Cut-free and Normalised Logic of Proofs

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The old question discussed by Gödel in 1933/38 concerning the intended provability semantics of the classical modal logic **S4** and intuitionistic logic **IPC** was finally settled by the logic of proofs introduced by Artemov (2001). The logic of proofs **Lp** is a natural extension of classical propositional logic by means of proof-carrying formulas. The operations of proofs in the logic of proofs suffice to recover the explicit provability of modal and intuitionistic logic.

In the logic of proofs **Lp** we can prove many results: e.g. the deduction theorem, the substitution lemma and the internalisation of proofs. Moreover, **Lp** can be shown to be sound and complete with respect to the modal logic **S4**, and with respect to Peano Arithmetic.

There also exists a version of **Lp** with an intuitionistic base, namely **Ilp**, introduced in Artemov (2002). Unsurprisingly, analogous results can be obtained in the logic of proofs with an intuitionistic base. Indeed, in **Ilp** too, we can prove the deduction theorem, the substitution lemma and the internalisation of proofs. Moreover, **Ilp** is sound and complete with respect to the modal logic **S4** with an intuitionistic base, and with respect to Heyting Arithmetic.

From a Gentzen-style point of view, we can formulate two similar sequent calculi for the two systems **Lp** and **Ilp**, respectively (see Artemov (2002)). Although simple and cut-free, these sequent calculi fail to satisfy certain properties that are standardly required from a "good" sequent calculus (in Poggiolesi (2010) and Wansing (1998), one can find a precise description of everything that is required from a "good" sequent calculus), namely their logical rules are not symmetric, their contraction rules are not shown to be admissible, and, most importantly, they do not satisfy the subformula property, which is to say they are not analytic calculi.

In this talk we aim at repairing this situation. We will do it in the following way and only for the system **Ilp**. We will firstly introduce a new sequent calculus **Gilp** for the intuitionistic logic of proofs and we will show that this calculus is cut-free, contraction-free and that its rules are symmetric. However **Gilp** does not satisfy the subformula property. In the light of this result, we will analyse the logic of proofs in detail and we will attempt to find the reason for its "resistance" to the analyticity. We will show that this reason is linked to the language of the logic of proofs. By consequence we will change the language of the logic of proofs and we will built a new sequent calculus **Gilp*** based on this new language. We will prove that (a fragment of) **Gilp*** realises any theorem of **Gilp**, and that also the converse holds. Finally, in order to show that in **Gilp*** the analyticity is finally reached, we will show that **Gilp*** is cut-free and that it enjoys a sort of normalisation. We will end the talk by arguing that this technique can also be used for finding an analytic sequent calculus for the system **Lp**.

References

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