Basis of Admissible Rules in the Implication-Negation Fragment of Super-Intuitionistic Logics

Petr Cintula* and George Metcalfe†

Following Lorenzen [7], a rule is said to be *admissible* for a logic (understood as a finitary structural consequence relation) if it can be added to a proof system for the logic without producing any new theorems. While the admissible rules of classical propositional logic CPC are also *derivable* – that is, CPC is *structurally complete* – this is not the case for non-classical (modal, many-valued, substructural, intermediate) logics in general. In particular, the study of admissible rules was stimulated by the discovery of admissible but underivable rules of intuitionistic propositional logic IPC such as the independence of premises rule:

$$\neg p \to (q \vee r) \;/\; (\neg p \to q) \vee (\neg p \to r).$$

The decidability of the set of admissible rules of IPC, posed as an open problem by Friedman in [3], was answered positively by Rybakov, who demonstrated also that this set has no finite *basis* (understood as a set of admissible rules that added to IPC produces all admissible rules) [12]. Nevertheless, following a conjecture by de Jongh and Visser, Iemhoff [6] and Rozière [11] established independently that an infinite basis is formed by the family of "Visser rules" ($n = 2, 3, \ldots$):

$$(\bigwedge_{i=1}^{n}(q_i \to p_i) \to (q_{n+1} \vee q_{n+2})) \vee r \;/\; \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(q_i \to p_i) \to q_j) \vee r.$$

More generally, the work of Rybakov [12] and Ghilardi [4, 5] has led to a reasonably comprehensive understanding of structural completeness and admissible rules for broad classes of intermediate logics. Kripke-frame based characterizations of *hereditarily structurally complete* (i.e., each extension of the logic is structurally complete) intermediate logics have been obtained by Citkin and Rybakov [2, 12].

Hereditary structural completeness for the implicational fragment of IPC was established by Prucnal [10], and the same proof method extends to the implication-conjunction and implication-conjunction-negation fragments [8]. Mints demonstrated hereditary structural completeness for implication-less fragments of IPC and showed moreover that any admissible underivable rule of IPC must contain both implication and disjunction [9]. Curiously, however, as observed by Wroński [13], the *implication-negation fragment* (equivalently, the implication-falsity fragment) – the logic of bounded BCKW-algebras – is not structurally complete. Consider, e.g., the following rule:

$$((\neg\neg p \to p) \to r), ((\neg\neg q \to q) \to r), (p \to \neg q) \;/\; r.$$

This rule is not derivable in IPC and therefore not in any of its fragments. The rule is also not admissible in IPC. However, it *is* admissible in the implication-negation fragment of this logic.

Hence the questions arise: Do there exist other admissible underivable rules for this fragment of a similar or quite different form? Do these admissible rules admit an elegant finite or infinite basis? Do they form a decidable set and if so, what is its complexity? What is the unification type of this fragment. The paper [1] answers these questions as follows:

*Institute of Computer Science, Academy of Sciences of the Czech Republic, Czech Republic, cintula@cs.cas.cz
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†Mathematics Institute, University of Bern, Switzerland, george.metcalfe@math.unibe.ch

- Elegant bases consisting of uniform infinite sequences of rules, similar to Wroński's example, are provided for the single and multiple conclusion admissible rules of the implication-negation fragment not only of IPC but of any intermediate logic.
- Kripke frame characterization is given of the (hereditarily) structurally complete intermediate logics with respect to the implication-negation fragment and used to show the lack of a finite basis for this fragment of IPC.
- It is shown that the unification type of the implication-negation fragment of any non-classical intermediate logic is finitary.

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