Generalized Sufficient Conditions for Modular Termination of Rewriting

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Abstract. Modular properties of term rewriting systems, i.e. properties which are preserved under disjoint unions, have attracted an increasing attention within the last few years. Whereas confluence is modular this does not hold true in general for termination. By means of a careful analysis of potential counterexamples we prove the following abstract result. Whenever the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two (finite) terminating term rewriting systems $\mathcal{R}_1$, $\mathcal{R}_2$ is non-terminating, then one of the systems, say $\mathcal{R}_1$, enjoys an interesting (undecidable) property, namely it is not termination preserving under non-deterministic collapses, i.e. $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is collapsing, i.e. contains a rule with a variable right hand side. This result generalizes known sufficient syntactical criteria for modular termination of rewriting and provides the basis for a couple of derived modularity results. Furthermore, we prove that the minimal rank of potential counterexamples in disjoint unions may be arbitrarily high which shows that interaction of systems in such disjoint unions may be very subtle. Finally, extensions and generalizations of our main results in various directions are discussed and sketched.

1 Introduction

The question whether properties of combinations of term rewriting systems (TRSs for short) are inherited from the corresponding properties of the constituent TRSs is of great importance, e.g. in the field of abstract data type specifications. In principle and also for efficiency reasons it is very useful to know whether a combined TRS has some property whenever this property already holds for the single 'modules'. A simple and natural way of such 'modular' constructions is given by the concept of 'direct sum' ([19]) or 'disjoint union'.[2] Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case the rule sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $\mathcal{R}_1 \oplus \mathcal{R}_2$. We shall also speak of the disjoint union

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[2] Roughly spoken, the concept of 'direct sum' as defined in [19] is slightly more general than that of 'disjoint union' because it allows for renaming function symbols in order to obtain disjointness.

of $\mathcal{R}_1$ and $\mathcal{R}_2$ using the implicit convention that $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be disjoint TRSs. A property P of TRSs is said to be *modular* if the following holds for all disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property P iff both $\mathcal{R}_1$ and $\mathcal{R}_2$ have property P. Toyama [19] has shown that confluence is modular. The termination property, however, is in general not modular as witnessed by the following counterexample of [19]:

Example 1. $\quad \mathcal{R}_1 \;:\; f(a,b,x) \to f(x,x,x) \qquad \mathcal{R}_2 \;:\; \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array}$

Clearly, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite derivation:

$$\begin{array}{rl} f(a,b,G(a,b)) & \to_{\mathcal{R}_1} f(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} f(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} f(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} \cdots \end{array}$$

Note, that in this example $\mathcal{R}_2$ is not confluent. Other more complicated examples by Klop & Barendregt as well as by Toyama gathered in [18] show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ may be non-terminating even if $\mathcal{R}_1$ and $\mathcal{R}_2$ are both terminating, confluent and interreduced. All these counterexamples have some common feature. Namely, one of the systems contains a duplicating rule, i.e. a rule $l \to r$ where some variable occurs strictly more often in r than in l, and the other system contains a collapsing rule $l' \to r'$, i.e. r' is a variable[3]. This observation was exploited by Rusinowitch [17] and Middeldorp [14] (see conditions (a)-(c) below). The counterexamples in [18], involving only confluent systems $\mathcal{R}_1$ and $\mathcal{R}_2$, contained non-left-linear rules which turned out to be essential as shown by Toyama, Klop and Barendregt [20] (see condition (d) below). These results may be summarized as follows:

Given two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$, their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating if $\mathcal{R}_1$, $\mathcal{R}_2$ are terminating and one of the following conditions is satisfied:

(a) Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a duplicating rule [17].
(b) Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a collapsing rule [17].
(c) One of the system $\mathcal{R}_1$, $\mathcal{R}_2$ contains neither collapsing nor duplicating rules [14].
(d) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are left-linear and confluent [20].

Condition (a) above implies that termination is modular for right-linear TRSs, in particular for string rewriting systems. Unfortunately, duplicating and collapsing rules occur quite often and naturally in many cases. Hence, practical applicability of conditions (a)-(c) is rather limited.

For further work on modular aspects of termination and of other important properties of TRSs we refer to [15], [11], [5], [13] and [16]. In this paper we shall provide a unifying structural approach to modularity of termination by means of a careful analysis of potential counterexamples. More details and missing proofs can be found in the extended version [6]. Extensions and further related results are presented in [8], [7].

[3] A system without collapsing rules is said to be *collapse-free*.

In the next section we briefly recall the basic notions, definitions and facts for TRSs needed later on. In section 3 the main results and their applications will be presented and discussed. In section 4 possible extensions and generalizations are sketched.

2 Preliminaries

2.1 Basic Notations and Definitions

We briefly recall the basic terminology needed for dealing with TRSs (cf. e.g. [10]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an obvious way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$. In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $V(t)$.

A *context* $C[, \ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ [4] where $\Box$ is a new special constant symbol. If $C[, \ldots,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the term obtained from $C[, \ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. For the set $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ of contexts we also write $\mathcal{CON}(\mathcal{F}, \mathcal{V})$. A *non-empty* context is a term from $\mathcal{CON}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ with $t \equiv C[s]$. If in addition $C[] \not\equiv \Box$ then s is a *proper* subterm of t. A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma) := \{x \in \mathcal{V} | \sigma x \not\equiv x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ (which is assumed to contain at least one constant) and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r), denoted by $l \to r$, with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$.[5]

Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\to_{\mathcal{R}^{\mathcal{F}}}$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$ and $t \equiv C[\sigma r]$. We also write $\to_{\mathcal{R}}$ or simply $\to$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$, respectively, is clear from the context. The symmetric, transitive and transitive-reflexive closures of $\to$ are denoted by $\leftrightarrow$, $\to^+$ and $\to^*$, respectively. A TRS $\mathcal{R}$ is *terminating* if $\to$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \to s_2 \to s_3 \to \cdots$.

A *partial ordering* $>$ on a set D is a transitive and irreflexive binary relation on D. A partial ordering $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is said to be *monotonic (w.r.t. the term structure)* if it possesses the *replacement property* $s > t \implies C[s] > C[t]$ for all

[4] The symbol '$\uplus$' denotes disjoint set union.

[5] This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

$s, t, C[]$. It is *stable (w.r.t. substitutions)* if $s > t \implies \sigma s > \sigma t$ for all s, t, σ. A *term ordering on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a monotonic and stable partial ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. A *reduction ordering* is a well-founded term ordering. A term ordering $>$ is said to be a *simplification ordering* if it additionally enjoys the subterm property $C[s] > s$ for any s and any non-empty context $C[]$.[6]

A TRS is *confluent* if ${}^* \leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^* \leftarrow$ and *locally confluent* if $\leftarrow \circ \rightarrow \subseteq \rightarrow^* \circ^* \leftarrow$.[7]

2.2 Disjoint Unions

The following notations and definitions for dealing with disjoint unions of TRSs mainly follow [19] and [15].

Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRSs with disjoint signatures $\mathcal{F}_1$, $\mathcal{F}_2$. Their *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ is the TRS $(\mathcal{R}_1 \uplus \mathcal{R}_2, \mathcal{F}_1 \uplus \mathcal{F}_2)$. A property $\mathcal{P}$ of TRSs is said to be *modular* if for all disjoint TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ the following holds: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property $\mathcal{P}$ iff both $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ have property $\mathcal{P}$.

Let $t \equiv C[t_1, \ldots, t_n]$ with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C[, \ldots,] \in \mathcal{CON}(\mathcal{F}_a, \mathcal{V})$ and $root(t_1), \ldots, root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. In this case the t_i's are the *principal subterms* or *principal aliens* of t. Note that every $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V}) \setminus (\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}))$ has a unique representation of the form $t \equiv C[\![t_1, \ldots, t_n]\!]$. The set of all *aliens* or *special subterms* of t is recursively defined in an obvious way. The *rank* of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ is given by

$$rank(t) = \begin{cases} 1 & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}) \\ 1 + max\{rank(t_i) | 1 \leq i \leq n\} & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}$$

An important basic fact about the rank of terms occurring in a $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation is the following ([19]): If $s \rightarrow^* t$ then $rank(s) \geq rank(t)$. Moreover, if $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ with $rank(s) = n$ then there exists a ground instance σs of s with $rank(\sigma s) = n$, too. [8] A (finite or infinite) derivation $D: s_1 \rightarrow s_2 \rightarrow s_3 \ldots$ is said to have rank n $(rank(D) = n)$ if n is the minimal rank of all the s_i's, i.e. $n = \min\{rank(s_i) | 1 \leq i\}$.

The topmost homogeneous part of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$, denoted by $top(t)$, is obtained from t by replacing all principal subterms by $\Box$, i.e.

$$top(t) = \begin{cases} t & \text{if} \quad rank(t) = 1 \\ C[, \ldots,] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}$$

Furthermore we shall use the abbreviations $\mathcal{GT}_\oplus$ for $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$, $\mathcal{GT}_\oplus^n$ for $\{t \in \mathcal{GT}_\oplus | rank(t) = n\}$ and $\mathcal{GT}_\oplus^{<n}$ for $\{t \in \mathcal{GT}_\oplus | rank(t) < n\}$. For the sake of better readability the function symbols from $\mathcal{F}_1$ are considered to be black and those of $\mathcal{F}_2$ to be white. Variables have no colour. A *top black (white)* term has a black (white) root symbol.

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[6] For the case that varyadic function symbols are allowed one additionally requires here the so-called 'deletion'-property (cf. Dershowitz [2]).

[7] Here, '$\circ$' denotes relation composition.

[8] This is easily verified by substituting appropriately $\mathcal{F}_1$- or $\mathcal{F}_2$-ground terms for those variables which occur in the 'deepest layer' of s.

For $s,t \in \mathcal{GT}_\oplus$ the one-step reduction $s \to t$ is said to be *inner* — denoted by $s \xrightarrow{i} t$ — if the reduction takes place in one of the principal subterms of s. Otherwise, we speak of an *outer* reduction step and write $s \xrightarrow{o} t$. A rewrite step $s \to t$ is *destructive at level 1* if the root symbols of s and t have different colours. The step $s \to t$ is *destructive at level* $n+1$ (for $n \geq 1$) if $s \equiv C[\![s_1,\ldots,s_j,\ldots,s_n]\!] \xrightarrow{i} C[s_1,\ldots,t_j,\ldots,s_n] \equiv t$ with $s_j \to t_j$ destructive at level n. Clearly, if a rewrite step is destructive (at some level) then the applied rewrite rule is collapsing, i.e. has a variable right-hand side. This is a basic fact which should be kept in mind subsequently.

For coding principal subterms, e.g. by new variables or constants, and for dealing with outer rewrite steps involving non-linear rules the following definitions are useful. For $s_1,\ldots,s_n,t_1,\ldots,t_n \in \mathcal{GT}_\oplus$ we write $\langle s_1,\ldots,s_n\rangle \propto \langle t_1,\ldots,t_n\rangle$ if $t_i \equiv t_j$ whenever $s_i \equiv s_j$, for all $1 \leq i < j \leq n$. The conjunction of $\langle s_1,\ldots,s_n\rangle \propto \langle t_1,\ldots,t_n\rangle$ and $\langle t_1,\ldots,t_n\rangle \propto \langle s_1,\ldots,s_n\rangle$ is denoted by $\langle s_1,\ldots,s_n\rangle \infty \langle t_1,\ldots,t_n\rangle$. The following basic properties of outer and inner reduction steps will be freely used in the sequel:

- If $s \xrightarrow{o} t$ then $s \equiv C[\![s_1,\ldots,s_n]\!], t \equiv C'[s_{i_1},\ldots,s_{i_m}]$ for some contexts $C[,\ldots,]$, $C'[,\ldots,]$, $i_1,\ldots,i_m \in \{1,\ldots,n\}$ and terms $s_1,\ldots,s_n \in \mathcal{T}_\oplus$. If moreover $s \xrightarrow{o} t$ is not destructive at level 1 then $t \equiv C'[\![s_{i_1},\ldots,s_{i_m}]\!]$.
- If $s \xrightarrow{i} t$ then $s \equiv C[\![s_1,\ldots,s_j,\ldots,s_n]\!]$ and $t \equiv C[s_1,\ldots,t_j,\ldots,s_n]$ for some context $C[,\ldots,], j \in \{1,\ldots,n\}$ and terms $s_1,\ldots s_n,t_j \in \mathcal{T}_\oplus$ with $s_j \to t_j$. If moreover $s \xrightarrow{i} t$ is not destructive at level 2 then $t \equiv C[\![s_1,\ldots,t_j,\ldots,s_n]\!]$.
- If $C[\![s_1,\ldots,s_n]\!] \xrightarrow{o} C'[\![s_{i_1},\ldots,s_{i_m}]\!], 1 \leq i_j \leq n$, $j \in \{1,\ldots,m\}$, by application of some rule then $C[\![t_1,\ldots,t_n]\!] \xrightarrow{o} C'[\![t_{i_1},\ldots,t_{i_m}]\!]$ by the same rule for all terms $t_1,\ldots,t_n$ with $\langle s_1,\ldots,s_n\rangle \propto \langle t_1,\ldots,t_n\rangle$.

3 Structural Properties of Minimal Counterexamples

3.1 Characterization of Minimal Counterexamples

Before formally stating and proving the main result we shall now illustrate the essential ideas and construction steps via an example from Drosten [4] which shows that termination need not be modular even for confluent TRSs.

Example 2.

$$\mathcal{R}_1 : \begin{array}{l} f(a,b,x) \to f(x,x,x) \\ f(x,y,z) \to c \\ a \to c \\ b \to c \end{array} \qquad \mathcal{R}_2 : \begin{array}{l} K(x,y,y) \to x \\ K(y,y,x) \to x \end{array}$$

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating and confluent, but their disjoint union is non-terminating. For instance, we have the following infinite derivation:

$$\begin{aligned} D : f(a,b,K(a,b,b)) &\xrightarrow{o}_{\mathcal{R}_1} f(K(a,b,b),K(a,b,b),K(a,b,b)) && (1)\\ &\xrightarrow{i}_{\mathcal{R}_2} f(a,K(a,b,b),K(a,b,b)) && (2)\\ &\xrightarrow{i}_{\mathcal{R}_1} f(a,K(c,b,b),K(a,b,b)) && (3)\\ &\xrightarrow{i}_{\mathcal{R}_1} f(a,K(c,c,b),K(a,b,b)) && (4)\\ &\xrightarrow{i}_{\mathcal{R}_2} f(a,b,K(a,b,b)) \xrightarrow{o}_{\mathcal{R}_1} \ldots && (5) \end{aligned}$$

Obviously, the crucial steps which enable this derivation to be infinite (and even cyclic) are the inner reductions (2)-(5), in particular the steps (2) and (5) which are destructive at level 2. They modify substantially the topmost homogeneous black[9] layer thereby enabling an outer reduction step previously not possible. The idea now is to abstract from the concrete form of these inner steps but retain the essential information which permits subsequent outer steps. For that purpose it is sufficient to consider the principal top white, i.e. $\mathcal{F}_2$-rooted, aliens and collect those top black, i.e. $\mathcal{F}_1$-rooted, terms to which the former may reduce. In other words, colour changing derivations issued by principal aliens are essential. The coding of the collected top black successors of some principal top white alien will be achieved by some new function symbol(s) which in a sense serve(s) for abstracting from the concrete form of white layers while keeping only the 'layer separating' information. Since in general also top black aliens hidden in deeper layers (cf. subsection 3.4 below) may eventually become principal top black aliens the whole process has to be performed in a recursive fashion in general (which is not necessary in the example). After this abstracting transformation process sequences of inner reduction steps like (2)-(5) above in the original derivation may be simulated by ('deletion' and) 'subterm' steps in the transformed derivation. In order to explain this in more detail let us choose H as a new (varyadic) layer separating function symbol. Then we get the transformed derivation

$$\begin{aligned}
D' : f(a,b,H(a,b,c)) &\xrightarrow{o}_{\mathcal{R}_1} f(H(a,b,c),H(a,b,c),H(a,b,c)) && (1')\\
&\xrightarrow{i}_{\mathcal{R}'_2} f(a,H(a,b,c),H(a,b,c)) && (2')\\
&\xrightarrow{i}_{\mathcal{R}'_2} f(a,H(b,c),H(a,b,c)) && (3')\\
&\equiv \quad f(a,H(b,c),H(a,b,c)) && (4')\\
&\xrightarrow{i}_{\mathcal{R}'_2} f(a,b,H(a,b,c)) \xrightarrow{o}_{\mathcal{R}_1} \ \ldots && (5')
\end{aligned}$$

where $\mathcal{R}_1$ is as above and $\mathcal{R}'_2 = \mathcal{R}^H_{sub} \cup \mathcal{R}^H_{del}$ with

$$\begin{aligned}
\mathcal{R}^H_{sub} &= \{H(x_1,\ldots,x_j,\ldots,x_n) \to x_j | 1 \le j \le n\}\ ,\\
\mathcal{R}^H_{del} &= \{H(x_1,\ldots,x_j,\ldots,x_n) \to H(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n) | 1 \le j \le n\}.
\end{aligned}$$

The top white principal alien $t := K(a,b,b)$ of the top black starting term $s := f(a,b,K(a,b,b))$ of D can be reduced (in arbitrarily many steps) to the top black successors a, b and c. Hence, the abstracting transformation of t yields $H(a,b,c)$ and the whole starting term s is transformed into $f(a,b,H(a,b,c))$. Furthermore, any outer step in D corresponds to an outer step in D' using the same rule. Any inner step in D which is not destructive at level 2, e.g. (3) and (4), corresponds in D' to a (possibly empty) sequence of inner $\mathcal{R}'_2$-steps not destructive at level 2 (here (3') and (4'), respectively). Any inner step in D which is destructive at level 2 (hence collapsing), e.g. (2) and (5), corresponds in D' to an $\mathcal{R}^H_{sub}$-step (here (2') and (5'), respectively).

[9] Remember that function symbols from $\mathcal{R}_1$ and $\mathcal{R}_2$ are considered to be black and white, respectively.

In order to stay within the usual scenario of fixed-arity function symbols we modify the above transformation by taking a new binary function symbol G and a new constant A instead of the varyadic symbol H. With the correspondence

$$H(t_1,\ldots,t_n) = \begin{cases} A & \text{if} \quad n = 0 \\ G(t_1, G(t_2, \ldots G(t_{n-1}, G(t_n, A)) \ldots)) & \text{if} \quad n > 0 \end{cases}$$

the above construction easily carries over and we obtain the derivation

$$\begin{array}{lll} D'': & f(a,b,G(a,G(b,G(c,A)))) & \\ & \xrightarrow{o}_{\mathcal{R}_1} f(G(a,G(b,G(c,A))),G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) & (1'') \\ & \xrightarrow{i}_{\mathcal{R}_2''} f(a,G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) & (2'') \\ & \xrightarrow{i}_{\mathcal{R}_2''} f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))) & (3'') \\ & \equiv \quad f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))) & (4'') \\ & \xrightarrow{i}_{\mathcal{R}_2''} f(a,b,G(a,G(b,G(c,A)))) \xrightarrow{o}_{\mathcal{R}_1} \ldots & (5'') \end{array}$$

Here, $\mathcal{R}_2''$ is to be interpreted as $\mathcal{R}_2'' = \mathcal{R}_{sub}^G$ with

$$\mathcal{R}_{sub}^G = \{G(x,y) \to x,\ G(x,y) \to y\},$$

i.e. deletion rules are not necessary any more. In the following formal presentation we shall use the latter transformation.

Definition 1. A TRS $\mathcal{R}$ is said to be *termination preserving under non-deterministic collapses* if termination of $\mathcal{R}$ implies termination of $\mathcal{R} \oplus \{G(x,y) \to x, G(x,y) \to y\}$.

Lemma 2. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two terminating disjoint TRSs such that*

$$D : s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ (involving only ground terms) of minimal rank, i.e. any derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of smaller rank is finite. Then we have:

(a) $rank(D) \geq 3$.
(b) Infinitely many steps in D are outer steps.
(c) Infinitely many steps in D are inner steps which are destructive at level 2.

Proof. **(a)** Follows from (c) since whenever $s_i \xrightarrow{i} s_{i+1}$ is destructive at level 2 then $rank(s_i) \geq 3$.
(b) Assume for a proof by contradiction that only finitely many steps in D are outer ones. W.l.o.g. we may further assume that no step in D is an outer one. Hence, for $s_1 \equiv C[\![t_1,\ldots,t_n]\!]$ all reductions in D are inner ones and take place below one of the positions of the t_i's. Since D is infinite we conclude by the pigeon hole principle that at least one of the t_i's initiates an infinite derivation whose rank is smaller than $rank(D)$. But this is a contradiction to the minimality assumption concerning $rank(D)$.
(c) For a proof by contradiction assume w.l.o.g. that no inner step in D is destructive at level 2. Then, with $\widetilde{s_i} := top(s_i)$ any outer step $s_i \xrightarrow{o} s_{i+1}$ in D yields $\widetilde{s_i} \to \widetilde{s_{i+1}}$ using the same rule from $\mathcal{R}_1 \oplus \mathcal{R}_2$ and for every inner step $s_i \xrightarrow{i} s_{i+1}$ we have $\widetilde{s_i} \equiv \widetilde{s_{i+1}}$. Assuming w.l.o.g. that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted, we can conclude by (b) that $\mathcal{R}_1$ is non-terminating which yields a contradiction. ∎

Next we formalize the transformation process illustrated above.

Definition 3. Let $\mathcal{R}_1, \mathcal{R}_2$ be two (finite) terminating disjoint TRSs, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ and $n \in N$ such that for every $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ there is no infinite $\mathcal{R}$-derivation starting with s. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$. Then the $\mathcal{F}_2$- (or *white*) *abstraction* is defined to be the mapping

$$\Phi : \mathcal{GT}^{\leq n}_{\oplus} \uplus \{t \in \mathcal{GT}^{n+1}_{\oplus} | root(t) \in \mathcal{F}_1\} \quad \longrightarrow \quad \mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$$

given by

$$\Phi(t) := \begin{cases} t & \text{if } t \in \mathcal{T}(\mathcal{F}_1) \\ A & \text{if } t \in \mathcal{T}(\mathcal{F}_2) \\ C[\![\Phi(t_1), \ldots, \Phi(t_m)]\!] & \text{if } t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{F}_1 \\ CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if } t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{F}_2 \end{cases}$$

with

$$\begin{aligned} SUCC^{\mathcal{F}_1}(t) &:= \{t' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | t \to^*_{\mathcal{R}} t', root(t') \in \mathcal{F}_1\}, \\ \Phi^*(M) &:= \{\Phi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Phi), \\ CONS(\langle\rangle) &:= A, \\ CONS(\langle s_1, \ldots, s_{k+1}\rangle) &:= G(s_1, CONS(\langle s_2, \ldots, s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1, \ldots, s_k\} &:= \langle s_{\pi(1)}, \ldots, s_{\pi(k)}\rangle, \end{aligned}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

Intuitively, for computing $\Phi(t)$ one proceeds top-down in a recusive fashion. Top black layers are left invariant whereas (for the case of top black t) the principal top white subterms are transformed by computing for every such top white subterm the set of possible top black successors, abstracting the resulting terms recursively, sorting the resulting set of abstracted terms and finally constructing again an ordinary term by means of using the new constant symbol A (for empty arguments sets) and the new binary function symbol G (for non-empty argument sets). The sorting process and the total ordering involved here are due to some proof-technical subtleties (cf. [6] for details).

In the following we shall implicitly use the convention that notions like *rank* or *inner* and *outer* reduction steps have to be interpreted w.r.t. some specific disjoint union which is clear from the context.

The next lemmas capture the important properties of the above defined abstracting transformation.

Lemma 4. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, n and Φ be given as in definition 3. Then, for any $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ and $root(s) \in \mathcal{F}_2$ we have:*

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \to^*_{\mathcal{R}'_2} \Phi(t),$$

where $\mathcal{R}'_2 := \mathcal{R}^G_{sub} := \{G(x,y) \to x, G(x,y) \to y\}$.

Proof. Straightforward using definition 3 (cf. [6]). ∎

Lemma 5. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Φ be given as in definition 3. Then, Φ is rank decreasing, i.e. for any $s \in dom(\Phi)$ we have $rank(\Phi(s)) \leq rank(s)$.*

Proof. By an easy induction on $rank(s)$ using the definition of Φ. ■

Inner and outer reduction steps in $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivations can be translated into corresponding (sequences of) inner and outer steps in the transformed derivation in $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ as described by

Lemma 6. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, $\mathcal{R}_2' = \mathcal{R}_{sub}^G = \{G(x,y) \to x, G(x,y) \to y\}$, n and the (white) $\mathcal{F}_2$-abstraction Φ be given as above. Then, for any $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$ we have:*

(a) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is not destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.

(b) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.

(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^{*}_{\mathcal{R}_2'} \Phi(t)$ with all steps not destructive at level 2.

(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^{+}_{\mathcal{R}_2'} \Phi(t)$ such that exactly one of these steps is destructive at level 2.

Proof. By case analysis using definition 3 (cf. [6]). ■

Now we are prepared to state and prove the main result of this section.

Theorem 7. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint (finite) TRSs which are both terminating such that their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then $\mathcal{R}_j$ is not termination preserving under non-deterministic collapses for some $j \in \{1,2\}$ and the other system $\mathcal{R}_k$, $k \in \{1,2\} \setminus \{j\}$ is collapsing. Moreover, the minimal rank of counterexamples in $\mathcal{R}_j \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is less than or equal to the minimal rank of counterexamples in $\mathcal{R}_1 \oplus \mathcal{R}_2$.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ be given as stated above. We consider a minimal counterexample, i.e. an infinite $\mathcal{R}$-derivation

$$D: \quad s_1 \to s_2 \to s_3 \to \quad \ldots$$

of minimal rank, let's say $n+1$. W.l.o.g. we may assume that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted ground terms having *rank* $n+1$. Since the preconditions of definition 3 are satisfied we may apply the white ($\mathcal{F}_2$-) abstraction function Φ to the s_i's. As it will be shown this yields an infinite $\mathcal{R}'$-derivation

$$D': \quad \Phi(s_1) \to^* \Phi(s_2) \to^* \Phi(s_3) \to^* \quad \ldots$$

where $\mathcal{R}' := \mathcal{R}_1 \oplus \mathcal{R}_2'$ with $\mathcal{R}_2' := \mathcal{R}_{sub}^G = \{G(x,y) \to x, G(x,y) \to y\}$. Using lemma 6 we conclude that for any step $s_j \to s_{j+1}$ in D we have

$$\begin{aligned} s_j \xrightarrow{o}_{\mathcal{R}_1} s_{j+1} &\Longrightarrow \Phi(s_j) \xrightarrow{o}_{\mathcal{R}_1} \Phi(s_{j+1}) \ , \\ s_j \xrightarrow{i}_{\mathcal{R}} s_{j+1} &\Longrightarrow \Phi(s_j) \xrightarrow{i}{}^{*}_{\mathcal{R}_2'} \Phi(s_{j+1}) . \end{aligned}$$

Hence, D' is indeed an $\mathcal{R}'$-derivation. Since according to lemma 2 (b) infinitely many steps in D are outer ones, the derivation D' is infinite, too. But this means that $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses. Moreover, lemma 2 (c) implies that $\mathcal{R}_2$ is collapsing. Lemma 5 finally implies $rank(D') \leq rank(D)$ which finishes the proof. ■

As an immediate consequence of this result we obtain

Corollary 8. *Termination (and hence also completeness) is modular for the class of (finite) TRSs which are termination preserving under non-deterministic collapses.*

By observing that $\mathcal{R}_{sub}^{G}$ is termination preserving under non-deterministic collapses we even get

Corollary 9. *The disjoint union of two (finite) terminating TRSs is again terminating whenever one of the systems is termination preserving under non-deterministic collapses and non-collapsing.*

The next result shows that the class of TRSs which are termination preserving under non-deterministic collapses comprises all non-duplicating TRSs.

Lemma 10. *Whenever a (finite) TRS is non-duplicating then it is termination preserving under non-deterministic collapses.*

Proof. Cf. [6]. ■

Theorem 7, corollaries 8, 9 and lemma 10 constitute a generalization of the main results of [17] and [14] (for finite TRSs). Theorem 7 corresponds nicely to the intuition that the existence of counterexamples crucially depends on 'non-deterministic collapsing' reduction steps. Hence, example 1 above is in a sense the simplest conceivable counterexample.

On the one side the general result stated in theorem 7 reveals an interesting structural property of potential counterexamples to modularity of termination. On the other side it is still rather abstract. The obvious question arising is which TRSs are indeed termination preserving under non-deterministic collapses. This question will be tackled next. Given an arbitrary TRS $\mathcal{R}$ it would be desirable to have a method for testing whether $\mathcal{R}$ is termination preserving under non-deterministic collapses. But it turns out that this is an undecidable property in general.

Theorem 11. *The property of TRSs to be termination preserving under non-deterministic collapses is undecidable.*

Proof. Cf. [6], [15]. ■

3.2 Derived Criteria for Modularity of Termination

In order to obtain easily verifiable sufficient conditions for the property of being termination preserving under non-deterministic collapses we shall now use the increasing interpretation method (cf. [9], [3]) for termination proofs of TRSs. Consider the situation where two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are given such that $\mathcal{R}_1$ is terminating. For verifying termination of $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ it then suffices to show that $\mathcal{R}_2$-rules can be 'simulated' by $\mathcal{R}_1$-rules. To be more precise, we get

Lemma 12. *Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be TRSs such that $\mathcal{R}_1$ is terminating. Moreover, let φ be an interpretation of $(\mathcal{F}_1 \cup \mathcal{F}_2)$-operations in terms of $\mathcal{F}_1$-operations which is the identity on $\mathcal{F}_1$ and which is strict[10] on $\mathcal{F}_2$. Then the union $(\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is terminating, too, provided that for every rule $l \to r \in \mathcal{R}_2$ we have $\varphi(l) \to_{\mathcal{R}_1}^{+} \varphi(r)$.*

[10] φ is a *strict* interpretation for $\mathcal{F}_2$, i.e. for any $f \in \mathcal{F}_2$ we have $V(f(x_1, \ldots, x_n)) \subseteq V(\varphi(f(x_1, \ldots, x_n)))$.

Proof. Cf. [6] ■

Of course, the method for proving termination according to the above lemma is rather restricted, because it requires in a sense that $\mathcal{R}_1 \cup \mathcal{R}_2$ terminates for the same reason as $\mathcal{R}_1$ alone. But in particular for the scenario of disjoint unions it is well-suited as we shall see now. Firstly, we need

Definition 13. A TRS $\mathcal{R}^{\mathcal{F}}$ is said to be *non-deterministically collapsing* if there exists a term $s[x,y] \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with $x,y \in \mathcal{V}$ such that $s[x,y] \to^+ x$ and $s[x,y] \to^+ y$, i.e. if some term can be reduced to two distinct variables.

Lemma 14. *If a TRS is non-deterministically collapsing then it is also termination preserving under non-deterministic collapses.*

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$ be a terminating and non-deterministically collapsing TRS. According to theorem 7 it suffices to show that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 = \{G(x,y) \to x,\, G(x,y) \to y\}$ is terminating. Since $\mathcal{R}_1^{\mathcal{F}_1}$ is non-deterministically collapsing there exists some term $s[x,y] \in \mathcal{T}(\mathcal{F}_1,\mathcal{V})$ with $x,y \in \mathcal{V}$ such that $s[x,y] \to^+ x$ and $s[x,y] \to^+ y$. W.l.o.g. we may further assume that x,y are the only variables appearing in $s[x,y]$. Now we interpret the function symbol G by $\varphi f = \lambda x,y\,.\,s[x,y]$ and simply apply lemma 12 the preconditions of which are satisfied. ■

Corollary 15. *Termination is modular for the class of (finite) TRSs which are non-deterministically collapsing.*

Next we consider cases where a terminating TRS $\mathcal{R}$ does not necessarily contain collapsing rules but remains terminating when such rules are added.

Definition 16. Let $\mathcal{R}^{\mathcal{F}}$ be a TRS and $f \in \mathcal{F}, \mathcal{F}' \subseteq \mathcal{F}$. Then, $\mathcal{R}^{\mathcal{F}}$ is said to be *f-simply terminating* if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}_{sub}^{f}$ with $\mathcal{R}_{sub}^{f} := \{f(x_1,\ldots,x_j,\ldots,x_n) \to x_j | 1 \le j \le n\}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is F'*-simply terminating* if $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}'} \mathcal{R}_{sub}^{f}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is *simply terminating* (cf. Kurihara & Ohuchi [13]) if $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}$-simply terminating, i.e. $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}} \mathcal{R}_{sub}^{f}$ is terminating.

Clearly, if a TRS $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}'$-simply terminating for some $\mathcal{F}' \subseteq \mathcal{F}$ then it is terminating, i.e. simple termination implies termination. Moreover we also obtain

Lemma 17. *Let $\mathcal{R}^{\mathcal{F}}$ be f-simply terminating for some $f \in \mathcal{F}$ of arity greater than 1. Then $\mathcal{R}^{\mathcal{F}\cup\mathcal{F}'}$ is $(\{f\}\cup\mathcal{F}')$-simply terminating for any $\mathcal{F}'$ with $\mathcal{F}'\cap\mathcal{F} = \emptyset$.*

Proof. Let $\mathcal{R}^{\mathcal{F}}$ be f-simply terminating for some $f \in \mathcal{F}$ with $arity(f) > 1$ and let $\mathcal{F}'$ be given with $\mathcal{F}' \cap \mathcal{F} = \emptyset$. W.l.o.g. we may assume that f has arity 2. We shall apply lemma 12 for proving that $\mathcal{R}^{\mathcal{F}\cup\mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating. For that purpose we interpret every $G \in \mathcal{F}'$ strictly in terms of $\mathcal{F}$-operations as follows (w.l.o.g. we assume that G is not a constant):

$$G(x_1,\ldots,x_n) = \begin{cases} f(x_1,x_1) & \text{if} \quad n = 1 \\ f(x_1, f(x_2,\ldots f(x_{n-1},x_n)\ldots)) & \text{if} \quad n > 1\,. \end{cases}$$

Now the assumptions of lemma 12 are clearly satisfied and we can conclude that $\mathcal{R}^{\mathcal{F}\cup\mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating. ■

Combining this result with lemma 14 and corollary 15 we obtain

Corollary 18. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two (finite) disjoint TRSs with $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1, 2$. Then the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (f_1- and f_2-simply) terminating, too.*

The intuition behind the notion of $\mathcal{F}$-simple termination is its close relationship to simplification orderings, an important subclass of reduction orderings which in practice are very often used for termination proofs. Simplification orderings are well-suited for that purpose due to the following result from Dershowitz [1] which we present in a slightly generalized version. [11]

Lemma 19. *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ terminates if there exists a simplification ordering $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $l \succ r$ for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$.*

Kurihara & Ohuchi [12] have shown that for finite TRSs simple termination can be characterized by means of simplification orderings.[12] By weakening the finiteness requirement we obtain the following result which is a slightly generalized version of a result in [12].

Lemma 20. *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ is ($\mathcal{F}-$) simply terminating if and only if there exists a simplification ordering $\succ$ with $l \succ r$ for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$.*

By specializing corollary 18 we finally obtain the main result from Kurihara & Ohuchi [12]:

Theorem 21. *Simple termination is a modular property of (finite) TRSs.*

This fact generalizes the well-known observation that common classes of (precedence[13] based) simplification orderings like recursive path orderings or recursive decomposition orderings exhibit a modular behaviour simply by combining the corresponding disjoint precedences.

In [12] theorem 21 is directly proved by means of a construction which has some similarity with our approach presented in the last section. Instead of our white (and black) abstraction function Kurihara & Ohuchi define a mapping called 'alien-replacement' which is tailored to some specific finite reduction sequence. Moreover their construction is in a sense incremental, but not rank-decreasing in general. To be more precise, consider some finite derivation

$$D: \quad s_0 \to s_1 \to s_2 \to \ldots \to s_m$$

in $\mathcal{R} := (\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1}) \oplus (\mathcal{R}_2 \cup \mathcal{R}_{sub}^{\mathcal{F}_2} \cup \mathcal{R}_{del}^{\mathcal{F}_2})$ with all s_i's top black and such that every $\mathcal{R}$-derivation starting from any (top white) principal alien of s_0 is finite. Then their 'alien replacement' construction for D essentially consists in (recursively) collecting, for any principal alien occurring in D, all direct descendants occurring in D and abstracting them via a new varyadic function symbol.

[11] The proof is based on Kruskal's tree theorem which roughly spoken states that any infinite sequence of terms from $\mathcal{T}(\mathcal{F})$ with $\mathcal{F}$ finite is self-embedding. Hence, the requirement that $\mathcal{R}$ must be finite can be weakened.

[12] In fact, problems concerning this finiteness requirement are neglected in [12].

[13] A precedence is a partial ordering on a set $\mathcal{F}$ of function symbols.

Using this transformation the $\mathcal{R}$-derivation D can be translated in a one-to-one manner into a corresponding $(\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1})$-derivation from which one can easily infer the modularity of simple termination using lemma 20.[14]

3.3 Minimal Counterexamples of Arbitrary Rank

Besides the features mentioned all counterexamples to modularity of termination presented above and in the literature (cf. [18]) have some more common property. Namely, the rank n of minimal counterexamples always equals 3. According to lemma 2 we must have $n \geq 3$. So, the question naturally arises whether this is a general phenomenon saying that, whenever the disjoint union of two terminating TRSs is non-terminating then there is a counterexamples having rank 3. This question is not only interesting by itself but also because many proofs concerning results on modular termination have to consider 'mixed' terms of arbitrary rank. In particular, the extremely complicated analysis performed in [20] for proving that completeness is modular for left-linear TRSs could be considerably simplified if counterexamples of rank 3 were always possible. Surprisingly (at least for the author) this is not the case as illustrated by

Example 3. $\quad \mathcal{R}_1 \;:\; f(x, g(x), \ldots, g^n(x), y) \to f(y, \ldots, y) \qquad \mathcal{R}_2 \;:\; \begin{array}{l} G(x) \to x \\ G(x) \to A \end{array}$

Here, f has arity $n+2$ and $g^n(x)$ stands for the n-fold application of g to x. Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. For instance, we have the following infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation[15]

$$\begin{array}{lll} D: & f((Gg)^n A, (Gg)^n A, \ldots (Gg)^n A) & \to_{\mathcal{R}_2} f(A, (Gg)^n A, \ldots, (Gg)^n A) \\ & & \to_{\mathcal{R}_2} f(A, g(Gg)^{n-1} A, \ldots, (Gg)^n A) \\ & & \to_{\mathcal{R}_2} f(A, gA, \ldots, (Gg)^n A) \\ & & \quad \vdots \\ & & \to_{\mathcal{R}_2} f(A, gA, g^2 A, \ldots, g^n A, (Gg)^n A) \\ & & \to_{\mathcal{R}_1} f((Gg)^n A, (Gg)^n A, \ldots, (Gg)^n A) \\ & & \to_{\mathcal{R}_2} \cdots \end{array}$$

of rank $2n+2$. A careful analysis of possible reductions shows that for this example $2n+2$ is the minimal rank of any conceivable non-terminating $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation. Moreover, it is straightforward to modify the example in such a way that only finite signatures with function symbols of (uniformly) bounded arities are involved. For instance, one may use a binary f' and encode $f(x_1, \ldots, x_n)$ by $f'(x_1, f'(x_2, \ldots, f'(x_{n-1}, x_n) \ldots))$.

Hence, we can conclude that for terminating disjoint TRSs with non-terminating disjoint union minimal counterexamples may have an arbitrarily high rank. This shows that the interaction in disjoint unions of TRSs may be very subtle, in particular concerning termination properties.

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[14] cf. [12], [13] for details; in fact, compared to [12], [13] contains a simplified and clarified version of 'alien replacement'.

[15] The notation used here should be self-explanatory. For example, $(Gg)^2(A)$ stands for $G(g(G(g(A))))$.

4 Extensions and Generalizations

There are a couple of directions in which our approach may be extended and generalized. But due to lack of space we cannot go into details (cf. [6]).

Firstly, the finiteness requirement concerning the TRSs involved in our main result and subsequently derived criteria can be considerably weakened. In fact, the reason for this finiteness condition was to ensure well-definedness of the black (or white) abstraction function Φ in definition 3. A closer look at the definition reveals that the essential property needed is that for any mixed term s of rank less than or equal n with n as in the definition, the set of possible successors of s, i.e. $\{s' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | s \to_{\mathcal{R}}^{+} s'\}$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ is finite. For that purpose it is sufficient to require that $\mathcal{R}$ is finitely branching, i.e. for any term $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ the one-step-successor set $\{s' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | s \to_{\mathcal{R}} s'\}$ is finite. In that case one may simply apply Kőnigs lemma.

Secondly — and more important in practice — one may also weaken the disjointness requirement. For instance it is possible in a rather straightforward manner to generalize our approach and the essential results to so-called 'combined systems with shared constructors' (cf. [13], [6]).

It is also possible to generalize most results in a uniform way to conditional term rewriting systems (CTRSs), in particular to so-called join CTRSs (cf. [8]). But for this purpose a careful analysis is necessary. As mentioned by Middeldorp (cf. [15]), the additional complications mainly arise from the fact that the fundamental property

$$s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t \quad \Longrightarrow \quad s \to_{\mathcal{R}_1} t \quad \vee \quad s \to_{\mathcal{R}_2} t \quad (*)$$

only holds for unconditional TRSs but not for CTRSs in general. This is due to the fact that for verifying the applicability of an $\mathcal{R}_1$-rule, i.e. for proving the corresponding instantiated conditions, rules from $\mathcal{R}_2$ may be crucial. Consider for instance

Example 4. Let $\mathcal{R}_1 = \{\ x \downarrow b \ \wedge \ x \downarrow c \ \Longrightarrow \ a \to a \ \}$ over $\mathcal{F}_1 = \{a, b, c\}$ and $\mathcal{R}_2 = \{G(x, y) \to x, G(x, y) \to y\}$ over $\mathcal{F}_2 = \{G, A\}$ be given. Here, we have $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ (by instantiating x with $G(b, c)$), but neither $a \to_{\mathcal{R}_1} a$ nor $a \to_{\mathcal{R}_2} a$.

Hence, this is also a very simple counterexample to modularity of termination for CTRSs, because both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (the reduction relation of $\mathcal{R}_1$ is empty). Moreover, the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation $a \to a \to a \to \ldots$ has rank one, a phenomenon which cannot occur in the unconditional case. Note that the system $\mathcal{R}_1$ above has the extra variable x in the condition part of its rule. Therefore it cannot be decreasing. When we forbid extra variables in the conditions then the minimal rank of potential counterexamples is at least 3 as it can be shown. But the fundamental property $(*)$ above may still be violated as shown by

Example 5. Let $\mathcal{R}_1 = \{\ x \downarrow a \ \wedge \ x \downarrow b \ \Longrightarrow \ f(x) \to f(x) \ \}$ over $\mathcal{F}_1 = \{a, b, f\}$ and $\mathcal{R}_2 = \{G(x, y) \to x, G(x, y) \to y\}$ over $\mathcal{F}_2 = \{G, A\}$ be given. Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly decreasing (and even reductive), hence terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating.

This example shows that – as mentioned in the introduction – the conditions

(a) Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a duplicating rule.
(c) One of the system $\mathcal{R}_1$, $\mathcal{R}_2$ contains neither collapsing nor duplicating rules.

are sufficient for ensuring modularity of termination only for unconditional TRSs, but not for CTRSs in general. In [15] it is shown that (a) and (c) are sufficient under the additional assumption that both systems are confluent.

In spite of these additional complications the essential features and results of our structural analysis of modular termination for the unconditional case can be generalized to CTRSs (cf. [8]). Due to lack of space we only want to mention one additional phenomenon which is different from the unconditional case. Namely, lemma 2 (c) which says that infinitely many steps in D are inner reductions which are destructive at level 2 does not hold for CTRSs in general. To wit, consider the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation $f(G(a,b)) \stackrel{o}{\to} f(G(a,b)) \stackrel{o}{\to} f(G(a,b)) \stackrel{o}{\to} \ldots$ in example 5 above.

Conclusion

We have presented a structural analysis of minimal counterexamples to modular termination of rewriting. It has been shown that the abstract property of TRSs to be termination preserving under non-deterministic collapses is crucial for the invariance of termination under disjoint unions. Although this property turns out to be undecidable in general it provides the basis for a couple of sufficient criteria for ensuring modularity of termination. Our general approach and the resulting sufficient conditions for modularity of termination generalize known results of [17], [14], [15], [12] and [13]. In particular, the basic ideas and constructions are also applicable to more general situations, namely for (non-disjoint) unions of TRSs with common constructors as well as for conditional TRSs. Moreover, a very simple class of examples has been presented which proves that the minimal rank of non-terminating derivations in disjoint unions of terminating TRSs may be arbitrarily high. This reflects in a sense the very subtle interaction of rewriting in disjoint unions and shows that arbitrarily complicated layer structures may be essential w.r.t. the termination behaviour.

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References

1. N. Dershowitz. A note on simplification orderings. *Information Processing Letters*, 9(5):212–215, 1979.
2. N. Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, pages 279–301, 1982.
3. N. Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3(1):69–116, 1987.
4. K. Drosten. *Termersetzungssysteme*. Informatik-Fachberichte 210. Springer, 1989. in German.
5. H. Ganzinger and R. Giegerich. A note on termination in combinations of heterogeneous term rewriting systems. *Bulletin of the European Association for Theoretical Computer Science*, 31:22–28, 1987.

6. B. Gramlich. A structural analysis of modular termination of term rewriting systems. SEKI Report SR-91-15, Dept. of Comp. Science, Univ. of Kaiserslautern, 1991.
7. B. Gramlich. Relating innermost, weak, uniform and modular termination of term rewriting systems. In *Conference on Logic Programming and Automated Reasoning, St. Petersburg*, Lecture Notes in Artificial Intelligence. Springer-Verlag, 1992. to appear.
8. B. Gramlich. Sufficient conditions for modular termination of conditional term rewriting systems (extended abstract). In *Proc. of the 3rd International Workshop on Conditional Term Rewriting Systems*, 1992. to appear.
9. G. Huet and D.C. Oppen. Equations and rewrite rules: A survey. In Ronald V. Book, editor, *Formal Languages, Perspectives And Open Problems*, pages 349–405. Academic Press, 1980.
10. J.W. Klop. Term rewriting systems. In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, volume I. Oxford University Press, 1990.
11. M. Kurihara and I. Kaji. Modular term rewriting systems and the termination. *Information Processing Letters*, 34:1–4, 1990.
12. M. Kurihara and A. Ohuchi. Modularity of simple termination of term rewriting systems. *Journal of IPS, Japan*, 34:632–642, 1990.
13. M. Kurihara and A. Ohuchi. Modularity of simple termination of term rewriting systems with shared constructors. Technical Report SF-36, Hokkaido University, Sapporo, 1990.
14. A. Middeldorp. A sufficient condition for the termination of the direct sum of term rewriting systems. In *Proceedings of the 4th IEEE Symposium on Logic in Computer Science*, pages 396–401, Pacific Grove, 1989.
15. A. Middeldorp. *Modular Properties of Term Rewriting Systems*. PhD thesis, Centre for Mathematics and Computer Science, Amsterdam, 1990.
16. A. Middeldorp and Y. Toyama. Completeness of combinations of constructor systems. In R.V. Book, editor, *Proc. of the 4th Int. Conf. on Rewriting Techniques and Applications*, volume 488 of *Lecture Notes in Computer Science*, pages 174–187. Springer, 1991.
17. M. Rusinowitch. On termination of the direct sum of term rewriting systems. *Information Processing Letters*, 26:65–70, 1987.
18. Y. Toyama. Counterexamples to termination for the direct sum of term rewriting systems. *Information Processing Letters*, 25:141–143, 1987.
19. Y. Toyama. On the Church-Rosser property for the direct sum of term rewriting systems. *Journal of the ACM*, 34(1):128–143, 1987.
20. Y. Toyama, J.W. Klop, and H.P. Barendregt. Termination for the direct sum of left-linear term rewriting systems. In N. Dershowitz, editor, *Proc. of the 3rd Int. Conf. on Rewriting Techniques and Applications*, volume 355 of *Lecture Notes in Computer Science*, pages 477–491. Springer, 1989.