On Modularity of Termination and Confluence Properties of Conditional Rewrite Systems

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Abstract. We investigate the modularity behaviour of termination and confluence properties of conditional term rewriting systems. In particular, we show how to obtain sufficient conditions for the modularity of weak termination, weak innermost termination, (strong) innermost termination, (strong) termination, confluence and completeness of conditional rewrite systems.

1 Introduction, Motivation and Overview

Starting with the seminal work of Toyama [Toy87b] the investigation of preservation properties of term rewriting systems (TRSs for short) under various forms of combinations has become a very interesting and active area of research. From a practical point of view this field has a great potential in applications of rewriting techniques since it provides the theoretical basis for a systematic construction of large systems of rewrite rules with some desired properties from smaller ones with corresponding properties. Vice versa, it is also crucial for analyzing properties of large systems by decomposing them into smaller units where corresponding properties are often easier to verify.

In [Toy87b] it was shown that confluence is indeed preserved under disjoint unions of (unconditional) TRSs whereas termination is not (cf. also [Toy87a]). This phenomenon was the starting point of a couple of investigations about how to obtain sufficient criteria for the preservation of termination, completeness (i.e. termination plus confluence) and of other interesting properties of TRSs under disjoint combinations (cf. e.g. [Rus87], [Mid89], [TKB89], [Mid90], [KO90], [Gra92a], [Gra92b], [Ohl93b]). Non-disjoint unions of TRSs with common constructors have been considered e.g. in [MT91], [KO92], [Gra92a], [Gra92b], [Ohl93b], [Ohl93a]. More general hierarchical combinations of TRSs have recently been dealt with in [Kri93], [Der93], [Kri94], [Gra93b], [FJ93]. Some preservation results for (disjoint and non-disjoint, but non-hierarchical) combinations of conditional TRSs (CTRSs for short) finally have been obtained in [Mid90] (cf. also [Mid93b]), [Mid93a], [Gra93c], [Ohl93b].

As shown in [TKB89] (by a very involved proof), completeness is preserved under disjoint union of left-linear (unconditional) TRSs. Instead of left-linearity

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one may also require a stronger confluence property. More precisely, termination (and hence also completeness) is preserved under disjoint unions of non-overlapping TRSs as well as of locally confluent overlay systems ([Gra92b], [Gra93b]). The crucial point is that for such TRSs (strong) innermost termination implies already (strong) termination. Recently we have been able to show that this latter property does indeed also hold for CTRSs ([Gra93a]). In the present paper we shall exploit this property and show how to obtain corresponding preservation results for disjoint unions of CTRSs. However, this generalization of [Gra92b] to the conditional case turned out to be pretty more complicated than expected, due to reasons which will be explained later on.

Before going into details let us give a summary of our main results (for join CTRSs admitting extra variables in the conditions):

- We give counterexamples showing that weak termination, weak innermost termination and (strong) innermost termination are not preserved under signature extensions in general (cf. Example 4 which contradicts Theorem 5.2 in [Mid93b]) whereas (strong) termination is indeed preserved under signature extensions (cf. Lemma 17).
- We give abstract sufficient conditions for the preservation of weak termination, weak innermost termination and (strong) innermost termination under signature extensions and more generally, for their modularity (cf. Theorem 19, Lemmas 20-26). In particular we show that these restricted termination properties are modular for confluent CTRSs and for CTRSs without extra variables (cf. Theorem 27).
- We show that (strong) termination and completeness are modular properties for non-overlapping CTRSs as well as for conditional overlay systems with joinable critical pairs (cf. Theorems 28, 29).

The rest of the paper is structured as follows. In the next section we shall introduce some necessary terminology. Then, in section 3, we shall summarize results of [Gra92b], [Gra93b], [Gra93b] on restricted and general termination and confluence properties of (C)TRSs. In the main section 4, we shall develop a couple of new preservation results for termination and confluence properties of CTRSs. Finally we summarize known and new results and discuss some open problems and related work.

2 Preliminaries

We assume familiarity with the basic theory for term rewriting (cf. e.g. [DJ90], [Klo92]). For brevity we shall make use of the following abbreviations which apply to TRSs (and, if sensible, also to terms): SN (strongly) terminating (strongly normalizing), WN weakly terminating (weakly normalizing), WIN weakly innermost terminating (weakly innermost normalizing), SIN (strongly) innermost terminating (strongly innermost normalizing), JCP (all) critical pairs are joinable, CR confluent (or equivalently, having the Church-Rosser property), WCR locally confluent, COMP complete, i.e. terminating and confluent, NE non-erasing (or variable-preserving), NO non-overlapping, OS overlay system, NF set of normal forms. Innermost reduction steps $s \to t$ are denoted by $s \underset{i}{\to} t$. In order to indicate the position p of the contracted redex in $s \to t$, the applied rule $l \to r$ and the matching substitution σ we shall also use the notation $s \to_{p,\sigma,l\to r} t$. If $p = \lambda$ (the root position) then we speak of a *root reduction step*. The identity substitution is denoted by *id*.

2.1 Basic Notions and Notations for Disjoint Unions

We adopt usual basic notions and notations for disjoint unions of rewrite systems as described e.g. in [Toy87b], [Mid90], [Mid93b], but repeat some conventions here for the sake of readability. When considering a disjoint union $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ we say that $\mathcal{R}_1^{\mathcal{F}_1}$ ($\mathcal{R}_2^{\mathcal{F}_2}$) is the black (white) system. Symbols from $\mathcal{F}_1$ ($\mathcal{F}_2$) are black (white). Variables have no colour. A black (white) term does not contain white (black) symbols. A term is top black (top white) if its top symbol is black (white). The *special subterms* (or *aliens*), *principal subterms* and the *rank* of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are defined as usual. If in a reduction step $s \to_{\mathcal{R}^{\mathcal{F}}} t$ the contracted redex is inside some principal alien of s, we speak of an *inner* step and write $s \xrightarrow{i}_{\mathcal{R}^{\mathcal{F}}} t$. Otherwise it is an *outer* step, which is denoted by $s \xrightarrow{o}_{\mathcal{R}^{\mathcal{F}}} t$. If s is a term with $rank(s) > 1$ then it has the form $s = C[\![s_1, \ldots, s_n]\!]$ where the s_i, $1 \le i \le n$ are the principal aliens of s and $C[, \ldots,]$ is a *context*, i.e. a term with 'holes' (represented by some fresh constant $\Box$). By $C[]_p$ we mean a context with one occurrence of $\Box$ at position p. Similarly, $C[]_\Pi$ denotes a context with $\Box$ at all positions from Π (which are required to be mutually disjoint). For terms $s_1, \ldots, s_n, t_1, \ldots, t_n$ we write $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ if $t_i = t_j$ whenever $s_i = s_j$, for all $1 \le i, j \le n$. This notion extends in straightforward way to substitutions. Following [Mid90], [Mid93b] we introduce some special notations in order to enable a compact treatment of 'degenerate' cases of '$t = C[\![t_1, \ldots, t_n]\!]$'. We write $C\langle, \ldots, \rangle$ for a term containing zero or more occurrences of $\Box$ and $C\{, \ldots, \}$ denotes a term different from $\Box$ itself, containing zero or more occurrences of $\Box$. If $t_1, \ldots, t_n$ are the (possibly zero) principal subterms of some term t (from left to right), then we write $t = C\{\{t_1, \ldots, t_n\}\}$ provided $t = C\{t_1, \ldots, t_n\}$. We write $t = C\langle\langle t_1, \ldots, t_n \rangle\rangle$ if $t = C\langle t_1, \ldots, t_n \rangle$ and either $C\langle, \ldots, \rangle \ne \Box$ and $t_1, \ldots, t_n$ are the principal subterms of t or $C\langle, \ldots, \rangle = \Box$ and $t \in \{t_1, \ldots, t_n\}$.

Lemma 1. *(cf. [Mid93b], Prop. 2.23) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs. Then every substitution σ can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black (white), σ_2 is top white (top black), and $\sigma_2 \propto \epsilon$.*

2.2 Conditional Term Rewriting Systems

Moreover, we need some basic terminology about conditional term rewriting systems (CTRSs) (cf. e.g. [DJ90], [Klo92], [Mid93b]).

Definition 2. A *CTRS* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set of *conditional rewrite rules* of the form $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies l \to r$ with $s_1, \ldots, s_n, t_1, \ldots, t_n, l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Moreover, we require $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$ as for unconditional TRSs, i.e. no variable left hand sides and no extra variables on the right hand side. Extra variables in conditions are allowed if not stated otherwise. Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant.

Depending on the interpretation of the equality sign in the conditions of rewrite rules, different reduction relations may be associated with a given CTRS as usual (yielding a *join*, *normal* or *semi-equational* CTRS). For the sake of readability we shall use in the following some compact notations for conditional rules and conjunctions of conditions. When writing $P \implies l \to r$ for some conditional rewrite rule then P stands for the conjunction of all conditions. Similarly, if P is $s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n$, then $P\downarrow$ means $s_1 \downarrow t_1 \ \wedge \ \ldots \ldots \wedge \ s_n \downarrow t_n$, and $\sigma(P)$ is to denote $\sigma(s_1) = \sigma(t_1) \ \wedge \ldots \wedge \ \sigma(s_n) = \sigma(t_n)$.

Definition 3. Let $\mathcal{R}$ be a join CTRS, and let $P_1 \implies l_1 \to r_1$ and $P_2 \implies l_2 \to r_2$ be two rewrite rules of $\mathcal{R}$ which have no variables in common. Suppose $l_1 = C[t]_p$ with $t \notin \mathcal{V}$ for some (possibly empty) context $C[]_p$ such that t and l_2 are unifiable with most general unifier σ, i.e. $\sigma(t) = \sigma(l_1/p) = \sigma(l_2)$. Then $\sigma(P_1) \wedge \sigma(P_2) \implies \sigma(C[r_2]) = \sigma(r_1)$ is said to be a *(conditional) critical pair* of $\mathcal{R}$. If the two rules are renamed versions of the same rule of $\mathcal{R}$, we do not consider the case $C[] = \Box$, i.e. we do not overlap a rule with itself at root position. A (conditional) critical pair $P \implies s = t$ is said to be *joinable* if $\sigma(s) \downarrow_{\mathcal{R}} \sigma(t)$ for every substitution σ with $\sigma(P)\downarrow$. A substitution σ which satisfies the conditions, i.e. for which $\sigma(P)\downarrow$ holds, is said to be *feasible*. Otherwise σ is *unfeasible*. Analogously, a (conditional) critical pair is said to be *feasible (unfeasible)* if there exists some (no) feasible substitution for it.

Note that testing joinability of conditional critical pairs is in general much more difficult than in the unconditional case since one has to consider all substitutions which satisfy the correspondingly instantiated conditions. Moreover, the critical pair lemma does not hold for CTRSs in general as shown e.g. by the following example.

Example 1. ([BK86]) Consider the join CTRS

$$\mathcal{R} = \begin{cases} x \downarrow f(x) \Longrightarrow f(x) \to a \\ \qquad\qquad\qquad\; b \to f(b)\,. \end{cases}$$

Here we get $f(b) \to a$ due to $b \downarrow f(b)$ and hence $f(f(b)) \to f(a)$. We also have $f(f(b)) \to a$ because of $f(b) \downarrow f(f(b))$. But a and $f(a)$ do not have a common reduct which is easily shown. Thus $\mathcal{R}$ is not locally confluent despite the lack of critical pairs.

Definition 4. Let $\mathcal{R}$ be a CTRS and let $\mathcal{R}_u$ be its unconditional version, i.e. $\mathcal{R}_u := \{l \to r \mid P \implies l \to r \in \mathcal{R}\}$. Then $\mathcal{R}$ is said to be *non-overlapping (NO)/ orthogonal / a (conditional) overlay system (OS)* if $\mathcal{R}_u$ is non-overlapping / orthogonal / an (unconditional) overlay system.

According to this definition Example 1 above shows that orthogonal CTRSs need not be confluent. But note that the CTRS $\mathcal{R}$ defined in Example 1 is not (strongly) innermost terminating.

The careful reader may have observed that the definition of being non-overlapping above is somehow rather restrictive. Namely, the case that there exist conditional critical pairs all of which are infeasible (and hence should not be 'properly critical') is not covered. Analogously, for a CTRS to be an overlay system one might allow critical pairs which are not overlays but require that all of them are infeasible, hence not 'properly critical'). These slightly generalized *semantical* versions of the properties of being non-overlapping and being an overlay system are treated in detail in [Gra93a]. For the sake of readability and due to lack of space we shall dispense here with an analogous treatment concerning preservation properties.[2] The other basic notions for unconditional TRSs introduced above generalize in a straightforward manner to CTRSs.

In the following we shall tacitly assume that all CTRSs considered are join CTRSs (which is the most important case in practice), except for cases where another kind of CTRSs is explicitly mentioned.

3 Restricted Termination and Confluence Properties of Conditional Term Rewriting Systems

Here we shall summarize some known and recently obtained new results on restricted termination and confluence properties of unconditional and conditional TRSs.

Theorem 5. *(cf. e.g. [Ros73], [O'D77], [Klo92]) Let $\mathcal{R}$ be an orthogonal, i.e. non-overlapping and left-linear (unconditional) TRS. Then $\mathcal{R}$ is confluent, and the following properties hold:*

(1a) $\forall t : [\, WIN(t) \implies SIN(t) \,]$.
(1b) $WIN(\mathcal{R}) \implies SIN(\mathcal{R})$.
(2a) $\forall t : [\, SIN(t) \implies SN(t) \,]$.
(2b) $SIN(\mathcal{R}) \implies SN(\mathcal{R})$.
(3) $\forall\, s, t : [\, s \xrightarrow[i]{} t \wedge SN(t) \implies SN(s) \,]$.
(4a) $NE(\mathcal{R}) \implies [\forall t : [\, WN(t) \implies SN(t) \,]]$.
(4b) $NE(\mathcal{R}) \implies [\, WN(\mathcal{R}) \implies SN(\mathcal{R}) \,]$.

For CTRSs the following technical result, which is a stronger local version of a result from [DOS88b], is crucial in order to obtain analogous relations between restricted and general termination and confluence properties of CTRSs.

Theorem 6. *([Gra93a]) Let $\mathcal{R}$ be a CTRS with $OS(\mathcal{R})$ and $JCP(\mathcal{R})$ and let s be a term with $SN(s)$. Furthermore let $C[\,]_\Pi$ be a context, and t, u, v be terms. Then we have: $u = C[s]_\Pi \to^* v \wedge s \to^* t \implies C[t]_\Pi \downarrow v$.*

Straightforward consequences of Theorem 6 are the following.

[2] In fact, concerning preservation properties there are indeed crucial differences between the usual syntactical and the slightly more general semantical versions of NO and OS. For instance, NO and OS are obviously modular which is not the case for the semantical version of OS.

Lemma 7. *([Gra93a]) Let $\mathcal{R}$ be a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$), and let s, t be terms with $s \rightarrow_{p,\sigma,P \Longrightarrow l \rightarrow r} t$. Furthermore let σ' be given with $\sigma \rightarrow^* \sigma'$, i.e. $\sigma(x) \rightarrow^* \sigma'(x)$ for all $x \in dom(\sigma)$, such that SN($\sigma(x)$) holds for all $x \in dom(\sigma)$. Then we have: $s = C[\sigma(l)]_p \rightarrow^* C[\sigma'(l)]_p \rightarrow_{p,\sigma',P \Longrightarrow l \rightarrow r} C[\sigma'(r)]_p$ (due to $\sigma'(P)\downarrow$) and $t = C[\sigma(r)]_p \rightarrow^* C[\sigma'(r)]_p$ for some context $C[]_p$.*

Corollary 8. *([Gra93a]) Let $\mathcal{R}$ be a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$), and let s be a term with SN(s). Then we have CR(s) and hence COMP(s), too.*

The termination assumption concerning s in this result is crucial as demonstrated by the following example.

Example 2. (Example 1 continued) Here $\mathcal{R} = \{x \downarrow f(x) \implies f(x) \rightarrow a,\ b \rightarrow f(b)\}$ is clearly an overlay system with joinable critical pairs (it is even non-overlapping). Moreover we have $f(f(b)) \rightarrow a$ and $f(f(b)) \rightarrow f(a)$ but not $a \downarrow f(a)$. Obvioully, SN($f(f(b))$) does not hold due to the presence of the rule $b \rightarrow f(b)$ in $\mathcal{R}$ (note that we even do not have SIN($f(f(b))$)).

Under the stronger assumption of global termination we get from Corollary 8 the following known critical pair criterion for confluence of conditional overlay systems.

Theorem 9. *([DOS88b]) A terminating CTRS which is an overlay system such that all its conditional critical pairs are joinable is confluent, hence complete.*

Concerning termination properties of CTRSs we have obtained the following results which generalize Theorem 5 to the conditional case and moreover, do not require left-linearity.

Theorem 10. *([Gra93a]) Let $\mathcal{R}$ be a non-overlapping CTRS. Then the following properties hold:*

(1a) $\forall t : [\, WIN(t) \implies SIN(t)\,]$.
(1b) $WIN(\mathcal{R}) \implies SIN(\mathcal{R})$.
(2a) $\forall t : [\, SIN(t) \implies SN(t)\,]$.
(2b) $SIN(\mathcal{R}) \implies SN(\mathcal{R})$.
(3) $\forall s,t : [\, s \underset{i}{\rightarrow} t \wedge SN(t) \implies SN(s)\,]$.
(4a) $NE(\mathcal{R}) \implies [\forall t : [\, WN(t) \implies SN(t)\,]]$.
(4b) $NE(\mathcal{R}) \implies [\, WN(\mathcal{R}) \implies SN(\mathcal{R})\,]$.

Moreover, it is possible to generalize Theorem 10(2) by allowing root overlaps but guaranteeing joinability of critical pairs.

Theorem 11. *([Gra93a]) For any CTRS $\mathcal{R}$ we have:*

(a) $OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \wedge SIN(\mathcal{R}) \implies SN(\mathcal{R}) \wedge CR(\mathcal{R})$, *and*
(b) $OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \implies [\forall s : [SIN(s) \implies SN(s) \wedge CR(s)]]$,

i.e. any (strongly) innermost terminating overlay system with joinable critical pairs is terminating and confluent, hence complete (part (a)) which also holds in the stronger local version (b).

4 Modularity Results for Disjoint Unions of CTRSs

Modular properties of (C)TRSs, i.e. properties which are preserved under disjoint unions, have attracted an increasing attention within the last few years.

Toyama [Toy87b] has shown that confluence is modular (for TRSs). The termination property, however, is in general not modular for TRSs (cf. [Toy87b], [Toy87a]).

When investigating the modularity behaviour of CTRSs the situation is much more complicated than for (unconditional) TRSs. For instance, as exhibited in [Mid90], the fundamental property of TRSs

$$s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t \quad \Longrightarrow \quad s \to_{\mathcal{R}_1} t \ \vee \ s \to_{\mathcal{R}_2} t$$

does not hold any more. This is due to the fact that when a rule of one of the systems is applied rules of the other system may be needed in order to satisfy the conditions. Nevertheless, confluence has turned out to be a modular property of (join) CTRSs, too, as shown by Middeldorp.

Theorem 12. *([Mid90], [Mid93b]) Confluence is a modular property of CTRSs.*

For some sufficient conditions for modularity of termination of CTRSs we refer to [Mid90] (cf. also [Mid93b]), [Mid93a], [Gra93c]). Here we shall concentrate on CTRSs satisfying some structural restrictions. In [Gra92b], [Gra93b] we have proved that any (strongly) innermost terminating, locally confluent (unconditional) overlay system is terminating, and hence confluent and complete. Moreover – by exploiting the fact that (strong) innermost termination is modular (for TRSs) in contrast to the non-modularity of (strong) termination – we have shown there how to derive from this result new modularity results and simplified proofs of known ones. Here we would also like to follow the same line of reasoning – exploiting Theorems 10 and 11 – in order to obtain derived modularity results for CTRSs. But again the situation is much more complicated for CTRSs than for TRSs since neither weak termination, nor weak innermost termination nor strong innermost termination are modular in general for CTRSs. This can be seen e.g. from the next example.

Example 3. (cf. [Mid93b] for similar examples)

$$\mathcal{R}_1 = \{\ x \downarrow b \ \wedge \ x \downarrow c \ \Longrightarrow \ a \to a \ \} \text{ over } \mathcal{F}_1 = \{a, b, c\},$$
$$\mathcal{R}_2 = \{G(x,y) \to x, G(x,y) \to y\} \text{ over } \mathcal{F}_2 = \{G, A\}.$$

Here, we have $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ by applying the $\mathcal{R}_1$-rule (x is substituted by $G(b,c)$), but neither $a \to_{\mathcal{R}_1} a$ nor $a \to_{\mathcal{R}_2} a$. Hence, a is an $\mathcal{R}_i$-normal form (for $i = 1, 2$) but not a normal form w.r.t. $\mathcal{R}_1 \oplus \mathcal{R}_2$. Obviously, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly (hence also weakly and innermost) terminating but their disjoint union is not. For instance, $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} \ldots$ is an infinite innermost derivation, and a does not have a normal from w.r.t. $(\mathcal{R}_1 \oplus \mathcal{R}_2)$.

4.1 Preservation Behaviour under Signature Extensions

From the observation in Example 3 above one might be tempted to conjecture (as it is done in [Mid93b]) that the preservation of normal forms, defined by (with $\mathcal{F} := \mathcal{F}_1 \uplus \mathcal{F}_2$, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$)

$$\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2): \quad \mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}) ,$$

should be a sufficient condition for the modularity of weak termination. But this is also not true in general.[3] The situation is even worse, since – surprisingly – it may happen that a weakly terminating CTRS may become not weakly terminating under the disjoint union with another 'empty' CTRS, i.e. simply by extending the signature without adding new rules. The same phenomenon may happen with weak and strong innermost termination as illustrated next.

Example 4. Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \left\{ \begin{array}{rcr} x \downarrow z \wedge z \downarrow y & \Longrightarrow & f(g(x,y)) \to f(g(x,y)) \\ & & g(x,x) \to x \\ & & g(x,a) \to c \\ & & g(x,b) \to c \\ & & g(x,f(y)) \to c \\ & & g(x,g(y,z)) \to c \\ & & g(a,x) \to c \\ & & g(b,x) \to c \\ & & g(f(y),x) \to c \\ & & g(g(y,z),x) \to c \\ & & c \to a \\ & & c \to b \end{array} \right.$$

over $\mathcal{F}_1 := \{f, g, a, b, c\}$ and $\mathcal{R}_2 := \emptyset$ over $\mathcal{F}_2 := \{G\}$ (with G unary). It is straightforward to verify that $\mathcal{R}_1^{\mathcal{F}_1}$ is weakly terminating, weakly innermost terminating and even strongly innermost terminating. The crucial point is that any arbitrary infinite $\mathcal{R}_1^{\mathcal{F}_1}$- derivation must contain rewrite steps using the first rule. But whenever this rule is applicable, the contracted redex cannot be innermost, since some proper subterm must then be reducible by the remaining rules which constitute a (strongly) terminating CTRS. Nevertheless, in the extended system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \{G\}}$ we get the cyclic (hence infinite) innermost $\mathcal{R}^{\mathcal{F}}$-derivation:

$$f(g(G(a),G(b))) \mathrel{\underset{i}{\to}}_{\mathcal{R}^{\mathcal{F}}} f(g(G(a),G(b))) \mathrel{\underset{i}{\to}}_{\mathcal{R}^{\mathcal{F}}} f(g(G(a),G(b))) \mathrel{\underset{i}{\to}}_{\mathcal{R}^{\mathcal{F}}} \cdots$$

by applying the first rule (instantiating the extra variable z by $G(c)$). Note moreover that there is no other way of reducing $f(g(G(a),G(b)))$ (all its proper subterms are in normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, and the second rule is clearly not applicable). Hence, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \{G\}}$ is neither strongly innermost terminating nor weakly innermost terminating nor weakly terminating.

[3] contradicting Theorem 5.2 in [Mid93b] (cf. also Theorem 4.3.20 in [Mid90]) the proof of which implicitly relies on the incorrect assumption $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2})$.

By a thorough analysis of abstraction and innermost reduction properties which allow to project reduction sequences on mixed terms to certain reduction sequences on pure terms we shall show below how the monotonicity of WN, WIN and SIN under signature extensions can be guaranteed. More generally, we also develop sufficient criteria for the modularity of these properties. This analysis heavily relies on some very useful terminology and technical results from [Mid93b].

Definition 13. (cf. Def. 3.2 in [Mid93b]) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$. Then the rewrite relation $\rightarrow_1$ is defined as follows: $s \rightarrow_1 t$ if there exists a rule $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \rightarrow r$ in $\mathcal{R}_1$, a context $C[]$ and a substitution σ such that $s = C[\sigma(l)]$, $t = C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i = 1, \ldots, n$, where $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ means that s_i and t_i are joinable using only $\xrightarrow{o}_1$-reduction steps. The relation $\rightarrow_2$ is defined analogously. The union of $\rightarrow_1$ and $\rightarrow_2$ is denoted by $\rightarrow_{1,2}$.

Lemma 14. *(cf. [Mid93b], Prop. 3.5) ('injective abstraction' is possible for $\xrightarrow{o}_1$-steps)*
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs, s, t be black terms and σ be a top white substitution with $\sigma(s) \xrightarrow{o}_1 \sigma(t)$. Then, for any substitution τ with $\sigma \propto \tau$, we have $\tau(s) \xrightarrow{o}_1 \tau(t)$.

Note that this result implies in particular that a step $\sigma(s) \xrightarrow{o}_1 \sigma(t)$ on mixed terms can be injectively abstracted into a 'pure' step $\tau(s) \rightarrow_{\mathcal{R}_1} \tau(t)$ by injectively replacing the maximal top white aliens of $\sigma(s)$ and $\sigma(t)$ by fresh variables. One may wonder whether such an injective abstraction is also possible for an arbitrary outer step $\sigma(s) \xrightarrow{o}_{\mathcal{R}_1} \sigma(t)$, where for verifying the conditions of applied $\mathcal{R}_1$-rules also inner $\mathcal{R}_1$-steps are allowed. This is not the case as shown next.

Example 5. Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \begin{cases} x \downarrow y \implies & f(x,y) \rightarrow x \\ & a \rightarrow b \end{cases}$$

over $\mathcal{F}_1 := \{f, a, b\}$ and $\mathcal{R}_2 := \emptyset$ over $\mathcal{F}_2 := \{G\}$ (with G unary). Here we have $f(G(a), G(b)) \xrightarrow{o}_{\mathcal{R}_1} G(a)$ (due to $G(a) \downarrow_{\mathcal{R}_1} G(b)$), but not $f(G(a), G(b)) \xrightarrow{o}_1 G(a)$ since for satisfying $G(a) \downarrow_{\mathcal{R}_1} G(b)$ we need an inner $\mathcal{R}_1$-step. Note that after injective abstraction of $s = f(G(a), G(b))$ and $t = G(a)$ into $f(x,y)$ and x, respectively, the reduction $f(x,y) \rightarrow_{\mathcal{R}_1} x$ is not possible any more.

The above example somehow suggests that by forbidding the possibility of inner reduction steps injective abstraction of reduction steps might still be possible. But even if all maximal top white aliens are irreducible, this is not possible in general.

Example 6. Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \begin{cases} x \downarrow z \wedge z \downarrow y \implies & f(x,y) \rightarrow x \\ & c \rightarrow a \\ & c \rightarrow b \end{cases}$$

with $\mathcal{F}_1 := \{f, a, b, c\}$, $\mathcal{R}_2 := \emptyset$, $\mathcal{F}_2 := \{G\}$ (with G unary). Here all proper subterms of $f(G(a), G(b))$ are ($\mathcal{R}^{\mathcal{F}}$-) irreducible and $f(G(a), G(b)) \overset{o}{\to}_{\mathcal{R}_1} G(a)$ but not $f(G(a), G(b)) \overset{o}{\to}_1 G(a)$ since for satisfying $G(a) \downarrow_{\mathcal{R}_1} z \wedge z \downarrow_{\mathcal{R}_1} G(b)$ we have to instantiate the extra variable z in the condition of the first rule by a mixed term of the form $G(u)$, e.g. $G(c)$, and to use inner $\mathcal{R}_1$-steps for establishing $G(a) \downarrow_{\mathcal{R}_1} G(u)$, $G(u) \downarrow_{\mathcal{R}_1} G(b)$. Note again that after injective abstraction of $s = f(G(a), G(b))$ and $t = G(a)$ into $f(x, y)$ and x, respectively, the reduction $f(x, y) \to_{\mathcal{R}_1} x$ is not possible any more.

Whereas in the above examples an injective abstraction of certain reduction steps on mixed terms to a corresponding reduction step on pure terms is not possible, a non-injective 'identifying' one, which replaces all maximal top white aliens by the *same* fresh variable, is indeed possible. This is shown next.

Lemma 15. *('identifying abstraction' is always possible)*
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs, s and t be black terms, σ be a top white substitution, and $\widetilde{\sigma}$ be defined by $\widetilde{\sigma}(x) := z$ for all $x \in dom(\sigma)$ where z is a 'fresh' variable, i.e. not occurring in s. Then we have:

(a) $\sigma(s) \overset{o}{\to}_{\mathcal{R}_1} \sigma(t) \quad \Longrightarrow \quad \widetilde{\sigma}(s) \overset{o}{\to}_{\mathcal{R}_1} \widetilde{\sigma}(t)$, *and*
(b) $\sigma(s) \overset{i}{\to}_{\mathcal{R}_1} \sigma(t) \quad \Longrightarrow \quad \widetilde{\sigma}(s) = \widetilde{\sigma}(t)$.

Proof. (b) is trivially satisfied by definition of $\overset{i}{\to}$ and of $\widetilde{\sigma}$. (a) can be proved by induction on n, the depth of rewriting (using the same proof structure as in [Mid93b] for Prop. 3.5, see Lemma 14 above). ∎

Corollary 16. *Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS, $\mathcal{F}'$ be a signature with $\mathcal{F} \subseteq \mathcal{F}'$, and $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Then we have: $s \to_{\mathcal{R}^{\mathcal{F}'}} t \implies s \to_{\mathcal{R}^{\mathcal{F}}} t$.*

Using Lemma 15 we are now able to show that at least termination (strong normalization) is preserved under signature extensions.

Lemma 17. *Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS and $\mathcal{F}'$ be a signature with $\mathcal{F} \subseteq \mathcal{F}'$. Then the following holds: SN($\mathcal{R}^{\mathcal{F}}$) $\iff$ SN($\mathcal{R}^{\mathcal{F}'}$).*

Proof. It clearly suffices to show: SN($\mathcal{R}^{\mathcal{F}}$) $\Longrightarrow$ SN($\mathcal{R}^{\mathcal{F}'}$). Hence, assuming SN($\mathcal{R}^{\mathcal{F}}$), we prove SN($s$) for all $s \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ by induction on $n = rank(s)$.[4] For $rank(s) = 1$ we get SN(s) by assumption. Let $rank(s) > 1$. If the root symbol of s is a new one, i.e. from $(\mathcal{F}' \setminus \mathcal{F})$, then SN($s$) follows by induction hypothesis, since rewrite steps in the top layer of s are impossible. If $root(s) \in \mathcal{F}$ then – again by induction hypothesis and the fact that $(\mathcal{F}' \setminus \mathcal{F})$-layers cannot collapse – any infinite $\mathcal{R}^{\mathcal{F}'}$-derivation starting with s would have to contain infinitely many outer $\mathcal{R}^{\mathcal{F}'}$-steps. But then identifying abstraction using Lemma 15 would yield an infinite $\mathcal{R}^{\mathcal{F}}$-derivation contradicting the assumption SN($\mathcal{R}^{\mathcal{F}}$). ∎

[4] Note that we may consider here $\mathcal{R}^{\mathcal{F}'}$ as disjoint union $\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ with $\mathcal{R}_1^{\mathcal{F}_1} = \mathcal{R}^{\mathcal{F}}$, $\mathcal{R}_2 = \emptyset$ and $\mathcal{F}_2 = \mathcal{F}' \setminus \mathcal{F}$.

In order to present sufficient criteria for the preservation of restricted termination properties under signature extensions and – more generally – under disjoint unions, we will introduce now some more notations, in particular for certain innermost reductions steps.

Definition 18. Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ be their disjoint union. If (for $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$) $s \rightarrow_{\mathcal{R}^{\mathcal{F}}} t$ by applying some $\mathcal{R}_1$-rule (where for satisfying the conditions also $\mathcal{R}_2$-rules are allowed) we denote this by $s \rightarrow_{\mathcal{R}_1^{\mathcal{F}}/\mathcal{R}^{\mathcal{F}}} t$ or simply $s \rightarrow_{\mathcal{R}_1/\mathcal{R}} t$. Furthermore, for $j = 1, 2$ we define the innermost reduction properties $\mathrm{IRP}_1(\mathcal{R}_j, \mathcal{R})$, $\mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R})$ and $\mathrm{IRP}_3(\mathcal{R}_j, \mathcal{R})$ by[5]

$$\begin{cases} \mathrm{IRP}_1(\mathcal{R}_j, \mathcal{R}) : \forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : \quad s \xrightarrow[i]{}_{\mathcal{R}_j} t \implies s \xrightarrow[i]{}_j t\,, \\ \mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R}) : \forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t \implies \exists t' \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \xrightarrow[i]{}_{\mathcal{R}_j} t' \\ \mathrm{IRP}_3(\mathcal{R}_j, \mathcal{R}) : \forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t \implies s \xrightarrow[i]{}_{\mathcal{R}_j} t\,. \end{cases}$$

Note that IRP_1 enables injective abstraction (via Lemma 14) which will be useful for establishing preservation results under signature extensions. Combined with IRP_2 or the stronger property IRP_3 it will turn out to capture the essence for obtaining modularity results later on.

Using the first innermost reduction property IRP_1 defined above we obtain a sufficient criterion for the preservation of WN, WIN and SIN under signature extensions as follows.

Theorem 19. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with $\mathcal{R}_2 = \emptyset$ (and $\mathcal{R}^{\mathcal{F}} := \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2}$) such that $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$ holds. Then we have the following equivalences:*

(a) $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1 \cup \mathcal{F}_2})$.
(b) $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1 \cup \mathcal{F}_2})$.
(c) $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1 \cup \mathcal{F}_2})$.

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ be given as above satisfying $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$. The '$\Longleftarrow$'-directions of (a), (b) and (c) are easy by Corollary 16.

(a) For proving $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}})$ we assume $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1})$ and show that every term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ has a normal form w.r.t. $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}}$ by induction on $rank(s)$. If $rank(s) = 1$, this is easy by the assumption $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1})$, Corollary 16 and the fact that $\mathcal{R}_2 = \emptyset$. If $rank(s) > 1$ then s has the form $s = C[\![s_1, \ldots, s_m]\!]$. If the top layer $C[, \ldots,]$ is white then by induction hypothesis and $\mathcal{R}_2 = \emptyset$ we get $\mathrm{WN}(s, \mathcal{R}_1^{\mathcal{F}})$. Otherwise, in the interesting case where $C[, \ldots,]$ is black, we know by induction hypothesis that every s_i $(1 \le i \le m)$ has a normal form w.r.t. $\mathcal{R}_1^{\mathcal{F}}$, let's say t_i. Hence, we get

$$s = C[\![s_1, \ldots, s_m]\!] \rightarrow^*_{\mathcal{R}_1^{\mathcal{F}}} C[t_1, \ldots, t_m] = C' \{\!\{ u_1, \ldots, u_n \}\!\}$$

[5] The notation used here for innermost reduction is slightly ambiguous (for the sake of readability). When writing $s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t$, $s \xrightarrow[i]{}_{\mathcal{R}_j} t$ or $s \xrightarrow[i]{}_j t$, we always mean that the reduced subterm is an innermost redex of s w.r.t. $\rightarrow_{\mathcal{R}}$ which is also an innermost redex of s w.r.t. $\xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}}$, $\xrightarrow[i]{}_{\mathcal{R}_j}$ or $\rightarrow_j$, respectively, since $\xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}}$, $\xrightarrow[i]{}_{\mathcal{R}_j}$ and $\rightarrow_j$ are subsets of $\rightarrow_{\mathcal{R}}$.

for some black context $C'\{,\ldots,\}$ and top white normal forms u_j w.r.t. $\mathcal{R}_1^{\mathcal{F}}$. Choosing fresh variables $x_1,\ldots,x_n$ injectively, i.e. $\langle u_1,\ldots,u_n\rangle \propto \langle x_1,\ldots,x_n\rangle$, we have $rank(C'\{x_1,\ldots,x_n\}) = 1$, hence by induction hypothesis $C'\{x_1,\ldots,x_n\}$ can be reduced in $\mathcal{R}_1^{\mathcal{F}_1}$ to a normal form $C''\langle x_{i_1},\ldots,x_{i_p}\rangle$. Thus we get

$$s \to^*_{\mathcal{R}_1^{\mathcal{F}}} C'\,\{\{\,u_1,\ldots,u_n\}\}\ \to^*_{\mathcal{R}_1^{\mathcal{F}}} C''\,\langle\!\langle\,u_{i_1},\ldots,u_{i_p}\rangle\!\rangle$$

with $u_{i_1},\ldots,u_{i_p} \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Now, denoting u_{i_j} by v_j, it suffices to show $C''\,\langle\!\langle\,v_1,\ldots,v_p\rangle\!\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. If this were not the case then there would exist an (outer) innermost $\mathcal{R}_1^{\mathcal{F}}$-reduction step of the form

$$C''\,\langle\!\langle\,v_1,\ldots,v_p\rangle\!\rangle \quad \overset{o}{\underset{i}{\to}}_{\mathcal{R}_1^{\mathcal{F}}} C'''\,\langle\!\langle\,v_{k_1},\ldots,v_{k_q}\rangle\!\rangle$$

with $1 \le k_l \le p$, $1 \le l \le q$. But then, due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1},\mathcal{R}_1^{\mathcal{F}})$, we could apply Lemma 14 which would yield (denoting x_{i_j} by y_j, $1 \le j \le p$)

$$C''\langle y_1,\ldots,y_p\rangle \to_{\mathcal{R}_1^{\mathcal{F}_1}} C'''\langle y_{k_1},\ldots,y_{k_q}\rangle\ .$$

But this is a contradiction to $C''\langle y_1,\ldots,y_p\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}_1})$. Hence we are done.

(b) For proving $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}})$ the same proof structure as for (a) can be used, but the argumentation is slightly different.

(c) For proving $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$ we assume $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1})$ and show by contradiction that for every term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ we have $\mathrm{SIN}(s,\mathcal{R}_1^{\mathcal{F}})$. Consider a counterexample which is minimal w.r.t. the subterm relation, i.e. a term s which has an infinite innermost $\mathcal{R}_1^{\mathcal{F}}$-derivation

$$(D) \quad s =: s_0 \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} s_1 \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} s_2 \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} \cdots$$

such that all proper subterms of s are strongly innermost terminating (w.r.t. $\mathcal{R}_1^{\mathcal{F}}$). Then, necessarily s is a top black term with $rank(s) > 1$. By the minimality assumption we know that there is some (first) innermost $\mathcal{R}_1^{\mathcal{F}}$-step

$$s_k = C_k[\![t_1,\ldots,t_n]\!] \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} C'\,\langle\!\langle\,t_{i_1},\ldots,t_{i_p}\rangle\!\rangle = s_{k+1}$$

in (D) which is a root reduction step. This implies in particular that all maximal top white aliens of s_k (as well as of all $s_{k'}$, $k' > k$) are $\mathcal{R}_1^{\mathcal{F}}$-irreducible. Hence all steps in

$$s_k \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+1} \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+2} \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} \cdots$$

are outer steps. Due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1},\mathcal{R}_1^{\mathcal{F}}))$ we may apply now Lemma 14 which yields an infinite innermost $\mathcal{R}_1^{\mathcal{F}_1}$ derivation

$$\widehat{s_k} \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}_1}} \widehat{s_{k+1}} \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}_1}} \widehat{s_{k+2}} \ \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}_1}} \cdots$$

by injectively abstracting the maximal top white aliens by fresh variables. But this is a contradiction to $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1})$. ■

4.2 Preservation Results for WN, WIN and SIN

Using the second innermost reduction property from Definition 18, we get the following preservation result for weak and weak innermost termination.

Lemma 20. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:*

(a) $IRP_2(\mathcal{R}_1, \mathcal{R}) \wedge IRP_2(\mathcal{R}_2, \mathcal{R}) \implies [\, WN(\mathcal{R}_1^{\mathcal{F}}) \wedge WN(\mathcal{R}_2^{\mathcal{F}}) \iff WN(\mathcal{R}^{\mathcal{F}})\,]$.
(b) $IRP_2(\mathcal{R}_1, \mathcal{R}) \wedge IRP_2(\mathcal{R}_2, \mathcal{R}) \implies [\, WIN(\mathcal{R}_1^{\mathcal{F}}) \wedge WIN(\mathcal{R}_2^{\mathcal{F}}) \iff WIN(\mathcal{R}^{\mathcal{F}})\,]$.

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ be given as above satisfying $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$. The '$\Longleftarrow$'-directions of the equivalences in (a) and (b)) are easy by Corollary 16.

(a) For proving $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WN}(\mathcal{R}_2^{\mathcal{F}}) \implies \mathrm{WN}(\mathcal{R}^{\mathcal{F}})$ we assume $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}})$, $\mathrm{WN}(\mathcal{R}_2^{\mathcal{F}})$ and show that every term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ has a normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$ by induction on $rank(s)$. If $rank(s) = 1$, this is easy by the assumptions $\mathrm{WN}(\mathcal{R}_j^{\mathcal{F}})$, $\mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R})$ (for $j = 1, 2$). If $rank(s) > 1$ then s has the form $s = C[\![s_1, \ldots, s_m]\!]$ with s top black w.l.o.g., and we know by induction hypothesis that every s_i $(1 \leq i \leq m)$ has a normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, let's say t_i. Hence, we get

$$s = C[\![s_1, \ldots, s_m]\!] \to^*_{\mathcal{R}^{\mathcal{F}}} C[t_1, \ldots, t_m] = C' \{\!\{\, u_1, \ldots, u_n \}\!\}$$

for some black context $C'\{, \ldots, \}$ and top white normal forms u_j w.r.t. $\mathcal{R}^{\mathcal{F}}$. By assumption, we can reduce $C' \{\!\{\, u_1, \ldots, u_n \}\!\}$ to some normal form w.r.t. $\mathcal{R}_1^{\mathcal{F}}$, i.e.

$$s \to^*_{\mathcal{R}^{\mathcal{F}}} C' \{\!\{\, u_1, \ldots, u_n \}\!\} \to^*_{\mathcal{R}_1^{\mathcal{F}}} C'' \langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$$

with $C'' \langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$ irreducible w.r.t. $\to_{\mathcal{R}_2/\mathcal{R}}$. Now it suffices to show $C'' \langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. If this were not the case then $C'' \langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$ would be innermost $\to_{\mathcal{R}_1^{\mathcal{F}}/\mathcal{R}^{\mathcal{F}}}$-reducible, hence by the assumption $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$ it would also be (innermost) $\to_{\mathcal{R}_1^{\mathcal{F}}}$-reducible, contradicting $C'' \langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Hence we are done.

(b) For proving $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}})$ the same proof structure as for (a) can be used, but the argumentation is slightly different. ■

An equivalent characterization of the precondition in Lemma 20 is given by the normal form property defined above. More precisely, we obtain the following result (the proof of which is omitted due to lack of space).

Lemma 21. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:* $NF(\mathcal{R}^{\mathcal{F}}) = NF(\mathcal{R}_1^{\mathcal{F}}) \cap NF(\mathcal{R}_2^{\mathcal{F}}) \iff IRP_2(\mathcal{R}_1, \mathcal{R}) \wedge IRP_2(\mathcal{R}_2, \mathcal{R})$.

Now, considering Lemma 20 above, the property $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ does not yet suffice for the equivalence $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})$. To see this, consider the following slightly modified version of Example 3.

Example 7. Consider the following disjoint CTRSs.

$\mathcal{R}_1 = \{x \downarrow b \wedge x \downarrow c \implies a \to a,\ a \to d\}$ over $\mathcal{F}_1 = \{a, b, c, d\}$,

$\mathcal{R}_2 = \{G(x,y) \to x, G(x,y) \to y\}$ over $\mathcal{F}_2 = \{G, A\}$.

Here, both $\mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ and $\mathcal{R}_2^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ are (strongly) innermost terminating – and even (strongly) terminating (it is easily shown that the first $\mathcal{R}_1$-rule is never applicable) – but their disjoint union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ is not (strongly) innermost terminating due to $a \xrightarrow[i]{}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \xrightarrow[i]{}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \xrightarrow[i]{}_{\mathcal{R}_1 \oplus \mathcal{R}_2} \ldots$. Nevertheless $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the normal form property NFP$(\mathcal{R}_1, \mathcal{R}_2)$: NF$(\mathcal{R}^{\mathcal{F}})$ = NF$(\mathcal{R}_1^{\mathcal{F}}) \cap$NF$(\mathcal{R}_2^{\mathcal{F}})$ or equivalently IRP$_2(\mathcal{R}_1, \mathcal{R}) \wedge$ IRP$_2(\mathcal{R}_2, \mathcal{R})$ (since a is now both $\mathcal{R}_1^{\mathcal{F}}$- and $\mathcal{R}^{\mathcal{F}}$-reducible due the presence of the rule $a \to d$ in $\mathcal{R}_1$).

Requiring instead of IRP$_2(\mathcal{R}_1, \mathcal{R}) \wedge$ IRP$_2(\mathcal{R}_2, \mathcal{R})$ the stronger property IRP$_3(\mathcal{R}_1, \mathcal{R}) \wedge$ IRP$_3(\mathcal{R}_2, \mathcal{R})$ accounts for this fact.

Lemma 22. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have: $IRP_3(\mathcal{R}_1, \mathcal{R}) \wedge IRP_3(\mathcal{R}_2, \mathcal{R}) \implies [\, SIN(\mathcal{R}_1^{\mathcal{F}}) \wedge SIN(\mathcal{R}_2^{\mathcal{F}}) \iff SIN(\mathcal{R}^{\mathcal{F}})\,]$.*

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ and $\mathcal{R}^{\mathcal{F}}$ be given as above satisfying IRP$_3(\mathcal{R}_1, \mathcal{R}) \wedge$ IRP$_3(\mathcal{R}_2, \mathcal{R})$. Now, the "$\Longleftarrow$"-direction is straightforward using Corollary 16. Vice versa, for proving SIN$(\mathcal{R}_1^{\mathcal{F}}) \wedge$ SIN$(\mathcal{R}_2^{\mathcal{F}}) \implies$ SIN$(\mathcal{R}^{\mathcal{F}})$ we proceed by contradiction assuming SIN$(\mathcal{R}_1^{\mathcal{F}})$ and SIN$(\mathcal{R}_2^{\mathcal{F}})$. Consider a minimal counterexample, i.e. an infinite innermost $\mathcal{R}^{\mathcal{F}}$-derivation $s_0 \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} s_1 \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} s_2 \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} \ldots$ such that no proper subterm of s_0 admits infinite innermost $\mathcal{R}^{\mathcal{F}}$-derivations. By the minimality assumption some step $s_k \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} s_{k+1}$ in the above derivation must be a root reduction step, let's say using a black rule. But then we know that all subsequent steps must also be $\to_{\mathcal{R}_1/\mathcal{R}}$-steps. Thus, by the assumption IRP$_3(\mathcal{R}_1, \mathcal{R})$ we can conclude that $s_k \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+1} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+2} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} \ldots$ is an infinite innermost $\mathcal{R}_1^{\mathcal{F}}$-derivation contradicting SIN$(\mathcal{R}_1^{\mathcal{F}})$. ■

Next we shall provide sufficient conditions for the innermost reduction properties IRP$_1$, IRP$_2$ and IRP$_3$. Having again a look at Examples 4 and 6 we see that in both cases the system $\mathcal{R}_1$ is non-confluent and has a rule with extra variables which seems to be essential. And indeed, forbidding extra variables or requiring confluence turns out to be crucial as will be shown next.

Lemma 23. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs without extra variables, with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then the innermost reduction properties $IRP_k(\mathcal{R}_j, \mathcal{R})$ hold for $j = 1, 2$ and $k = 1, 2, 3$.*

Proof. It suffices to show the following for $j = 1, 2$:

$$\forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t \implies s \xrightarrow[i]{}_{\mathcal{R}_j} t \implies s \xrightarrow[i]{}_{j} t$$

This is straightforward by induction on the depth of rewriting, and exploiting the absence of extra variables (since the step $s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t$ is innermost and due to the absence of extra variables we know that for verifying the conditions of the applied $\mathcal{R}_j$-rule only outer $\to_j$-steps are possible). ■

For the case of confluent CTRSs we need two more technical lemmas from [Mid93b].

Lemma 24. *(cf. Propositions 3.6 and 3.8 in [Mid93b]) Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint confluent CTRSs and $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$. Then $\rightarrow_{1,2}$ is confluent and $\downarrow_{1,2}$ coincides with $\overset{*}{\leftrightarrow}_{\mathcal{R}}$.*

Lemma 25. *(cf. Prop. 3.13 in [Mid93b]) Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint, confluent CTRSs, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $s_1, \ldots, s_n, t_1, \ldots, t_n$ be black terms. Then, for every substitution σ with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ for $i = 1, \ldots, n$ there exists a substitution τ such that $\sigma \rightarrow^*_{1,2} \tau$ and $\tau(s_i) \downarrow^o_1 \tau(t_i)$ for $i = 1, \ldots, n$.*

Lemma 26. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint confluent CTRSs, with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then the innermost reduction properties $IRP_k(\mathcal{R}_j, \mathcal{R})$ hold for $j = 1, 2$ and $k = 1, 2, 3$.*

Proof. It suffices to show the following for $j = 1, 2$:

$$\forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : \; s \; \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t \implies s \; \underset{i}{\rightarrow}_{\mathcal{R}_j} t \implies s \; \underset{i}{\rightarrow}_j t$$

Now consider a step $s \; \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t$ using some $\mathcal{R}_j$-rule $P \implies l \rightarrow r$ with matching substitution σ (which may also instantiate the extra variables in the conditions P). Hence, we have $\sigma(u) \downarrow_{\mathcal{R}} \sigma(v)$ for all conditions $u \downarrow v$ in P. By Lemma 24 and Lemma 25 we obtain the existence of some substitution τ with $\sigma \rightarrow^*_{1,2} \tau$ and $\tau(u) \downarrow^o_1 \tau(v)$ for all $u \downarrow v$ in P. Since the step $s \; \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t$ is innermost we know that σ is irreducible for all $x \in dom(\sigma) \cap \mathcal{V}(l)$, hence σ and τ coincide on $\mathcal{V}(l)$. Thus we get $\sigma(l) = \tau(l)$, $\sigma(r) = \tau(r)$ which implies $s \; \underset{i}{\rightarrow}_j t$ as desired. ■

4.3 Modularity Results

By combining Theorem 19 and the Lemmas 20, 22, 23 and 26 we now obtain the following modularity results (which obviously also cover the case of signature extensions).

Theorem 27. *Weak termination, weak innermost termination and (strong) innermost termination are modular properties for confluent CTRSs and for CTRSs without extra variables (in the conditions).*

Let us come back now to the question under which conditions termination and completeness are modular for CTRSs. Combining different results now enables us to prove the following.

Theorem 28. *Termination and completeness are modular for the class of non-overlapping CTRSs.*

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint non-overlapping and terminating CTRSs. Applying Theorem 9 yields confluence of $\mathcal{R}_i$ for $i = 1, 2$. By assumption we know in particular that both systems are (strongly) innermost terminating. Hence, by Theorem 27 we get that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (strongly) innermost terminating, too. The property of being non-overlapping is obviously modular for CTRSs. Hence $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (strongly) innermost terminating and non-overlapping. Finally, applying Theorem 10 yields (strong) termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ which – again by Theorem

9 – implies confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Hence, $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a confluent, (strongly) terminating and non-overlapping CTRS. Vice versa, assume that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-overlapping and terminating, hence complete. Then we know that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-overlapping and terminating, hence complete. ■

Similarly, we can show that termination and completeness are modular for the class of conditional overlay systems with joinable critical pairs.

Theorem 29. *Termination and completeness are modular for the class of conditional overlay systems with joinable critical pairs.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint terminating, conditional overlay systems with joinable critical pairs. Applying Theorem 9 yields confluence of $\mathcal{R}_i$ for $i = 1, 2$. By assumption we know in particular that both systems are (strongly) innermost terminating. Hence, by Theorem 27 we get that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (strongly) innermost terminating, too. The property of being a conditional overlay system is obviously modular for CTRSs. Hence $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a (strongly) innermost terminating, conditional overlay system. Now, in order to be able to apply Theorem 11 for inferring (strong) termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ we need to establish joinability of all (conditional) critical pairs of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent we know by Theorem 12 that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, too. Hence, in particular, all critical pairs of $\mathcal{R}_1 \oplus \mathcal{R}_2$ must be joinable. Applying Theorem 11 now yields that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a (strongly) terminating and confluent conditional overlay system (with joinable critical pairs, of course). Vice versa, assume that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a conditional overlay system with joinable critical pairs which is terminating, hence confluent and complete. Then we know that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating conditional overlay systems. By Theorem 12 confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ implies confluence of both $\mathcal{R}_1$ and $\mathcal{R}_2$ (hence in particular also joinability of critical pairs). ■

Note that – compared to the unconditional case – the proofs of Theorem 28 and Theorem 29 are more complicated. This is due to the fact that both local confluence and joinability of all critical pairs are not modular for CTRSs in general ([Mid93b]) as well as (strong) innermost termination (cf. Example 4 above).

5 Discussion, Related Work and Open Problems

Our results above and the counterexamples falsifying a couple of quite obvious and tempting conjectures demonstrate once more the inherent complexity and intricacy of CTRSs. Allowing extra variables in the conditions gives rise to some additional complications and phenomena. We have also seen that some termination and confluence properties of CTRSs depend on each other in a very subtle manner (cf. e.g. Theorems 27, 28, 29 and their proofs). Compared to unconditional TRSs the compositional behaviour of CTRSs under disjoint unions and even under signature extensions has turned out to be much more complicated (cf. e.g. Example 4). In particular, the properties of having (only) joinable critical pairs (JCP), being locally confluent (WCR) and being weakly, weakly innermost and (strongly) innermost terminating (WN, WIN and SIN, respectively) are in general not preserved under disjoint unions, in contrast to the unconditional case.

In the presentation we have focussed on join CTRSs which – from an operational point of view – is the most interesting type. It needs some further investigations (but should not be too difficult) to find out which of our results do also hold for semi-equational CTRSs. Note for instance, that the counterexample 4 does not work anymore for the semi-equational case.

Due to the aforementioned bad compositional behaviour of CTRSs concerning joinability of (all) critical pairs, local confluence and (strong) innermost termination, it is not straightforward to generalize our modularity results, in particular Theorems 28 and 29 to less restrictive non-disjoint combination mechanisms. Up to now – and as far as we know – the only comparable positive results in this direction have been obtained in [Mid93a] for 'combinations of conditional constructor systems', using a quite different proof technique. There, the disjointness requirement is weakened but extra variables are not allowed[6] and only a proper subclass of (conditional) overlay systems, namely (conditional) constructor systems, is considered. It remains to be seen to what extent our analysis here – combined with ideas of [Mid93b] – can be generalized to such non-disjoint combinations of CTRSs.

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[6] In fact, in [Mid93a] it is conjectured that the main results, namely preservation of weak normalization, semi-completeness and completeness under combinations of conditional constructor systems, do also hold when allowing extra variables (in the conditions of the rules).

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