2, \ldots, s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1, \ldots, s_k\}) &:= \langle s_{\pi(1)}, \ldots, s_{\pi(k)}\rangle, \end{aligned}$$

[10] Here our general assumption that extra variables in the conditions of rules are forbidden is crucial! See also example 2.

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

Intuitively, for computing $\Phi(t)$ one proceeds top-down in a recursive fashion. Top black layers are left invariant whereas (for the case of top black t) the principal top white subterms are transformed by computing for every such top white subterm the set of possible top black successors, abstracting the resulting terms recursively, sorting the resulting set of abstracted terms and finally constructing again an ordinary term by means of using the new constant symbol A (for empty arguments sets) and the new binary function symbol G (for non-empty argument sets). The sorting process and the total ordering involved here are due to some proof-technical subtleties which are explained and detailed in [4]. Note moreover, that the finiteness assumption in definition 7 concerning the involved CTRSs ensures that Φ is well-defined since the set of possible top black successors of top white principal aliens is finite. In fact, this finiteness assumption can be considerably weakened as shown in [4], [5].

Now we can formalize some properties of the above defined abstracting transformation which are technically crucial for proving the main structural result.

Lemma 8. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, n and Φ be given as in definition 7. Then, for any $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ and $root(s) \in \mathcal{F}_2$ we have:*

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \to^*_{\mathcal{R}'_2} \Phi(t),$$

where $\mathcal{R}'_2 := \mathcal{R}^G_{sub} := \{G(x,y) \to x, G(x,y) \to y\}$.

Proof. By an easy case analysis according to the definition of Φ. ■

Lemma 9. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Φ be given as in definition 7. Then, Φ is rank decreasing, i.e. for any $s \in dom(\Phi) := \mathcal{GT}^{\leq n}_{\oplus} \uplus \{t \in \mathcal{GT}^{n+1}_{\oplus} | root(t) \in \mathcal{F}_1\}$ we have $rank(\Phi(s)) \leq rank(s)$.*

Proof. By an easy induction on $rank(s)$ using the definition of Φ. ■

Lemma 10. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, $\mathcal{R}'_2 = \mathcal{R}^G_{sub} = \{G(x,y) \to x, G(x,y) \to y\}$, $\mathcal{R}' = \mathcal{R}_1 \uplus \mathcal{R}'_2$ be given. Moreover, let n and the $\mathcal{F}_2$-abstraction Φ be given as in definition 7. Then, for any $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$ we have:*

(a) If $s \xrightarrow{o}_{\mathcal{R}} t$ using an $\mathcal{R}_1$-rule is not destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}'} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.

(b) If $s \xrightarrow{o}_{\mathcal{R}} t$ using an $\mathcal{R}_1$-rule is destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}'} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.

(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(t)$ with all steps not destructive at level 2.

(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^+_{\mathcal{R}'_2} \Phi(t)$ such that exactly one of these steps is destructive at level 2.

Proof. The general proof structure is as follows: We show by induction over the depth of rewriting steps the implication

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \to^*_{\mathcal{R}'} \Phi(s) \ .$$

The case distinction for inner and outer as well as for destructive and non-destructive reduction steps proceeds according to the definition of Φ yielding a proof of (a)-(d).[11] ■

Finally we obtain the general structure theorem for CTRSs.

Theorem 11. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint (finite) CTRSs which are both terminating such that their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then $\mathcal{R}_j$ is not termination preserving under non-deterministic collapses for some $j \in \{1,2\}$ and the other system $\mathcal{R}_k$, $k \in \{1,2\} \setminus \{j\}$, is collapsing. Moreover, the minimal rank of counterexamples in $\mathcal{R}_j \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is less than or equal to the minimal rank of counterexamples in $\mathcal{R}_1 \oplus \mathcal{R}_2$.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ be given as stated above. We consider a minimal counterexample, i.e. an infinite $\mathcal{R}$-derivation

$$D: \quad s_1 \to s_2 \to s_3 \to \quad \ldots$$

of minimal rank, let's say $n+1$. W.l.o.g. we may assume that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted ground terms having *rank* $n+1$. Since the preconditions of definition 7 are satisfied we may apply the white ($\mathcal{F}_2$-) abstraction function Φ to the s_i's. Using lemma 10 we conclude that this yields an $\mathcal{R}'$-derivation

$$D': \quad \Phi(s_1) \to^* \Phi(s_2) \to^* \Phi(s_3) \to^* \quad \ldots$$

where $\mathcal{R}' := \mathcal{R}_1 \oplus \mathcal{R}'_2$ with $\mathcal{R}'_2 := \mathcal{R}^G_{sub} = \{G(x,y) \to x, G(x,y) \to y\}$. From lemma 10 we know that for any step $s_j \to s_{j+1}$ in D we have

$$\begin{aligned} s_j \xrightarrow{o}_{\mathcal{R}} s_{j+1} &\Longrightarrow \Phi(s_j) \xrightarrow{o}_{\mathcal{R}'} \Phi(s_{j+1}) \ , \\ s_j \xrightarrow{i}_{\mathcal{R}} s_{j+1} &\Longrightarrow \Phi(s_j) \xrightarrow{i}{}^*_{\mathcal{R}'} \Phi(s_{j+1}). \end{aligned}$$

Since according to lemma 5 (b) infinitely many steps in D are outer ones it follows that D' is an infinite $\mathcal{R}'$-derivation. But this means that $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses. Moreover, under the assumption that $\mathcal{R}_2$ is non-collapsing the $\mathcal{F}_2$-abstraction of principal subterms of a minimal counterexample always yields the constant A which implies that the transformed infinite derivation is an $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$. Thus $\mathcal{R}_2$ must be collapsing. Lemma 9 finally implies $rank(D') \leq rank(D)$ which finishes the proof. ■

[11] Note that the assumption of having no extra variables in the conditions is important because this would cause problems with the *rank* of instantiated condition terms. In that case substitution of the extra variables in the condition part which are implicitly existentially quantified might yield terms of arbitrarily high *rank* which in turn might prevent Φ from being well-defined for these instantiated terms.

This general result can now be exploited for deriving a couple of sufficient criteria for modular termination of CTRSs. For instance, termination (and hence also completeness) is modular for the class of (finite) CTRSs which are termination preserving under non-deterministic collapses as well as for the class of collapse-free CTRSs[12] (cf. [12]). It is also modular for the class of (finite) CTRSs which are non-deterministically collapsing where a CTRS $\mathcal{R}^{\mathcal{F}}$ is said to be *non-deterministically collapsing* if some term can be reduced to two distinct variables.

Definition 12. Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS and $f \in \mathcal{F}, \mathcal{F}' \subseteq \mathcal{F}$. Then, $\mathcal{R}^{\mathcal{F}}$ is said to be *f-simply terminating* if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ with $\mathcal{R}^{f}_{sub} := \{f(x_1, \ldots, x_j, \ldots, x_n) \to x_j | 1 \leq j \leq n\}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is *$\mathcal{F}'$-simply terminating* if $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}'} \mathcal{R}^{f}_{sub}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is *simply terminating* if $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}$-simply terminating (see [10]), i.e. if $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}} \mathcal{R}^{f}_{sub}$ is terminating.

Clearly, simple termination implies termination for CTRSs. Moreover, we easily obtain the following results.

Lemma 13. *Let $\mathcal{R}^{\mathcal{F}}$ be a f-simply terminating CTRS for some $f \in \mathcal{F}$ of arity greater than 1. Then $\mathcal{R}^{\mathcal{F} \cup \mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating for any $\mathcal{F}'$ with $\mathcal{F}' \cap \mathcal{F} = \emptyset$.*

Corollary 14. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two (finite) disjoint CTRSs with $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1, 2$. Then the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (f_1- and f_2-simply) terminating, too.*

By specializing corollary 14 we finally obtain

Theorem 15. *Simple termination is modular for the class of (finite) CTRSs $\mathcal{R}^{\mathcal{F}}$ such that there exists at least one function symbol $f \in \mathcal{F}$ of arity greater than 1.*

Kurihara & Ohuchi [10] have shown that for finite unconditional TRSs simple termination can be characterized by means of simplification orderings. This yields in slightly generalized form:

Lemma 16. *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ is $(\mathcal{F}-)$ simply terminating if and only if there exists a simplification ordering $\succ$ with $l \succ r$ for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$.*

An analogous characterization of simple termination for CTRSs does not seem to be possible in a straightforward manner. Obviously, any (finite) simply terminating CTRS can be shown to be terminating by some simplification ordering, but not vice-versa in general. To see this, let us have again a look on example 3.

[12] Note that for collapse-free CTRSs the finiteness assumption in definition 7 is not necessary for ensuring well-definedness of the abstraction function Φ.

Example 4. (example 3 continued)
Consider $\mathcal{R}_1 = \{\ x \downarrow a \ \wedge \ x \downarrow b \implies f(x) \to f(x)\ \}$ over the extended signature $\mathcal{F}_1 = \{a, b, f, G\}$, with G binary. Here the reduction relation induced by $\mathcal{R}_1$ is empty, hence any simplification ordering trivially suffices for ensuring termination of $\mathcal{R}_1$. But, due to the non-termination of $\mathcal{R}_1 \cup \{f(x) \to x, G(x,y) \to x, G(x,y) \to y\}$, $\mathcal{R}_1$ is not simply terminating .

In order to provide a better intuition about simple termination of CTRSs we relate this notion to other ones. For that purpose we need the following definitions and abbreviations:

A (finite) CTRS $\mathcal{R}$ is *simplifying (SIMP)* (cf. [8]) if there exists a simplification ordering $>$ satisfying $l > r, s_i, t_i, 1 \leq i \leq n$ for all rules $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \to r \in \mathcal{R}$. $\mathcal{R}$ is *terminating (or strongly normalizing) with a simplification ordering (SOSN)* if there exists a simplification ordering $>$ with $\to_{\mathcal{R}} \subseteq >$. It is *simply reducing (SRED)* if there exists a simplification ordering $>$ with $l > r$ for every rule $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \to r \in \mathcal{R}$, i.e. if the corresponding unconditional TRS can be oriented by some simplification ordering. For terminating (or strongly normalizing), decreasing and simply terminating CTRSs we use the abbreviations *SN*, *DEC* and *SISN*, respectively. Then we obtain the following easily verifiable result.

Lemma 17. *For any (finite) CTRS $\mathcal{R}$ we have:*

(a) $SIMP(\mathcal{R}) \implies SRED(\mathcal{R}) \implies SISN(\mathcal{R}) \implies SOSN(\mathcal{R}) \implies SN(\mathcal{R})$.
(b) All implications in (a) are proper.
(c) $SIMP(\mathcal{R}) \implies DEC(\mathcal{R}) \implies SN(\mathcal{R})$.
(d) The properties $SRED(\mathcal{R}), SISN(\mathcal{R})$ *and* $SOSN(\mathcal{R})$ *are incomparable with* $DEC(\mathcal{R})$.

Note in particular, that every simplifying CTRS is clearly simply terminating but not vice-versa in general. Simple termination even does not imply decreasingness as shown e.g. by the CTRS consisting of the single rule $a \downarrow b \implies a \to a$.

Finally us mention some aspects not yet handled. Firstly our modularity results have only been proved for join CTRS. But it should (at least be partially) possible to extend them to the semi-equational case. Note that again new subtle effects may occur in semi-equational CTRSs. For instance, lemma 5 (a) does not hold any more. To wit, consider

Example 5. $\mathcal{R}_1 = \{\ b \overset{*}{\leftrightarrow} c \implies a \to a\ \}$, $\mathcal{R}_2 = \{G(x,y) \to x, G(x,y) \to y\}$.

Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (and decreasing) but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. We even have a counterexample of *rank* 1, namely $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \ldots$ because we have $b \leftarrow G(b,c) \to c$ in $\mathcal{R}_2$, hence $b \overset{*}{\leftrightarrow}_{\mathcal{R}_2} c$. Considered as join CTRS, $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating, however!

Secondly, the variable conditions required for CTRSs should be investigated in more detail. It should be possible to allow extra variables in the conditions. Extra variables in right hand sides seem to be much more difficult to handle than extra variables only in the conditions (see the discussion in [13]).

Moreover, the relationship between the results in [12] and ours remains to be clarified.

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