On the Structure of Herbrand-Disjunctions

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Herbrand-Disjunctions are a most basic analytic proof system for first-order logic. Non-analytic proof systems, like the sequent calculus with cuts, have the advantage of providing shorter proofs. It is well-known that the increase in proof-length when computing an Herbrand-Disjunction from a proof with cuts can be as high as the hyperexponential function 2_n where $2_0 = 1$ and $2_{n+1} = 2^{2_n}$, see [4].

It is clear that most large Herbrand-Disjunctions cannot be obtained from small proofs with cuts. This observation is an instance of a more general phenomenon that arises whenever we are given a large set (here: the Herbrand-Disjunctions of length at most $2_{P(n)}$ for some fixed polynomial P and n being the logical complexity of the formula) and a small set (here: proofs with cut of length at most $P(n)$) of descriptions of elements of the large set. Other instances of this phenomenon include the fact that a sequence of randomly chosen boolean functions does not possess polynomial-size circuits [1] or that the recursive sets are a measure-zero subset of all sets.

This talk will concentrate on the question which Herbrand-Disjunctions can be obtained from a proof with cuts and which cannot. While it is obvious that, for resulting from a proof with cuts, it is necessary that the Herbrand-Disjunction contains some kind of regularity, it is not clear what kind of regularity that is. I shall present some results (and work in progress) towards a combinatorial charaterization of the structure of Herbrand-Disjunctions arising from cut-elimination.

In particular: it is possible to read off a regular tree grammar (the natural generalisation of regular (string) grammars to trees [2]) G from a proof π s.t. every Herbrand-Disjunction obtainable from π by the standard set of proof reduction rules for cut-elimination is a subset of the (regular tree) language defined by G, see [3].

References

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