From Hypersequents to Parallel Processes

Francesco A. Genco[1]
(joint work with Agata Ciabattoni
and partly with Federico Aschieri)

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FAKULTÄT FÜR INFORMATIK
Faculty of Informatics

logics Logical Methods in Computer Science

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Proof theory of non-classical logics

Analytic and modular calculi for classes of logics

- Proof search
- Prove meta-logical properties in a constructive way

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(sequents, hypersequents, labelled sequents,
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Embeddings

- Expressiveness relations between formalisms
- Transfer of results (avoiding repetitions and mistakes)

[Wansing, 1998], [Fitting, 2012],
[Goré and Ramanayake, 2012], [Ramanayake 2015, 2016]. . .

Sequent calculus [Gentzen, 1935]

$$\Gamma \Rightarrow \Delta$$

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$$\Gamma \Rightarrow \Delta$$

Some sequent rules:

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (l\wedge)$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (cut)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (lw)$$

The method of structural rules

To obtain analytic and modular calculi

- Fix an analytic base calculus
- Define a translation from axioms
  to (analyticity-preserving) rules
- Obtain a **general systematic framework**

Large classes of logics captured
[Ciabattoni et al., 2008]
[https://www.logic.at/tinc/webaxiomcalc/]

Beyond sequents

Sequents are **simple** and **versatile**

Beyond sequents

Sequents are **simple** and **versatile**

but **not enough** to define modular analytic proof systems **for many interesting logics**

Consider the axioms for **intermediate logics**:

$$
\left\{
\begin{array}{ll}
\neg\neg A \lor \neg A & \textit{Jankov} \\
(A \to B) \lor (B \to A) & \textit{Gödel} \\
A \lor (A \to (B \lor (B \to C))) & Bd_2 \\
\vdots &
\end{array}
\right.
$$

No sequent structural rule can capture these axioms [Ciabattoni et al., 2012, Ann. Pure Appl. Logic]

More structure

The ***linearity axiom*** characterises *Gödel logic*

$$(A \to B) \vee (B \to A)$$

More structure

The ***linearity axiom*** characterises *Gödel logic*

$$(A \to B) \lor (B \to A)$$

We can define structural rules based on the syntax of this axiom using two (simple) generalisations of sequents:

HYPERSEQUENTS		**SYSTEMS OF RULES**
[Mints, 1968] [Pottinger, 1983] [Avron, 1987]	and	[Negri, 2014]

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

Multiset of sequents (interpreted disjunctively)

$$\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

Multiset of sequents (interpreted disjunctively)

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

We can represent the *linearity axiom* as

$$(A \to B) \vee (B \to A)$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

Multiset of sequents (interpreted disjunctively)

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

We can represent the *linearity axiom* as

$$\Rightarrow A \rightarrow B \mid \; \Rightarrow B \rightarrow A$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

Multiset of sequents (interpreted disjunctively)

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We can represent the *linearity axiom* as

$$A \Rightarrow B \mid B \Rightarrow A$$

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Multiset of sequents (interpreted disjunctively)

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

We can represent the *linearity axiom* as

$$A \Rightarrow B \mid B \Rightarrow A$$

and transform this into the rule

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init. \qquad \dfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A) \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\ (EC)$$

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$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{B \Rightarrow B}\ init. \qquad \cfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A) \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{B \Rightarrow B}\ init. \qquad \cfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A) \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\ (EC)$$

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$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B \Rightarrow B}\ init. \qquad \overline{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A) \mid \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid {\color{red}A}, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid {\color{red}A}, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2} \ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init. \qquad \dfrac{}{{\color{red}A} \Rightarrow A}\ init.}{{\color{red}A} \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\ (\to r)}{\Rightarrow A \to B \mid \Rightarrow (A \to B) \lor (B \to A)}\ (\lor r)}{\Rightarrow (A \to B) \lor (B \to A) \mid \Rightarrow (A \to B) \lor (B \to A)}\ (\lor r)}{\Rightarrow (A \to B) \lor (B \to A)}\ (EC)$$

Example of hypersequent derivation

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Systems of rules [Negri, 2014, J. Logic Comput]

Set of rules with:

- **order constraints**
- **shared formula meta-variables**

Very expressive formalism

Systems of rules on labelled sequents (*e.g.*, "$xRy, \Gamma \Rightarrow \Delta, y : A$") capture all normal modal logics formalised by Sahlqvist formulae

Systems of rules

Set of rules with:

- **order constraints**
- **shared formula meta-variables**

We will consider
purely syntactical two-level systems:

$$\frac{\Gamma_1^1 \Rightarrow \Delta_1^1 \;\ldots\; \Gamma_1^{n_1} \Rightarrow \Delta_1^{n_1}}{\Gamma_1 \Rightarrow \Delta_1}\;(top_1) \qquad\qquad \frac{\Gamma_k^1 \Rightarrow \Delta_k^1 \;\ldots\; \Gamma_k^{n_k} \Rightarrow \Delta_k^{n_k}}{\Gamma_k \Rightarrow \Delta_k}\;(top_k)$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\frac{\Gamma \Rightarrow \Delta \qquad\qquad \ldots \qquad\qquad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\;(bottom)$$

Example of system

We represent the axiom $(A \to B) \vee (B \to A)$ as the system

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1)}{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta\end{matrix}} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_2)}{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta\end{matrix}}}{\Gamma \Rightarrow \Delta}\ (ec)$$

Example of system of rules derivation

$$
\dfrac{
\begin{array}{c}
\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\,(com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
}{\Gamma \Rightarrow \Delta}\,(ec)
$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \to B}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \to A}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r)
}{\Rightarrow (A \to B) \vee (B \to A)}\,(ec)
$$

Example of system of rules derivation

$$
\dfrac{
\begin{array}{c}
\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1) \\ \vdots \\ \color{red}{\Gamma \Rightarrow \Delta}
\end{array}
\qquad
\begin{array}{c}
\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1) \\ \vdots \\ \color{red}{\Gamma \Rightarrow \Delta}
\end{array}
}{\Gamma \Rightarrow \Delta}\ (ec)
$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init.}{A \Rightarrow B}\ (com_1)}{\Rightarrow A \to B}\ (\to r)}{\color{red}{\Rightarrow (A \to B) \vee (B \to A)}}\ (\vee r)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ init.}{B \Rightarrow A}\ (com_2)}{\Rightarrow B \to A}\ (\to r)}{\color{red}{\Rightarrow (A \to B) \vee (B \to A)}}\ (\vee r)
}{\Rightarrow (A \to B) \vee (B \to A)}\ (ec)
$$

Example of system of rules derivation

$$\dfrac{\begin{array}{c}\dfrac{B,\Gamma_1 \Rightarrow \Delta_1}{A,\Gamma_1 \Rightarrow \Delta_1}\,(com_1)\\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\dfrac{A,\Gamma_2 \Rightarrow \Delta_2}{B,\Gamma_2 \Rightarrow \Delta_2}\,(com_1)\\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(ec)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \to B}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \to A}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r)$$

$$\dfrac{\Rightarrow (A \to B) \vee (B \to A) \qquad \Rightarrow (A \to B) \vee (B \to A)}{\Rightarrow (A \to B) \vee (B \to A)}\,(ec)$$

Example of system of rules derivation

$$
\dfrac{
\begin{array}{c}
\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{{\color{red}A}, \Gamma_1 \Rightarrow \Delta_1}\;(com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{{\color{red}A}, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\;(com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
}{\Gamma \Rightarrow \Delta}\;(ec)
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{B \Rightarrow B}\;init.
}{{\color{red}A} \Rightarrow B}\;(com_1)
}{\Rightarrow A \to B}\;(\to r)
}{\Rightarrow (A \to B) \vee (B \to A)}\;(\vee r)
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{{\color{red}A} \Rightarrow A}\;init.
}{B \Rightarrow A}\;(com_2)
}{\Rightarrow B \to A}\;(\to r)
}{\Rightarrow (A \to B) \vee (B \to A)}\;(\vee r)
}{\Rightarrow (A \to B) \vee (B \to A)}\;(ec)
$$

Example of system of rules derivation

$$\dfrac{\begin{array}{c}\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1) \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\,(com_1) \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,(ec)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \to B}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \to A}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(ec)$$

Example of system of rules derivation

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\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
}{\Gamma \Rightarrow \Delta}\ (ec)
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{B \Rightarrow B}\ init.
}{A \Rightarrow B}\ (com_1)
}{\Rightarrow A \to B}\ (\to r)
}{\Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)
\qquad
\dfrac{
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}{\Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)
}{\Rightarrow (A \to B) \vee (B \to A)}\ (ec)
$$

A casual resemblance?

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}}{\Gamma \Rightarrow \Delta}\vdots \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}}{\Gamma \Rightarrow \Delta}\vdots}{\Gamma \Rightarrow \Delta}$$

The two formalisms seem to be related

Can we formalise this intuition in its full generality?

Where does this lead to?

The Embedding

[Ciabattoni and Genco. Journal version in preparation.]

Rule translation

- Any hypersequent rule can be rewritten as a two-level system of rules
- Any two-level system of rules can be rewritten as a hypersequent rule

$$\frac{\mathcal{G} \mid \Gamma'_1 \Rightarrow \Delta'_1 \qquad \dots \qquad \mathcal{G} \mid \Gamma'_k \Rightarrow \Delta'_k}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n}$$

$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\Downarrow$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma'_i \Rightarrow \Delta'_i\}_{1 \le i \le k}$

$$\cfrac{\cfrac{\mathcal{S}_1}{\Gamma_1 \Rightarrow \Delta_1} \atop {\vdots \atop \Gamma \Rightarrow \Delta} \qquad \cdots \qquad \cfrac{\mathcal{S}_n}{\Gamma_n \Rightarrow \Delta_n} \atop {\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

Rule translation

- Any hypersequent rule can be rewritten as a two-level system of rules
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$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\Uparrow$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma'_i \Rightarrow \Delta'_i\}_{1 \leq i \leq k}$

$$\cfrac{\cfrac{\cfrac{\mathcal{S}_1}{\Gamma_1 \Rightarrow \Delta_1}}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \cdots \qquad \cfrac{\cfrac{\mathcal{S}_n}{\Gamma_n \Rightarrow \Delta_n}}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

Derivation translation

Any hypersequent derivation can be translated into a two-level systems derivation using corresponding rules, and vice versa

- Individual translations are quite simple and natural
- The order of rule applications is preserved
- The general proof, on the other hand, is complex:
  - non-locality of systems of rules
  - general form of rules

Example of derivation translation

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
\quad
\dfrac{
\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
\quad
\dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid \Rightarrow B \to (A \wedge B)}\,(\to r)
}{\Rightarrow A \to (A \wedge B) \mid \Rightarrow B \to (A \wedge B)}\,(\to r)
}{\Rightarrow A \to (A \wedge B) \mid \Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B)) \mid \Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B))}\,(EC)
$$

$$\Updownarrow$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow A \to (A \wedge B)}\,(\to r)
}{\Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B))}\,(\vee r)
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow B \to (A \wedge B)}\,(\to r)
}{\Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \to (A \wedge B)) \vee (B \to (A \wedge B))}\,(ec)
$$

How far does this go?

The two formalisms are equivalent
w.r.t. intermediate logics

Nonetheless, the embedding does not depend
on the logical rules in an essential way

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w.r.t. intermediate logics

Nonetheless, the embedding does not depend
on the logical rules in an essential way

$\Downarrow$

It can be naturally extended to other calculi
(*e.g.*, the hypersequent calculi for modal logics
in [Kurokawa, 2013][Lahav, 2013][Indrzejczak, 2015])

Applications of the Embedding

Systems of rules made local

By the embedding we can
represent any two-level system of rules in a local form:

$$
\dfrac{\dfrac{\dfrac{B,\Gamma_1 \Rightarrow \Delta_1}{A,\Gamma_1 \Rightarrow \Delta_1}}{\Gamma \Rightarrow \Delta}\vdots \qquad \dfrac{\dfrac{A,\Gamma_2 \Rightarrow \Delta_2}{B,\Gamma_2 \Rightarrow \Delta_2}}{\Gamma \Rightarrow \Delta}\vdots}{\Gamma \Rightarrow \Delta}
\quad \rightsquigarrow \quad
\dfrac{\mathcal{G} \mid B,\Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A,\Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A,\Gamma_1 \Rightarrow \Delta_1 \mid B,\Gamma_2 \Rightarrow \Delta_2}
$$

Systems of rules made analytic

Given **any structural two-level system of rules** we can:

1. Translate the system of rules into a hypersequent rule [embedding]

Systems of rules made analytic

Given **any structural two-level system of rules** we can:

1. Translate the system of rules into a hypersequent rule [embedding]
2. Apply the *completion* procedure [Ciabattoni et al., 2008, LICS]

Systems of rules made analytic

Given **any structural two-level system of rules** we can:

1. Translate the system of rules into a hypersequent rule
   [embedding]
2. Apply the *completion* procedure
   [Ciabattoni et al., 2008, LICS]
3. Translate the result back
   [embedding]

Systems of rules made analytic

Given **any structural two-level system of rules** we can:

1. Translate the system of rules into a hypersequent rule
   [embedding]
2. Apply the *completion* procedure
   [Ciabattoni et al., 2008, LICS]
3. Translate the result back
   [embedding]

To obtain an **analytic** 2**-level system of rules**

Hypersequents made natural

The embedding provides a connection
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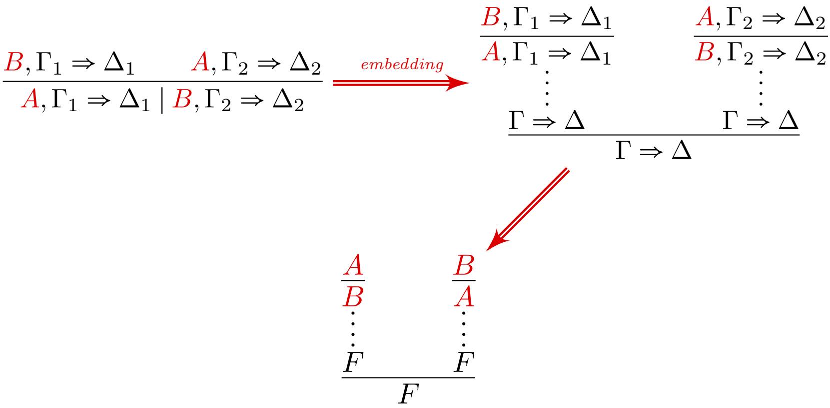

The Computational Meaning

The computational meaning of hypersequents

[Avron, 1991]

Intermediate logics formalised by **hypersequent calculi**
could serve as base for **parallel λ-calculi**

The computational meaning of hypersequents

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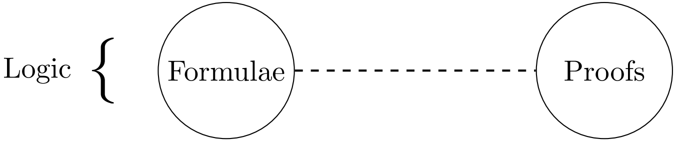
Intermediate logics formalised by **hypersequent calculi** could serve as base for **parallel λ-calculi**

The Problem

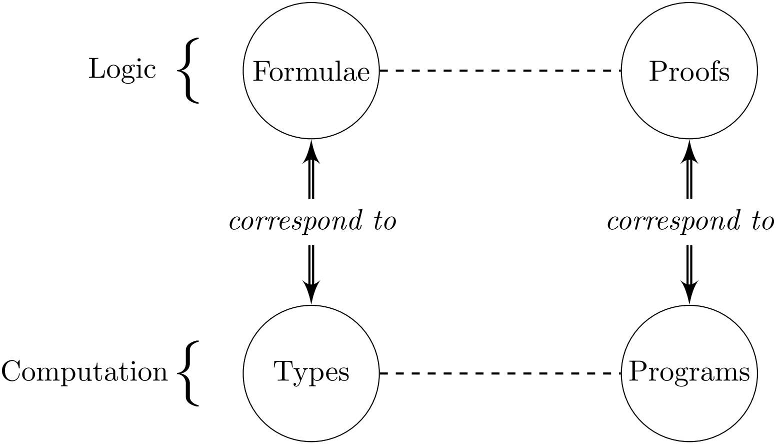
Find **computational interpretations** for these logics

$$\begin{array}{ccccc} \text{Logic} & \Longrightarrow & \text{Hypersequents} & \overset{\textit{Embedding}}{\Longrightarrow} & \text{Systems of Rules} \\ \Uparrow & & & & \Downarrow \\ \lambda\text{-Calculus} & & \overset{\textit{Curry–Howard Correspondence}}{\Longleftarrow} & & \text{N.D.} \end{array}$$

The Curry–Howard correspondence [Howard, 1980]



The Curry–Howard correspondence [Howard, 1980]



The implication rules, for example

$$\frac{t : A \to B \qquad u : A}{tu : B}$$

$$\frac{\begin{array}{c}[x : A] \\ \vdots \\ t : B\end{array}}{\lambda x.t : A \to B}$$

Moving to Gödel logic

The linearity rule

$$\dfrac{\begin{array}{c}\dfrac{A}{B}\\ \vdots\\ F\end{array} \qquad\qquad \begin{array}{c}\dfrac{B}{A}\\ \vdots\\ F\end{array}}{F}$$

Moving to Gödel logic

The linearity rule

$$\dfrac{\begin{matrix}\dfrac{u:A}{eu:B}\\ \vdots \\ s:F\end{matrix} \qquad\qquad \begin{matrix}\dfrac{v:B}{ev:A}\\ \vdots \\ t:F\end{matrix}}{F}$$

Moving to Gödel logic

The linearity rule

$$\dfrac{\begin{array}{c}\dfrac{u : A}{eu : B} \\ \vdots \\ s : F\end{array} \qquad \begin{array}{c}\dfrac{v : B}{ev : A} \\ \vdots \\ t : F\end{array}}{s \parallel_e t : F}$$

Moving to Gödel logic

The linearity rule

$$\dfrac{\dfrac{[e : A \to B] \quad u : A}{eu : B} \qquad \dfrac{[e : B \to A] \quad v : B}{ev : A}}{\dfrac{\begin{array}{c}\vdots \\ s : F\end{array} \qquad \begin{array}{c}\vdots \\ t : F\end{array}}{s \parallel_e t : F}}$$

A derivation, a program

$$\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_1 x : (B \to A) \to F} \qquad [e : B \to A]^2}{(\pi_1 x)e : F}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_0 x : (A \to B) \to F} \qquad [e : A \to B]^2}{(\pi_0 x)e : F} \qquad (\pi_1 x)e : F}{(\pi_0 x)e \parallel_e (\pi_1 x)e : F}\,2}{\lambda x.(\pi_0 x)e \parallel_e (\pi_1 x)e : ((A \to B) \to F) \wedge ((B \to A) \to F) \to F}\,1$$

A derivation, a program

$$\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_1 x : (B \to A) \to F} \qquad [e : B \to A]^2}{(\pi_1 x)e : F}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_0 x : (A \to B) \to F} \qquad [e : A \to B]^2}{(\pi_0 x)e : F} \qquad (\pi_1 x)e : F}{(\pi_0 x)e \parallel_e (\pi_1 x)e : F}\;2}{\lambda x.(\pi_0 x)e \parallel_e (\pi_1 x)e : ((A \to B) \to F) \wedge ((B \to A) \to F) \to F}\;1$$

A derivation, a program

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$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_0 x : (A \to B) \to F} \qquad [e : A \to B]^2}{(\pi_0 x)e : F} \qquad (\pi_1 x)e : F}{(\pi_0 x)e \parallel_e (\pi_1 x)e : F}\ {}^{2}}{\lambda x.(\pi_0 x)e \parallel_e (\pi_1 x)e : ((A \to B) \to F) \wedge ((B \to A) \to F) \to F}\ {}^{1}$$

A derivation, a program

$$\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_1 x : (B \to A) \to F} \qquad [e : B \to A]^2}{(\pi_1 x)e : F}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_0 x : (A \to B) \to F} \qquad [e : A \to B]^2}{(\pi_0 x)e : F} \qquad (\pi_1 x)e : F}{(\pi_0 x)e \parallel_e (\pi_1 x)e : F}\, 2}{\lambda x.(\pi_0 x)e \parallel_e (\pi_1 x)e : ((A \to B) \to F) \wedge ((B \to A) \to F) \to F}\, 1$$

A derivation, a program

$$\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_1 x : (B \to A) \to F} \qquad [e : B \to A]^2}{(\pi_1 x)e : F}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_0 x : (A \to B) \to F} \qquad [e : A \to B]^2}{(\pi_0 x)e : F} \qquad (\pi_1 x)e : F}{(\pi_0 x)e \parallel_e (\pi_1 x)e : F}\ 2}{\color{red}{\lambda x.(\pi_0 x)e \parallel_e (\pi_1 x)e} : ((A \to B) \to F) \wedge ((B \to A) \to F) \to F}\ 1$$

A derivation, a program

$$\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_1 x : (B \to A) \to F} \qquad [e : B \to A]^2}{(\pi_1 x)e : F}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : ((A \to B) \to F) \wedge ((B \to A) \to F)]^1}{\pi_0 x : (A \to B) \to F} \qquad [e : A \to B]^2}{(\pi_0 x)e : F} \qquad (\pi_1 x)e : F}{(\pi_0 x)e \parallel_e (\pi_1 x)e : F}\ {\scriptstyle 2}}{\lambda x.(\pi_0 x)e \parallel_e (\pi_1 x)e : ((A \to B) \to F) \wedge ((B \to A) \to F) \to F}\ {\scriptstyle 1}$$

Normalisation, computation

Normalisation procedure

- removing detours
- subformula property

Proof transformation steps

$$\Downarrow$$

Steps of the **computation**

Implication reduction, for example

$$
\dfrac{\dfrac{\begin{array}{c}\Gamma_0, [x : A]^1 \\ \vdots \\ u : B\end{array}}{\lambda x.u : A \to B}\,1 \qquad \begin{array}{c}\Gamma_1 \\ \mathcal{P} \\ t : A\end{array}}{(\lambda x.u)t : B}
\qquad \mapsto \qquad
\begin{array}{c} \Gamma_1 \\ \mathcal{P} \\ \Gamma_0,\ t : A \\ \vdots \\ u[t/x] : B \end{array}
$$

Cross reductions

$$\mathcal{C}[e\,u] \parallel_e \mathcal{D}[e\,v]$$

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{P}_1\\ \dfrac{A}{B}\\ \vdots\\ F\end{array} \qquad \begin{array}{c}\Delta\\ \mathcal{P}_2\\ \dfrac{B}{A}\\ \vdots\\ F\end{array}}{F}\ e$$

Cross reductions

$$\mathcal{C}[e\,u] \parallel_e \mathcal{D}[e\,v]$$

$$\mathcal{D}[u^{e'\langle\overline{z}\rangle/\overline{y}}] \qquad\qquad \mathcal{C}[v^{e'\langle\overline{y}\rangle/\overline{z}}]$$

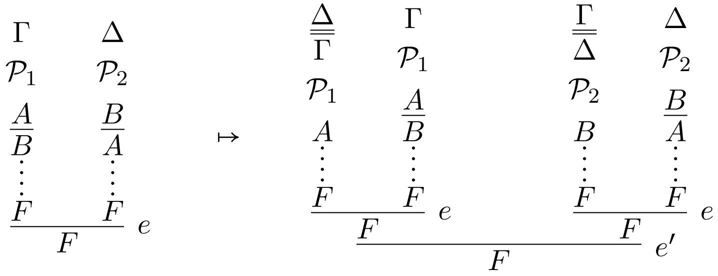
$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{P}_1\\ \dfrac{A}{B}\\ \vdots\\ F\end{array} \qquad \begin{array}{c}\Delta\\ \mathcal{P}_2\\ \dfrac{B}{A}\\ \vdots\\ F\end{array}}{F}\; e \qquad\qquad \begin{array}{c}\Delta\\ =\!=\\ \Gamma\\ \mathcal{P}_1\\ A\\ \vdots\\ F\end{array} \qquad\qquad \begin{array}{c}\Gamma\\ =\!=\\ \Delta\\ \mathcal{P}_2\\ B\\ \vdots\\ F\end{array}$$

$$F \qquad\qquad e'$$

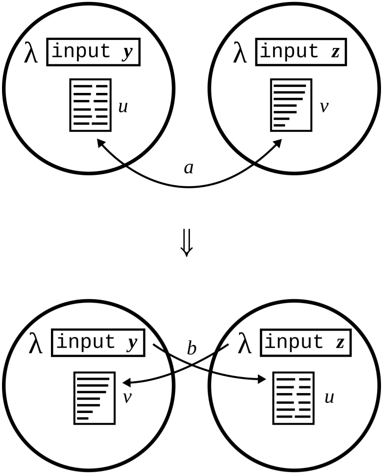
Cross reductions

$$\mathcal{C}[e\,u] \parallel_e \mathcal{D}[e\,v] \qquad \mapsto$$

$$(\mathcal{D}[u^{e'\langle\overline{z}\rangle/\overline{y}}] \parallel_e \mathcal{C}[e\,u]) \parallel_{e'} (\mathcal{C}[v^{e'\langle\overline{y}\rangle/\overline{z}}] \parallel_e \mathcal{D}[e\,v])$$



Communications



Results

Curry–Howard correspondence
for propositional Gödel logic

[Aschieri, Ciabattoni and Genco. Submitted.]

- Normalisation
- Subformula property
- Meaningful computational reductions
  (*e.g.*, in terms of optimisation via code mobility)

Future work

Find other **computational interpretations** of logics formalised by hypersequent calculi

$$x^A : A \qquad \frac{\begin{matrix}[x^A : A] \\ \vdots \\ u : B\end{matrix}}{\lambda x^A u : A \to B} \qquad \frac{t : A \to B \quad u : A}{tu : B}$$

$$\frac{u : A \quad t : B}{\langle u, t\rangle : A \wedge B} \qquad \frac{u : A \wedge B}{u\,\pi_0 : A} \qquad \frac{u : A \wedge B}{u\,\pi_1 : B}$$

$$\frac{\begin{matrix}[a^{A \to B} : A \to B] \\ \vdots \\ u : C\end{matrix} \qquad \begin{matrix}[a^{B \to A} : B \to A] \\ \vdots \\ v : C\end{matrix}}{u \parallel_a v : C}$$

$$\frac{\Gamma \vdash u : \bot}{\Gamma \vdash \mathrm{efq}_P(u) : P} \qquad \text{with } P \text{ atomic, } P \neq \bot.$$

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