Some Critical Remarks on Incompatibility Semantics

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1 Introduction

In his fifth Locke Lecture (2006)[1] Robert Brandom has presented a new type of semantics for propositional classical and modal logic ('incompatibility semantics') that is embedded in his quite general programme ('inferential pragmatism') addressing major challenges to analytic philosophy. Inferential pragmatism is an important, comprehensive, and widely discussed contribution to contemporary philosophy. This is not the place to comment on this programme in general. Rather, we want to draw attention to a particular problem with the semantic framework suggested by Brandom, arising from a misinterpretation of an allegedly central technical result. This misunderstanding has strong repercussions on the philosophical significance of incompatibility semantics.

The main features that Brandom ascribes to his incompatibility semantics can be briefly summarized as follows:

(1) it is based on the notion of *material incompatibility* of (interpreted) sentences, rather than on their truth;

(2) it is *strongly intensional*, treating conjunction and, in particular, negation on a par with the modal operator 'necessarily';

(3) it is *holistic* and *non-compositional* in the sense that the meaning of a given compound formula (sentence) F is not determined by the semantic interpretants of the subformulas and connectives occurring in F;

(4) it nevertheless enjoys *recursive projectibility*, i.e., incompatibilities between logically complex formulas are determined by the incompatibilities between formulas that are less complex;

(5) it *refutes* the claim that holistic semantics *cannot account for* the projectibility and systematicity of a language, and hence also not for its *learnability*.

*This work is supported by Eurocores-ESF/FWF grant I143-G15 (LogICCC-LoMoReVI).

[1]The lectures are accessible at `http://www.pitt.edu/~rbrandom/` and published as Chapter 5 of (Brandom, 2008). I first learned about this exciting endeavor from a very stimulating invited talk of Brandom at *Logica 2008*.

As indicated, Brandom's presentation of incompatibility semantics is not only motivated from a broader philosophical perspective, but also comes fully equipped with a corresponding formal machinery. It thus certainly deserves the attention not only of philosophers, but also of mathematical logicians. Indeed, regarding claims 3, 4, and 5, above, Brandom argues that

> [...] holism *within* each level of constructional complexity is entirely compatible with recursiveness *between* levels. [...] The system I am describing allows us to *prove* it. (In this context, proof is the word made flesh.) The semantic values of all the logically compound sentences are computable entirely from the values of *less complex* sentences.
> (Brandom, 2008), p.135

The main purpose of this contribution is to point out an important gap in Brandom's argument that mainly concerns the intended meaning of negated sentences as 'minimal Aristotelean contraries'. If our analysis is correct, claim 4 and consequently also claim 5 remains unsubstantiated.

The rest of the paper is organized as follows. We start with a short review of Brandom's axioms for incoherence and incompatibility (Section 2). This is followed by a discussion of Brandom's concepts of holism and of recursive projectibility in Section 3. Section 4 addresses what I perceive as the major problem of incompatibility semantics. Throughout Sections 3 and 4 we provide direct citations of (Brandom, 2008) to enable also readers that are not familiar with Brandom's original presentation to judge the adequateness and fairness of our criticism. We conclude in Section 6 with the suggestion to consider logical dialogue games as a pragmatist, analytic, and inferentialist alternative to incompatibility semantics.

2 Incompatibility semantics in a nutshell

Incompatibility semantics is defined for a classical propositional language enriched by a standard modal operator $\Box$[2]. A language L_P is a set of formulas that contains a (finite or infinite) set P of propositional variables and is closed under subformulas: $\neg F \in L_P$ implies $F \in L_P$, $F \wedge G \in L_P$ implies $F, G \in L_P$, and $\Box F \in L_P$ implies $F \in L_P$. Whenever no particular set of variables P is referred to, we will suppress the subscript. For sake of conciseness, let us call any subset of L a *theory*. (We emphasize that no closure under logical consequence is implied: 'theory' is just our abbreviation for 'element of the powerset of L'.) Brandom presents his framework in axiomatic form. The basic semantic notion is '*incoherence*': theories are classified as either incoherent or else coherent. The set of all incoherent theories is called *incoherence frame* Inc. If we add further formulas to an incoherent theory it remains incoherent. In other words, Inc has to satisfy the following monotonicity condition:

Axiom (Persistence): $X \in \text{Inc}$ and $X \subseteq Y$ implies $Y \in \text{Inc}$.

[2] We use the signs $\neg$, $\wedge$, and $\Box$, instead of Brandom's N, K, and L, respectively. Moreover we stick to the usual infix notation for conjunction.

Given Persistence, a frame induces an *incompatibility function* I mapping theories into sets of theories by stipulating

Axiom (Partition): $X \cup Y \in \mathrm{Inc}$ iff $X \in I(Y)$.

Note that, instead of starting with an incoherence frame, one could start by specifying for any theory X the set of theories $I(X)$ that are incompatible with X. Obviously X is incoherent iff X is incompatible with itself ($X \in I(X)$). Thus the 'Partition Axiom' just amounts to an alternative presentation of incoherence[3]. The function I supports a concise definition of corresponding notion of entailment: a theory X *(incompatibility) entails* a finite set of formulas Y if everything that is incompatible with all formulas in Y is also incompatible with X.

Definition (Entailment): $X \models_{\mathrm{Inc}} Y$ iff $\bigcap_{F \in Y} I(\{F\}) \subseteq I(X)$.

This more general form of an entailment relation, where a finite set of formulas, not just a single formula, appears on the right hand side, is just like in Gentzen's sequent calculus **LK** for classical logic; i.e. the right hand side is to be interpreted disjunctively, while the formulas on the left hand side are to be interpreted conjunctively. As usual, we will write $F_1, \ldots, F_n \models_{\mathrm{Inc}} G_1, \ldots, G_m$ instead of $\{F_1, \ldots, F_n\} \models_{\mathrm{Inc}} \{G_1, \ldots, G_m\}$.

Brandom specifies the semantics of logical connectives by the following axioms:

Axiom (Negation): $X \cup \{\neg F\} \in \mathrm{Inc}$ iff $X \models_{\mathrm{Inc}} F$.

Axiom (Conjunction): $X \cup \{F \wedge G\} \in \mathrm{Inc}$ iff $X \cup \{F, G\} \in \mathrm{Inc}$.

Axiom (Necessity): $X \cup \{\Box F\} \in \mathrm{Inc}$ iff $X \in \mathrm{Inc}$ or $\exists Y \notin I(X)\colon Y \not\models_{\mathrm{Inc}} F$.

Disjunction ($\vee$) and implication ($\rightarrow$) are defined from $\neg$ and $\wedge$ as in classical logic.

Incompatibility semantics relates to traditional Tarski-style semantics as summarized in the following theorem, where $\models_{S5}$ is the standard entailment relation (defined over Kripke models) for the modal logic $S5$.

Theorem 1. *For all theories X and formulas F: $X \models_{S5} F$ iff $X \models_{\mathrm{Inc}} F$ for all incompatibility frames* Inc.

Incidentally, the modal component of the language will not really concern us much here. The problem that we want to point out and analyze below arises already for the classical propositional connectives.

[3]One might argue that the presented axioms are incomplete in a rather trivial sense: certainly the empty set of formulas is to be regarded as coherent if one wants to avoid interpretations in which *every* theory is incoherent and every formula is incompatible with itself. However Brandom explicitly admits also the degenerate case where $\{\} \in \mathrm{Inc}$.

3 Holism and recursive projectibility

As already mentioned in the introduction, Brandom insists on the *holism* and *non-compositionality* implied by the way he sets up his semantic framework. Indeed, the axioms for negation and for necessity take a non-predicative form. They refer to entailment and therefore to incompatibilities between theories that are not mentioned on the left hand sides of the equivalences. At least *prima facie*, it is not clear whether the axioms for $\neg F$ and for $\Box F$ are equivalent to conditions that only involve F and the context X, since incompatibility entailment implicitly refers to *all* theories over the given language and not just to those consisting of formulas with lower logical complexity than F and the members of X. (Only the axiom for conjunction amounts to a recursively checkable condition in the obvious way.) Indeed, Brandom states:

> Crucial to the compositionality of meaning is that the semantic values of logically complex sentences be reducible to the semantic values of their constituents. In the framework of incompatibility logic, however, meaning is *holistic*, and so [...] reduction cannot proceed sentence by sentence. (Brandom, 2008), p. 147

In light of the fact that incompatibility entailment (for the language without $\Box$) coincides with the classical entailment relation as specified by Gentzen's well known introduction rules of **LK** in a manner that only refers to the immediate subformulas of introduced formulas, this claim is somewhat problematic. After all, well known work in *proof theoretic semantics*—see, e.g., (Kahle & Schroeder-Heister, 2006)—where the logical rules of an appropriate (cut-free) sequent calculus or, equivalently, of a normalizing natural deduction system, are regarded as specifications of the *meaning* of logical connectives, can hardly be classified as *holistic* semantics. Nevertheless, it may be granted that there is a sense in which the axioms presented in Section 2 can be read as a 'holistic specification' of the semantics of the logical language L.

More importantly, the above quoted passage continues as follows:

> What we want instead is to show how the frame for a language with logically complex sentences can be reduced to the frame for a syntactically less complex fragment of the language. (Brandom, 2008), p. 147

Risking the charge of pettiness, in order to argue that my analysis below is on target, I have to point out that in the only relevant interpretation of this central claim, the first occurrence of 'the frame' in the quoted sentence must be read as 'any frame', whereas the second occurrence of 'the frame' must be read as 'some frame'. This should be uncontroversial: after all, employing incompatibility semantics to interpret a concrete formal language L amounts to the specification of a concrete frame for L. Certainly *all* frames for L, i.e., all structures where Inc and I satisfy the above axioms, are to be taken as candidates for interpretation.

In any case, also the passage on p. 135 of (Brandom, 2008), cited in the introduction of the present paper, makes clear: recursive projectibility can only be established by showing that the problem of computing the semantic value of a logically compound formula $F \in L$ in any given frame Inc for L can be reduced to computing the semantic values of less complex formulas (not necessarily subformulas of F) in frames that arise from Inc by restriction to the relevant sublanguages of L. Brandom thinks that the key to showing this is the notion of *inferentially conservative extensions* of frames, to be explored in the next section.

4 Problems with inferential conservativeness

Brandom defines a language L' to be a *proper extension* of L if $L \subseteq L'$ and all atomic formulas of L' are already in L. Strictly speaking, Brandom's formulation of the axioms reviewed in Section 2 implies that *all* formulas over a set P of atomic formulas (propositional variables) that can be recursively defined using the connectives $\neg$, $\wedge$, and $\Box$ have to be present in the language over which the frame is defined. (E.g., according to the axiom for conjunction, $\{F, G\} \in \text{Inc}$ implies $F \wedge G \in \text{Inc}$ and thus the underlying language L has to be closed under conjunction.) Consequently, there were no relevant proper extensions of L, but L itself. However, from his somewhat idiosyncratic definition of a language (reviewed Section 2) and from the comments following the definition of 'proper extension' (Brandom, 2008), p. 147, it is clear that Brandom wants to refer to sets of formulas that are closed under subformulas, but that are not necessarily closed under connecting already present formulas by the logical connectives. The simplest way to save Brandom's intentions is to assume that the axioms for negation, conjunction, and necessity are augmented by the conditions 'if $\neg F \in L$', 'if $F \wedge G \in L$', and 'if $\Box F \in L$', respectively. This assumption allows us to follow Brandom in speaking of languages and corresponding frames that need not be closed under applying logical connectives. A frame Inc$'$ for a language L', that properly extends L, is called *inferentially conservative with respect to* a frame Inc for L if for all $X, Y \subseteq L$ we have $X \models_{\text{Inc}} Y$ iff $X \models_{\text{Inc}'} Y$.

Brandom realizes that there are problems with the desired uniqueness of inferentially conservative extensions of frames in case of infinite theories. Therefore he defines the notion of a 'determined frame' as follows:

> Let L' be a proper extension of L and Inc be a frame for L. The *frame for L' determined by* Inc is the smallest frame for L' that is IC [inferentially conservative] with respect to Inc. (Brandom, 2008), p. 148

At first sight the use of the definite article, implying uniqueness, seems problematic. However 'smallest' here does *not* mean 'not contained in any other frame with the relevant property', but rather, as clarified by a preceding remark,

> [. . .] 'smallest' has the sense of, contained in every other frame for L' that is IC with respect to Inc. (Brandom, 2008), p. 148

This allows Brandom to announce:

> We now show that the determined frame exists. (If it does exist, it is immediate from the definition that it is unique.)
> (Brandom, 2008), p. 148

It is indeed not difficult to establish the existence of frames that are inferentially conservative over a given frame for a proper sublanguage. Uniqueness, as indicated, is less straightforward. (Brandom credits his research assistant Alp Aker for crucial technical contributions.)

Theorem 2. *For every proper extension L' of a language L and for every frame* Inc *for L there exists a frame* Inc'_d *for L' that is determined by* Inc*.*

The problem with Theorem 2 is not that it were wrong but that it does not match the purpose for which it was motivated. Remember from Section 3 that Brandom claims that although incompatibility semantics is holistic and non-compositional, it nevertheless enjoys recursive projectibility. To support this claim formally the following assertion, that is a kind of *inverse* to Theorem 2, has to be considered:

(Strong projectibility): *For every proper extension L' of a language L, every frame* Inc' *for L' is determined by some frame* Inc *over L.*

Arguably, it might suffice to establish the following slightly weaker version:

(Weak projectibility): *For every frame* Inc' *over L'_P there is some frame* Inc *over $L_P = P$, such that* Inc *the frame for L' is determined by* Inc*.*

However, it is not difficult to show that both forms of projectibility fail. (A concrete counter example will be specified below.) In contrast to Brandom's claims—e.g., in the cited passages on p. 135 and on p. 147 of (Brandom, 2008)—the semantic values, i.e. the incompatibilities as specified by a given frame, of logically compound formulas are *not* computable from the values of less complex formulas, in general. In fact, already a simple counting argument should make clear that there is *no deterministic way at all* in which a given frame for L' can be reduced to a frame over a proper sublanguage L: in general, if L' is a proper extension of L, there are strictly more different frames for L' than for L. Thus there is no surjective function from the set of frames for L to the set of frames for L'. To illustrate the problem with Brandom's suggestion of using inferential conservativeness to establish recursive projectibility consider the following simple example.

Let the language $L_P = P$ consist of just two atomic sentences (propositional variables), say, $P = \{a, b\}$. If we exclude the possibility that the empty set is incoherent and assume that neither a nor b is self-incompatible, just two possible frames, i.e., two different sets of incoherent subsets of L_P remain:

- $\mathrm{Inc}_1 = \{\{a, b\}\}$, meaning that a and b are incompatible,
- $\mathrm{Inc}_2 = \{\}$, meaning that a and b are not incompatible.

By the definition of (incompatibility) entailment, we have $a \not\models_{\mathrm{Inc}_1} b$ and $b \not\models_{\mathrm{Inc}_1} a$, but $a \models_{\mathrm{Inc}_2} b$ and $b \models_{\mathrm{Inc}_2} a$. Moreover, note that for *all* frames Inc over L_P and all $F, G \in L_P$ we have: $F \models_{\mathrm{Inc}} G$ iff $\{F, G\} \notin \mathrm{Inc}$. The fact that compatibility (i.e. non-incompatibility) between formulas implies entailment already hints at a problem. However, at this point, one might still be satisfied with the remark that the strange coincidence of compatibility and entailment is due to the absence of negation from the language. Let us therefore consider the proper extension $L'_P = \{a, b, \neg a, \neg b\}$ of L_P. It is easy to check that the following two frames Inc'_1 and Inc'_2 over L'_P are inferentially conservative with respect to Inc_1 and Inc_2, respectively:

- Inc'_1 consists of those theories over L'_P that contain $\{a, b\}$, or $\{\neg a, a\}$, or $\{\neg b, b\}$. (Note that, because of Persistence, a frame cannot just consist in the three exhibited two-element sets. We always have to close of with respect to supersets to obtain a frame.)

- Inc'_2 consists of those theories over L'_P that contain $\{a, \neg b\}$, or $\{\neg a, b\}$, or $\{\neg a, a\}$, or $\{\neg b, b\}$.

Since Inc'_1 is inferentially conservative over Inc_1, we still have $a \not\models_{\mathrm{Inc}'_1} b$ and $b \not\models_{\mathrm{Inc}'_1} a$. More generally, we obtain $F \not\models_{\mathrm{Inc}'_1} G$ for all $F, G \in L'_P$, where $F \neq G$. In contrast, for Inc'_2, we obtain $a \models_{\mathrm{Inc}'_2} b$ and $b \models_{\mathrm{Inc}'_2} a$, as required for the inferential conservativeness of Inc'_2 with respect to Inc_2. Moreover we have $\neg a \models_{\mathrm{Inc}'_2} \neg b$ as well as $\neg b \models_{\mathrm{Inc}'_2} \neg a$ by the form of Inc'_2 and the axioms. Note that in all frames each formula is incompatible with its negation (assuming, of course, that it is in the language). Moreover all frames that are inferentially conservative over Inc_1 have to contain $\{a, b\}$, since $\{a, b\} \in \mathrm{Inc}$ is equivalent to $\{a, b\} \models_{\mathrm{Inc}} \{\}$ for all frames Inc. Therefore Inc'_1 is the smallest (and in fact the only) frame for L'_P that is inferentially conservative with respect to Inc_1. Thus it is *determined* by Inc_1, according to Brandom's definition. The case for Inc'_2 is similar: it is the only frame for L'_P that is inferentially conservative over Inc_2. Consequently Inc'_2 is *determined* by Inc_2.

For our argument it is important to recognize that the above examples of determined frames leave many possible frames for L'_P as *not determined by any frame* for L_P. In particular, it is straightforward to check that in the only frame Inc for L_P where $a \models_{\mathrm{Inc}} b$ and $b \not\models_{\mathrm{Inc}} a$ the atomic formula a is self-incompatible, which implies that $a \models_{\mathrm{Inc}} \{\}$. But it is not difficult to specify a frame Inc' for L'_P where $a \models_{\mathrm{Inc}'} b$ and $b \not\models_{\mathrm{Inc}'} a$ as well as $a \not\models_{\mathrm{Inc}} \{\}$: let Inc' consist of just those theories that contain $\{a, \neg b\}$ or $\{a, \neg a\}$ or $\{b, \neg b\}$. Inc' is not inferentially conservative over *any* frame for L_P and thus cannot be recursively projected or in any other systematic way reduced to any frame over L_P. This provides the concrete counter example to (weak and strong) projectibility, announced earlier. As already indicated above, the central fact that not all frames over languages with negation can be reduced to frames without negation can also be established by checking that there are just 5 different frames for L_P, while there exist more than twice as many different frames for L'_P, each inducing a different entailment relation.

5 Consequences of the non-reducibility of frames

As we have seen in the last section, recursive projectibility, i.e. the claim that incompatibilities between logically complex formulas are determined by the incompatibilities between formulas that are less complex, cannot be maintained. Independently of the contents of Theorem 2, the presented examples show that there are frames over simple, finite languages with negation for which the incompatibilities are not determined by any frame over a language without negation. In other words, even the knowledge of the semantic status of *all* formulas without negation does not suffice to determine the semantic status of formulas with negation. Coming back to Brandom's five claims about incompatibility semantics, formulated in the Introduction of this paper, this means that claim 4 (recursive projectibility) cannot be maintained. Since claim 5, namely that the properties of incompatibility semantics allow to refute the assertion that holistic semantics cannot account for the learnability of a language, depends on the validity of claim 4, it remains unsubstantiated as well.

We emphasize that the outlined problem arises already for non-modal languages. Moreover, the axiom for conjunction stipulates a direct reduction of the status of a formula $F \wedge G$ to that of F and G, respectively, and thus does not contribute to the 'holism' or to the intensional character of incompatibility semantics. Brandom insists in defining disjunction and (material) implication in terms of negation and conjunction, just like in classical logic: $F \vee G =_{df.} \neg(\neg F \wedge \neg G)$ and $F \rightarrow G =_{df.} \neg(F \wedge \neg G)$. Therefore, as far as non-modal languages are concerned, the problem rests with the suggested semantics of negation. However, in light of claim 2 (intentionality), claim 3 (holism and non-compositionality), but also claim 5 (learnability), it might be deemed odd that Brandom does not consider more direct, alternative routes to provide meaning for disjunction and, in particular, implication. Of course, if the only aim were to characterize ordinary classical logic or the simplest modal logic, $S5$, in terms of incoherence/incompatibility, then restricting attention to negation and conjunction is an obvious move. However, why should one insist on classical logic or on $S5$ in the wider context of inferentialism and analytic pragmatism? Indeed, some of Brandom's remarks, in particular in his reply to an attempt to bring incompatibility semantics for modal logic closer to Kripke semantics (Göcke, Pleitz, & Wulfen, 2008)[4], indicate that Brandom were happy to go beyond just $S5$. Concerning the non-modal fragment of the language there are corresponding remarks on intuitionism in (Brandom,

[4]This is not the place to analyze the interesting ideas of Göcke, Pleitz, and von Wulfen. However, let me emphasize that some of their suggestions are clearly mistaken for precisely the same reason that invalidates some of Brandom's claims: worlds in a Kripke structure cannot be identified with sets of atomic sentences that are maximally coherent with respect to an incompatibility frame, since those sets, even jointly, do not determine the semantic status of negated sentences and of modal sentences.

2008). But remember that, while $\neg F$ abbreviates $F \to \bot$, falsum ($\bot$), disjunction, conjunction, and implication are not inter-definable in intuitionistic logic. In any case, I cannot imagine that Brandom really wants to claim that, e.g., the stipulation that $F \vee G$ abbreviates $\neg(\neg F \wedge \neg G)$ is all one must or should say about the learnability of disjunctive sentences, granted that we know how to learn the meanings of negation and conjunction. Consequently, the case for claim 5 would remain incomplete, even if recursive projectibility could be established.

There seems to be an easy way out of the quagmire of frames for negation-free languages: just insist that all *literals*, i.e. all atomic formulas and all their negations are always present in a language. However this suggestion does not fit well with Brandom's explicit intention to treat classical propositional connectives in an intensional manner (cf. claim 2), on a par with the modal operator, and without involving the notions of truth or falsity. Given the 'recursive' axiom for conjunction and the definition of disjunctions and implications as abbreviations of negated conjunctions, we are left with just a rather trivial notational variant of ordinary Tarskian truth functional semantics for classical logic. To see this, we identify classical interpretations, i.e. assignments of *true* and *false* to propositional variables $\{p_1, p_2, \ldots\}$, with maximally coherent subsets of literals $\{p_1, \neg p_1, p_2, \neg_2, \ldots\}$. More formally, any classical truth value assignment $v : P \mapsto \{true, false\}$ induces the set $\Phi(v) = \{p \mid v(p) = true\} \cup \{\neg p \mid v(p) = false\}$. Clearly Φ amounts to a bijection between truth value assignments and maximally coherent sets of literals, since every set that contains both p and $\neg p$ for some propositional variable $p \in P$ is incoherent, and since every coherent set of literals that contains neither p nor $\neg p$ can be extended to another coherent set by adding either p or $\neg p$. In other words: there remains only a superficial difference between talk about (Tarskian) interpretations satisfying a formula F and talk about maximally coherent subsets of literals that incompatibility-entail F, respectively. This can hardly be accepted as a way to save incompatibility semantics as intended.

6 Hints on dialogue game semantics

After this largely negative assessment of some central aspects of incompatibility semantics, we want to end at a more positive note by briefly suggesting that Brandom's fascinating project of providing a formal semantics that fits well into the wider frame of inferentialism and analytic pragmatism could well be based on quite different notions. In fact, I can think of many different ways of building formal semantics on pragmatist and inferentialist grounds. At least one concept of this kind is ready to be picked up from the literature: dialogue game semantics.

Already in the late 1950s Paul Lorenzen suggested to specify the meaning of logical connectives by reference to what we are obliged and entitled to do in confrontational dialogues involving logically complex assertions (Lorenzen, 1960). More precisely, Lorenzen suggested a strategic game that starts with the assertion of a sentence (formula) F by a proponent **P** to be challenged by an opponent **O** and proceeds by systematic attack and defense moves referring to relevant subformulas of F. The rules guiding this game take the following form:

Implication rule: If **P** asserts $F \to G$ then **O** is entitled to attack by asserting F in reply, which in turn obliges **P** to assert G as well.

Conjunction rule: If **P** asserts $F \wedge G$ then **O** is entitled to attack by obliging **P** to assert F, likewise **O** is entitled to attack by obliging **P** to assert G.

Disjunction rule: If **P** asserts $F \vee G$ then **O** is entitled to attack by obliging **P** to either assert F or to assert G, where the choice is up to **P**.

The rules have been formulated just for **P** as defender as **O** as attacker, but the implication rule entails that the roles may switch. Accordingly, analogous rules also hold for inverted roles, i.e. for **P** challenging assertions of **O**. $\neg F$ is defined as $F \to \bot$, where $\bot$ is a formula that can never be defended successfully; i.e. whoever asserts $\bot$ looses the dialogue game. Otherwise **P** wins the game if **O** attacks a sentence that **O** herself has already asserted previously. If furthermore certain structural rules (Lorenzen's *Rahmenregeln*), regulating the succession of attacks and defenses, are imposed, one can show that **P** has a winning strategy for exactly those initial assertions that are intuitionistically valid, see (Felscher, 1985). Already Lorenzen and his collaborators considered different versions of the game—for further references see (Felscher, 1986; Krabbe, 1985)—and it soon became clear that not only intuitionistic logic, but rather a large variety of logics, including classical logic and modal logics, can be characterized in a similar manner. Moreover, systematic correspondences between logical dialogue games and sequent as well as hypersequent based proof theory have been explored, e.g. in (Krabbe, 1985; Fermüller, 2003; Fermüller & Metcalfe, 2009). Those correspondences incidentally relate dialogue games also to the programme of proof theoretic semantics, mentioned briefly in Section 3. In the 1970s a particularly interesting variant of Lorenzen's ideas has been developed by Robin Giles in an attempt to provide 'tangible meaning' to logical connectives and atomic assertions as they arise in reasoning within theories of physics (Giles, 1974, 1977). Giles combines a Lorenzen style dialogue game with a betting scheme on the results of elementary experiments associated with atomic sentences that may show dispersion; i.e. the same experiment may yield a different result upon repetition—only a subjective success probability is stipulated. Giles showed that a strategy, that guarantees that no money is lost in average when arguing and betting accordingly, exists if and only if the originally asserted sentence corresponds to a formula that is valid in infinitely value Łukasiewicz logic. This result has later been generalized to further so-called t-norm based fuzzy logics and, again, connected to proof theoretical investigations (Ciabattoni, Fermüller, & Metcalfe, 2005; Fermüller & Metcalfe, 2009; Fermüller, 2009).

Admittedly, these short and vastly incomplete hints at dialogical approaches to logic cannot replace a serious investigation that seeks to clarify whether, why, and how the largely *implicit* reference to normative, pragmatic, and inferentialist concepts in dialogue games can be made *explicit*. I can only hope that my critical remarks on incompatibility semantics don't deter any friend of Brandom's general approach to semantics and to reasoning—among whom I certainly count myself—from taking up this suggestion.

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