Generalized Sufficient Conditions for Modular Termination of Rewriting

Bernhard Gramlich*

Fachbereich Informatik
Universität Kaiserslautern
Postfach 3049
6750 Kaiserslautern
Germany
gramlich@informatik.uni-kl.de

Abstract. Modular properties of term rewriting systems, i.e. properties which are preserved under disjoint unions, have attracted an increasing attention within the last few years. Whereas confluence is modular this does not hold true in general for termination. By means of a careful analysis of potential counterexamples we prove the following abstract result. Whenever the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two (finitely branching) terminating term rewriting systems $\mathcal{R}_1$, $\mathcal{R}_2$ is non-terminating, then one of the systems, say $\mathcal{R}_1$, enjoys an interesting (undecidable) property, namely it is not termination preserving under non-deterministic collapses, i.e. $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is collapsing, i.e. contains a rule with a variable right hand side. This result generalizes known sufficient criteria for modular termination of rewriting and provides the basis for a couple of derived modularity results. Furthermore, we prove that the minimal rank of potential counterexamples in disjoint unions may be arbitrarily high which shows that interaction of systems in such disjoint unions may be very subtle. Finally, extensions and generalizations of our main results in various directions are discussed. In particular, we show how to generalize the main results to some restricted form of non-disjoint combinations of term rewriting systems, namely for 'combined systems with shared constructors'.

Key words: term rewriting systems, termination, combination, disjoint union, modularity

1 Introduction

The question whether properties of combinations of term rewriting systems (TRSs for short) are inherited from the corresponding properties of the constituent TRSs is of great importance, e.g. in the field of abstract data type specifications. In principle and also for efficiency reasons it is very useful to know

* This research was supported by the 'Deutsche Forschungsgemeinschaft, SFB 314 (D4-Projekt)'.

whether a combined TRS has some property whenever this property already holds for the single 'modules'. A simple and natural way of such 'modular' constructions is given by the concept of 'direct sum' ([29]) or 'disjoint union' which of course covers only a very special kind of combinations. Roughly spoken, the concept of 'direct sum' as defined in [29] is slightly more general than that of 'disjoint union' because it allows for renaming function symbols in order to obtain disjointness. Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case the rule sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $\mathcal{R}_1 \oplus \mathcal{R}_2$. We shall also speak of the disjoint union of $\mathcal{R}_1$ and $\mathcal{R}_2$ using the implicit convention that $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be disjoint TRSs. A property P of TRSs is said to be *modular* if the following holds for all disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property P iff both $\mathcal{R}_1$ and $\mathcal{R}_2$ have property P. Toyama [29] has shown that confluence is modular. The termination property, however, is in general not modular as witnessed by the following counterexample of [29]:

Example 1. $\mathcal{R}_1 : \quad f(a,b,x) \to f(x,x,x) \qquad \mathcal{R}_2 : \quad G(x,y) \to x$
$G(x,y) \to y$

Clearly, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite derivation:

$$\begin{aligned} f(a,b,G(a,b)) &\to_{\mathcal{R}_1} f(G(a,b),G(a,b),G(a,b)) \\ &\to_{\mathcal{R}_2} f(a,G(a,b),G(a,b)) \\ &\to_{\mathcal{R}_2} f(a,b,G(a,b)) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

Note, that in this example $\mathcal{R}_2$ is not confluent. Other more complicated examples by Klop & Barendregt as well as by Toyama gathered in [28] show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ may be non-terminating even if $\mathcal{R}_1$ and $\mathcal{R}_2$ are both terminating, confluent and interreduced. All these counterexamples have some common feature. Namely, one of the systems contains a duplicating rule, i.e. a rule $l \to r$ where some variable occurs strictly more often in r than in l, and the other system contains a collapsing rule $l' \to r'$, i.e. r' is a variable. This observation was exploited by Rusinowitch [26] and Middeldorp [18] (see conditions (a)-(c) below). The counterexamples in [28], involving only confluent systems $\mathcal{R}_1$ and $\mathcal{R}_2$, contain non-left-linear rules and admit critical overlaps strictly below the root which turned out to be essential as shown by Toyama, Klop and Barendregt [30] (see (d) below) and Gramlich [9], respectively (see (e) below). These results may be summarized as follows:

Given two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$, their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating if $\mathcal{R}_1$, $\mathcal{R}_2$ are terminating and one of the following conditions is satisfied:

(a) Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a duplicating rule [26].
(b) Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains a collapsing rule [26].
(c) One of the system $\mathcal{R}_1$, $\mathcal{R}_2$ contains neither collapsing nor duplicating rules [18].

(d) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are left-linear and confluent [30].
(e) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent overlay systems [9].

As discussed in [21] conditions (a)-(c) together with example 1 provide a complete analysis for the termination of the disjoint union of two terminating TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ in terms of the distribution of collapsing and duplicating rules among $\mathcal{R}_1$ and $\mathcal{R}_2$.

Condition (a) above implies that termination is modular for right-linear TRSs, in particular for string rewriting systems. Unfortunately, duplicating and collapsing rules occur quite often and naturally in many cases. Hence, practical applicability of conditions (a)-(c) is rather limited.

Some other interesting modularity properties of TRSs related to normal forms are investigated by Middeldorp in [17]. These results are generalized to (various versions of) conditional TRSs in [19], [20]. In particular, concerning the termination property it turns out that condition (b) above is still sufficient for ensuring termination of the disjoint union of conditional TRSs, but (a) and (c) are shown to hold only under the additional requirement that R_1 and R_2 are confluent ([21]).

Another interesting line of research is pursued by Kurihara & Kaji [14] where modular properties of TRSs w.r.t. a modified reduction relation are investigated. Essentially, this so-called 'modular reduction' requires that, given a disjoint union of several 'module' TRSs, successive reduction steps have to be performed in the same module as long as possible, i.e. until a normal form w.r.t. this module is obtained. Reduction to a normal form in one module is considered to be one step of 'modular reduction'.

Ganzinger & Giegerich [6] consider the termination property in restricted combinations of heterogeneous, i.e. many-sorted TRSs, where the involved signatures do not have to be completely disjoint. The disjointness requirement of combinations of TRSs is also relaxed in recent investigations of Kurihara & Ohuchi [16], Middeldorp & Toyama [23], Middeldorp [22], Ohlebusch [24] and Gramlich [9].

The rest of this paper which is an extended version of [8] based on [7] is structured as follows. In the next section we briefly recall the basic notions, definitions and facts for TRSs needed later on. In section 3 the main results and their applications will be presented and discussed. In section 4 possible extensions and generalizations are developed.

2 Preliminaries

2.1 Basic Notations and Definitions

We briefly recall the basic terminology needed for dealing with TRSs (e.g. [13], [4]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an obvious way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$. In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $V(t)$.

A *context* $C[, \ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ (the symbol '$\uplus$' denotes disjoint set union) where $\Box$ is a new special constant symbol. If $C[, \ldots,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the term obtained from $C[, \ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. For the set $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ of contexts we also write $\mathcal{CON}(\mathcal{F}, \mathcal{V})$. A *non-empty* context is a term from $\mathcal{CON}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ with $t \equiv C[s]$. If in addition $C[] \not\equiv \Box$ then s is a *proper* subterm of t. A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma) := \{x \in \mathcal{V} | \sigma x \not\equiv x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r), denoted by $l \to r$, with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$. This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\to_{\mathcal{R}^{\mathcal{F}}}$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$ and $t \equiv C[\sigma r]$. We also write $\to_{\mathcal{R}}$ or simply $\to$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$, respectively, is clear from the context. The symmetric, transitive and transitive-reflexive closures of $\to$ are denoted by $\leftrightarrow$, $\to^+$ and $\to^*$, respectively.

A TRS $\mathcal{R}$ is *terminating* if $\to$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \to s_2 \to s_3 \to \cdots$. A *partial ordering* $>$ on a set D is a transitive and irreflexive binary relation on D. A partial ordering $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is said to be *monotonic (w.r.t. the term structure)* if it possesses the *replacement property* $s > t \Longrightarrow C[s] > C[t]$ for all $s, t, C[]$. It is *stable (w.r.t. substitutions)* if $s > t \Longrightarrow \sigma s > \sigma t$ for all s, t, σ. A *term ordering on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a monotonic and stable partial ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. A *reduction ordering* is a well-founded term ordering. A term ordering $>$ is said to be a *simplification ordering* if it additionally enjoys the subterm property $C[s] > s$ for any s and any non-empty context $C[]$. For the case that varyadic function symbols are allowed one additionally requires here the so-called 'deletion'-property (cf. Dershowitz [2]). The *homeomorphic embedding relation* $\trianglelefteq$ on terms is recursively defined by $s \equiv f(s_1, \ldots, s_m) \trianglelefteq g(t_1, \ldots, t_n) \equiv t$ if either $s \trianglelefteq t_i$ for some $i \in \{1, \ldots, n\}$ or $f \equiv g$ and $s_j \trianglelefteq t_{i_j}$ for all $j \in \{1, \ldots, m\}$ and some $i_1, \ldots, i_m$ with $1 \le i_1 < i_2 < \cdots < i_m \le n$. A TRS $\mathcal{R}$ is said to be *self-embedding* if there exists a self-embedding $\mathcal{R}$-derivation, i.e. a reduction sequence $s_1 \to s_2 \to s_3 \to \cdots$ with $s_i \trianglelefteq s_j$ for some i, j with $i < j$.

A TRS is confluent if ${}^*\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\leftarrow$ and locally confluent if $\leftarrow \circ \to \subseteq \to^* \circ {}^*\!\leftarrow$ ('$\circ$' denotes relation composition). A confluent and terminating TRS is said to be *convergent* or *complete*.

2.2 Disjoint Unions

The following notations and definitions for dealing with disjoint unions of TRSs mainly follow [29] and [21].

Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRSs with disjoint signatures $\mathcal{F}_1$, $\mathcal{F}_2$. Their *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ is the TRS $(\mathcal{R}_1 \uplus \mathcal{R}_2, \mathcal{F}_1 \uplus \mathcal{F}_2)$. A property $\mathcal{P}$ of TRSs is said to be *modular* if for all disjoint TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ the following holds: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property $\mathcal{P}$ iff both $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ have property $\mathcal{P}$. Note that for proving modularity in most interesting cases the 'only-if' part is trivial, whereas the 'if-part' is the real problem.

Let $t \equiv C[t_1, \ldots, t_n]$, $n \geq 1$, with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C[, \ldots,] \in \mathcal{CON}(\mathcal{F}_a, \mathcal{V})$ and $root(t_1), \ldots, root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. In this case the t_i's are the *principal subterms* or *principal aliens* of t. Note that every $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V}) \setminus (\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}))$ has a unique representation of the form $t \equiv C[\![t_1, \ldots, t_n]\!]$. The set of all *aliens* of t can be recursively defined in an obvious manner.

The *rank* of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ is defined by

$$rank(t) = \begin{cases} 1 & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}) \\ 1 + max\{rank(t_i) | 1 \leq i \leq n\} & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}$$

An important basic fact about the rank of terms occurring in a $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation is the following ([29]): If $s \to^* t$ then $rank(s) \geq rank(t)$. Moreover, if $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ with $rank(s) = n$ then there exists a ground instance σs of s with $rank(\sigma s) = n$, too. This is easily verified by substituting appropriately $\mathcal{F}_1$- or $\mathcal{F}_2$-ground terms for those variables which occur in the 'deepest layer' of s. A (finite or infinite) derivation $D: \ s_1 \to s_2 \to s_3 \ldots$ is said to have rank n $(rank(D) = n)$ if n is the minimal rank of all the s_i's, i.e. $n = \min\{rank(s_i) | 1 \leq i\}$.

The topmost homogeneous part of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$, denoted by $top(t)$, is obtained from t by replacing all principal subterms by $\Box$, i.e.

$$top(t) = \begin{cases} t & \text{if} \quad rank(t) = 1 \\ C[, \ldots,] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}$$

Furthermore (for $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$) we shall sometimes make use of the abbreviations $\mathcal{T}(\mathcal{F})^n$ for $\{t \in \mathcal{T}(\mathcal{F}) | rank(t) = n\}$ and $\mathcal{T}(\mathcal{F})^{\leq n}$ for $\{t \in \mathcal{T}(\mathcal{F}) | rank(t) \leq n\}$. For the sake of better readability the function symbols from $\mathcal{F}_1$ are considered to be black and those of $\mathcal{F}_2$ to be white. Variables have no colour. A *top black (white)* term has a black (white) root symbol.

The one-step reduction $s \to t$ in a disjoint union is said to be *inner* — denoted by $s \xrightarrow{i} t$ — if the reduction takes place in one of the principal subterms of s. Otherwise, we speak of an *outer* reduction step and write $s \xrightarrow{o} t$. A rewrite step $s \to t$ is *destructive at level 1* if the root symbols of s and t have different colours. The step $s \to t$ is *destructive at level* $n+1$ (for $n \geq 1$) if $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \xrightarrow{i} C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level n. Clearly, if a rewrite step is destructive (at some level) then the applied rewrite rule is collapsing, i.e. has a variable right-hand side. This is a basic fact which should be kept in mind subsequently.

Following [21] we introduce some special notations in order to enable a compact treatment of 'degenerate' cases of '$t \equiv C[\![t_1, \ldots, t_n]\!]$'. First the notion of context is extended. We write $C\langle, \ldots, \rangle$ for a term containing zero or more occurrences of $\Box$ and $C\{, \ldots, \}$ denotes a term different from $\Box$ itself, containing zero or more occurrences of $\Box$. If $t_1, \ldots, t_n$ are the (possibly zero) principal subterms of some term t (from left to right), then we write $t \equiv C\{\!\{t_1, \ldots, t_n\}\!\}$ provided $t \equiv C\{t_1, \ldots, t_n\}$. We write $t \equiv C\langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ if $t \equiv C\langle t_1, \ldots, t_n\rangle$ and either $C\langle, \ldots, \rangle \not\equiv \Box$ and $t_1, \ldots, t_n$ are the principal subterms of t or $C\langle, \ldots, \rangle \equiv \Box$ and $t \in \{t_1, \ldots, t_n\}$.

For coding principal subterms, e.g. by new variables or constants, and for dealing with outer rewrite steps involving non-linear rules the following definitions are useful. For terms $s_1, \ldots, s_n, t_1, \ldots, t_n$ we write $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$ if $t_i \equiv t_j$ whenever $s_i \equiv s_j$, for all $1 \leq i < j \leq n$. The following basic properties of outer and inner reduction steps will be freely used in the sequel:

- If $s \xrightarrow{o} t$ then $s \equiv C\{\!\{s_1, \ldots, s_n\}\!\}, t \equiv C'\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$ for some contexts $C\{, \ldots, \}, C'\langle, \ldots, \rangle, i_1, \ldots, i_m \in \{1, \ldots, n\}$ and terms $s_1, \ldots, s_n$. If moreover $s \xrightarrow{o} t$ is not destructive at level 1 then $t \equiv C'\{\!\{s_{i_1}, \ldots, s_{i_m}\}\!\}$.
- If $C\{\!\{s_1, \ldots, s_n\}\!\} \xrightarrow{o} C'\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle, 1 \leq i_j \leq n, j \in \{1, \ldots, m\}$, by application of some rule then $C\{t_1, \ldots, t_n\} \xrightarrow{o} C'\langle t_{i_1}, \ldots, t_{i_m}\rangle$ by the same rule for all terms $t_1, \ldots, t_n$ with $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$.
- If $s \xrightarrow{i} t$ then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!]$ and $t \equiv C[s_1, \ldots, t_j, \ldots, s_n]$ for some context $C[, \ldots,], j \in \{1, \ldots, n\}$ and terms $s_1, \ldots, s_n, t_j$ with $s_j \to t_j$. If moreover $s \xrightarrow{i} t$ is not destructive at level 2 then $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$.

3 Structural Properties of Minimal Counterexamples

3.1 Characterization of Minimal Counterexamples

Before formally stating and proving the main result we shall now illustrate the essential ideas and construction steps via an example from Drosten [5] which shows that termination need not be modular even for confluent TRSs.

Example 2.

$$\mathcal{R}_1 : \quad \begin{aligned} f(a,b,x) &\to f(x,x,x) \\ f(x,y,z) &\to c \\ a &\to c \\ b &\to c \end{aligned} \qquad \mathcal{R}_2 : \quad \begin{aligned} K(x,y,y) &\to x \\ K(y,y,x) &\to x \end{aligned}$$

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating and confluent, but their disjoint union is non-terminating. For instance, we have the following infinite derivation:

$$\begin{array}{rlll} D: & f(a,b,K(a,b,b)) & \xrightarrow{o}_{\mathcal{R}_1} f(K(a,b,b),K(a,b,b),K(a,b,b)) & (1) \\ & & \xrightarrow{i}_{\mathcal{R}_2} f(a,K(a,b,b),K(a,b,b)) & (2) \\ & & \xrightarrow{i}_{\mathcal{R}_1} f(a,K(c,b,b),K(a,b,b)) & (3) \\ & & \xrightarrow{i}_{\mathcal{R}_1} f(a,K(c,c,b),K(a,b,b)) & (4) \\ & & \xrightarrow{i}_{\mathcal{R}_2} f(a,b,K(a,b,b)) & (5) \\ & & \xrightarrow{o}_{\mathcal{R}_1} \cdots & \end{array}$$

Obviously, the crucial steps which enable this derivation to be infinite (and even cyclic) are the inner reductions (2)-(5), in particular the steps (2) and (5) which are destructive at level 2. They modify substantially the topmost homogeneous black [2] layer thereby enabling an outer reduction step previously not possible. The idea now is to abstract from the concrete form of these inner steps but retain the essential information which permits subsequent outer steps. For that purpose it is sufficient to consider the principal top white, i.e. $\mathcal{F}_2$-rooted, aliens and collect those top black, i.e. $\mathcal{F}_1$-rooted, terms to which the former may reduce. In other words, colour changing derivations issued by principal aliens are essential. The coding of the collected top black successors of some principal top white alien will be achieved by some new function symbol(s) which in a sense serve(s) for abstracting from the concrete form of white layers while keeping only the 'layer separating' information. Since in general also top black aliens hidden in deeper layers (cf. subsection 3.4 below) may eventually become principal top black aliens the whole process has to be performed in a recursive fashion in general (which is not necessary in the example). After this abstracting transformation process sequences of inner reduction steps like (2)-(5) above in the original derivation may be simulated by ('deletion' and) 'subterm' steps in the transformed derivation. In order to explain this in more detail let us choose H as a new (varyadic) layer separating function symbol. Then we get the transformed derivation

$$\begin{array}{rlll} D': & f(a,b,H(a,b,c)) & \xrightarrow{o}_{\mathcal{R}_1} f(H(a,b,c),H(a,b,c),H(a,b,c)) & (1') \\ & & \xrightarrow{i}_{\mathcal{R}'_2} f(a,H(a,b,c),H(a,b,c)) & (2') \\ & & \xrightarrow{i}_{\mathcal{R}'_2} f(a,H(b,c),H(a,b,c)) & (3') \\ & & \equiv \quad f(a,H(b,c),H(a,b,c)) & (4') \\ & & \xrightarrow{i}_{\mathcal{R}'_2} f(a,b,H(a,b,c)) & (5') \\ & & \xrightarrow{o}_{\mathcal{R}_1} \cdots & \end{array}$$

where $\mathcal{R}_1$ is as above and $\mathcal{R}'_2 = \mathcal{R}^H_{sub} \cup \mathcal{R}^H_{del}$ with

$$\mathcal{R}^H_{sub} = \{H(x_1,\ldots,x_j,\ldots,x_n) \to x_j | 1 \le j \le n\} \ ,$$
$$\mathcal{R}^H_{del} = \{H(x_1,\ldots,x_j,\ldots,x_n) \to H(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n) | 1 \le j \le n\}.$$

[2] Remember that function symbols from $\mathcal{R}_1$ and $\mathcal{R}_2$ are considered to be black and white, respectively.

The top white principal alien $t := K(a,b,b)$ of the top black starting term $s := f(a,b,K(a,b,b))$ of D can be reduced (in arbitrarily many steps) to the top black successors a, b and c. Hence, the abstracting transformation of t yields $H(a,b,c)$ and the whole starting term s is transformed into $f(a,b,H(a,b,c))$. Furthermore, any outer step in D corresponds to an outer step in D' using the same rule. Any inner step in D which is not destructive at level 2, e.g. (3) and (4), corresponds in D' to a (possibly empty) sequence of inner $\mathcal{R}_2'$-steps not destructive at level 2 (here (3') and (4'), respectively). Any inner step in D which is destructive at level 2 (hence collapsing), e.g. (2) and (5), corresponds in D' to an $\mathcal{R}_{sub}^H$-step (here (2') and (5'), respectively).

In order to stay within the usual scenario of fixed-arity function symbols we modify the above transformation by taking a new binary function symbol G and a new constant A instead of the varyadic symbol H. With the correspondence

$$H(t_1,\ldots,t_n) = \begin{cases} A & \text{if} \quad n = 0 \\ G(t_1, G(t_2, \ldots G(t_{n-1}, G(t_n, A)) \ldots)) & \text{if} \quad n > 0 \end{cases}$$

the above construction easily carries over and we obtain the derivation

$$\begin{array}{lll} D'' : & f(a,b,G(a,G(b,G(c,A)))) & \\ & \xrightarrow{o}_{\mathcal{R}_1} f(G(a,G(b,G(c,A))),G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) & (1'') \\ & \xrightarrow{i}_{\mathcal{R}_2''} f(a,G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) & (2'') \\ & \xrightarrow{i}_{\mathcal{R}_2''} f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))) & (3'') \\ & \equiv \quad f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))) & (4'') \\ & \xrightarrow{i}_{\mathcal{R}_2''} f(a,b,G(a,G(b,G(c,A)))) & (5'') \\ & \xrightarrow{o}_{\mathcal{R}_1} \ldots & \end{array}$$

Here, $\mathcal{R}_2''$ is to be interpreted as $\mathcal{R}_2'' = \mathcal{R}_{sub}^G$ with

$$\mathcal{R}_{sub}^G = \{G(x,y) \to x,\ G(x,y) \to y\},$$

i.e. deletion rules are not necessary any more. In the following formal presentation we shall use the latter transformation.

Definition 1. A TRS $\mathcal{R}$ is said to be *termination preserving under non-deterministic collapses* if termination of $\mathcal{R}$ implies termination of $\mathcal{R} \oplus \{G(x,y) \to x, G(x,y) \to y\}$.

Lemma 2. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two terminating disjoint TRSs such that*

$$D : s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ (involving only ground terms) of minimal rank, i.e. any derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of smaller rank is finite. Then we have:

(a) $rank(D) \geq 3$.
(b) Infinitely many steps in D are outer steps.

(c) Infinitely many steps in D are inner reductions which are destructive at level 2.

Proof. (a) Follows from (c) since whenever $s_i \xrightarrow{i} s_{i+1}$ is destructive at level 2 then $rank(s_i) \geq 3$.

(b) Assume for a proof by contradiction that only finitely many steps in D are outer ones. W.l.o.g. we may further assume that no step in D is an outer one. Hence, for $s_1 \equiv C[\![t_1, \ldots, t_n]\!]$ all reductions in D are inner ones and take place below one of the positions of the t_i's. Since D is infinite we conclude by the pigeon hole principle that at least one of the t_i's initiates an infinite derivation whose rank is smaller than $rank(D)$. But this is a contradiction to the minimality assumption concerning $rank(D)$.

(c) For a proof by contradiction assume w.l.o.g. that no inner step in D is destructive at level 2. Then, with $\widetilde{s_i} := top(s_i)$ any outer step $s_i \xrightarrow{o} s_{i+1}$ in D yields $\widetilde{s_i} \to \widetilde{s_{i+1}}$ using the same rule from $\mathcal{R}_1 \oplus \mathcal{R}_2$ and for every inner step $s_i \xrightarrow{i} s_{i+1}$ we have $\widetilde{s_i} \equiv \widetilde{s_{i+1}}$. Assuming w.l.o.g. that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted, we can conclude by (b) that $\mathcal{R}_1$ is non-terminating which yields a contradiction.

■

Next we formalize the transformation process illustrated above.

Definition 3. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two (finite) terminating disjoint TRSs, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $n \in N$ such that for every $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ there is no infinite $\mathcal{R}$-derivation starting with s. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A, G\})$. Then the $\mathcal{F}_2$- (or *white*) *abstraction* is defined to be the mapping

$$\Phi : \mathcal{T}(\mathcal{F})^{\leq n} \uplus \{t \in \mathcal{T}(\mathcal{F})^{n+1} | root(t) \in \mathcal{F}_1\} \quad \longrightarrow \quad \mathcal{T}(\mathcal{F}_1 \uplus \{A, G\})$$

given by

$$\Phi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1) \\ A & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_2) \\ C[\![\Phi(t_1), \ldots, \Phi(t_m)]\!] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{F}_1 \\ CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{F}_2 \end{cases}$$

with

$$\begin{aligned} SUCC^{\mathcal{F}_1}(t) &:= \{t' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | t \to_{\mathcal{R}}^* t', root(t') \in \mathcal{F}_1\}, \\ \Phi^*(M) &:= \{\Phi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Phi), \\ CONS(\langle\rangle) &:= A, \\ CONS(\langle s_1, \ldots, s_{k+1}\rangle) &:= G(s_1, CONS(\langle s_2, \ldots, s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1, \ldots, s_k\}) &:= \langle s_{\pi(1)}, \ldots, s_{\pi(k)}\rangle, \end{aligned}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

Intuitively, for computing $\Phi(t)$ one proceeds top-down in a recusive fashion. Top black layers are left invariant whereas (for the case of top black t) the principal top white subterms are transformed by computing for every such top white subterm the set of possible top black successors, abstracting the resulting terms recursively, sorting the resulting set of abstracted terms and finally constructing again an ordinary term by means of using the new constant symbol A (for empty arguments sets) and the new binary function symbol G (for non-empty argument sets). The sorting process and the total ordering involved here are due to some proof-technical subtleties which will become clear later on. For illustration let us consider again example 2.

Example *(example 2 continued)*
Here the white abstraction of the s_i's in the original derivation D yields e.g. (using alphabetical sorting)

$$\begin{aligned}
\Phi(s_1) &= \Phi(f(a,b,K(a,b,b))) = f(a,b,\Phi(K(a,b,b))) \\
&= f(a,b,CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(K(a,b,b)))))) \\
&= f(a,b,CONS(SORT(\Phi^*(\{a,b,c\})))) = f(a,b,CONS(SORT(\{a,b,c\}))) \\
&= f(a,b,CONS(\langle a,b,c\rangle)) = f(a,b,G(a,G(b,G(c,A)))) \quad , \\
\Phi(s_3) &= \Phi(f(a,K(a,b,b),K(a,b,b))) = f(a,\Phi(K(a,b,b),\Phi(K(a,b,b)))) \\
&= f(a,CONS(SORT(\Phi^*(\{a,b,c\}))),CONS(SORT(\Phi^*(\{a,b,c\})))) \\
&= f(a,CONS(SORT(\{a,b,c\})),CONS(SORT(\{a,b,c\}))) \\
&= f(a,CONS(\langle a,b,c\rangle),CONS(\langle a,b,c\rangle)) \\
&= f(a,G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) \quad \text{and} \\
\Phi(s_4) &= \Phi(f(a,K(c,b,b),K(a,b,b))) = f(a,\Phi(K(c,b,b),\Phi(K(a,b,b)))) \\
&= f(a,CONS(SORT(\Phi^*(\{b,c\}))),CONS(SORT(\Phi^*(\{a,b,c\})))) \\
&= f(a,CONS(SORT(\{b,c\})),CONS(SORT(\{a,b,c\}))) \\
&= f(a,CONS(\langle b,c\rangle),CONS(\langle a,b,c\rangle)) = f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))).
\end{aligned}$$

Note that the subterm rewrite step $\Phi(s_3) \to \Phi(s_4)$ reducing $G(a,G(b,G(c,A)))$ to $G(b,G(c,A))$ would not have been possible if we had sorted $\{b,c\}$ as $\langle b,c\rangle$ and $\{a,b,c\}$ as $\langle c,b,a\rangle$.

In the following we shall implicitly use the convention that notions like *rank* or *inner* and *outer* reduction steps have to be interpreted w.r.t. some specific disjoint union which is clear from the context.

The next lemmas capture the important properties of the above defined abstracting transformation.

Lemma 4. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Φ be given as in definition 3. Then, Φ is rank decreasing, i.e. for any $s \in dom(\Phi)$ we have $rank(\Phi(s)) \leq rank(s)$.*

Proof. By an easy induction on $rank(s)$ using the definition of Φ. ∎

Reduction steps in $(\mathcal{R}_1 \oplus \mathcal{R}_2)$ can be translated in corresponding (sequences of) reduction steps in $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ as follows.

Lemma 5. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, n and Φ be given as in definition 3. Then, for any $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ and $root(s) \in \mathcal{F}_2$ we have:*

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \to^*_{\mathcal{R}'_2} \Phi(t),$$

where $\mathcal{R}'_2 := \mathcal{R}^G_{sub} = \{G(x,y) \to x, G(x,y) \to y\}$.

Proof. Let s, t be given with $rank(s) \leq n$, $root(s) \in \mathcal{F}_2$ and $s \to t$. We distinguish two cases. For $rank(s) = 1$ we have $s, t \in \mathcal{T}(\mathcal{F}_2)$, hence $\Phi(s) = A = \Phi(t)$ by definition of Φ. If $rank(s) > 1$ then s has the form $s \equiv C[\![s_1, \ldots, s_n]\!]$. If $s \to t$ is destructive at level 1 then we get $t \equiv s_j$ for some $j \in \{1, \ldots, n\}$ with $t \in SUCC^{\mathcal{F}_1}(s)$, hence $\Phi(s) \to^+_{\mathcal{R}'_2} \Phi(t)$ by definition of Φ. Otherwise, t is top white, too, and we obtain $SUCC^{\mathcal{F}_1}(s) \supseteq SUCC^{\mathcal{F}_1}(t)$, hence $\Phi^*(SUCC^{\mathcal{F}_1}(s)) \supseteq \Phi^*(SUCC^{\mathcal{F}_1}(t))$. By definition of $SORT$ and $CONS$, finally, we get $\Phi(s) \to^*_{\mathcal{R}'_2} \Phi(t)$. Note that the sorting process involved here is needed for ensuring that $\Phi(t)$ is homeomorphically embedded in $\Phi(s)$, or more precisely, that $\Phi(t)$ can be obtained from $\Phi(s)$ by applying subterm rules from $\mathcal{R}^G_{sub}$. ■

Lemma 6. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, $\mathcal{R}'_2 = \mathcal{R}^G_{sub} = \{G(x,y) \to x, G(x,y) \to y\}$, n and the (white) $\mathcal{F}_2$-abstraction Φ be given as above. Then, for any $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$ we have:*

(a) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is not destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.

(b) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.

(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(t)$ with all steps not destructive at level 2.

(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^+_{\mathcal{R}'_2} \Phi(t)$ such that exactly one of these steps is destructive at level 2.

Proof. Under the assumptions of the lemma assume that $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ are given with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$.

(a) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is not destructive at level 1 then we have $s \equiv C\{\!\{s_1, \ldots, s_m\}\!\}$, $t \equiv C'\{\!\{s_{i_1}, \ldots, s_{i_k}\}\!\}$, $1 \leq i_j \leq m$, $1 \leq j \leq k$ for some contexts C, C'. By definition of Φ this implies $\Phi(s) = C\{\!\{\Phi(s_1), \ldots, \Phi(s_m)\}\!\}$ and $\Phi(t) = C'\{\!\{\Phi(s_{i_1}), \ldots, \Phi(s_{i_k})\}\!\}$, hence also $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule because of $\langle s_1, \ldots, s_m \rangle \propto \langle \Phi(s_1), \ldots, \Phi(s_m) \rangle$. Clearly, $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ is not destructive at level 1, too.

(b) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is destructive at level 1 then we have $s \equiv C[\![s_1, \ldots, s_m]\!]$, $t \equiv s_j$ for some j with $1 \leq j \leq m$ and some context C. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = \Phi(s_j)$, hence also $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule because of $\langle s_1, \ldots, s_m \rangle \propto \langle \Phi(s_1), \ldots, \Phi(s_m) \rangle$. Clearly, $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ is destructive at level 1, too.

(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then we have $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!]$, $t \equiv C[\![s_1, \ldots, s'_j, \ldots, s_m]\!]$, $s_j \rightarrow_{\mathcal{R}} s'_j$ for some j with $1 \leq j \leq m$ and some context C. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_j), \ldots, \Phi(s_m)]\!]$ and
$\Phi(t) = C[\![\Phi(s_1), \ldots, \Phi(s'_j), \ldots, \Phi(s_m)]\!]$. Since s_j, s'_j are top white, i.e. $\mathcal{F}_2$-rooted, we get $\Phi(s_j) = A = \Phi(s'_j)$ for the case $s_j \in \mathcal{T}(\mathcal{F}_2)$ and $\Phi(s_j) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s_j)))), \Phi(s'_j) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s'_j))))$, otherwise. Since $s_j \rightarrow_{\mathcal{R}} s'_j$ this implies
$SUCC^{\mathcal{F}_1}(s_j) \supseteq SUCC^{\mathcal{F}_1}(s'_j)$, hence $\Phi(s_j) \rightarrow^*_{\mathcal{R}'_2} \Phi(s'_j)$ and also $\Phi(s) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(t)$ with no step destructive at level 2.

(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then we have $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!]$, $t \equiv C[s_1, \ldots, s'_j, \ldots, s_m]$ with $s_j \rightarrow_{\mathcal{R}} s'_j$ colour changing for some j with $1 \leq j \leq m$ and some context C. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_j), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = C[\Phi(s_1), \ldots, \Phi(s'_j), \ldots, \Phi(s_m)]$. Moreover, $s'_j \in SUCC^{\mathcal{F}_1}(s_j)$, hence $\Phi(s) \xrightarrow{i}{}^+_{\mathcal{R}'_2} \Phi(t)$. In this derivation there is exactly one (inner) step which is destructive at level 2, namely the last one.

■

Now we are prepared to state and prove the main result of this section.

Theorem 7. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint (finite) TRSs which are both terminating such that their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then $\mathcal{R}_j$ is not termination preserving under non-deterministic collapses for some $j \in \{1, 2\}$ and the other system $\mathcal{R}_k$, $k \in \{1, 2\} \setminus \{j\}$ is collapsing. Moreover, the minimal rank of counterexamples in $\mathcal{R}_j \oplus \{G(x, y) \rightarrow x, G(x, y) \rightarrow y\}$ is less than or equal to the minimal rank of counterexamples in $\mathcal{R}_1 \oplus \mathcal{R}_2$.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ be given as stated above. We consider a minimal counterexample, i.e. an infinite $\mathcal{R}$-derivation

$$D: \quad s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \quad \ldots$$

of minimal rank, let's say $n+1$. W.l.o.g. we may assume that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted ground terms having *rank* $n+1$. Since the preconditions of definition 3 are satisfied we may apply the white ($\mathcal{F}_2$-) abstraction function Φ to the s_i's. As it will be shown this yields an infinite $\mathcal{R}'$-derivation

$$D': \quad \Phi(s_1) \rightarrow^* \Phi(s_2) \rightarrow^* \Phi(s_3) \rightarrow^* \quad \ldots$$

where $\mathcal{R}' := \mathcal{R}_1 \oplus \mathcal{R}'_2$ with $\mathcal{R}'_2 := \mathcal{R}^G_{sub} = \{G(x, y) \rightarrow x, G(x, y) \rightarrow y\}$.
Using lemma 6 we conclude that for any step $s_j \rightarrow s_{j+1}$ in D we have

$$s_j \xrightarrow{o}_{\mathcal{R}_1} s_{j+1} \Longrightarrow \Phi(s_j) \xrightarrow{o}_{\mathcal{R}_1} \Phi(s_{j+1}) \,,$$
$$s_j \xrightarrow{i}_{\mathcal{R}} s_{j+1} \Longrightarrow \Phi(s_j) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(s_{j+1}).$$

Hence, D' is indeed an $\mathcal{R}'$-derivation. Since according to lemma 2 (b) infinitely many steps in D are outer ones, the derivation D' is infinite, too. But this means that $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses. Moreover, lemma 2 (c) implies that $\mathcal{R}_2$ is collapsing. This can also be inferred more directly by observing that for non-collapsing $\mathcal{R}_2$ the $\mathcal{F}_2$-abstraction of principal subterms of a minimal counterexample always yields the constant A which implies that the transformed infinite derivation is an $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$. Lemma 4 finally implies $rank(D') \leq rank(D)$ which finishes the proof. ■

As immediate consequences of this result we obtain

Corollary 8. *Termination is modular for the class of (finite) TRSs which are termination preserving under non-deterministic collapses.*

Corollary 9. *The disjoint union of two (finite) terminating TRSs is again terminating whenever one of the systems is termination preserving under non-deterministic collapses and non-collapsing.*

As is easily verified the finiteness assumption in definition 3 concerning the TRSs involved can be completely dropped if only non-collapsing TRSs are considered. Hence we also get

Corollary 10. *([26]) Termination is modular for the class of TRSs which are non-collapsing.*

The next result shows that the class of TRSs which are termination preserving under non-deterministic collapses comprises all non-duplicating TRSs.

Lemma 11. *Whenever a TRS is non-duplicating then it is termination preserving under non-deterministic collapses.*

Proof. Let $\mathcal{R}_1$ be a non-duplicating and terminating TRS. Then consider $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 := \mathcal{R}^G_{sub} := \{G(x,y) \to x, G(x,y) \to y\}$. We define the term ordering $>$ on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ by lexicographically combining $\to^+_{\mathcal{R}_1}$ and the ordering $>_G$ which counts occurrences of G as follows: $s >_G t \ :\Longleftrightarrow \ oc(G,s) >_{nat} oc(G,t)$, $s =_G t \ :\Longleftrightarrow \ oc(G,s) =_{nat} oc(G,t)^3$ and $> := lex(>_G, \to^+_{\mathcal{R}_1})$. The form of $\mathcal{R}_2$ and the fact that $\mathcal{R}_1$ is non-duplicating implies $s \to_{\mathcal{R}} t \implies s \geq_G t$ and $s \to_{\mathcal{R}_2} t \implies s >_G t$. Since both $\to^+_{\mathcal{R}_1}$ and $>_G$ are well-founded term orderings the lexicographic combination $>$ is well-founded, too. Moreover, $>$ is monotonic w.r.t. replacement and $> \cap \to^+_{\mathcal{R}_1}$ is stable w.r.t. substitutions. Hence it suffices to show $l > r$ for any rule $l \to r \in \mathcal{R}$. The case $l \to_{\mathcal{R}_2} r$ is trivial. For $l \to_{\mathcal{R}_1} r$ we have $l \geq_G r$, $l \to^+_{\mathcal{R}_1} r$ and hence $l > r$. This shows that $\mathcal{R}$ is terminating, i.e. $\mathcal{R}_1$ is termination preserving under non-deterministic collapses. ■

[3] Here, $oc(f,s)$ yields the number of occurrences of the symbol f in the term s. By $>_{nat}$ and $=_{nat}$ we mean the usual ordering and equality on natural numbers.

The above results constitute a generalization of the main results of [26] and [18] (for finite TRSs).

Theorem 7 corresponds nicely to the intuition that the existence of counterexamples crucially depends on 'non-deterministic collapsing' reduction steps. Hence, example 1 above is in a sense the simplest conceivable counterexample.

On the one side the general result stated in theorem 7 reveals an interesting structural property of potential counterexamples to modularity of termination. On the other side it is still rather abstract. The obvious question arising is which TRSs are indeed termination preserving under non-deterministic collapses. This question will be tackled next. In section 4 we shall show how the finiteness condition concerning the TRSs involved can be weakened.

Given an arbitrary TRS $\mathcal{R}$ it would be desirable to have a method for testing whether $\mathcal{R}$ is termination preserving under non-deterministic collapses. But it turns out that this is an undecidable property in general.

Theorem 12. *The property of TRSs to be termination preserving under non-deterministic collapses is undecidable.*

Proof sketch: This result is an implicit consequence of the proof of the fact that termination is an undecidable property of disjoint unions of terminating TRSs as shown by Middeldorp and Dershowitz (cf. [21]).[4] Roughly spoken the construction proceeds as follows: Given an arbitrary TRS $\mathcal{R}$, another TRS $\mathcal{R}_1$ is constructed by appropriately combining $\mathcal{R}$ with the system $\mathcal{R}_2 := \{f(a,b,x) \to f(x,x,x)\}$ of the introductory example 1 in such a way that $\mathcal{R}_1$ is terminating notwithstanding the fact that $\mathcal{R}$ may be non-terminating. Moreover, choosing $\mathcal{R}_2 := \{G(x,y) \to x,\ G(x,y) \to y\}$, it can be shown that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating if and only if $\mathcal{R}$ is terminating. Since for arbitrary TRSs termination is known to be undecidable (cf. [11]) it follows that the property of TRSs to be termination preserving under non-deterministic collapses is undecidable, too. ∎

3.2 The Increasing Interpretation Method

In order to obtain easily verifiable sufficient conditions for the property of being termination preserving under non-deterministic collapses we shall now use a general method for termination proofs — namely the well-founded mapping method (cf. [12], [3]) — and adapt it to the scenario of disjoint unions.

Let $\mathcal{R}$ be a TRS over some signature $\mathcal{F}$. For proving termination of $\mathcal{R}$ it suffices to exhibit a well-founded partial ordering $>$ on $\mathcal{T}(\mathcal{F})$ satisfying

$$(1) \quad \forall s,t \in \mathcal{T}(\mathcal{F}) : \quad s \to_{\mathcal{R}} t \quad \Longrightarrow \quad s > t\,.$$

[4] Middeldorp states in [21] that this result has been independently obtained by Dershowitz.

The well-founded mapping method suggests to take a well-founded partial ordering $>_D$ on some set D and some termination function $\tau : \mathcal{T}(\mathcal{F}) \longrightarrow D$ for defining $>$ by

$$(2) \quad s > t \quad :\Longleftrightarrow \quad \tau(s) >_D \tau(t) .$$

This method is specialized to the increasing interpretation (cf. [12]) or monotone algebra (cf. [31]) method by taking D to be an $\mathcal{F}$-algebra and τ to be the unique $\mathcal{F}$-homomorphism from $\mathcal{T}(\mathcal{F})$ to D. Then (1) is guaranteed by

$$(3) \quad \forall s,t \in \mathcal{T}(\mathcal{F})\, \forall f \in \mathcal{F} : \quad s > t \quad \Longrightarrow \quad f(\ldots,s,\ldots) > f(\ldots,t,\ldots)$$

and

$$(4) \quad \forall l \to r \in \mathcal{R}\, \forall \sigma,\ \sigma\, \mathcal{T}(\mathcal{F})\text{−ground substitution} : \quad \sigma(l) > \sigma(r) .$$

Let us now consider the scenario where two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are given such that $\mathcal{R}_1$ is terminating. For proving termination of $\mathcal{R}_1 \cup \mathcal{R}_2$ we apply the increasing interpretation method as follows: Choose D to be $\mathcal{T}(\mathcal{F}_1)$ considered as $\mathcal{F}$-algebra $\mathcal{D}$ with $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$-operations are interpreted as in the term algebra $\mathcal{T}(\mathcal{F}_1)$ and every $\mathcal{F}_2$-operation is interpreted in some fixed way in terms of $\mathcal{F}_1$-operations, i.e.

$$f^{\mathcal{D}} := \lambda x_1, \ldots, x_n . f(x_1, \ldots, x_n) \quad \text{for} \quad f \in \mathcal{F}_1$$

and

$$f^{\mathcal{D}} := \lambda x_1, \ldots, x_n . t_f \,, t_f \in \mathcal{T}(\mathcal{F}_1, \{x_1, \ldots, x_n\}) \quad \text{for} \quad f \in \mathcal{F}_2 .$$

Hence, the unique homomorphism $\varphi : \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2) \to \mathcal{D}$ is given by $\varphi(f) = f^{\mathcal{D}}$. As well-founded partial ordering $>_D$ on $D = \mathcal{T}(\mathcal{F}_1)$ we take $>_D := \to^+_{\mathcal{R}_1}$. For this case (3) and (4) specialize to

$$(3') \quad \forall s,t \in \mathcal{T}(\mathcal{F}_1)\, \forall f \in \mathcal{F}_2 : \quad s \to^+_{\mathcal{R}_1} t \quad \Longrightarrow \quad (\varphi f)(\ldots,s,\ldots) \to^+_{\mathcal{R}_1} (\varphi f)(\ldots,t,\ldots)$$

and

$$(4') \quad \forall l \to r \in \mathcal{R}_2\, \forall \sigma, \sigma\, \mathcal{T}(\mathcal{F}_1)\text{−ground substitution} : \quad \varphi(\sigma l) \to^+_{\mathcal{R}_1} \varphi(\sigma r) .$$

Now, it is easily verified that $(3')$ is satisfied whenever φ is a *strict* interpretation for $\mathcal{F}_2$, i.e. for any $f \in \mathcal{F}_2$ we have $V(f(x_1, \ldots, x_n)) \subseteq V(\varphi(f(x_1, \ldots, x_n)))$. For verifying $(4')$ it suffices to show that $\mathcal{R}_2$-rules can be 'simulated' by $\mathcal{R}_1$-rules. To be more precise, we get

Lemma 13. *Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be TRSs such that $\mathcal{R}_1$ is terminating. Moreover, let φ be an interpretation of $(\mathcal{F}_1 \cup \mathcal{F}_2)$-operations in terms of $\mathcal{F}_1$-operations which is the identity on $\mathcal{F}_1$ and which is strict on $\mathcal{F}_2$. Then the union $(\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is terminating, too, provided that for every rule $l \to r \in \mathcal{R}_2$ we have $\varphi(l) \to^+_{\mathcal{R}_1} \varphi(r)$.*

An easy consequence of this result is the following

Corollary 14. *Whenever a TRS $\mathcal{R}^{\mathcal{F}}$ is terminating then $\mathcal{R}^{\mathcal{F}'}$ is terminating, too, for any enriched signature $\mathcal{F}' \supseteq \mathcal{F}$.*

Note that if $\mathcal{F}$ does not contain symbols of arity > 1 and $\mathcal{F}$ contains a symbol of arity > 1 then this result is not a straightforward consequence of lemma 13 (since the required strict interpretations do not exist) but can be easily proved directly.

Of course, the method for proving termination according to the above lemma is rather restricted, because it requires in a sense that $\mathcal{R}_1 \cup \mathcal{R}_2$ terminates for the same reason as $\mathcal{R}_1$ alone. But in particular for the scenario of disjoint unions it is well-suited as we shall see now.

3.3 Derived Criteria for Modularity of Termination

Concrete sufficient criteria for modularity of termination are easily obtained by combining the previous considerations with corollary 8. Firstly, we need

Definition 15. A TRS $\mathcal{R}^{\mathcal{F}}$ is said to be *non-deterministically collapsing* if there exists a term $s[x,y] \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with $x,y \in \mathcal{V}$ such that $s[x,y] \rightarrow^{+} x$ and $s[x,y] \rightarrow^{+} y$, i.e. if some term can be reduced to two distinct variables.

Lemma 16. *If a TRS is non-deterministically collapsing then it is also termination preserving under non-deterministic collapses.*

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$ be a terminating and non-deterministically collapsing TRS. We have to show that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 = \{G(x,y) \rightarrow x,\ G(x,y) \rightarrow y\}$ is terminating. Since $\mathcal{R}_1^{\mathcal{F}_1}$ is non-deterministically collapsing there exists some term $s[x,y] \in \mathcal{T}(\mathcal{F}_1,\mathcal{V})$ with $x,y \in \mathcal{V}$ such that $s[x,y] \rightarrow^{+}_{\mathcal{R}_1} x$ and $s[x,y] \rightarrow^{+}_{\mathcal{R}_1} y$. W.l.o.g. we may further assume that x,y are the only variables appearing in $s[x,y]$. Now we interpret the function symbol G by $\varphi f = \lambda x,y \, . \, s[x,y]$ and simply apply lemma 13 the preconditions of which are satisfied. ∎

Corollary 17. *Termination is modular for the class of (finite) TRSs which are non-deterministically collapsing.*

Next we consider cases where a terminating TRS $\mathcal{R}$ does not necessarily contain collapsing rules but remains terminating when such rules are added.

Definition 18. Let $\mathcal{R}^{\mathcal{F}}$ be a TRS and $f \in \mathcal{F}, \mathcal{F}' \subseteq \mathcal{F}$. Then, $\mathcal{R}^{\mathcal{F}}$ is said to be *f-simply terminating* if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ with $\mathcal{R}^{f}_{sub} = \{f(x_1,\ldots,x_j,\ldots,x_n) \rightarrow x_j \mid 1 \leq j \leq n\}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is *$\mathcal{F}'$-simply terminating* if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{\mathcal{F}'}_{sub}$ (with $\mathcal{R}^{\mathcal{F}'}_{sub} = \bigcup_{f \in \mathcal{F}'} \mathcal{R}^{f}_{sub}$) is terminating. $\mathcal{R}^{\mathcal{F}}$ is *simply terminating* if $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}$-simply terminating, i.e. $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{\mathcal{F}}_{sub}$ is terminating.

Clearly, if a TRS $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}'$-simply terminating for some $\mathcal{F}' \subseteq \mathcal{F}$ then it is terminating, i.e. simple termination implies termination.

As a straightforward consequence of lemma 16 we get

Corollary 19. *If $\mathcal{R}^{\mathcal{F}}$ is f-simply terminating for some $f \in \mathcal{F}$ with $arity(f) \geq 2$ then $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ is (terminating and) termination preserving under non-deterministic collapses.*

Combining this result with corollary 8 we obtain the following.

Corollary 20. *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two (finite) disjoint TRSs with $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1, 2$. Then the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (f_1- and f_2-simply) terminating, too.*

If $\mathcal{R}^{\mathcal{F}}$ is a TRS with $arity(f) \leq 1$ for all $f \in \mathcal{F}$ then $\mathcal{R}^{\mathcal{F}}$ is obviously non-duplicating, hence termination preserving under non-deterministic collapses according to lemma 11. This leads to

Corollary 21. *If $\mathcal{R}^{\mathcal{F}}$ is simply terminating then $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}_{sub}^{\mathcal{F}}$ is (terminating and) termination preserving under non-deterministic collapses.*

which together with corollary 8 implies the following main result from Kurihara & Ohuchi [15]:

Theorem 22. *Simple termination is a modular property of (finite) TRSs.*

The intuition behind the notion of $\mathcal{F}$-simple termination is its close relationship to simplification orderings, an important subclass of reduction orderings which in practice are very often used for termination proofs. Simplification orderings are well-suited for that purpose due to the following result from Dershowitz [1].

Lemma 23. *A (finite) TRS $\mathcal{R}^{\mathcal{F}}$ over some signature $\mathcal{F}$ terminates if there exists a simplification ordering $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $l \succ r$ for every rule $l \rightarrow r \in \mathcal{R}^{\mathcal{F}}$.*

Note that the proof is based on Kruskal's tree theorem which roughly spoken states that any infinite sequence of terms containing only a finite number of distinct symbols is self-embedding. Hence, the requirement in the above (and the following) lemma that $\mathcal{R}$ must be finite can be weakened. It suffices that for any $\mathcal{R}^{\mathcal{F}}$-derivation the terms occurring in it contain only finitely many distinct symbols. This is satisfied e.g. for (possibly infinite) $\mathcal{R}^{\mathcal{F}}$ and finite $\mathcal{F}$, but also for more general cases (cf. [25]).

It is easy to show that simple termination and termination by means of simplification orderings are related as follows.

Lemma 24. *A (finite) TRS $\mathcal{R}^{\mathcal{F}}$ over some signature $\mathcal{F}$ is ($\mathcal{F}-$) simply terminating if and only if there exists a simplification ordering $\succ$ with $l \succ r$ for every rule $l \rightarrow r \in \mathcal{R}^{\mathcal{F}}$.*

Theorem 22 together with the above lemma 24 generalize the well-known observation that common classes of (precedence[5] based) simplification orderings like recursive path orderings or recursive decomposition orderings exhibit a modular behaviour simply by combining the corresponding disjoint precedences.

[5] A precedence is a partial ordering on a set $\mathcal{F}$ of function symbols.

In [15] theorem 22 is directly proved by means of a construction which has some similarity with our approach presented in the last section. Instead of our white (and black) abstraction function Kurihara & Ohuchi define a mapping called 'alien-replacement'[6] which is tailored to some specific finite reduction sequence. Moreover their construction is in a sense incremental, but not rank-decreasing in general. To be more precise, consider some finite derivation

$$D : \quad s_0 \to s_1 \to s_2 \to \ldots \to s_m$$

in $\mathcal{R} := (\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1}) \oplus (\mathcal{R}_2 \cup \mathcal{R}_{sub}^{\mathcal{F}_2} \cup \mathcal{R}_{del}^{\mathcal{F}_2})$ with all s_i's top black and such that every $\mathcal{R}$-derivation starting from any (top white) principal alien of s_0 is finite. Then their 'alien replacement' construction for D essentially consists in (recursively) collecting, for any principal alien occurring in D, all direct descendants occurring in D and abstracting them via a new varyadic function symbol. Using this transformation the $\mathcal{R}$-derivation D can be translated in a one-to-one manner into a corresponding $(\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1})$-derivation from which one can easily infer the modularity of simple termination using the following result which is closely related to lemma 24.

Lemma *([15],[16]) Let $\mathcal{R}^{\mathcal{F}}$ be a TRS. Then there exists a simplification ordering $\succ$ with $l \succ r$ for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$ if and and only if the rewrite relation associated to $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}_{sub}^{\mathcal{F}}(\cup \mathcal{R}_{del}^{\mathcal{F}})$[7] is irreflexive.*

3.4 Minimal Counterexamples of Arbitrary Rank

Besides the features mentioned all counterexamples to modularity of termination presented above and in the literature (cf. [28]) have some more common property. Namely, the rank n of minimal counterexamples always equals 3. According to lemma 2 we must have $n \geq 3$. So, the question naturally arises whether this is a general phenomenon saying that, whenever the disjoint union of two terminating TRSs is non-terminating then there is a counterexamples having rank 3. This question is not only interesting by itself but also because many proofs concerning results on modular termination have to consider 'mixed' terms of arbitrary rank. In particular, the extremely complicated analysis performed in [30] for proving that completeness is modular for left-linear TRSs could be considerably simplified if counterexamples of rank 3 were always possible. Surprisingly this is not the case as illustrated by

Example 3. $\quad \mathcal{R}_1 : \quad f(x, g(x), y) \to f(y, y, y) \qquad \mathcal{R}_2 : \quad G(x) \to x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad G(x) \to A$

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating, but $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. For instance, we have the following infinite $\mathcal{R}$-derivation

[6] cf. [15], [16] for details; in fact, compared to [15], [16] contains a simplified and clarified version of 'alien replacement'.

[7] Deletion rules are necessary if function symbols are allowed to be varyadic.

$$
\begin{aligned}
D: \quad f(G(g(A)),G(g(A)),G(g(A))) &\to_{\mathcal{R}_2} \quad f(A,G(g(A)),G(g(A))) \\
&\to_{\mathcal{R}_2} \quad f(A,g(A),G(g(A))) \\
&\to_{\mathcal{R}_1} \quad f(G(g(A)),G(g(A)),G(g(A))) \\
&\to_{\mathcal{R}_2} \cdots
\end{aligned}
$$

of rank 4. By analyzing for which mixed terms s, t it is possible that $s \to_{\mathcal{R}} t$ and $s \to_{\mathcal{R}} g(t)$ one can show that the minimal rank of a non-terminating $\mathcal{R}$-derivation is exactly 4.

Moreover, example 3 can be easily generalized in order to show that the rank of minimal counterexamples may be arbitrarily high.

Example 4.

$$
\mathcal{R}_1 \ : \ f(x, g(x), \ldots, g^n(x), y) \to f(y, \ldots, y) \qquad \mathcal{R}_2 \ : \ \begin{array}{l} G(x) \to x \\ G(x) \to A \end{array}
$$

Here, f has arity $n+2$ and $g^n(x)$ stands for the n-fold application of g to x. Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. For instance, we have the following infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation[8]

$$
\begin{aligned}
D: \quad f((Gg)^n A, (Gg)^n A, \ldots (Gg)^n A) &\to_{\mathcal{R}_2} f(A, (Gg)^n A, \ldots, (Gg)^n A) \\
&\to_{\mathcal{R}_2} f(A, g(Gg)^{n-1} A, \ldots, (Gg)^n A) \\
&\to_{\mathcal{R}_2} f(A, gA, \ldots, (Gg)^n A) \\
&\ \ \vdots \\
&\to_{\mathcal{R}_2} f(A, gA, g^2 A, \ldots, g^n A, (Gg)^n A) \\
&\to_{\mathcal{R}_1} f((Gg)^n A, (Gg)^n A, \ldots, (Gg)^n A) \\
&\to_{\mathcal{R}_2} \cdots
\end{aligned}
$$

of rank $2n+2$. Again a careful analysis of possible reductions shows that for this example $2n+2$ is the minimal rank of any conceivable non-terminating $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation. Moreover, it is straightforward to modify the above examples in such a way that only finite signatures with function symbols of (uniformly) bounded arities are involved. For instance, one may use a binary f' and the encoding $f'(x_1, f'(x_2, \ldots, f'(x_{n-1}, x_n) \ldots))$ for $f(x_1, \ldots, x_n)$.

Hence, we can conclude that for terminating disjoint TRSs with non-terminating disjoint union minimal counterexamples may have an arbitrarily high rank. This shows that the interaction in disjoint unions of TRSs may be very subtle, in particular concerning termination properties.

4 Extensions and Generalizations

4.1 Non-Self-Embedding Systems

In this subsection we shall consider only finite TRSs in order to simplify the discussion. We have seen that – as a consequence of our main result – simple

[8] The notation used here should be self-explanatory. For example, $(Gg)^2(A)$ stands for $G(g(G(g(A))))$.

termination is modular. In other words, termination of a disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ can be shown by a simplification ordering if and only if this holds already for $\mathcal{R}_1$ and $\mathcal{R}_2$. Now simplification orderings are closely connected to the self-embedding property of TRSs. According to Kruskal's tree theorem the property of being non-self-embedding implies termination. Furthermore, simple termination is sufficient for being non-self-embedding. Hence, a natural question is to ask whether termination is also modular for non-self-embedding systems, or in slightly modified form: Is the property of being non-self-embedding a modular one? Having again a closer look on example 1 with $\mathcal{R}_1 = \{f(a,b,x) \to f(x,x,x)\}$, $\mathcal{R}_2 = \{G(x,y) \to x\ , G(x,y) \to y\}$ it is clear that $\mathcal{R}_1$ is terminating, but cannot be simply terminating because it is self-embedding as witnessed e.g. by the one-step-derivation $f(a,b,f(a,b,b)) \to_{\mathcal{R}_1} f(f(a,b,b),f(a,b,b),f(a,b,b))$. Now consider the following modified version of example 1:

Example 5.

$$\mathcal{R}_1 : \quad \begin{array}{l} f(a,b,x) \to h(x,x,x) \\ h(a,b,x) \to f(x,x,x) \end{array} \qquad \mathcal{R}_2 : \quad \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array}$$

Clearly, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating and even non-self-embedding as can be easily shown, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite (and hence self-embedding) derivation:

$$\begin{array}{rl} f(a,b,G(a,b)) & \to_{\mathcal{R}_1} h(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} h(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} h(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} f(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} f(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} f(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} \cdots \end{array}$$

Thus, we may conclude that termination is not modular in general for non-self-embedding TRSs and that the property of being non-self-embedding is not a modular one. Note, that this reveals a gap between simply terminating and non-self-embedding systems. In fact, every simply terminating TRS is non-self-embedding, but not vice-versa because we have e.g. in $\mathcal{R}_1 \cup \mathcal{R}^f_{sub}$ with $\mathcal{R}_1$ as above:

$$\begin{array}{rl} f(a,b,f(a,b,b)) & \to \ \ h(f(a,b,b),f(a,b,b),f(a,b,b)) \\ & \to^+ h(a,b,f(a,b,b)) \\ & \to \ \ f(f(a,b,b),f(a,b,b),f(a,b,b)) \\ & \to^+ f(a,b,f(a,b,b)) \\ & \to \ \ \cdots \end{array}$$

Hence, both implications

$$\mathcal{R} \text{ simply terminating} \quad \Longrightarrow \quad \mathcal{R} \text{ non-self-embedding} \quad \Longrightarrow \quad \mathcal{R} \text{ terminating}$$

cannot be reversed. This is well-known for the latter one (cf. e.g. Dershowitz [3]) but it is nowhere mentioned in the literature for the first one. Moreover, the gap between non-self-embedding and simply terminating TRSs exists even for TRSs which contain only unary function symbols, hence for string rewriting systems. To this end consider the system $\mathcal{R} := \{g(g(x)) \to h(f(h(x))),\ h(h(x)) \to g(f(g(x)))\}$ over $\mathcal{F} := \{f, g, h\}$. Here, $\mathcal{R}$ is easily shown to be non-self-embedding but it is not (f-)simply terminating because we have for instance the following infinite (cyclic), hence self-embedding derivation in $\mathcal{R} \cup \mathcal{R}^f_{sub}$:

$$g(g(x)) \to h(f(h(x))) \to h(h(x)) \to g(f(g(x))) \to g(g(x)) \to \ \cdots .$$

4.2 Weakening the Finiteness Requirement

Most results presented so far which rely on our main theorem 7 required that the involved TRSs have only finitely many rewrite rules. This assumption can be considerably weakened as it will be shown now. In fact, the reason for this finiteness condition was to ensure well-definedness of the white (or black) abstraction function Φ in definition 3. A closer look at the definition reveals that the essential property needed is that for any mixed (ground) term s of *rank* less than or equal n with n as in the definition, the set of possible successors of s, i.e. $SUCC(s) := \{s' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | s \to^+_{\mathcal{R}} s'\}$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, is finite. For that purpose it is sufficient to require that $\mathcal{R}$ is finitely branching, i.e. for any term $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ the one-step-successor set $\{s' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | s \to_{\mathcal{R}} s'\}$ is finite. In that case one may simply apply Kőnigs lemma. The following result provides a characterization of the property of TRSs to be finitely branching.

Lemma 25. *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ is finitely branching if and only if for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$ there are only finitely many different rules in $\mathcal{R}^{\mathcal{F}}$ with the same left hand side l.* [9]

Proof. Consider an arbitrary ground term s and possible $\mathcal{R}^{\mathcal{F}}$-reductions. Clearly, there are only finitely many different left hand sides of rules in $\mathcal{R}^{\mathcal{F}}$ which can match some subterm of s. Hence, the set of one-step-successors of s can be infinite only in the case that there are infinitely many different rules in $\mathcal{R}^{\mathcal{F}}$ with the same left hand side. The only-if-direction of the lemma is trivial. ∎

Corollary 26. *The property of (possibly infinite) TRSs to be finitely branching is modular.*

Hence, all results basing on our main theorem 7 can be generalized by requiring the involved TRSs to be only finitely branching instead of finite. Note that the signature may still be infinite. This case is only problematic if simplification orderings are used for trying to prove termination. For infinite TRSs over infinite signatures the lemmas 23 and 24 do not hold any more in general because Kruskal's tree theorem can no longer be applied. Hence, if an infinite TRS $\mathcal{R}^{\mathcal{F}}$ over some infinite signature $\mathcal{F}$ can be oriented by some simplification ordering this does not necessarily imply termination of $\mathcal{R}^{\mathcal{F}}$ any more.

[9] Note that rules which can be obtained from one another by renaming variables are considered to be equal.

The restriction to finitely branching TRSs is essential for our basic construction to apply as can be seen from the following example involving a non-finitely branching TRS over some infinite signature.

Example 6. Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be given with $\mathcal{R}_2 = \{H(x, y, y) \to x,\ H(y, x, y) \to x\}$, $\mathcal{F}_2 = \{H, A\}$, $\mathcal{F}_1 = \{f_0, f_1, f_2, \ldots\} \cup \{0, 1, 2, \ldots\} \cup \{\omega\}$ and

$$\begin{array}{lllll}
\mathcal{R}_1 & : & f_0(0,1,x) \to f_1(x,x,x) & & 0 \to 2\ \ 0 \to 4\ \ 0 \to 6\ \ldots \\
& & f_1(2,3,x) \to f_2(x,x,x) & & 1 \to 3\ \ 1 \to 5\ \ 1 \to 7\ \ldots \\
& & f_2(4,5,x) \to f_3(x,x,x) & & 0 \to \omega \\
& & \quad\vdots \quad\ \to \quad \vdots & & 1 \to \omega
\end{array}$$

Here, $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating ($\mathcal{R}_2$ is even confluent) but $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating as can be seen from the infinite $\mathcal{R}$-derivation (note that if we do not insist on confluence of $\mathcal{R}_2$ the example can be simplified by omitting the two $\mathcal{R}_1$-rules involving ω and taking $\mathcal{R}_2 = \{H(x, y) \to x, H(x, y) \to y\}$):

$$\begin{array}{rl}
& f_0(H(0,1,1), H(0,1,1), H(0,1,1)) \to_{\mathcal{R}}^{+} f_0(0, 1, H(0,1,1)) \\
\to_{\mathcal{R}_1} & f_1(H(0,1,1), H(0,1,1), H(0,1,1)) \to_{\mathcal{R}}^{+} f_1(2, 3, H(0,1,1)) \\
\to_{\mathcal{R}_1} & f_2(H(0,1,1), H(0,1,1), H(0,1,1)) \to_{\mathcal{R}}^{+} f_2(4, 5, H(0,1,1)) \\
\to_{\mathcal{R}_1} & f_3(H(0,1,1), H(0,1,1), H(0,1,1)) \to_{\mathcal{R}}^{+} f_3(6, 7, H(0,1,1)) \quad \ldots
\end{array}$$

The infinity of $\mathcal{R}_1$ is essential for the existence of this counterexample because for every finite subset $\mathcal{R}_1'$ of $\mathcal{R}_1$ the disjoint union $\mathcal{R}_1' \oplus \mathcal{R}_2$ is again terminating. The abstracting transformation underlying theorem 7 is not applicable here since it would yield infinite terms. For instance, $H(0, 1, 1)$ has the infinitely many different $\mathcal{F}_1$-rooted successors $\{0, 2, 4, \ldots\} \cup \{1, 3, 5, \ldots\} \cup \{\omega\}$ in $\mathcal{R}$. Hence, the whole transformation process would yield an infinite derivation consisting of infinite terms. Nevertheless, $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses, because for $\mathcal{R}' := \mathcal{R}_1 \oplus \{G(x, y) \to x, G(x, y) \to y\}$ we have e.g.

$$\begin{array}{rl}
& f_0(G(0,1), G(0,1), G(0,1)) \to_{\mathcal{R}'}^{+} f_0(0, 1, G(0,1)) \\
\to_{\mathcal{R}_1} & f_1(G(0,1), G(0,1), G(0,1)) \to_{\mathcal{R}'}^{+} f_1(2, 3, G(0,1)) \\
\to_{\mathcal{R}_1} & f_2(G(0,1), G(0,1), G(0,1)) \to_{\mathcal{R}'}^{+} f_2(4, 5, G(0,1)) \\
\to_{\mathcal{R}_1} & f_3(G(0,1), G(0,1), G(0,1)) \to_{\mathcal{R}'}^{+} f_3(6, 7, G(0,1)) \quad \ldots
\end{array}$$

This means that the conclusion of theorem 7 holds for this example although we cannot apply 7 due to the required finiteness condition.

In fact, we conjecture that it should be possible to completely drop any finiteness assumption in theorem 7 and derived results.

4.3 Weakening the Disjointness Requirement

For practical purposes the invariance of properties of TRSs under non-disjoint unions is very important, too. In general, most interesting properties do not exhibit such an invariant behaviour under arbitrary non-disjoint unions. But for certain restricted variants of combinations some results are known (e.g. [6], [23], [16], [22], [24]). We shall now investigate for which cases our results can be generalized.

4.3.1 Hierarchical Combinations

One natural kind of non-disjoint union of TRSs is a hierarchical combination in the following sense. Let some TRS $\mathcal{R}_1 \subseteq \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ over some signature $\mathcal{F}_1$ and some TRS $\mathcal{R}_2 \subseteq \mathcal{T}(\mathcal{F}_2, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ over the signature $\mathcal{F}_1 \uplus \mathcal{F}_2$ be given. Since the left hand sides of $\mathcal{R}_2$ do not contain $\mathcal{F}_1$-symbols, $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ may be considered as a hierarchical extension of $\mathcal{R}_1$. Now, in general such a hierarchical combination of TRSs clearly does not preserve termination of its constituents but perhaps under some further restrictions.

In fact, for well-known classes of simplification orderings like the recursive path ordering (RPO) (cf. [3]) such an invariance result for the termination property may be achieved. To be more precise, let us assume that $\mathcal{R}_1, \mathcal{R}_2$ are given as above such that termination of both systems can be shown by appropiate RPOs induced by precedences $>_{\mathcal{F}_1}$ and $>_{\mathcal{F}_2 \cup \mathcal{F}_1}$, respectively. Then it is easy to prove (by structural induction according to the definition of the RPO) that $>_{\mathcal{F}_2 \cup \mathcal{F}_1} = >_{\mathcal{F}_2 \cup \mathcal{F}_1} |_{\mathcal{F}_2 \times \mathcal{F}_2}$ may be assumed w.l.o.g., i.e. relations of $>_{\mathcal{F}_2 \cup \mathcal{F}_1}$ involving an $\mathcal{F}_1$-symbol are not really necessary for ensuring termination of $\mathcal{R}_2$. Hence, it follows that the union $\mathcal{R}_1 \cup \mathcal{R}_2$ is terminating, too, simply by taking the RPO induced by the union of the two precedences.

This kind of reasoning should also be possible for other classes of precedence based simplication orderings. An obvious question arising therefrom is whether the termination property is also preserved under such hierarchical combinations under the weaker assumption that termination of $\mathcal{R}_1$ and $\mathcal{R}_2$ can be shown by some simplification ordering. Unfortunately, this is not the case as shown by the following simple

Example 7. Let $\mathcal{R}_1$: $a \to b$ and $\mathcal{R}_2$: $h(x,x) \to h(a,b)$ be given over signatures $\mathcal{F}_1 = \{a,b\}$, $\mathcal{F}_2 = \{h\}$. Then both $\mathcal{R}_1$ and $\mathcal{R}_2$ are simply terminating. This is obvious for $\mathcal{R}_1$ and easy to show for $\mathcal{R}_2$ by considering $\mathcal{R}_2' := \mathcal{R}_2 \cup \mathcal{R}_{sub}^H = \{h(x,x) \to h(a,b), h(x,y) \to x, h(x,y) \to y\}$. But $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ is non-terminating. For instance we have $h(b,b) \to_{\mathcal{R}_2} h(a,b) \to_{\mathcal{R}_1} h(b,b) \to \cdots$.

4.3.2 Non-Disjoint Unions with Common Constructors

In practice the necessity of considering non-disjoint unions of TRSs often comes from the fact that some class of function symbols naturally occurs in several distinct component TRSs. This is for instance the case with constructors.

Definition 27. ([23]) A *constructor system (CS)* is a TRS $\mathcal{R}^{\mathcal{F}}$ with the property that $\mathcal{F}$ can be partitioned into $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ such that every left hand side $f(s_1, \ldots, s_n)$ of a rewrite rule from $\mathcal{R}^{\mathcal{F}}$ satisfies $f \in \mathcal{D}$ and $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.[10] Function symbols in $\mathcal{D}$ are called *defined symbols* and those in $\mathcal{C}$ *constructors*.

[10] This definition of constructor system corresponds to what is usually meant when one speaks of a constructor discipline (for specifying functions).

Middeldorp and Toyama have shown in [23] that completeness is preserved under the union of constructor systems with disjoint sets of defined symbols (and common set of constructor symbols). In fact, a slightly more general result is proved in [23].

Kurihara & Ohuchi ([16]) investigate another notion of combining TRSs with common constructors.

Definition 28. ([16]) A TRS $\mathcal{R}^{\mathcal{F}}$ with a fixed partition of $\mathcal{F}$ into $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ is said to be a *TRS with constructors* provided that for any rule $l \to r \in \mathcal{R}^{\mathcal{F}}$ we have $root(l) \in \mathcal{D}$. Given two TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ with constructors over signatures $\mathcal{F}_1 = \mathcal{C} \uplus \mathcal{D}_1$, $\mathcal{F}_2 = \mathcal{C} \uplus \mathcal{D}_2$, the TRS $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ over the signature $\mathcal{F} = \mathcal{C} \uplus (\mathcal{D}_1 \uplus \mathcal{D}_2)$ is called the *combined system with shared constructors* $\mathcal{C}$.

Of course, every union of constructor systems with disjoint sets of defined symbols (and common set of constructor symbols) is a combined system with shared constructors, but not vice-versa. For combined systems with shared constructors Kurihara & Ohuchi [16] have generalized in a straightforward manner their main result from [15], namely modularity of simple termination.

Our structural analysis of potential counterexamples presented in the last section can also be generalized from the disjoint union case to the case of combined systems with shared constructors. This will be done now. Firstly we have to adapt and extend some terminology from the disjoint union case.

Let us assume in the following that $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ with $\mathcal{F}_1 = \mathcal{C} \uplus \mathcal{D}_1$, $\mathcal{F}_2 = \mathcal{C} \uplus \mathcal{D}_2$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ are finite TRSs with constructors such that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a combined system with shared constructors $\mathcal{C}$. The defined function symbols from $\mathcal{D}_1$ and $\mathcal{D}_2$ are considered to be *black* and *white*, respectively. Constructor symbols (and variables) are considered to be *transparent* and are painted according to the context above the actual position. This means that if a constructor symbol appears at the root of a term then it is considered to be transparent. Otherwise, its colour is (recursively) the same as the colour of its predecessor (in tree representation). A term s is said to be *black* (*white*) if it does not contain white (black) symbols. It is said to be *top black* (resp. *top white*, *top transparent* if its root symbol is black (resp. white, transparent). If s contains symbols from both $\mathcal{D}_1$ and $\mathcal{D}_2$ it is said to be *mixed*. If s is a mixed top black (top white) term then it has a unique representation of the form $s \equiv C[\![s_1, \ldots, s_n]\!]$ with $C[, \ldots,]$ a black (white) context[11] and $root(s_i) \in \mathcal{D}_2$ ($root(s_i) \in \mathcal{D}_1$) for $i = 1, \ldots, n$ where the s_i's are the *principal subterms* or *principal aliens* of s. If s is a top transparent term (containing black or white symbols) we get a unique representation of the form $s \equiv C[\![s_1, \ldots, s_n]\!]$ with $C[, \ldots,]$ a transparent constructor context and $root(s_i) \in \mathcal{D}_1 \cup \mathcal{D}_2$ for $i = 1, \ldots, n$. For the sake of brevity and in order to avoid confusion we shall denote such a constructor context $C[, \ldots,]$ by $\widehat{C}[, \ldots,]$ in the

[11] In order to be consistent with the terminology of black (white) terms the 'hole' $\Box$ used for describing contexts has to be considered as a (transparent) constructor symbol here.

sequel. The *rank* of a term s is defined by

$$rank(s) = \begin{cases} 0 & \text{if} \quad s \in \mathcal{T}(\mathcal{C}, \mathcal{V}) \\ 1 & \text{if} \quad s \in (\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V})) \setminus \mathcal{T}(\mathcal{C}, \mathcal{V}) \\ max\{rank(s_i) | 1 \leq i \leq n\} & \text{if} \quad s \equiv \widehat{C}[\![s_1, \ldots, s_n]\!] \\ 1 + max\{rank(s_i) | 1 \leq i \leq n\} & \text{if} \quad s \equiv C[\![s_1, \ldots, s_n]\!] \, . \end{cases}$$

Note that an outermost transparent constructor context does not contribute to the rank of a term. Reduction steps (with $\mathcal{R}$) are again rank decreasing, i.e. $s \to t$ implies $rank(s) \geq rank(t)$ as it is easily verified. An ($\mathcal{R}$-) derivation $D : s_1 \to s_2 \to \cdots$ is said to have rank n if n is the minimal rank of all the s_i's. For a top black (or top white) term s an $\mathcal{R}$-reduction step $s \to t$ is said to be *inner* and denoted by $s \xrightarrow{i} t$ if it is of the form $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[s_1, \ldots, t_j, \ldots, s_n]$ with $s_j \to t_j$. Otherwise it is an *outer* step which is denoted by $s \xrightarrow{o} t$. For a top transparent term s an $\mathcal{R}$-reduction step $s \to t$ is said to be *inner* and denoted by $s \xrightarrow{i} t$ if it is of the form $s \equiv \widehat{C}[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to \widehat{C}[s_1, \ldots, t_j, \ldots, s_n]$ with $s_j \xrightarrow{i} t_j$. Otherwise it is an *outer* step which is denoted by $s \xrightarrow{o} t$. For a top black (top white) term s an $\mathcal{R}$-reduction step $s \to t$ is *destructive at level* 1 if the root symbols of s and t have different colours, i.e. if $root(s) \in \mathcal{D}_1$, $root(t) \in \mathcal{D}_2 \cup \mathcal{C} \cup \mathcal{V}$ ($root(s) \in \mathcal{D}_2$, $root(t) \in \mathcal{D}_1 \cup \mathcal{C} \cup \mathcal{V}$). The step $s \to t$ is *destructive at level* $n+1$ ($n \geq 1$) if it is of the form $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[s_1, \ldots, t_j, \ldots, s_n]$ with $s_j \to t_j$ destructive at level n. For a top transparent term s a step $s \to t$ is *destructive at level* n if it is of the form $s \equiv \widehat{C}[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to \widehat{C}[s_1, \ldots, t_j, \ldots, s_n]$ with $s_j \to t_j$ destructive at level n. Note that reduction steps which are destructive at a level greater than 1 may essentially change the layer structure of terms and create new reduction possibilities. Note moreover that whenever a step is destructive (at some level) then the applied rule $l \to r$ is collapsing (i.e. $r \in \mathcal{V}$) or its right hand side has a constructor as root symbol (i.e. $root(r) \in \mathcal{C}$). Such a rule $l \to r$ is said to be *constructor-lifting*. A TRS with constructors is said to be *constructor-lifting* if it contains a constructor-lifting rule. The fact that a destructive behaviour of reduction steps may not only be caused by collapsing but also by constructor-lifting rules constitutes a crucial difference to the disjoint union case.[12] For instance, there exist minimal counterexamples of rank 2. A simple example of this kind due to Seifert [27] is obtained by slightly modifying Toyama's basic counterexample as follows.

Example 8.

$$\mathcal{R}_1 : \quad f(a, b, x) \to f(x, x, x) \qquad \mathcal{R}_2 : \quad \begin{array}{l} B \to a \\ B \to b \end{array}$$

with $\mathcal{C} = \{a, b\}$, $\mathcal{D}_1 = \{f\}$, $\mathcal{D}_2 = \{B\}$. Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating but the combined system is non-terminating. Consider e.g. the infinite derivation

$$D : f(a, b, B) \to_{\mathcal{R}_1} f(B, B, B) \to_{\mathcal{R}_2} f(a, B, B) \to_{\mathcal{R}_2} f(a, b, B) \to \cdots$$

[12] This fact has not been fully taken into account in [7] where the presentation is partially incorrect as noticed by R.Seifert [27].

which obviously has rank 2. Note that $\mathcal{R}_2$ is non-collapsing but both its rules are constructor-lifting. This enables the second and third step in D above to be destructive at level 2. Note moreover that whenever a reduction step $s \to t$ is destructive at level 2 then $rank(s) \geq 2$.

Another important observation for combined systems with shared constructors which is implicitly used in the sequel is the following. Consider an infinite (ground) derivation $D: s_1 \to s_2 \to s_3 \cdots$ in a combined system of minimal rank, let's say $n+1$ $(n \geq 1)$, where w.l.o.g. we may assume s_1 to be top black. Then s_2 is also top black with $rank(s_2) = n+1$ or else s_2 is a top transparent term of the form $s_2 = \widehat{C}[\![t_1, \ldots, t_n]\!]$. In that case all aliens t_j, $1 \leq j \leq n$ which are top white have $rank(t_j) \leq n$. But at least one of the aliens t_k, $1 \leq k \leq n$ is top black, has again rank $n+1$ and initiates an infinite derivation that corresponds to a subsequence of D. Hence one may even assume that in every step $s_i \to s_{i+1}$ of D a top black alien of rank $n+1$ of s_i is reduced which (for $i > 1$) in the previous step was modified (then this step was not destructive at level 1) or introduced (then the previous step was destructive at level 1 by application of a constructor-lifting rule).

Now we are prepared for generalizing our structural analysis of minimal counterexamples for the disjoint union case to the case of non-disjoint combinations of TRSs with shared constructors. We shall omit most proofs since they are very similar to the proofs of the corresponding results for the disjoint union case taking into account the above considerations.

Lemma 29. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be terminating TRSs with constructors such that*

$$D: s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in the combined system $\mathcal{R}$ (involving only ground terms) of minimal rank. Then we have:

(a) $rank(D) \geq 2$.
(b) Infinitely many steps in D are outer steps.
(c) Infinitely many steps in D are inner reductions which are destructive at level 2.

Definition 30. Let $\mathcal{R}_1, \mathcal{R}_2$ be (finite) terminating TRSs with constructors over signatures $\mathcal{F}_1 = \mathcal{C} \uplus \mathcal{D}_1$ and $\mathcal{F}_2 = \mathcal{C} \uplus \mathcal{D}_2$, respectively, with $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}_1 \uplus \mathcal{D}_2$ and $n \in N$ such that for every $s \in \mathcal{T}(\mathcal{F})$ with $rank(s) \leq n$ there is no infinite derivation in the combined system $\mathcal{R}$ starting with s. Moreover, let $\mathcal{T}(\mathcal{F})^{\leq n} := \{t \in \mathcal{T}(\mathcal{F}) | rank(t) \leq n\}$, $\mathcal{T}(\mathcal{F})^{n+1}_{\mathcal{D}_1} := \{t \in \mathcal{T}(\mathcal{F}) | rank(t) = n+1, root(t) \in \mathcal{D}_1\} \cup \{t \equiv \widehat{C}[\![t_1, \ldots, t_n]\!] | max\{rank(t_i) | root(t_i) \in \mathcal{D}_1\} = n+1, max\{rank(t_i) | root(t_i) \in \mathcal{D}_2\} \leq n\}$. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A, G\})$. Then the $\mathcal{D}_2$- (or *white*) *abstraction* is defined to be the mapping

$$\Psi : \mathcal{T}(\mathcal{F})^{\leq n} \uplus \mathcal{T}(\mathcal{F})^{n+1}_{\mathcal{D}_1} \longrightarrow \mathcal{T}(\mathcal{F}_1 \uplus \{A, G\})$$

given by

$$\Psi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{C}) \\ \widehat{C}[\![\Psi(t_1), \ldots, \Psi(t_m)]\!] & \text{if} \quad t \equiv \widehat{C}[\![t_1, \ldots, t_m]\!] \\ t & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1), root(t) \in \mathcal{D}_1 \\ C[\![\Psi(t_1), \ldots, \Psi(t_m)]\!] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{D}_1 \\ CONS(SORT(\Psi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], root(t) \in \mathcal{D}_2 \end{cases}$$

with

$$\begin{aligned} SUCC^{\mathcal{F}_1}(t) &:= \{t' \in \mathcal{T}(\mathcal{F}) | t \rightarrow^*_{\mathcal{R}} t', root(t') \in \mathcal{F}_1\}, \\ \Psi^*(M) &:= \{\Psi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Psi), \\ CONS(\langle\rangle) &:= A, \\ CONS(\langle s_1, \ldots, s_{k+1}\rangle) &:= G(s_1, CONS(\langle s_2, \ldots, s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1, \ldots, s_k\}) &:= \langle s_{\pi(1)}, \ldots, s_{\pi(k)}\rangle, \end{aligned}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

In the following we shall implicitly use the convention that notions like *rank* or *inner* and *outer* reduction steps have to be interpreted w.r.t. some specific combined system with shared constructors which should be clear from the context.

Lemma 31. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Ψ be given as in definition 30. Then, Ψ is rank decreasing, i.e. for any $s \in dom(\Psi) = \mathcal{T}(\mathcal{F})^{\leq n} \uplus \mathcal{T}(\mathcal{F})^{n+1}_{\mathcal{D}_1}$ we have $rank(\Psi(s)) \leq rank(s)$.*

Lemma 32. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$, n and Ψ be given as in definition 30. Then, for any $s, t \in \mathcal{T}(\mathcal{F})^{\leq n}$ with $root(s) \in \mathcal{D}_2$ we have:*

$$s \rightarrow_{\mathcal{R}} t \quad \Longrightarrow \quad \Psi(s) \rightarrow^*_{\mathcal{R}'_2} \Psi(t),$$

where $\mathcal{R}'_2 := \mathcal{R}^G_{sub} = \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$.

Lemma 33. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{R}'_2 = \mathcal{R}^G_{sub} = \{G(x,y) \rightarrow x,\, G(x,y) \rightarrow y\}$, n and the $\mathcal{D}_2$-abstraction Ψ be given as above. Then, for any $s, t \in \mathcal{T}(\mathcal{F})$ with $rank(s) \leq n+1$, $root(s) \in \mathcal{D}_1$ and $s \rightarrow_{\mathcal{R}} t$ we have:*

(a) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is not destructive at level 1 then $\Psi(s) \xrightarrow{o}_{\mathcal{R}_1} \Psi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.
(b) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is destructive at level 1 then $\Psi(s) \xrightarrow{o}_{\mathcal{R}_1} \Psi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.
(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then $\Psi(s) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Psi(t)$ with all steps not destructive at level 2.
(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then $\Psi(s) \xrightarrow{i}{}^+_{\mathcal{R}'_2} \Psi(t)$ such that exactly one of these steps is destructive at level 2.

Theorem 34. *Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ be a (finite) combined system with shared constructors $\mathcal{C}$ such that both systems $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating and such that $\mathcal{R}$ is non-terminating. Then $\mathcal{R}_j$ is not termination preserving under non-deterministic collapses for some $j \in \{1,2\}$ and the other system $\mathcal{R}_k$, $k \in \{1,2\} \setminus \{j\}$ is collapsing or constructor-lifting. Moreover, the minimal rank of counterexamples in $\mathcal{R}_j \cup \{G(x,y) \to x, G(x,y) \to y\}$ (considered as a combined system with shared constructors $\mathcal{C}$) is less than or equal to the minimal rank of counterexamples in $\mathcal{R}$.*

Corollary 35. *If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a (finite) combined system with shared constructors such that $\mathcal{R}_1$, $\mathcal{R}_2$ are terminating TRSs which are termination preserving under non-deterministic collapses then $\mathcal{R}$ is terminating, too.*

Corollary 36. *If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a combined system with shared constructors such that $\mathcal{R}_1$, $\mathcal{R}_2$ are terminating and neither collapsing nor constructor-lifting then $\mathcal{R}$ is terminating, too.*

Corollary 37. *If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a (finite) combined system with shared constructors such that $\mathcal{R}_1$, $\mathcal{R}_2$ are terminating and one of them is termination preserving under non-deterministic collapses and neither collapsing nor constructor-lifting then $\mathcal{R}$ is terminating, too.*

Lemma 38. *If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a (finite) combined system with shared constructors such that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-deterministically collapsing then $\mathcal{R}$ is terminating if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating, too.*

Corollary 39. *Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ be a (finite) combined system with shared constructors with $f_1 \in \mathcal{D}_1, f_2 \in \mathcal{D}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1, 2$. Then $\mathcal{R}$ is (f_1- and f_2-simply) terminating, too.*

In [16] Kurihara & Ohuchi have generalized their main result from [15]. In slightly modified form it may be stated as follows.

Theorem 40. *(cf. [16]) A (finite) combined system $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with shared constructors is simply terminating if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are simply terminating.*

Note that this result is not a direct consequence of our results above - in contrast to what has been stated in [7]. The reason is that if we consider some (finite) combined system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$ with $\mathcal{F}_1 = \mathcal{D}_1 \uplus \mathcal{C}$, $\mathcal{F}_2 = \mathcal{D}_2 \uplus \mathcal{C}$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, then simple termination of $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ clearly implies that both $\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{D}_1} \cup \mathcal{R}_{sub}^{\mathcal{C}}$ and $\mathcal{R}_2^{\mathcal{F}_2} \cup \mathcal{R}_{sub}^{\mathcal{D}_2} \cup \mathcal{R}_{sub}^{\mathcal{C}}$ are (terminating and) termination preserving under non-deterministic collapses. But the TRS $(\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{D}_1} \cup \mathcal{R}_{sub}^{\mathcal{C}}) \cup (\mathcal{R}_2^{\mathcal{F}_2} \cup \mathcal{R}_{sub}^{\mathcal{D}_2} \cup \mathcal{R}_{sub}^{\mathcal{C}})$ is (in general) no longer a combined system with shared constructors. Hence, corollary 35 cannot be directly applied. But it is indeed possible to extend our approach to cover such cases, too, namely as follows. We start the analysis for combinations of (finite) TRSs of the form $(\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{F}_1}) \cup (\mathcal{R}_2^{\mathcal{F}_2} \cup \mathcal{R}_{sub}^{\mathcal{F}_2})$, with $\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$ a combined system with shared constructors and modify our basic construction appropiately. To be more, in definition 30, lemma 29 and lemma 33 we replace $\mathcal{R}_1$ by $\mathcal{R}_1^* = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{F}_1}$, $\mathcal{R}_2$ by $\mathcal{R}_2^* = \mathcal{R}_2^{\mathcal{F}_2} \cup \mathcal{R}_{sub}^{\mathcal{F}_2}$ and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ by $\mathcal{R}^* = \mathcal{R}_1^* \cup \mathcal{R}_2^*$ which yields in particular $SUCC^{\mathcal{F}_1}(t) := \{t' \in \mathcal{T}(\mathcal{F}) | t \rightarrow_{\mathcal{R}^*}^* t', root(t') \in \mathcal{F}_1\}$. Then it is easy to verify that lemmas 29 and 33 also hold for $\mathcal{R}_1^*$, $\mathcal{R}_2^*$ and $\mathcal{R}^*$. This is in turn sufficient for a direct proof of theorem 40 as follows. Suppose there is an infinite $\mathcal{R}^*$-derivation of minimal rank

$$D: \quad s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \quad \ldots$$

Using the arguments given between example 8 and lemma 29 one may assume w.l.o.g. that $root(s_i) \in \mathcal{F}_1$ (eliminating the outer transparent contexts). By the modified versions of lemmas 29 and 33 the infinite $\mathcal{R}^*$-derivation D can be translated into an infinite $(\mathcal{R}_1^* \cup \mathcal{R}_{sub}^G)$-derivation. But this contradicts the assumption that $\mathcal{R}_1$ is simply terminating (cf. corollary 21).

Note that the finiteness condition required in theorem 34 and derived results can be weakened by the same line of reasoning as presented in subsection 4.2 for the disjoint union case.

Finally, before concluding, let us mention that our basic approach for analysing minimal counterexamples can also be generalized to the case of conditional (join) TRSs as detailed in [10].

Conclusion

We have presented a structural analysis of minimal counterexamples to modular termination of rewriting. It has been shown that the abstract property of TRSs to be termination preserving under non-deterministic collapses is crucial for the invariance of termination under disjoint unions. Although this property turns out to be undecidable in general it provides the basis for a couple of sufficient criteria for ensuring modularity of termination. The basic ideas and constructions of our approach are also applicable to more general situations. In particular, we have shown how to generalize it for (non-disjoint) unions of TRSs with shared constructors. Moreover, a very simple class of examples has been presented which proves that the minimal rank of non-terminating derivations in disjoint unions of terminating TRSs may be arbitrarily high. This reflects in a sense the very subtle interaction of rewriting in disjoint unions and shows that arbitrarily complicated layer structures may be essential w.r.t. the termination behaviour.

Acknowledgements: I would like to thank J. Avenhaus, K. Madlener, A. Sattler-Klein, J. Steinbach and C.-P. Wirth for useful comments on early versions of this paper. Moreover I am grateful to R. Seifert for revealing some errors in [7]. And finally I would like to thank A. Middeldorp for many fruitful discussions on modularity topics as well as the anonymous referees for some constructive comments.

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