Combining supervaluation and degree based reasoning under vagueness

Christian G. Fermüller and Robert Kosik

Technische Universität Wien, Austria

Abstract. Two popular approaches to formalize adequate reasoning with vague propositions are usually deemed incompatible: On the one hand, there is supervaluation with respect to precisification spaces, which consist in collections of classical interpretations that represent admissible ways of making vague atomic statements precise. On the other hand, t-norm based fuzzy logics model truth functional reasoning, where reals in the unit interval $[0,1]$ are interpreted as degrees of truth. We show that both types of reasoning can be combined within a single logic **SŁ**, that extends both: Łukasiewicz logic **Ł** and (classical) **S5**, where the modality corresponds to '... is true in all complete precisifications'. Our main result consists in a game theoretic interpretation of **SŁ**, building on ideas already introduced by Robin Giles in the 1970s to obtain a characterization of **Ł** in terms of a Lorenzen style dialogue game combined with bets on the results of binary experiments that may show dispersion. In our case the experiments are replaced by random evaluations with respect to a given probability distribution over permissible precisifications.

1 Introduction

Providing adequate logical calculi for systematic reasoning about vague information is a major challenge in the intersection of logic, AI, and computer science. Many different models of reasoning with vague notions are on the market. In fact, the literature on so-called theories of vagueness is almost unsurmountable large and still fast growing. (We refer to the book [20], the reader [21], and the more recent collection [1] for further references.) Still, one can single out two quite different approaches as particularly popular—albeit popular in different communities. On the one hand, there is *fuzzy logic* 'in Zadeh's narrow sense' (see, e.g., [15, 17]) focusing on the study of truth functional logics, based on a (potential) continuum of degrees of truth, usually identified with the real closed unit interval $[0,1]$. On the other hand, there is the concept of *supervaluation* (see, e.g., [11, 20, 27]), which maintains that vague statements have to be evaluated with respect to all their admissible precisifications. The slogan 'truth is supertruth' in the latter context entails the thesis that a logical formula built up from vague atomic propositions is true if and only if it is true in each of its (classical) precisifications. Whereas this is often understood as a vindication of classical logic (even) in contexts of vague information[1], all degree based fuzzy logics agree upon the rejection of the logical

[1] One should mention that the extent to which supervaluation leads to classical validity and consequence relations is hotly debated. See, in particular [22] for proofs that compactness,

validity of *tertium non datur* ($A \vee \neg A$). Consequently it is not surprising that supervaluation and degree based reasoning, respectively, are deemed fundamentally incompatible.

We think that a formal assessment, relating the scope and limits of both types of logical reasoning with each other, is essential for judging their adequateness for applications in information processing (as argued in [9]). As a first step towards such an evaluation we seek to identify a common formal framework for degree based logics and supervaluation. The main purpose of this paper is to show that supervaluation as well as t-norm based fuzzy logics can be interpreted as referring to classical precisifications at different levels of formula evaluation. The resulting semantic framework allows to *combine* both forms of reasoning within a single logic. The main tool for achieving an appropriate mathematical analysis of corresponding analytic reasoning is borrowed from the dialogue based characterization of infinite-valued Łukasiewicz logic developed by Robin Giles in the 1970s [13, 14]. In particular, our extension of Giles's game for **Ł** will lead us to a tableau style calculus for the evaluation of formulas over precisification spaces.

We like to emphasize that it is *not* the purpose of this work to (just) introduce yet another modal extension of a particular fuzzy logic. Rather we seek to derive an adequate logic for the combination of supervaluation and degree based reasoning from first principles about the formalization of vague notions and propositions. Still, a short comparison with similar extensions of fuzzy logics will be presented in Section 5.

We also point out that throughout the paper we only deal with propositional logics.

2 Supervaluation, Sorites, and t-norm based fuzzy logics

The use of supervaluation to obtain a semantics for languages that accommodate (also) *vague* propositions was introduced by Kit Fine in [11] and has remained an important point of reference for investigations into logic and vagueness ever since (see, e.g., [20, 27, 28, 9]. The main idea is to evaluate propositions not simply with respect to classical interpretations—i.e., assignments of the truth values 0 ('false') and 1 ('true') to atomic statements—but rather with respect to a whole *space* Π of (possibly) partial interpretations. For every partial interpretation I in Π, Π is required to contain also a classical interpretation I' that extends I. I' is called an *admissible (complete) precisification* of I. A proposition is called *supertrue* in Π if it evaluates to 1 in all admissible precisifications, i.e., in all classical (i.e., complete) interpretations contained in Π.

Example 1. To illustrate supervaluation let us briefly describe how the famous *Sorites paradox* (see, e.g, [20, 28, 3]) is solved in this context. Suppose that h_i stands for the proposition "i (properly arranged) sand-corns make a heap of sand". Let us further agree that h_1 is false (i.e., a single sand-corn is not a heap of sand) but that h_{10000} is true. The paradox consists in the fact that also the proposition $h_i \supset h_{i-1}$—read: "If i sand-corns make a heap then also $i-1$ sand-corns make a heap"—seems to be true for each $i > 1$. However, from these implicative propositions and h_{10000} we can derive h_1 using modus ponens only. In other words, classical logic is at variance with the above mentioned,

upwards and downwards Löwenheim-Skolem, and recursive axiomatizability fail for 'natural' supervaluation based consequence relations.

seemingly innocent intuitions. Using supervaluation we can easily accommodate the intuition that h_1 is definitely false while h_{10000} is definitely true by assigning 0 to h_1 and 1 to h_{10000} in each admissible precisification in a corresponding space Π. On the other hand, at least one statement h_i, where $1 < i < 10000$ is taken to be *vague*, i.e., neither definitely false nor definitely true. This means that for some i the space Π contains admissible precisifications I and I' such that h_i evaluates to 0 in I, but to 1 in I'. Assuming further that h_j is true in an admissible precisification, whenever already h_i is true there for some $i < j$, we obtain that there is an i such that $h_i \supset h_{i-1}$ is not true in all interpretations contained in Π. In other words: while h_{10000} is supertrue and h_1 is superfalse, at least one statement of the form $h_i \supset h_{i-1}$ is neither supertrue nor superfalse.

Note that the inference from h_{10000} to h_1 is blocked since the conclusion of modus ponens is guaranteed to be supertrue only if both premises are supertrue. In fact, supervaluationists like to identify truth with supertruth and thus feel justified in claiming to have 'saved' classical logic also in context of vague propositions. (See, e.g., [20].)

Note, that no reference to (strictly) partial interpretations is needed to determine which propositions are supertrue. The partial interpretations represent additional information that is used to model the semantics of modal operators like 'definitely' or 'indefinitely'. However, we will not investigate such operators here and thus may simplify the notion of a space Π by assuming that Π contains admissible complete precisifications, i.e., classical interpretations only. We will use the term 'precisification space' henceforth for such structures.

One complaint about the above analysis of the Sorites paradox focuses on the fact that we seem to have good reasons to insist that 'taking away one sand-corn from a heap does not result in a non-heap' formalized as $h_i \supset h_{i-1}$ is, if not simply true, at least *almost true* for all $i > 1$. Supervaluation itself does not accommodate this intuition. In contrast, *fuzzy logics* 'in Zadeh's narrow sense' are often claimed to solve the Sorites paradox while respecting all mentioned intuitions. Indeed, in fuzzy logics one may assign an intermediary truth value, close to 1 to all instances of $h_i \supset h_{i-1}$. Using a properly generalized (from $\{0,1\}$ to $[0,1]$) truth function for implication and generalized modus ponens, respectively, one may still block the inference from h_{10000} to h_1, even if h_{10000} is interpreted as definitely true (1) and h_0 as as definitely false 0. (For a detailed analysis of Sorites in the context of t-norm based fuzzy logics we refer to [18].)

Supervaluation and fuzzy logics can be viewed as capturing contrasting, but individually coherent intuitions about the role of logical connectives in vague statements. Consider a sentence like

(*) "The sky is blue and is not blue".

When formalized as $b\&\neg b$, (*) is superfalse in all precisification spaces, since either b or $\neg b$ is evaluated to 0 in each precisification. This fits Kit Fine's motivation in [11] to capture 'penumbral connections' that prevent any mono-colored object from having two colors at the same time. According to Fine's intuition the statement "The sky is blue" absolutely contradicts the statement "The sky is not blue", even if neither statement is definitely true or definitely false. Consequently (*) is judged as definitely false, although admittedly composed of vague sub-statements. On the other hand, by asserting (*) one may intend to convey the information that both component statements are true *only to* *some degree*, different from 1 but also from 0. Under this reading and certain 'natural' choices of truth functions for & and $\neg$ the statement $b\&\neg b$ is *not* definitely false, but receives some intermediary truth value.

We are motivated by the fact that, although supervaluation is usually deemed incompatible with fuzzy logics, one may (and should) uncover a substantial amount of common ground between both approaches to reasoning under vagueness. This common ground becomes visible if one relates the (in general intermediary) truth value of an atomic proposition p as stipulated in fuzzy logics to the 'density' of those interpretations in a precisification space Π that assign 1 to p.

Example 2. Let $h_1,\ldots,h_{10000}$ be as in Example 1 and let these h_i be the only atomic propositions taken into consideration. We define a corresponding precisification space Π as follows: Π consists in the set of all classical interpretations I, that fulfill the following conditions, which model 'penumbral connections' in the sense of [11]. (We write $I(p)$ for the value $\in \{0,1\}$ that is assigned to proposition p in I).

1. $I(h_1) = 0$ and $I(h_{10000}) = 1$
2. $i \leq j$ implies $I(h_i) \leq I(h_j)$ for all $i, j \in \{1,\ldots,10000\}$

The first condition makes h_1 superfalse and h_{10000} supertrue in Π. The second condition captures the assumption that, if some precisification declares i sand-corns to form a heap, then, for all $j \geq i$, j sand-corns also form a heap under the same precisification. Note that supervaluation leaves the semantic status of all statements $h_i \supset h_{i-1}$, where $1 < i \leq 10000$, undecided. However, we can observe that $I(h_i \supset h_{i-1}) = 1$ in *all but one* of the, in total, 99999 interpretations I in Π, whenever $1 < i \leq 10000$. It is thus tempting to say that Π itself (but not supervaluation!) respects the intuition that $h_i \supset h_{i-1}$—informally read as "taking away one sand-corn from a heap still leaves a heap"—is (at least) 'almost true'. Once one accepts the idea that truth may come in degrees, it seems natural to identify what could be called the 'global truth value of h_i with respect to Π' with the fraction of admissible precisifications $I \in S$ where $I(h_i) = 1$. We thus obtain $\frac{i-1}{99999}$ as global truth value of h_i, here.

Following this example we will use *global truth values* $\in [0,1]$ to make information explicit that is implicit in precisification spaces, but is not used in supervaluation. A simple way to extract a global truth value for an atomic proposition p from a given precisification space Π is suggested by Example 2: just divide the number of interpretations I in Π that assign 1 to p by the total number of interpretations in Π. This is feasible if Π is represented by a finite set or multiset of interpretations. (For related ideas underlying the so-called 'voting semantics' of fuzzy logics, we refer to [26, 12].) More generally, since we view the interpretations in Π as corresponding to different ways of making all atomic propositions precise, it seems natural not just to count those precisifications, but to endow Π with a probability measure μ on the σ-algebra formed by all subsets of precisifications in Π, where μ is intended to represent the relative plausibility (or 'frequency in non-deterministic evaluations') of different precisifications. Suppose, e.g., that in refining Example 2 we want to model the intuition that a 'cut-off' point n between heaps and non-heaps—i.e., an n where $I(h_n) \neq I(h_{n+1})$—is more plausibly assumed to be near $n = 100$ than near $n = 9500$. Then we may take

$\mu(\mathscr{J}_{\sim 100})$ to be higher than $\mu(\mathscr{J}_{\sim 9500})$, where $\mathscr{J}_{\sim n}$ denotes the set of all interpretations I where the 'cut-off' point is near n, in the sense that $I(h_{n-c}) = 0$ but $I(h_{n+c}) = 1$ for some fixed smallish c, say $c = 10$.

Note that if we insist on *truth functional* semantics, then we cannot simply extend the above method for extracting truth values from Π from atomic propositions to logically complex propositions. E.g., in general, the fraction of interpretations I in a finite precisification space Π for which $I(p\&q) = 1$ is not uniquely determined by the fractions of interpretations that assign 1 to p and q, respectively.

Obviously, the question arises which truth functions should be used for the basic logical connectives. For this we follow Hájek (and many others) in making the following 'design choices' (see, e.g., [15, 17]):

1. The truth function for conjunction is a continuous, commutative, associative, and monotonically non-decreasing function $* : [0,1]^2 \mapsto [0,1]$, where $0 * x = 0$ as well as $1 * x = x$. In other words: $*$ is a continuous t-norm.
2. The residuum $\Rightarrow_*$ of the t-norm $*$—i.e., the unique function $\Rightarrow_*$: $[0,1]^2 \mapsto [0,1]$ satisfying $x \Rightarrow_* y = \sup\{z \mid x * z \leq y\}$—serves as the truth function for implication.
3. The truth function for negation is defined as $\lambda x[x \Rightarrow_* 0]$.

Given a continuous t-norm $*$ with residuum $\Rightarrow_*$, one obtains a fuzzy logic $\mathbf{L}(*)$ based on a language with binary connectives $\supset$ (implication), & (strong conjunction), constant $\bot$ (falsum), and defined connectives $\neg A =_{def} A \supset \bot$, $A \wedge B =_{def} A\&(A \supset B)$, $A \vee B =_{def} ((A \supset B) \supset B) \wedge ((B \supset A) \supset A)$ (negation, weak conjunction and disjunction, respectively) as follows. A *valuation* for $\mathbf{L}(*)$ is a function v assigning to each propositional variable a truth value from the real unit interval $[0,1]$, uniquely extended to v^* for formulas by:

$$v^*(A\&B) = v^*(A) * v^*(B), \;\; v^*(A \supset B) = v^*(A) \Rightarrow_* v^*(B), \;\; v^*(\bot) = 0$$

Formula F is valid in $\mathbf{L}(*)$ iff $v^*(F) = 1$ for all valuations v^* pertaining to the t-norm $*$.

Three fundamental continuous t-norms and their residua are:

	t-norm	associated residuum
Łukasiewicz	$x *_{\mathbf{Ł}} y = \sup\{0, x+y-1\}$	$x \Rightarrow_{\mathbf{Ł}} y = \inf\{1, 1-x+y\}$
Gödel	$x *_{\mathbf{G}} y = \inf\{x,y\}$	$x \Rightarrow_{\mathbf{G}} y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ otherwise} \end{cases}$
Product	$x *_{\mathbf{P}} y = x \cdot y$	$x \Rightarrow_{\mathbf{P}} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$

Any continuous t-norm is obtained by an ordinal sum construction based on these three (see, [25, 15]). The logics $\mathbf{L}(*_{\mathbf{Ł}})$, $\mathbf{L}(*_{\mathbf{G}})$, and $\mathbf{L}(*_{\mathbf{P}})$, are called Łukasiewicz logic **Ł**, Gödel logic **G**, and Product logic **P**, respectively.

The mentioned logics have different features that render them adequate for different forms of applications. E.g., Gödel logic **G**, is the only t-norm based logic, where the truth value of every formula A only depends on the relative order of truth values of atomic subformulas of A, but not on the absolute values of these subformulas. However, Example 2 suggests another *desideratum*, that we formulate as an additional design choice:

4. Small changes in $v^*(A)$ or $v^*(B)$ result in, at most, small changes in $v^*(A \supset B)$. More precisely: the truth function $\Rightarrow_*$ for implication is continuous.

Design choices 1-4 jointly determine a unique logic:

Proposition 1. *Ł is the only logic of type* $\mathbf{L}(*)$*, where* $\Rightarrow_*$ *is continuous.*

Proof. Let $x, y, u \in [0, 1]$. For any continuous t-norm $*$ we have (see [16]):

- If $x < u \leq y$ and $u = u * u$ is idempotent then $(y \Rightarrow_* x) = x$.
- If $y \leq x$ then $(y \Rightarrow_* x) = 1$.

Putting $y = u$ in these inequalities we get for idempotent u:

- $(u \Rightarrow_* x) = x$ for $x < u$
- $(u \Rightarrow_* x) = 1$ for $x \geq u$

It follows that $\Rightarrow_*$ is not continuous at (u, u) if u is idempotent and $0 < u < 1$. By the ordinal sum representation of [25] each continuous t-norm is the generalized sum of order isomorphic copies of the Łukasiewicz and product t-norms. In this construction boundaries of an interval are mapped to idempotent elements. It follows that the only continuous t-norms with no idempotent elements except 0 and 1 are given by a single interval whose boundaries are mapped to 0 and 1. The corresponding t-norms are order isomorphic to Łukasiewicz or product t-norm respectively.

The residuum $x \Rightarrow_* y$ of product t-norm is not continuous at $(0, 0)$. Hence the only continuous t-norms with continuous implication are order isomorphic to the Lukasiewicz t-norm. The unique corresponding logic is Łukasiewicz logic **Ł**. ◇

Note that we have used the same symbols for classical conjunction, negation, and implication as for their respective counterparts in t-norm based fuzzy logics. In principle, one might keep the classical logical vocabulary apart from the logical vocabulary for fuzzy logics in defining a logic that combines supervaluation with t-norm based valuations. However, the results in Section 3, below, can be seen as a justification of our choice of a unified logical syntax for the logic SŁ that extends Łukasiewicz logic, but incorporates also classical logic. The crucial link between classical and $*_{\text{Ł}}$- based valuation over precisification spaces is obtained by making the concept of supertruth explicit also in our language. For this we introduce the (unary) connective S—read: "It is supertrue that ..."—which will play the role of an **S5**-like modal operator. Modal extensions of fuzzy logics have already been studied in other contexts; see, e.g., chapter 8 of [15] and [8]. However SŁ is different from the modal extensions of **Ł** studied by Hájek, Godo, Esteva, Montagna, and others, since it combines classical reasoning with many-valued reasoning in a different way, as will get clear below. (See also Section 5.)

Formulas of SŁ are built up from the propositional variables $p \in V = \{p_1, p_2, \ldots\}$ and the constant $\bot$ using the connectives & and $\supset$. The additional connectives $\neg$, $\wedge$, and $\vee$ are defined as explained above. In accordance with our earlier (informal) semantic considerations, a precisification space is formalized as a triple $\langle W, e, \mu \rangle$, where $W = \{\pi_1, \pi_2, \ldots\}$ is a non-empty (countable) set, whose elements π_i are called *precisification points*, e is a mapping $W \times V \mapsto \{0, 1\}$, and μ is a probability measure on the σ-algebra formed by all subsets of W. Given a precisification space $\Pi = \langle W, e, \mu \rangle$ a *local truth value* $\|A\|_\pi$ is defined for every formula A and every precisification point $\pi \in W$ inductively by

$$\|p\|_\pi = e(\pi, p), \text{ for } p \in V \tag{1}$$
$$\|\bot\|_\pi = 0 \tag{2}$$
$$\|A\&B\|_\pi = \begin{cases} 1 \text{ if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 1 \\ 0 \text{ otherwise} \end{cases} \tag{3}$$
$$\|A \supset B\|_\pi = \begin{cases} 1 \text{ if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 0 \\ 0 \text{ otherwise} \end{cases} \tag{4}$$
$$\|\mathsf{S}A\|_\pi = \begin{cases} 1 \text{ if } \forall \sigma \in W : \|A\|_\sigma = 1 \\ 0 \text{ otherwise} \end{cases} \tag{5}$$

Local truth values are classical and do not depend on the underlying t-norm $*_{\text{Ł}}$. In contrast, the *global truth value* $\|A\|_\Pi$ of a formula A is defined by

$$\|p\|_\Pi = \mu(\{\pi \in W | e(\pi, p) = 1\}), \text{ for } p \in V \tag{6}$$
$$\|\bot\|_\Pi = 0 \tag{7}$$
$$\|A\&B\|_\Pi = \|A\|_\Pi *_{\text{Ł}} \|B\|_\Pi \tag{8}$$
$$\|A \supset B\|_\Pi = \|A\|_\Pi \Rightarrow_{\text{Ł}} \|B\|_\Pi \tag{9}$$
$$\|\mathsf{S}A\|_\Pi = \|\mathsf{S}A\|_\pi \text{ for any } \pi \in W \tag{10}$$

Note that $\|\mathsf{S}A\|_\pi$ is the same value (either 0 or 1) for all $\pi \in W$. In other words: 'local' supertruth is in fact already global; which justifies the above clause for $\|\mathsf{S}A\|_\Pi$. Also observe that we could have used clauses 8 and 9 also to define $\|A\&B\|_\pi$ and $\|A \supset B\|_\pi$ since the (global) t-norm based truth functions coincide with the (local) classical ones, when restricted to $\{0, 1\}$. (However that might have obscured their intended meaning.)

A formula A is called *valid* in S**Ł** if $\|A\|_\Pi = 1$ for all precisification spaces Π. We identify a logic with the set of its valid formulas. Note that for any atom p the formula $\mathsf{S}p \supset p$ is valid, but that $\mathsf{S}A \supset A$ is *not valid* in general for compound formulas A. In other words, S**Ł** is not closed under substitution.

Proposition 2. S**Ł** *restricted to formulas without occurrences of* S *coincides with* **Ł**. *On the other hand* $\{A \mid \mathsf{S}A \in \mathsf{S}\text{Ł}\}$ *coincides with* **S5**.

Proof. The first part of the claim follows immediately from clauses 7, 8 and 9, above; and the fact that all values $v(p_i) \in [0, 1]$ for some propositional variable $p_i \in \{p_1, \ldots p_n\}$ can be obtained as $\mu(\{\pi \in W | e(\pi, p) = 1\})$ for a suitable precisification space $\langle W, e, \mu \rangle$ where W and e correspond to all 2^n assignments of 0 or 1 to the p_i.

The second part follows from clauses 1-5 and 10 using the well known fact that in any Kripke model $\langle W, R, e \rangle$ for **S5**—where W is the set of possible worlds, R is the accessibility relation, and e the mapping that assigns 0 or 1 to each propositional variable in each world—R can be assumed to be the total relation $W \times W$. $\diamond$

We have the following finite model property.

Proposition 3. *A formula F is valid in* S**Ł** *if and only if F is valid in all those precisification spaces $\langle W, e, \mu \rangle$ where W is finite.*

Proof. Let $\Pi = \langle W, e, \mu \rangle$ and let $V_F = \{p_1, \ldots, p_n\}$ be the propositional variables occurring in F. Moreover, let $\mathscr{B}_F$ be the set of all classical truth value assignments $I : V_F \mapsto \{0,1\}$. We write $I = e(\pi)$ if $\forall p \in V_F : I(p) = e(\pi, p)$ and define a new precisification space $\Pi^f = \langle W^f, e', \mu' \rangle$ as follows:

- $W_f = \{I \in \mathscr{B}_F \mid \exists \pi \in W : I = e(\pi)\}$
- $e'(I, p) = e(\pi, p)$, where $I = e(\pi)$
- $\mu'(\{I\}) = \mu(\{\pi \mid I = e(\pi)\})$, which uniquely extends to all subsets of W_f.

It is straightforward to check that $\|F\|_\Pi = \|F\|_{\Pi^f}$. Thus we have shown that in evaluating F it suffices to consider precisification spaces with at most $2^{p(F)}$ precisification points, where $p(F)$ is the number of different propositional variables occurring in F. $\Diamond$

3 Reasoning via dialogue games

We have defined the logic S**Ł** in an attempt to relate supervaluation and 'fuzzy valuation' in a common framework based on precisification spaces. But we have not yet said anything about proof systems or—more generally—about formal *reasoning* in this context. We conjecture that a Hilbert-style calculus for S**Ł** can be obtained by extending any system for **Ł** with the following axioms

$$
\begin{array}{llll}
A1: & \mathsf{S}(A \vee \neg A) & A2: & \mathsf{S}A \vee \neg \mathsf{S}A \\
A3: & \mathsf{S}(A \supset B) \supset (\mathsf{S}A \supset \mathsf{S}B) & A4: & \mathsf{S}p \supset p, \text{ for atoms } p \\
A5: & \mathsf{S}A \supset \mathsf{S}\mathsf{S}A & A6: & \mathsf{S}\mathsf{S}A \supset \mathsf{S}A
\end{array}
$$

and the Necessitation Rule $\frac{A}{\mathsf{S}A}$ for supertruth. However, mainly due to space constraints, we defer a corresponding soundness and completeness proof to an extended version of this paper and concentrate on an analysis of S**Ł** that seems more revealing with respect to its intended semantics and also more important from a computational point of view. Building on an extension of (a variant of) Robin Giles's dialogue and betting game for **Ł** (see [13, 14, 10]) we provide a game based characterization of S**Ł**. Our game will be seen to correspond to a tableau style system for analytic reasoning over given precisification spaces. It consists of two largely independent building blocks:

(1) Betting for random verifications. Assume that two players—let's say me and you—agree to pay 1€ to the opponent player for each assertion of an atomic statement, which is false according to a randomly chosen admissible precisification. More formally, given a precisification space $\Pi = \langle W, e, \mu \rangle$ the *risk value* $\langle p \rangle_\Pi$ associated with a propositional variable p is defined as $\langle p \rangle_\Pi = \mu(\{\pi \in W | e(\pi, p) = 0\})$; moreover we define $\langle \bot \rangle_\Pi = 1$. Note that $\langle p \rangle_\Pi$ corresponds to the probability (as determined by μ) of having to pay 1€, when asserting p.

Let $p_1, p_2, \ldots, q_1, q_2, \ldots$ denote atomic statements, i.e., propositional variables or $\bot$. By $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ we denote an *elementary state* in the game, where I assert each of the q_i in the multiset $\{q_1, \ldots, q_n\}$ of atomic statements and you, likewise, assert each atomic statement $p_i \in \{p_1, \ldots, p_m\}$. To illustrate this notions consider the elementary state $[p \| q, q]$. According to the outlined arrangement, we have to evaluate p once, and q twice in randomly chosen precisifications. If, e.g., all three evaluations result in 0 then I owe you 2€ and you owe me 1€, implying a total loss of 1€ for me.

The *risk* associated with a multiset $P = \{p_1, \ldots, p_m\}$ of atomic formulas is defined as $\langle p_1, \ldots, p_m \rangle_\Pi = \sum_{i=1}^m \langle p_i \rangle_\Pi$. The risk $\langle\rangle_\Pi$ associated with the empty multiset is 0. Note that $\langle P \rangle_\Pi$ thus denotes the average amount of money that I expect to have to pay to you according to the above arrangements if I have asserted the atomic formulas in P. The risk associated with an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ is calculated from my point of view. Therefore the condition $\langle p_1, \ldots, p_m \rangle_\Pi \geq \langle q_1, \ldots, q_n \rangle_\Pi$, which we will call *success condition*, expresses that I do not expect any loss (but possibly some gain) when betting on the truth of atomic statements as explained above. Returning to our example of the elementary state $[p \| q, q]$, I expect an average loss of 0.5€ with respect to $\Pi = \langle W, e, \mu \rangle$, where μ is the uniform contribution over a finite set of precisification points W with $|\{\pi \in W \mid e(\pi, r) = 1\}| = |\{\pi \in W \mid e(\pi, r) = 0\}|$ for $r = p$ and $r = q$, implying $\langle p \rangle_\Pi = \langle q \rangle_\Pi = 0.5$. If for some alternative precisification space Π' we have $\langle p \rangle_{\Pi'} = 0.8$ and $\langle q \rangle_{\Pi'} = 0.3$ then my average loss is negative; more precisely, I can expect a gain of 0.2€ in average.

(2) A dialogue game for the analysis of complex formulas. We follow Giles and Paul Lorenzen (see, e.g., [23]) in constraining the meaning of connectives by reference to rules of a dialogue game that proceeds by systematically reducing arguments about compound formulas to arguments about their subformulas.

For the sake of clarity, we first assume that formulas are built up from propositional variables and $\bot$ using the connectives $\supset$ and S only. (Note that in **Ł**, and therefore also in S**Ł**, one can define strong conjunction and consequently also all other connectives using $A \& B =_{def} (A \supset (B \supset \bot)) \supset \bot)$. However, we will present a more direct analysis of conjunction and disjunction, below.)

The dialogue rule for implication can be stated as follows (cf. [13, 14]):

$(R_\supset)$ If I assert $A \supset B$ then, whenever you choose to attack this statement by asserting A, I have to assert also B. (And *vice versa*, i.e., for the roles of me and you switched.)

Note that a player may also choose not to attack the opponent's assertions of $A \supset B$. This rule reflects the idea that the *meaning of implication* entails the principle that an assertion of "If A then B." obliges one to assert also B if the opponent in a dialogue grants (i.e., asserts) A.

The dialogue rule for the supertruth modality involves a relativization to specific precisification points:

(R_S) If I assert $\mathsf{S}A$ then I also have to assert that A holds at any precisification point π that you may choose. (And *vice versa*, i.e., for the roles of me and you switched.)

Let us henceforth use A^π as shorthand for 'A holds at the precisification point π' and speak of A as a *formula indexed* by π, accordingly. Note that using rule (R_S) entails that we have to deal with indexed formulas also in rule $(R_\supset)$. However, we don't have to change the rule itself, which will turn out to be adequate independently of the kind of evaluation—degree based or supervaluation based—that we aim at in a particular context. Rather, we only need to stipulate that in applying $(R_\supset)$ the precisification point index of $A \supset B$ (if there is any) is inherited by the subformulas A and B. If, on the other hand, we apply rule (R_{S}) to an already indexed formula $(\mathsf{S}A)^{\rho}$ then the index ρ is overwritten by whatever index π is chosen by the opponent player; i.e., we have to continue with the assertion A^{π}. Of course, we also have to account for indices of formulas in elementary states. This is achieved in the obvious way: we simply augment the definition of *risk* (with respect to $\Pi = \langle W, e, \mu \rangle$) by $\langle p^{\pi} \rangle_{\Pi} = 1 - e(\pi, p)$. In other words, the probability of having to pay 1€ for claiming that p holds at the precisification point π is 0 if p is true at π and 1 if p is false at π.

To simplify notations we will use the special *global index* ε ($\notin W$) to indicate that a formula is *not* referring to a particular precisification point. Thus *every* formula is indexed now, but A^{ε} means that A is asserted 'globally', i.e., without reference to a particular precisification.

We use $[A_1^{\pi_1}, \ldots, A_m^{\pi_m} \| B_1^{\rho_1}, \ldots, B_n^{\rho_n}]$ to denote an arbitrary (not necessarily elementary) *state* of the game, where $\{A_1^{\pi_1}, \ldots, A_m^{\pi_m}\}$ is the multiset of formulas that are currently asserted by you, and $\{B_1^{\rho_1}, \ldots, B_n^{\rho_n}\}$ is the multiset of formulas that are currently asserted by me. (Note that this implies, that we don't care about the order in which formulas are asserted.)

A *move initiated by me* (m-move) in state $[\Gamma \| \Delta]$ consists in my picking of some non-atomic formula A^{π} from the multiset Γ and proceeding as follows:

- If $A^{\pi} = (A_1 \supset A_2)^{\pi}$ then I may either *attack* by asserting A_1^{π} in order to force you to assert A_2^{π} in accordance with $(R_{\supset})$, or *admit* A^{π}. In the first case the successor state is $[\Gamma', A_2^{\pi} \| \Delta, A_1^{\pi}]$, in the second case it is $[\Gamma' \| \Delta]$, where $\Gamma' = \Gamma - \{A^{\pi}\}$.
- If $A^{\pi} = \mathsf{S}B^{\pi}$ then I choose an arbitrary $\sigma \in W$ thus forcing you to assert B^{σ}. The successor state is $[\Gamma', B^{\sigma} \| \Delta]$, where $\Gamma' = \Gamma - \{A^{\pi}\}$.

A *move intiated by you* (y-move) is symmetric, i.e., with the roles of me and you interchanged. A run of the game consists in a sequence of states, each resulting from a move in the immediately preceding state, and ending in an elementary state $[p_1^{\pi_1}, \ldots, p_m^{\pi_m} \| q_1^{\rho_1}, \ldots, q_n^{\rho_n}]$. I *succeed* in this run if this final state fulfills the success condition, i.e., if

$$\sum_{j=1}^{n} \langle q_j^{\rho_j} \rangle_{\Pi} - \sum_{i=1}^{m} \langle p_i^{\pi_i} \rangle_{\Pi} \leq 0. \tag{11}$$

The term at the left hand side of inequality 11 is my *expected loss* at this state. In other words, I succeed if my expected loss is 0 or even negative, i.e., in fact a gain.

As mentioned above, other connectives can be reduced to implication and *falsum*. However, using the corresponding definitions directly hardly results in dialogue rules that are as natural as $(R_{\supset})$. In the following we will formulate dialogue rules only from my point of view, with the implicit understanding that the corresponding rule for you is completely symmetric. For conjunction *two* candidate rules seem natural:

$(R_{\wedge})$ If I assert $A_1 \wedge A_2$ then I have to assert also A_i for any $i \in \{1, 2\}$ that you may choose.

$(R_{\wedge'})$ If I assert $A_1 \wedge' A_2$ then I have to assert also A_1 as well as A_2.

Rule $(R_{\wedge})$ is dual to the following natural candidate for a disjunction rule:

($R_\vee$) If I assert $A_1 \vee A_2$ then I have to assert also A_i for some $i \in \{1,2\}$ that I myself may choose.

Moreover it is clear how ($R_\wedge$) generalizes to a rule for universal quantification. Note that the modality S can be seen as a kind of universal quantifier over corresponding classical propositions at all precisification points; which is reflected in the form of the rules ($R_\wedge$) and (R_{S}), respectively.

It follows already from results in [13, 14] that rules ($R_\wedge$) and ($R_\vee$) are adequate for weak conjunction and disjunction in **Ł**, respectively. $\wedge$ and $\vee$ are also called 'lattice connectives' in the context of fuzzy logics, since their truth functions are given by

$$v^*(A \wedge B) = \inf\{v^*(A), v^*(B)\} \quad \text{and} \quad v^*(A \vee B) = \sup\{v^*(A), v^*(B)\}.$$

The question arises, whether we can use the remaining rule ($R_{\wedge'}$) to characterize strong disjunction (&). However, rule ($R_{\wedge'}$) is inadequate in the context of our betting scheme for random evaluation in a precisification space. The reason for this is that we have to make sure that for any (not necessarily atomic) assertion we make, we risk a *maximal* loss of 1€. It is easy to see that rules ($R_\supset$), ($R_\wedge$), ($R_\vee$), and (R_{S}) comply with this constraint; however if I assert $p \wedge' q$ and we play according to ($R_{\wedge'}$), then I end up with an expected loss of 2€, in case both p and q are superfalse. There is a simply way to redress this situation to obtain a rule that is adequate for (&): Allow any player who asserts $A_1 \& A_2$ to hedge her possible loss by asserting $\bot$ instead; which of course corresponds to the obligation to pay 1€ (but not more) in the resulting final state. We thus obtain:

($R_\&$) If I assert $A_1 \& A_2$ then I either have to assert also A_1 as well as A_2, or else I have to assert $\bot$.

All discussed rules induce definitions of corresponding *moves* in the game, analogously to the case of ($R_\supset$) and (R_{S}), illustrated above.

4 Adequacy of the game

To prove that the game presented in Section 3 indeed characterizes logic **SŁ**, we have to analyse all possible runs of the game starting with some arbitrarily complex assertion by myself. A *strategy* for me will be a tree-like structure, where a branch represents a possible run resulting from particular choices made by myself, taking into account all of your possible choices in (y- or m-moves) that are compatible with the rules. We will only have to look at strategies for *me* and thus call a strategy *winning* if I succeed in all corresponding runs (according to condition 11).

Remember that by Proposition 3 we can assume that the set W of the underlying precisification space $\Pi = \langle W, e, \mu \rangle$ is finite. The construction of strategies can be viewed as systematic proof search in an analytic tableau calculus with the following rules:

$$\frac{[\Gamma \| \Delta, (A_1 \supset A_2)^\pi]}{[\Gamma, A_1^\pi \| \Delta, A_2^\pi] \mid [\Gamma \| \Delta]}(\supset_y) \qquad \frac{[\Gamma, (A_1 \supset A_2)^\pi \| \Delta]}{[\Gamma, A_2^\pi \| \Delta, A_1^\pi]}(\supset_m^1) \qquad \frac{[\Gamma, (A_1 \supset A_2)^\pi \| \Delta]}{[\Gamma \| \Delta]}(\supset_m^2)$$

$$\frac{[\Gamma \| \Delta, (A_1 \& A_2)^\pi]}{[\Gamma \| \Delta, A_1^\pi, A_2^\pi]}(\&_y^1) \qquad \frac{[\Gamma \| \Delta, (A_1 \& A_2)^\pi]}{[\Gamma \| \Delta, \bot^\pi]}(\&_y^2) \qquad \frac{[\Gamma, (A_1 \& A_2)^\pi \| \Delta]}{[\Gamma, A_1^\pi, A_2^\pi \| \Delta] \mid [\Gamma, \bot^\pi \| \Delta]}(\&_m)$$

$$\frac{[\Gamma \| \Delta, (A_1 \wedge A_2)^\pi]}{[\Gamma \| \Delta, A_1^\pi] \mid [\Gamma \| \Delta, A_2^\pi]}(\wedge_y) \qquad \frac{[\Gamma, (A_1 \wedge A_2)^\pi \| \Delta]}{[\Gamma, A_1^\pi \| \Delta]}(\wedge_m^1) \qquad \frac{[\Gamma, (A_1 \wedge A_2)^\pi \| \Delta]}{[\Gamma, A_2^\pi \| \Delta]}(\wedge_m^2)$$

$$\frac{[\Gamma \| \Delta, (A_1 \vee A_2)^\pi]}{[\Gamma \| \Delta, A_1^\pi]}(\vee_y^1) \qquad \frac{[\Gamma \| \Delta, (A_1 \vee A_2)^\pi]}{[\Gamma \| \Delta, A_2^\pi]}(\vee_y^2) \qquad \frac{[\Gamma, (A_1 \vee A_2)^\pi \| \Delta]}{[\Gamma, A_1^\pi \| \Delta] \mid [\Gamma, A_2^\pi \| \Delta]}(\vee_m)$$

$$\frac{[\Gamma \| \Delta, (\mathsf{S}A)^\pi]}{[\Gamma \| \Delta, A^{\pi_1}] \mid \ldots \mid [\Gamma \| \Delta, A^{\pi_n}]}(\mathsf{S}_y) \qquad \frac{[\Gamma, (\mathsf{S}A)^\pi \| \Delta]}{[\Gamma, A^\rho \| \Delta]}(\mathsf{S}_m)$$

In all rules π can denote any index, including the global index ε. In rule (S_y) we assume that $W = \{\pi_1, \ldots, \pi_m\}$ and in rule (S_m) the index ρ can be any element of W. Note that, in accordance with the definition of a strategy for *me*, your choices in the moves induce branching, whereas for my choices a single successor state that is compatible with the dialogue rules is chosen.

The finiteness assumption for W is not needed in proving the following theorem.

Theorem 1. *A formula F is valid in* SŁ *if and only if for every precisification space Π I have a winning strategy for the game starting in state $[\|F]$.*

Proof. Note that every run of the game is finite. For every final elementary state $[p_1^{\pi_1}, \ldots, p_m^{\pi_m} \| q_1^{\rho_1}, \ldots, q_n^{\rho_n}]$ the success condition says that we have to compute the risk $\sum_{j=1}^n \langle q_j^{\rho_j} \rangle_\Pi - \sum_{i=1}^m \langle p_i^{\pi_i} \rangle_\Pi$, where $\langle r^\pi \rangle_\Pi = \mu(\{\rho \in W | e(\rho, r) = 0\})$ if $\pi = \varepsilon$ and $\langle r^\pi \rangle_\Pi = 1 - e(\pi, r)$ otherwise, and check whether the resulting value (in the following denoted by $\langle p_1^{\pi_1}, \ldots, p_m^{\pi_m} \| q_1^{\rho_1}, \ldots, q_n^{\rho_n} \rangle_\Pi$) is ≤ 0 to determine whether I 'win' the game. To obtain my minimal final risk (i.e., my minimal expected loss) that I can enforce in any given state S by playing according to an optimal strategy, we have to take into account the supremum over all risks associated with the successor states to S that you can enforce by a choice that you may have in a (y- or m-)move S. On the other hand, for any of my choices I can enforce the infimum of risks of corresponding successor states. In other words, we prove that we can extend the definition of *my expected loss* from elementary states to arbitrary states such that the following conditions are satisfied:

$$\langle \Gamma, (A \supset B)^\pi \| \Delta \rangle_\Pi = \inf\{\langle \Gamma \| \Delta \rangle_\Pi, \langle \Gamma, B^\pi \| A^\pi, \Delta \rangle_\Pi\} \tag{12}$$
$$\langle \Gamma, (A \& B)^\pi \| \Delta \rangle_\Pi = \sup\{\langle \Gamma, A^\pi, B^\pi \| \Delta \rangle_\Pi, \langle \Gamma, \bot^\pi \| \Delta \rangle_\Pi\} \tag{13}$$
$$\langle \Gamma, (A \wedge B)^\pi \| \Delta \rangle_\Pi = \inf\{\langle \Gamma, A^\pi \| \Delta \rangle_\Pi, \langle \Gamma, B^\pi \| \Delta \rangle_\Pi\} \tag{14}$$
$$\langle \Gamma, (A \vee B)^\pi \| \Delta \rangle_\Pi = \sup\{\langle \Gamma, A^\pi \| \Delta \rangle_\Pi, \langle \Gamma, B^\pi \| \Delta \rangle_\Pi\} \tag{15}$$

for assertions by you and, for my own assertions:

$$\langle \Gamma \| (A \supset B)^\pi, \Delta \rangle_\Pi = \sup\{\langle \Gamma, A^\pi \| B^\pi, \Delta \rangle_\Pi, \langle \Gamma \| \Delta \rangle_\Pi\} \tag{16}$$
$$\langle \Gamma \| (A \& B)^\pi, \Delta \rangle_\Pi = \inf\{\langle \Gamma \| A^\pi, B^\pi, \Delta \rangle_\Pi, \langle \Gamma \| \bot, \Delta \rangle_\Pi\} \tag{17}$$
$$\langle \Gamma \| (A \wedge B)^\pi, \Delta \rangle_\Pi = \sup\{\langle \Gamma \| A^\pi, \Delta \rangle_\Pi, \langle \Gamma \| B^\pi, \Delta \rangle_\Pi\} \tag{18}$$
$$\langle \Gamma \| (A \vee B)^\pi, \Delta \rangle_\Pi = \inf\{\langle \Gamma \| A^\pi, \Delta \rangle_\Pi, \langle \Gamma \| B^\pi, \Delta \rangle_\Pi\} \tag{19}$$

Furthermore we have

$$\langle \Gamma \,\|\, (\mathsf{S}A)^{\pi}, \Delta \rangle_{\Pi} = \sup_{\rho \in W} \{ \langle \Gamma \,\|\, A^{\rho}, \Delta \rangle_{\Pi} \} \tag{20}$$

$$\langle \Gamma, (\mathsf{S}A)^{\pi} \,\|\, \Delta \rangle_{\Pi} = \inf_{\rho \in W} \{ \langle \Gamma, A^{\rho} \,\|\, \Delta \rangle_{\Pi} \} \tag{21}$$

We have to check that $\langle \cdot \,\|\, \cdot \rangle_{\Pi}$ is well-defined; i.e., that conditions 12-21 together with the definition of my expected loss (risk) for elementary states indeed can be simultaneously fulfilled and guarantee uniqueness. To this aim consider the following generalisation of the truth function for SŁ to multisets Γ of indexed formulas:

$$\|\Gamma\|_{\Pi} =_{def} \sum_{A^{\pi} \in \Gamma, \pi \neq \varepsilon} \|A\|_{\pi} + \sum_{A^{\varepsilon} \in \Gamma} \|A\|_{\Pi}.$$

Note that

$$\|A\|_{\Pi} = \|\{A^{\varepsilon}\}\|_{\Pi} = 1 \text{ iff } \langle \,\|\, A^{\varepsilon} \rangle_{\Pi} \leq 0.$$

In words: A is valid in SŁ iff my risk in the game starting with my assertion of A is non-positive. Moreover, for elementary states we have

$$\langle p_1^{\pi_1}, \ldots, p_m^{\pi_m} \,\|\, q_1^{\rho_1}, \ldots, q_n^{\rho_n} \rangle_{\Pi} = n - m + \|p_1^{\pi_1}, \ldots, p_m^{\pi_m}\|_{\Pi} - \|q_1^{\rho_1}, \ldots, q_n^{\rho_n}\|_{\Pi}.$$

We generalize the risk function to arbitrary states by

$$\langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} =_{def} |\Delta| - |\Gamma| + \|\Gamma\|_{\Pi} - \|\Delta\|_{\Pi}$$

and check that it satisfies conditions 12-21. We only spell out two cases. To avoid case distinctions let $\|A\|_{\varepsilon} =_{def} \|A\|_{\Pi}$. For condition 12 we have

$$\begin{aligned}
&\langle \Gamma, (A \supset B)^{\pi} \,\|\, \Delta \rangle^{*}_{\Pi} = |\Delta| - |\Gamma| - 1 + \|\Gamma\|_{\Pi} + \|(A \supset B)\|_{\pi} - \|\Delta\|_{\Pi} \\
&= \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} - 1 + \|(A \supset B)\|_{\pi} = \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} - 1 + (\|A\|_{\pi} \Rightarrow_{\text{Ł}} \|B\|_{\pi}) \\
&= \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} - 1 + \inf\{1, 1 - \|A\|_{\pi} + \|B\|_{\pi}\} = \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} - 1 + \inf\{1, 1 + \langle B^{\pi} \,\|\, A^{\pi} \rangle^{*}_{\Pi}\} \\
&= \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} + \inf\{0, \langle B^{\pi} \,\|\, A^{\pi} \rangle^{*}_{\Pi}\} = \inf\{\langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi}, \langle \Gamma, B^{\pi} \,\|\, A^{\pi}, \Delta \rangle^{*}_{\Pi}\}
\end{aligned}$$

For condition 20 we have

$$\begin{aligned}
&\langle \Gamma \,\|\, (\mathsf{S}A)^{\pi}, \Delta \rangle^{*}_{\Pi} = |\Delta| + 1 - |\Gamma| + \|\Gamma\|_{\Pi} - \|\Delta\|_{\Pi} - \|\mathsf{S}A\|_{\pi} \\
&= \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} + 1 - \|\mathsf{S}A\|_{\pi} = \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} + 1 - \inf_{\rho \in W}\{\|A\|_{\rho}\} \\
&= \langle \Gamma \,\|\, \Delta \rangle^{*}_{\Pi} + \sup_{\rho \in W}\{\|A\|_{\rho}\} = \sup_{\rho \in W}\{\langle \Gamma, A^{\rho} \,\|\, \Delta \rangle^{*}_{\Pi}\}
\end{aligned}$$

◇

Remark 1. It already follows from a well known general theorem ('saddle point theorem') about finite games with perfect information that conditions 12-21 uniquely extend any given risk assignment from final states to arbitrary states. However, our proof above yields more information, namely that the extended risk function indeed matches the semantics of logic SŁ, as defined in Section 2.

By a *regulation* we mean an assignment of game states to labels '*you move next*' and '*I move next*' that constrain the possible runs of the game in the obvious way. A regulation is *consistent* if the label '*you (I) move next*' is only assigned to states where such a move is possible, i.e., where I (you) have asserted a non-atomic formula. As a simple but nice corollary to our proof of Theorem 1, we obtain:

Corollary 1. *The total expected loss* $\langle \Gamma \| \Delta \rangle^*_\Pi$ *that I can enforce in a game over* Π *starting in state* $[\Gamma \| \Delta]$ *only depends on* Γ*,* Δ*, and* Π*. In particular, it is the same for every consistent regulation that may be imposed on the game.*

5 Remarks on related work

Various kinds of modal extensions of fuzzy logics have been considered in the literature. E.g., chapter 8 of the central monograph [15] presents the family $\mathbf{S5}(L)$ for t-norm based fuzzy logics L by letting the truth value $e(w,p)$ assigned to a proposition p at a world $w \in W$ of a Kripke model range over $[0,1]$ instead of $\{0,1\}$. The truth function for the modality $\Box$ is given by $\|\Box A\|_w = \inf_{v \in W} \|A\|_v$. Of course, $\Box$, thus defined, behaves quite differently from supertruth S. In particular $\Box A \vee \neg \Box A$ is not valid. On the other hand, $\triangle A \vee \neg \triangle A$ is valid for the widely used 'definiteness' operator $\triangle$ as axiomatized in [2]. However also $\triangle$ over **Ł** is quite different from S in **SŁ**, as can be seen by considering the distribution axiom $\triangle(A \vee B) \supset (\triangle A \vee \triangle B)$ of [2]: Replacing $\triangle$ by S yields a formula that is not valid in **SŁ**. For the same reason the modal extensions of logic **MTL** considered in [5] are not able to express 'supertruth'.

Yet another type of natural extension of fuzzy logics arises when one considers the propositional operator *Pr* for 'It is probable that ...'. In [15, 19, 8] model structures that are essentially like our precisification spaces are used to specify the semantics of *Pr*. More exactly, one defines $\|Pr(A)\|_w = \mu(\{w \in W \mid e(w,A) = 1\})$, which implies that, for atomic propositions p, $Pr(p)$ is treated like p itself in **SŁ**. However, supertruth S cannot be expressed using *Pr*, already for the simple reason that the syntax of the mentioned 'fuzzy probability logics' does not allow for nesting of *Pr*. Moreover, classical and degree based connectives are separated at the syntactic level; whereas our dialogue game based analysis justifies the syntactic identification of both types of connectives in the context of precisification spaces.

Our way to define evaluation over a precisification space is also related to ideas of Dorothy Egdington [7]. However, while Edgington also refers to 'truth on proportions of precisifications', she insists on evaluations that are not truth functional.

Finally we mention that some of the ideas underlying our presentation of **SŁ** are already—at least implicitly—present in [10]. However no corresponding formal definitions or results have been presented there.

6 Conclusion and future work

We have presented an analysis of logical reasoning with vague propositions that incorporates two seemingly different approaches to semantics: supervaluation and degree based valuation. The resulting logic **SŁ** has been characterized as the set of those formulas which a player can assert in a natural dialogue+betting game over precisification spaces, without having to expect a loss of money.

The agenda for related future work includes the 'lifting' of our tableau style evaluation system to a hypersequent calculus, that abstracts away from particular underlying precisification spaces. This will lead to a proof system related to the calculi in [24] and in [4], and should be a good basis for exploring also other t-norm based evaluations over precisification spaces. Moreover we want to investigate the extension of SŁ by further modal operators that seem relevant in modelling propositional attitudes arising in contexts of vagueness.

References

1. J.C. Beall (ed.): *Liars and Heaps*. Oxford University Press 2003.
2. Matthias Baaz: Infinite-valued Gödel logics with 0-1-projections and relativizations. In: *Gödel 96. Kurt Gödel's Legacy*, Springer LNL 6 (1996), 23-33.
3. Linda C. Burns: *Vagueness: an Investigation Into Natural Language and the Sorites Paradox*, Dordrecht, Kluwer Academic Publishers, 1991.
4. A. Ciabattoni, C.G. Fermüller, G. Metcalfe: Uniform Rules and Dialogue Games for Fuzzy Logics. In: *LPAR 2004*, F. Baader, A. Voronkov (eds.), Springer LNAI 3452 (2005), 496-510.
5. A. Ciabattoni, F. Montagna, G. Metcalfe: Adding Modalities to MTL and its Extensions. *Linz Symposium 2005*, to appear.
6. R. Cignoli, I.M.L. D'Ottaviano, D. Mundici: Algebraic Foundations of Many-valued Reasoning. Trends in Logic, Volume 7. Kluwer Academic Publishers, Dordrecht, 1999.
7. D. Edgington: Validity, Uncertainty and Vagueness. *Analysis* 52/4 (1992), 193-204.
8. T. Flaminio, F. Montagna: A logical and algebraic treatment of conditional probability. Arch. Math. Logic 44, 245-262 (2005).
9. Christian G. Fermüller: Theories of Vagueness Versus Fuzzy Logic: Can Logicians Learn from Philosophers? *Neural Network World Journal* 13(5) (2003), 455-466.
10. Christian G. Fermüller: Revisiting Giles's Game. *Logic, Games and Philosophy: Foundational Perspectives*, Prague Colloquium October 2004, to appear.
11. Kit Fine: Vagueness, Truth and Logic, *Synthése* 30, 265-300, 1975.
12. B.R. Gaines. Foundations of fuzzy reasoning. *International Journal of Man-Machine Studies*, 8:623–668, 1976.
13. Robin Giles: A non-classical logic for physics. *Studia Logica* 33, vol. 4, (1974), 399-417.
14. Robin Giles: A non-classical logic for physics. In: R. Wojcicki, G. Malinkowski (Eds.) *Selected Papers on Łukasiewicz Sentential Calculi*. Polish Academy of Sciences, 1977, 13-51.
15. Petr Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, 1998.
16. Petr Hájek: Basic fuzzy logic and BL-algebras. *Soft Computing* 2 (1998), 124-128
17. Petr Hájek: Why fuzzy logic?. In *A Companion to Philosophical Logic* (D. Jackquette, ed.), Blackwell, 2002, 595-606.
18. Petr Hájek, Vilém Novák: The sorites paradox and fuzzy logic. *Intl. J. of General Systems* 32/4 (2003) 373-383
19. Petr Hájek, Lluis Godo, Francesc Esteva: Fuzzy logic and probability. *11th Conference on Uncertainty in Artificial Intelligence (UAI-95*, Morgan Kaufmann 1995, 237-244.
20. Keefe, Rosanna: *Theories of Vagueness*, Cambridge University Press, 2000.
21. Rosanna Keefe, Peter Smith (eds.): *Vagueness: A Reader*, Massachusetts, MIT Press, 1987.
22. Philip Kremer and Michael Kremer: Some Supervaluation-based Consequence Relations. *Journal of Philosophical Logic* 32(3), June 2003, 225-244.
23. P. Lorenzen: Logik und Agon. In *Atti Congr. Internaz. di Filosofia*, Sansoni, 1960, 187-194.
24. G. Metcalfe, N. Olivetti, D. Gabbay: Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic* 6(3), 578-613, 2005.
25. P.S. Mostert, A.L. Shields: On the structure of semigroups on a compact manifold with boundary. *Annals of Mathematics* 65 (1957), 117-143.
26. Jeff Paris: Semantics for Fuzzy Logic Supporting Truth Functionality. In: *Discovering the World with Fuzzy Logic*. Physica-Verlag, Heidelberg, 2000, 82-104.
27. Achille Varzi: Vagueness, Logic, and Ontology, *The Dialogue 1*, 135-154.
28. Timothy Williamson: *Vagueness*, London, Routledge, 1994.