On Matrices, Nmatrices, and Games

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Abstract

Hintikka's semantic game for classical logic is generalized to the family of all finite valued matrices. This in turn serves as a springboard for developing game semantics for all propositional formulas with respect to arbitrary finite non-deterministic matrices. In this approach a new concept of non-deterministic valuation, called 'liberal valuation', emerges that augments the usually employed static and dynamic valuations in a natural manner. Liberal valuation is shown to correspond to unrestricted semantic games, while the characterization of static and dynamic valuations involves certain restrictions of the game that are handled by an interactive pruning procedure.

1 Introduction

In this paper we connect two lines of research that seem to be hardly related at all at a first glance: semantic games, as introduced by Hintikka, on the one hand side, and deterministic as well as non-deterministic matrix based semantics on the other hand side. Hintikka's characterization of classical logic in terms of a competitive game between a Proponent **P**[1], who seeks to verify that a given formula is true in a given model, and an Opponent **O**, who challenges the Proponent's claim. While various extensions and variants of Hintikka's original game, that cover a broad range of nonclassical logics, have been described in the literature, it has so far seemingly never been attempted to uniformly define semantic games with respect to arbitrary finite truth tables. After briefly revisiting Hintikka's game in Section 2, we will generalize, in Section 3, the classical semantic game to so-called $\mathcal{M}$-games that correspond to valuations with respect to an arbitrary finite matrix $\mathcal{M}$; i.e., to a collection of finite truth tables. However $\mathcal{M}$-games will only serve as an intermediate station towards a more ambitious goal. In [2, 3] Arnon and Lev have introduced the concept of non-deterministic matrices, which, following Avron and his collaborators, we will call Nmatrices. This concept turned out to be a useful and versatile

[1] Various other names are used in the literature to refer to the pair of players of a semantic game: e.g., Verifier/Falsifier, Abelard/Eloise, or, as in [20], simply player I/player II. The use of Proponent/Opponent originates with Lorenzen [16, 17], who introduced a different, but related logical game already in the late 1950s.

tool, in particular in the context of proof theoretic investigations. Here we will look at Nmatrices from a purely semantic point of view, motivated by the question whether $\mathcal{M}$-games can be extended to non-deterministic truth tables. Two versions of Nmatrix based semantics are traditionally studied: (1) static valuations, where a logical connective may be associated with more than one (ordinary, i.e., deterministic) truth table, but where the chosen truth table remains the same throughout the valuation of a given formula; (2) dynamic valuations, where the chosen truth table may vary during the valuation, as long as the same truth value is assigned to identical subformulas. As we will show in Section 4, for every Nmatrix semantics $\mathcal{N}$ there is a corresponding $\mathcal{N}$-game that looks exactly like an $\mathcal{M}$-game: each rule refers to a connective and a truth value and specifies choices by **P** and **O** in a format that can be directly extracted from the corresponding truth table. However, it turns out that, without further restrictions, the Proponent's winning strategies in an $\mathcal{N}$-game neither match static nor dynamic valuations. These winning strategies rather give rise to a further concept of non-deterministic valuation, introduced as 'liberal valuation' here. In liberal valuations different truth values may be assigned to different occurrences of subformulas, even if they are identical. We will argue that liberal valuations are interesting and useful even independently of semantic games. But we provide characterizations of dynamic and static valuations in terms of $\mathcal{N}$-games as well. We show that certain pruning processes, to be applied to the unrestricted game viewed as a tree, lead to restricted $\mathcal{N}$-games that are adequate for dynamic or static valuations, depending on the specific version of pruning. The pruning process can be described as a series of interactions between the two players, thus sticking with the spirit of game semantics.

Arguably, game semantics displays its full power only at the first order level. Nevertheless we will remain in the realm of propositional logics here to keep the exposition reasonably concise. We do not expect any complications in generalizing $\mathcal{N}$-games to a broad family of quantifiers, namely so-called distribution quantifiers and their non-deterministic cousins. However, a clarification of this conjecture and a detailed exposition of non-deterministic distribution quantifiers via games calls for future work, as indicated in the conclusion of this paper.

2 Generalizing Hintkka's game

We begin by revisiting Hintikka's well known semantic game (also known as evaluation game) for classical logic. In that game, a player whom we will call the Proponent **P** seeks to defend the claim that a formula F is true in a given model M, while an Opponent **O** attempts to refute this claim. Since we are only interested in propositional connectives here, M can be identified with an assignment of 'true' or 'false' to propositional variables. At each stage of the game one of the two players asserts a subformula of F. The game is initiated by **P**'s assertion of F and proceeds in accordance with the following rules that refer to the outermost connective of the currently asserted formula.

($R_\wedge$) If **P** asserts $G \wedge H$ then **O** attacks by pointing either to the left or to the right subformula. At the next stage **P** has to assert G or H, according to **O**'s choice.

($R_\vee$) If **P** asserts $G \vee H$ then at the next stage **P** has to assert either G or H at her own choice.

($R_\neg$) If **P** asserts $\neg G$ then the roles of the two players are switched and the game continues with **O** asserting G.

Remark 1 Although usually omitted for Hintikka's games, a rule for implication can be compiled from $G \to H =_{\text{df.}} \neg G \vee H$:

($R_\to$) If **P** asserts $G \to H$ then **P** has a choice between continuing the game with **O**'s assertion of G after switching the players' roles or, alternatively, to continue with her own (**P**'s) assertion of H (without switching roles).

In Section 3, we will introduce the concept of $\mathcal{M}$-games that allows one to formulate rules in a many-valued setting, even for connectives that cannot be defined in terms of conjunction, disjunction, and negation.

Note that, although a role switch may occur due to negation, we have not specified what happens at a stage where **O** asserts a formula. Actually, this is not necessary since the roles are understood to be perfectly dual. In other words, we simply interchange **P** and **O** to obtain the rules for assertions of **O**. In any case, the game ends when an atomic formula A is asserted. If A is true in M then the player who asserts A wins (and the other player loses). Clearly we have a finite extensive-form zero-sum game of perfect information. We will consider only games of this type throughout the paper. The central fact about Hintikka's game can be expressed as follows.

Theorem 1 (Hintikka[2]) *A closed formula F is true in a 'model M iff* **P** *has a winning strategy in the semantic game starting with* **P***'s assertion of F.*

The game provides an alternative to standard Tarskian semantics: instead of referring to a recursively defined evaluation function based on truth tables, we refer to winning strategies in a two-person extensive form game. The full potential of the game theoretic approach to truth is borne out by various generalizations of classical first-order logic that arise by considering natural variations on the standard semantic game described above. Most prominently we obtain IF-logic by admitting that the players may have imperfect information about the previous choices at a given state of the game. This in particular allows one to model, e.g., branching quantifiers and so-called slashed quantifiers and connectives (see, e.g., [19, 15]).

Remark 2 Also modal logics can be characterized by appropriate semantic games (see, e.g., [8]). A less well-known, but also very fertile game theoretic approach to semantics is Giles's game for Łukasiewicz logic [11, 12, 9]. Giles developed his game semantics for reasoning under uncertainty independent from

[2]We refer to [20] for a comprehensive presentation of classical game semantics.

Hintkka. In fact he referred to Lorenzen's dialogue game for intuitionistic logic in formulating the logical rules of his game for infinitely-valued Łukasiewicz logic. However, with hindsight, it is clear that Giles's game is a species of semantic game, rather than a Lorenzen style dialogue game. For a recent exposition that makes the relation to Hintikka's game transparent and that explores one of the many applications beyond standard Łukasiewicz logic we refer to [10].

We want to generalize Hintkka's classical game in yet a further, seemingly still unexplored direction by designing games that correspond to arbitrary finite valued valuations of formulas as well as to valuations with respect to non-deterministic matrices (Nmatrices) [2, 3, 5]. Nmatrix semantics generalizes ordinary matrix semantics for many-valued logics in a natural and interesting way by allowing for choices between different truth values, instead of relying on strictly functional dependencies with respect to truth values of subformulas, in evaluating a given formula. Therefore we do not deal with two independent ways of generalizing classical game semantics here; we rather devise games for finite valued valuations in order to generalize those games further to match Nmatrix semantics. Nevertheless we argue that games for finite valued logics are interesting in their own right. For example, one may go on to study the effects of imperfect information for those games and design many-valued versions of IF-logic in that manner. To do so one should investigate rules for so-called distribution quantifiers. However, in order to keep the focus on the principles of deterministic and non-deterministic matrix semantics and their relation to game semantics, we will confine ourselves to propositional languages here.

Before going on to finite valued valuations, we briefly illustrate by a simple example the fact that non-determinism naturally pops up when playing around with semantic games, even if we stick with the two classical truth values. We have pointed out that the rules in Hintikka's game are perfectly symmetric for the two players. But we may ask what happens if we break this symmetry.

Example 1 *Let ud keep the rules for conjunction and for disjunction as they are. Note that we in fact have four rules so far:* $(R^{\mathbf{P}}_{\wedge})$, $(R^{\mathbf{O}}_{\wedge})$, $(R^{\mathbf{P}}_{\vee})$, $(R^{\mathbf{O}}_{\vee})$, *where* $(R^{\mathbf{P}}_{\wedge})$ *and* $(R^{\mathbf{P}}_{\vee})$ *are exactly as formulated under the names* $(R_{\wedge})$ *and* $(R_{\vee})$ *above, and* $(R^{\mathbf{O}}_{\wedge})$ *and* $(R^{\mathbf{O}}_{\vee})$ *arise from those rules by simply swapping* **P** *and* **O**. *Also the rule* $(R^{\mathbf{P}}_{\neg})$ *shall remain unchanged: if we are in a state where* **P** *asserts* $\neg F$*, the game continues in a state where* **O** *asserts* F*. However, we will alter* $(R^{\mathbf{O}}_{\neg})$ *as follows. If* **O** *asserts* $\neg F$ *she will be given a choice: she may either let the game continue with* **P***'s assertion of* F *(as in the standard rule) or ignore the negation sign and continue the game with her assertion of* F*. What is the effect of liberalizing the negation rule for* **O** *in this manner? Clearly, we have changed the semantics of negation. But can we relate this 'liberal' form of negation to any known logic? Note that we did not change the winning conditions: we still refer to an assignment of either* **t** *(true) or* **f** *(false) to atomic formulas and stipulate that the player asserting an atomic formula* A *(in the last stage of the game) wins (and the other player loses) if* A *is true and loses (and the other player wins) if* A *is false. Therefore we can still define logical validity and consequence in the usual way, by referring to games for all* *classical assignments. In other words our new game does define a logic; but it seems to be quite different from classical logics. For example, it is easy to see that* **P** *does not any longer have a winning strategy for every game starting with his assertion of $A \vee \neg A$. It turns out that winning strategies in the modified game correspond to valuations with respect to the non-deterministic truth tables*

	$\neg$
t	$\{\mathbf{f}\}$
f	$\{\mathbf{f}, \mathbf{t}\}$

and

		$\vee$
t	**t**	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{t}\}$
f	**t**	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

.

As we will see in Section 4, actually three different forms of Nmatrix based valuations emerge. They correspond to different (respectively missing) restrictions on the application of rules like ($R^{\mathbf{O}}_{\neg}$). For the current example ($A \vee \neg A$, where A is atomic) the three types of valuations coincide.

3 Games for finite valued deterministic valuations

The generalization of classical logic arising from truth functions over sets of truth values beyond $\{\mathbf{f}, \mathbf{t}\}$ is an old idea that has been explored in countless articles and books (see, e.g., [13, 18, 14, 6].

Remark 3 To some, the realm of finite-valued propositional actually seems a too simple playground to warrant yet further investigation. However, we emphasize that many apparently interesting claims about classical logics, like e.g. pointing out that different types of proof systems (sequent calculus, tableau, natural deduction, clause form translation plus resolution, etc.) can be translated into each other, are in fact empty if not formulated against a background of a wider class of logics that allows one to view point to point connections as particular instances of a general, schematically defined relation. It is in this latter spirit that we introduce a generalized concept of semantic games that fits the family of all logics that have a finite truth table semantics.

We largely follow the handbook article [6] in providing basic definitions that support formal investigations of the class of all finitely valued logics at an appropriately abstract level.

Definition 1 (Syntax) *A* propositional language $\mathcal{L}$ *consists of a countably infinite set* PV *of* propositional variables*, a finite set* $\mathrm{PC}_{\mathcal{L}}$ *of* propositional constants[3]*, and a non-empty finite set* $\mathrm{OP}_{\mathcal{L}}$ *of* propositional connectives*. Each* $\diamond \in \mathrm{OP}_{\mathcal{L}}$ *has a fixed finite arity* $ar(\diamond) \geq 1$*. The* formulas $\mathrm{FORM}_{\mathcal{L}}$ *over* $\mathcal{L}$ *are built up from* PV *and* $\mathrm{PC}_{\mathcal{L}}$ *using* $\mathrm{OP}_{\mathcal{L}}$*, as usual. The elements of* $\mathrm{PV} \cup \mathrm{PC}_{\mathcal{L}}$ *are called* atomic formulas*. By $sf(F)$ we denote the set of all subformulas of F, including F itself.*

We will usually drop the adjective 'propositional' from the above notions.

[3]Propositional constants could alternatively be treated as 0-ary connectives.

Throughout this paper we use $\mathcal{V}$ to denote the set of truth values. $\mathcal{V}$ is always finite here and contains at least two elements.

Definition 2 (Matrix semantics) *Given a language $\mathcal{L}$ we associate a function $\widehat{\diamond} : \mathcal{V}^n \to \mathcal{V}$ with each $\diamond \in \mathrm{OP}_\mathcal{L}$, where $ar(\diamond) = n$. $\widehat{\diamond}$ is also referred to as* truth table *for the connective $\diamond$. The set $\{\widehat{\diamond} : \diamond \in \mathrm{OP}_\mathcal{L}\}$ is augmented by a function that fixes a truth value $\widehat{C} \in \mathcal{V}$ for each $C \in \mathrm{PC}_\mathcal{L}$ to form the* matrix *$\mathcal{M}$ for $\mathcal{L}$.*

A partial valuation in $\mathcal{M}$ over an assignment $\alpha : \mathrm{PV} \to \mathcal{V}$ *is a function $v^\alpha_\mathcal{M} : \mathcal{F} \to \mathcal{V}$, where $\mathcal{F} \subseteq \mathrm{FORM}_\mathcal{L}$ is closed under subformulas, that satisfies the following conditions.*

(a) $v^\alpha_\mathcal{M}(F) = \alpha(F)$ *if* $F \in \mathrm{PV}$,

(b) $v^\alpha_\mathcal{M}(F) = \widehat{F}$ *if* $F \in \mathrm{PC}_\mathcal{L}$,

(c) $v^\alpha_\mathcal{M}(\diamond(F_1, \ldots, F_n)) = \widehat{\diamond}(v^\alpha_\mathcal{M}(F_1), \ldots, v^\alpha_\mathcal{M}(F_n))$ *if* $\diamond \in \mathrm{OP}_\mathcal{L}$ *with* $ar(\diamond) = n$.

If $\mathcal{F} = \mathrm{FORM}_\mathcal{L}$ then $v^\alpha_\mathcal{M}$ is a (total) valuation.

Proposition 1 *Every assignment α induces a unique partial valuation $v^\alpha_\mathcal{M}$ in $\mathcal{M}$ over α on its domain. Therefore every partial valuation $v^\alpha_\mathcal{M}$ can be uniquely extended to a total valuation over the same assignment α.*

Remark 4 We could go on to single out a set of designated truth values and define corresponding notions of *validity* and *logical consequence* with respect to a given matrix. However there is no need to do so here, since semantic games are only concerned with valuations. This entails that we are not interested in (deductive) logics here, but rather in the semantics of logical connectives.

Different kinds of generic *external proof systems* have been developed for finite valued logics. (By 'external' one refers to the presence of labels or signs, in addition to formulas in the system; see [6]). They are best described by reference to *signed formula expressions*. While we are not interested in proof systems here, our games are close enough to profit from (a somewhat simplified version[4] of) this formal meta-language for finite valued logics.

Definition 3 (Signed formula expression) *For every $F \in \mathrm{FORM}_\mathcal{L}$ and every $w \in \mathcal{V}$ the expression $w{:}F$ is called a* signed formula. Signed formula expressions *are defined inductively as follows.*

- *Every signed formula is a signed formula expression.*
- *If ϕ and ψ are signed formula expressions then so are $\phi \mathbin{\mathbb{A}} \psi$, $\phi \mathbin{\mathbb{V}} \psi$, $\phi \supset \psi$, $\phi \equiv \psi$, and $\neg\phi$.*

[4]In the full language of signed formula expressions, presented in [6], signs consists not of single truth values, but rather of sets of truth values. While leading to more compact expressions in general, this does not increase the expressiveness of the formalism. Moreover it could easily cause confusions when we move on to Nmatrices, where sets of truth values are employed in a related, but nevertheless somewhat different manner.

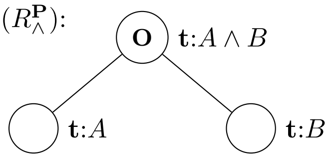
The semantics of signed formula expression refers to a given matrix semantics $\mathcal{M}$ over $\mathcal{L}$ and an assignment $\alpha : \mathrm{PV} \to \mathcal{V}$.

- *We call a signed formula $w{:}F$* true in $\mathcal{M}$ under α *if $v^{\alpha}_{\mathcal{M}}(F) = w$;*
- *$\phi \mathbin{\mathbb{A}} \psi$, $\phi \mathbin{\mathbb{V}} \psi$, $\phi \supset \psi$, $\phi \equiv \psi$, and $\neg\phi$ are true or false according to the truth or falsity of the subexpressions ϕ and ψ, where the meaning of the meta-connectives is that of ordinary classical conjunction, disjunction, implication, equivalence, and negation, respectively.*

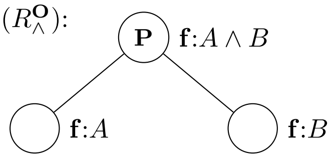
If $\phi \equiv \psi$ is true in $\mathcal{M}$ under all assignments then ϕ and ψ are called equivalent *in $\mathcal{M}$.*

At first sight it might not be obvious how to generalize Hintikka's game to an arbitrary matrix $\mathcal{M}$. Any combinations of the three basic operations 'choice by Proponent **P**', 'choice by Opponent **O**', and 'role switch between **P** and **O**', clearly do not lead beyond classical logic. We only obtain rules for further classical connectives in this manner. However a closer look at 'role switch' guides the way. Note that it is not enough to know the current (sub)formula at a given state of the game; we also have to keep track of all role switches. In other words, at any give state of the game we have to record whether **P** wants to show that the current (sub)formula is true (as in the beginning of the game) or whether she wants to show that it is false (like **O** in the beginning of the game). It therefore seems natural to depict, e.g., the conjunction rules $(R^{\mathbf{P}}_{\wedge})$ and $(R^{\mathbf{O}}_{\wedge})$ as *rule trees*:

$(R^{\mathbf{P}}_{\wedge})$:

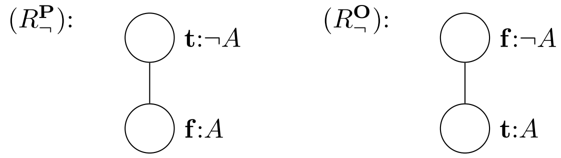


$(R^{\mathbf{O}}_{\wedge})$:



where the labels **P** and **O** at the root nodes indicate which player's turn it is to choose the successor state. Note that these labels do *not* indicate which player is the one who asserts $A \wedge B$. This job is rather taken over by the truth value signs now: $\mathbf{t}{:}A \wedge B$ means that **P** wants to show that $A \wedge B$ is true, whereas $\mathbf{f}{:}A \wedge B$ means that **P** wants to show that $A \wedge B$ is false. The latter fact was originally expressed by saying that it is **O** who asserts $A \wedge B$ (see Section 2). This is an important observation for the generalized semantic game, introduced below: we will no longer speak of a formula asserted by some player, but rather about a *signed formula* $w{:}F$, where **P** seeks to verify that $v^{\alpha}_{\mathcal{M}}(F) = w$ with respect to the given assignment α and **O** seeks to falsify this claim.

Trees for the disjunction rules $(R^{\mathbf{P}}_{\vee})$ and $(R^{\mathbf{O}}_{\vee})$ are obtained by replacing $\wedge$ by $\vee$, **P** by **O**, and **t** by **f** in the trees for $(R^{\mathbf{P}}_{\wedge})$ and $(R^{\mathbf{O}}_{\wedge})$, respectively. The negation rules look as follows:

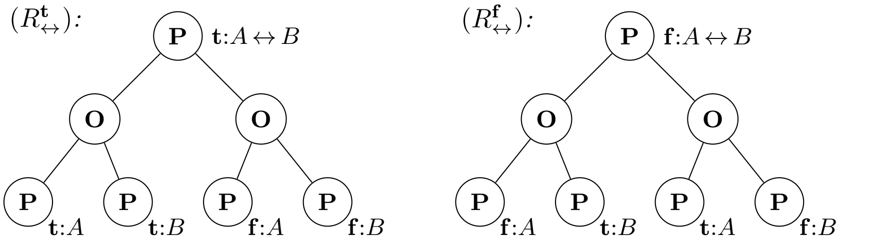


Remark 5 Note that we have not labeled the root nodes by either **P** or **O**. The role switch is solely reflected by the switch of truth value signs. Since there is only one successor state in each case it does not matter to which player (if any) we want to attribute the corresponding move.

As already explained, the truth value sign that we attach to the current formula of a game state determines the roles of two players. Therefore the superscripts **P** and **O** in the rule names are redundant and are in fact somewhat misleading now. Remember, that instead of talking about some player's assertion of a formula, we rather talk about a *signed formula* that **P** wants to verify and **O** wants to falsify. The rules are completely determined by the truth value sign and the connective. Consequently the superscripts in the rule names will from now on be replaced by the truth value exhibited in the signed formula at the root of the rule tree.

Remark 6 While the disjunction and conjunction rules of Hintikka's game only exhibit choices to be made either solely by **P** or solely by **O**, in general choices by both players are involved in a full round of the game, as illustrated for the equivalence connective ($\leftrightarrow$) in Example 2.

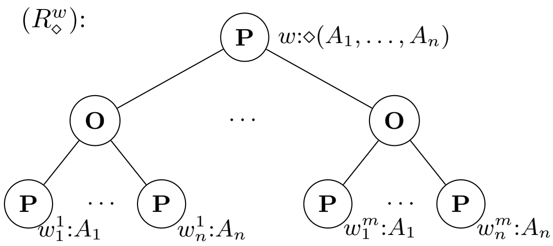
Example 2 *To understand the following rules for classical equivalence ($\leftrightarrow$) just remember that equivalence can be expressed by combining conjunction, disjunction, and negation. More precisely, the following signed formula expression are valid for classical semantics:* $\mathbf{t}{:}A{\leftrightarrow}B \equiv ((\mathbf{t}{:}A \mathbin{\&} \mathbf{t}{:}B) \vee (\mathbf{f}{:}A \mathbin{\&} \mathbf{f}{:}B))$ *and* $\mathbf{f}{:}A{\leftrightarrow}B \equiv ((\mathbf{f}{:}A \mathbin{\&} \mathbf{t}{:}B) \vee (\mathbf{t}{:}A \mathbin{\&} \mathbf{f}{:}B))$*. These expressions suggest the following game rule trees:*



The rules for $\leftrightarrow$ in Example 2 already indicate the general form of a semantic game rule tree for an n-ary connective $\diamond$ and some truth value $w \in \mathcal{V}$ exhibited in Figure 1.

Remark 7 (on Figure 1). Like in Hintikka's original game, rules are understood as *schemes*: we assume $A_1, \ldots, A_n$ to be pairwise distinct propositional variables and have to instantiate those variables accordingly in any *application* of $(R^w_\diamond)$ to a concrete formula $F = \diamond(F_1, \ldots, F_n)$. Each such application

Figure 1: General rule format (‘rule tree’) for connective $\diamond$ and truth value w:



where $w^i_j \in \mathcal{V}$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

corresponds to a *round* of the game, where in any round first **P** may choose between different options of defending her claim that F evaluates to w. These options correspond to the **O**-labeled successor nodes of the **P**-labeled root of the rule tree. We will simply speak of **O**-nodes and **P**-nodes from now on. In the second part of a round it is **O**’s turn to pick a successor node and thus a particular signed subformula $w^i_j{:}A_j$. The fact that the corresponding nodes are **P**-nodes indicates that, unless we have reached an atomic formula, a new round (of the same general type) follows at such states.

Remark 8 We shall look at a rule from **P**’s point of view. Therefore **P**’s options correspond to disjunction and **O**’s options correspond to conjunction at the meta-level. It may also happen that according to the truth table $\widehat{\diamond}$ a formula of the form $\diamond(F_1, \ldots, F_n)$ can never evaluate to a particular truth value w. In that case the rule $(R^w_\diamond)$ consists solely of the **P**-node labeled by $w{:}\diamond(F_1, \ldots, F_n)$.

As made clear in Example 2 and in Remark 8, we want the rule tree to match certain disjunctions of conjunctions of signed formulas that are equivalent to the relevant signed formula. For this purpose we employ complete disjunctive normal forms as specified in the following definition.

Definition 4 *The* complete disjunctive normal form *with respect to a matrix $\mathcal{M}$ for the signed formula $w{:}\diamond(A_1, \ldots, A_n)$ is the signed formula expression*

$$w{:}\diamond(A_1, \ldots, A_n) \quad \equiv \quad \bigvee_{\substack{(w_1, \ldots, w_n) \in \mathcal{V}^n \\ \widehat{\diamond}(w_1, \ldots, w_n) = w}} \quad \bigwedge_{1 \leq i \leq n} w_i{:}A_i \quad .$$

Remark 9 Strictly speaking, complete disjunctive normal forms are only unique up to the order of disjuncts and of conjuncts within a disjunct. However it is convenient to assume that this order is fixed. (E.g., by respecting the lexicographic order on tuples of truth values, based on some fixed linear order on $\mathcal{V}$.) This allows us to speak of *the* complete disjunctive normal and of *the* $\mathcal{M}$-game based on a matrix $\mathcal{M}$ (in Definition 6, below).

Definition 5 *A rule of* $(R^w_\diamond)$*, as exhibited in Figure 1, is said to* correspond to the complete disjunctive normal form *if the labels* $w_i{:}A^j_i$ *of the* **P***-nodes (leaves) of the rule tree match the conjuncts of the signed formula expression exhibited in Definition 4.*

Remark 10 In general, there are many equivalent disjunctive normal forms for a given connective and truth value. One may want to base the game on shorter normal forms instead of the complete ones specified in Definition 4, whenever available. Such shorter normal forms can be obtained from the complete norm forms by appealing to various kind of general laws for signed formula expressions (as explained in [6]). In fact, we could alternatively also specify rules where first **O** and only then **P** make their respective choices. Such rules correspond to conjunctive, rather than to disjunctive normal forms. Moreover, we could have rules of both types in a single semantic game. This will in many cases lead to more compact rules. For sake of clarity, we will not care about such optimizations here and stick with rules that correspond to complete disjunctive norm forms. (However our results can straightforwardly be generalized to cover other types of rules as well.)

Remark 11 Remember that the complete disjunctive normal form for the signed formula $w{:}\diamond(A_1, \ldots, A_n)$ is empty if $\widehat{\diamond}(v_1, \ldots, v_n) \neq w$ for all $v_1, \ldots, v_n \in \mathcal{V}$. **P**, of course, loses the game when such a state is reached.

After these preparations, we can now specify semantic games matching matrix semantics as trees that are composed of appropriate instances of rule trees.

Definition 6 ($\mathcal{M}$-game) *Given a formula* $F \in \mathrm{FORM}_\mathcal{L}$*, a truth value* $w \in \mathcal{V}$*, and an assignment* $\alpha : \mathrm{PV} \to \mathcal{V}$ *the* $\mathcal{M}$-game for $w{:}F$ under α *is defined inductively as an ordered tree as follows.*

- *If* F *is atomic then the game consists of a single* **P***-node labeled by* $w{:}F$.
  - *If* $F \in \mathrm{PV}$ *then* **P** *wins if* $\alpha(F) = w$*, otherwise* **O** *wins.*
  - *If* $F \in \mathrm{PC}_\mathcal{L}$ *then* **P** *wins if* $\widehat{F} = w$*, otherwise* **O** *wins.*
- *If* $F = w{:}\diamond(F_1, \ldots, F_n)$ *then the game begins with an instance of the tree shown in Figure 1, where* $(R^w_\diamond)$ *corresponds to the complete disjunctive normal form (specified in Definition 4): the* **P***-node at the root is labeled with* $w{:}F$ *and the* **P***-nodes at the bottom are labeled with signed formulas* $w^i_j{:}F_j$*, obtained by instantiating the corresponding* $w^i_j{:}A_j$ *in the rule tree for for* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*. Any such node labeled with* $w^i_j{:}F_j$ *is the root of an* $\mathcal{M}$*-game for* $w_i{:}F_j$ *under* α*. If the disjunctive normal form is empty, then the* **P***-node labeled by* $w{:}F$ *is a game at which* **O** *wins and* **P** *loses.*

Strategies in extensive form games are usually defined as functions from histories to moves. For our purposes it is convenient to identify a strategy for **P** with a sub-tree of the game that singles out a particular successor to each

P-node, but leaves the possible moves of **O**, represented by the successor nodes of an **O**-node, unchanged. (Strategies for **O** could be defined likewise; but we do not care about them here.)

Definition 7 ($\mathcal{M}$-game strategy) *A strategy τ for* **P** *in an $\mathcal{M}$-game for $w{:}F$ under α arises from the tree representing the $\mathcal{M}$-game by removing all except one successor (***O***-)node to each internal* **P***-node in the tree. If* **P** *wins at every leaf node of τ then τ is called a* winning strategy for **P**.

The adequateness of $\mathcal{M}$-games is expressed by the following theorem.

Theorem 2 *Given a language $\mathcal{L}$ and a matrix $\mathcal{M}$, the following are equivalent for every formula F, truth value $v \in \mathcal{V}$ and assignment α.*

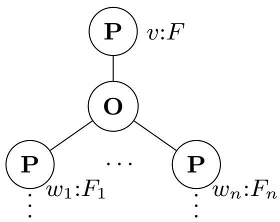
(1) **P** *has a winning strategy τ for the $\mathcal{M}$-game for $v{:}F$ under α.*

(2) *$v^{\alpha}_{\mathcal{M}}(F) = v$, where $v^{\alpha}_{\mathcal{M}}$ is the (unique) valuation over α.*

Proof.
(1) $\Rightarrow$ (2): Given τ, we define $v_\tau(G) = w$ whenever there is a **P**-node in τ labeled by $w{:}G$. We have $v_\tau(F) = v$, since the root of τ is labeled by $v{:}F$. It remains to show that v_τ is the partial valuation induced by α in $\mathcal{M}$ on $\mathrm{sf}(F)$ and therefore can be extended to the total evaluation $v^{\alpha}_{\mathcal{M}}$ by Proposition 1. We proceed by induction on the depth $d(\tau)$ of τ.

- $d(\tau) = 0$: This means that τ consists of a **P**-node labeled by $v{:}F$. Since τ is a winning strategy we have either $F \in \mathrm{PV}$ and $v_\tau(F) = \alpha(F) = v$ or $F \in \mathrm{PC}_{\mathcal{L}}$ and $v_\tau(F) = \widehat{F} = v$, as required.

- $d(\tau) > 0$: By definition, τ picks out (via a particular **O**-node) a disjunct of the complete disjunctive normal form to which the relevant rule $(R^v_\diamond)$ corresponds. It therefore looks as follows:



where, $F = \diamond(F_1, \ldots, F_n)$ and $\widehat{\diamond}(w_1, \ldots, w_n) = v$. By the induction hypothesis, the sub-trees of τ that are rooted in the lower **P**-nodes, labeled by $w_i{:}F_i$, for $i = 1, \ldots, n$, induce the partial valuations v^i_τ over α on $\mathcal{M}$, that are defined on $\mathrm{sf}(F_i)$ for $i = 1, \ldots, n$, respectively. Moreover we have $v^i_\tau(F_i) = w_i$. Since partial valuations are completely determined by α and $\mathcal{M}$, we have $v^i_\tau(G) = v^j_\tau(G)$ for every $G \in \mathrm{sf}(F_i) \cap \mathrm{sf}(F_j)$. We therefore can extend all v^i_τ for $i = 1, \ldots, n$, to the common partial valuation v_τ in $\mathrm{sf}(F) = \{F\} \cup \bigcup_{1 \le i \le n} \mathrm{sf}(F_i)$, where $v_\tau(F) = \widehat{\diamond}(w_1, \ldots, w_n) = v$, as required.

(2) $\Rightarrow$ (1): We show by induction on F that any given valuation $v^\alpha_\mathcal{M}$, where $v^\alpha_\mathcal{M}(F) = v$, induces an $\mathcal{M}$-game winning strategy τ for **P** rooted in $v{:}F$.

- If F is atomic, then τ consists just in a single **P**-node labeled by $v{:}F$.
- Suppose $F = \diamond(F_1, \ldots, F_n)$: By the induction hypothesis $v^\alpha_\mathcal{M}(F_i) = w_i$ there is a winning strategy τ_i for **P** in the $\mathcal{M}$-game for $w_i{:}F_i$ under α for every $i = 1, \ldots, n$. If $\widehat{\diamond}(w_1, \ldots, w_n) = v$ then there is a disjunct $\bigwedge\!\!\!\bigwedge_{1 \leq i \leq n} w_i{:}F_i$ in the complete disjunctive normal form for $v{:}\diamond(F_1, \ldots, F_n)$. Since the rules of the game correspond to complete disjunctive normal forms, this means that we obtain the required winning strategy τ as follows. Join the root (**P**-)nodes of $\tau_1, \ldots, \tau_n$ labeled by $w_1{:}F_1, \ldots, w_n{:}F_n$, respectively, by a introducing a common parent **O**-node, and add as a parent node to this **O**-node a **P**-node labeled by $v{:}F$. □

Remark 12 Väänänen [20] describes how Hintikka's semantic game can be modified to obtain a model existence game for classical logic without changing the basic form of the rules for the logical connectives. The model existence game, in turn, is related to a (signed analytic) tableau calculus for classical logic. Roughly speaking, a winning strategy in the model existence game for the player who seeks to show that there exists an interpretation in which F evaluates to **f** can be extracted from every fully expanded, but non-closed tableau for a formula F (and therefore with the signed formula **f**:F at its root). The same holds if **f** is replaced by **t**. This connection between semantic games and signed analytic tableaux can be generalized to finite valued logics. (See, e.g., [14] for a tableau calculus of the appropriate type.) In any case, the relation between the rules of $\mathcal{M}$-games and logical rules for external proof systems of various types should not be surprising, since they are all based on normal forms for signed formulas, as explained in detail in [6].

4 Nmatrices and corresponding games

As already indicated in Section 2, non-deterministic matrices (*Nmatrices*) generalize standard matrix semantics in a fertile manner, motivated by a host of interesting applications. The concept was introduced in [2] and has been studied in many papers by Arnon Avron and his collaborators since. (We refer to the handbook article [5] for a recent overview.) Nmatrices induce two different forms of evaluation for propositional languages as specified in Definition 1. The following definition closely follows Definition 8 of Section 3, but differs in some inessential details from the terminology used by Avron and his colleagues.

Definition 8 (Nmatrix semantics) *Given a language $\mathcal{L}$ we associate a function $\widetilde{\diamond} : \mathcal{V}^n \to 2^\mathcal{V} \setminus \{\emptyset\}$ with each $\diamond \in \mathrm{OP}_\mathcal{L}$, where $ar(\diamond) = n$. $\widetilde{\diamond}$ is also referred to as* non-deterministic truth table *for the connective $\diamond$. The set $\{\widetilde{\diamond} : \diamond \in \mathrm{OP}_\mathcal{L}\}$ is augmented by a function that fixes a non-empty set of truth values $\widetilde{C} \subseteq \mathcal{V}$ for each $C \in \mathrm{PC}_\mathcal{L}$ to form the* Nmatrix $\mathcal{N}$ *for $\mathcal{L}$.*

A partial dynamic valuation in $\mathcal{N}$ *over an assignment $\alpha : \mathrm{PV} \to \mathcal{V}$ is a function $dv^\alpha_\mathcal{N} : \mathcal{F} \to \mathcal{V}$, where $\mathcal{F} \subseteq \mathrm{FORM}_\mathcal{L}$ is closed under subformulas, that satisfies the following conditions.*

(a) $dv^{\alpha}_{\mathcal{N}}(F) = \alpha(F)$ *if* $F \in \mathrm{PV}$,

(b) $dv^{\alpha}_{\mathcal{N}}(F) \in \widetilde{F}$ *if* $F \in \mathrm{PC}_{\mathcal{L}}$,

(c) $dv^{\alpha}_{\mathcal{N}}(\diamond(F_1, \ldots, F_n)) \in \widetilde{\diamond}(dv^{\alpha}_{\mathcal{N}}(F_1), \ldots, dv^{\alpha}_{\mathcal{N}}(F_n))$ *if* $\diamond \in \mathrm{OP}_{\mathcal{L}}$*,* $ar(\diamond) = n$*.*

If $\mathcal{F} = \mathrm{FORM}_{\mathcal{L}}$ *then* $dv^{\alpha}_{\mathcal{N}}$ *is a* (total) valuation*.*

A (partial) static valuation $sv^{\alpha}_{\mathcal{N}}$ in $\mathcal{N}$ over assignment α *is a (partial) dynamic valuation in* $\mathcal{N}$ *over* α *that additionally satisfies the following functionality principle: for each* $\diamond \in \mathrm{OP}_{\mathcal{L}}$ *and for every* $G_1, \ldots, G_n, F_1, \ldots F_n \in \mathrm{FORM}_{\mathcal{L}}$

$$sv^{\alpha}_{\mathcal{N}}(\diamond(G_1, \ldots, G_n) = sv^{\alpha}_{\mathcal{N}}(\diamond(F_1, \ldots, F_n)) \text{ if } sv^{\alpha}_{\mathcal{N}}(G_i) = sv^{\alpha}_{\mathcal{N}}(F_i), i = 1, \ldots, n.$$

Like an ordinary matrix based valuation, every Nmatrix based valuation assigns a definite truth value to each formula. However now different such valuations may extend the same assignment and result in different truth values getting assigned to the same formula.

Note that an ordinary matrix corresponds to the special case of an Nmatrix, where $\widetilde{\diamond}(v_1, \ldots, v_n)$ is a singleton set for every $\diamond \in \mathrm{OP}_{\mathcal{L}}$ and $v \in \mathcal{V}$. Clearly dynamic and static valuation coincide in that case.

Definition 9 *Given an Nmatrix* $\mathcal{N}$ *for* $\mathcal{L}$*, a matrix* $\mathcal{M}$ *for* $\mathcal{L}$ *is said to be a* determination *of* $\mathcal{N}$ *if* $\widehat{\diamond}(v_1, \ldots, v_n) \in \widetilde{\diamond}(v_1, \ldots, v_n)$ *for every* $\diamond \in \mathrm{OP}_{\mathcal{L}}$ *and* $\widehat{C} \in \widetilde{C}$ *for every* $C \in \mathrm{PC}_{\mathcal{L}}$*, where* $\widehat{\diamond}$ *and* $\widetilde{\diamond}$ *denote the truth tables for* $\diamond$ *in* $\mathcal{M}$ *and in* $\mathcal{N}$*, respectively.*

This notion allows us to formulate the following observation compactly.

Proposition 2 *For every formula* $F \in \mathrm{FORM}_{\mathcal{L}}$ *and every assignment* α *there is a static valuation* $sv^{\alpha}_{\mathcal{N}}$ *in Nmatrix* $\mathcal{N}$ *over* α *such that* $sv^{\alpha}_{\mathcal{N}}(F) = v$ *iff* $v^{\alpha}_{\mathcal{M}}(F) = v$ *the valuation* $v^{\alpha}_{\mathcal{M}}$ *over* α *in some determination* $\mathcal{M}$ *of* $\mathcal{N}$*.*

Together with Theorem 2, Proposition 2 entails the following characterization of static valuations.

Corollary 1 *For every formula* $F \in \mathrm{FORM}_{\mathcal{L}}$ *and every assignment* α *there is a static valuation* $sv^{\alpha}_{\mathcal{N}}$ *over* α *in* $\mathcal{N}$ *such that* $sv^{\alpha}_{\mathcal{N}}(F) = v$ *iff* **P** *has a winning strategy in some* $\mathcal{M}$*-game for* $v{:}F$ *under* α*, where* $\mathcal{M}$ *is a determination of* $\mathcal{N}$*.*

To characterize dynamic semantics, as well as to introduce yet a further type of interpretation based on Nmatrices, we prefer an alternative route to corresponding semantic games. We can still use signed formula expressions to talk formally about Nmatrix semantics. But since the same assignment may induce different dynamic valuations, we have to define their truth differently.

Definition 10 (Signed formulas for Nmatrices) *A signed formula* $w{:}F$ *is* true in an Nmatrix $\mathcal{N}$ with respect to a dynamic valuation $dv^{\alpha}_{\mathcal{N}}$ *if* $dv^{\alpha}_{\mathcal{N}}(F) = w$*. The generalization of this definition to signed formula expressions is like in Definition 3.*

For our purposes it suffices that we can express truth conditions for signed formulas in Nmatrix semantics in close analogy to complete disjunctive normal forms in matrix semantics.

Definition 11 *The* complete disjunctive truth condition $\bar{\Delta}^w_{\diamond(F_1,\ldots,F_n)}$ *with respect to an Nmatix $\mathcal{N}$ for the signed formula $w{:}\diamond(F_1,\ldots,F_n)$ is given by*

$$\bar{\Delta}^w_{\diamond(F_1,\ldots,F_n)} \quad = \quad \bigvee_{\substack{(w_1,\ldots,w_n)\in\mathcal{V}^n \\ w\in\tilde{\diamond}(w_1,\ldots,w_n)}} \quad \bigwedge_{1\le i\le n} w_i{:}F_i \quad .$$

The following fact follows directly from the definition of a dynamic valuation.

Proposition 3 *Let $F=\diamond(F_1,\ldots,F_n)\in\mathrm{FORM}_{\mathcal{L}}$, $w\in\mathcal{V}$, and $\mathcal{N}$ an Nmatrix for $\mathcal{L}$. With respect to every partial dynamic valuation $dv^{\alpha}_{\mathcal{N}}$, if $w{:}F$ is true, then also its corresponding complete disjunctive truth condition $\bar{\Delta}^w_{\diamond(F_1,\ldots,F_n)}$ is true. Conversely, if $\bar{\Delta}^w_{\diamond(F_1,\ldots,F_n)}$ is true with respect to some partial dynamic valuation $dv^{\alpha}_{\mathcal{N}}$ defined on $sf(\bar{\Delta}^w_{\diamond(F_1,\ldots,F_n)})\subseteq\bigcup_{1\le i\le n} sf(F_i)$, then $w{:}F$ is true with respect to some extension of $dv^{\alpha}_{\mathcal{N}}$ that is defined also on F.*

We define unrestricted $\mathcal{N}$-games based on complete disjunctive truth conditions. While only certain sub-games ('restricted $\mathcal{N}$-games') will correspond to dynamic and static valuation, respectively, unrestricted $\mathcal{N}$-games are of independent interest as well: they generalize $\mathcal{M}$-games (Definition 6) straightforwardly and therefore appear to be a natural concept, at least from a game semantic point of view. Moreover, we argue that unrestricted $\mathcal{N}$-games codify a new version of non-deterministic matrix evaluation that may well turn out to be useful for various applications, as briefly indicated in Example 4, below.

Definition 12 (Unresticted $\mathcal{N}$-games) *Given a formula $F\in\mathrm{FORM}_{\mathcal{L}}$, a truth value $w\in\mathcal{V}$, and an assignment $\alpha:\mathrm{PV}\to\mathcal{V}$, the* unrestricted $\mathcal{N}$-game for $w{:}F$ under α *is defined inductively as the following ordered tree.*

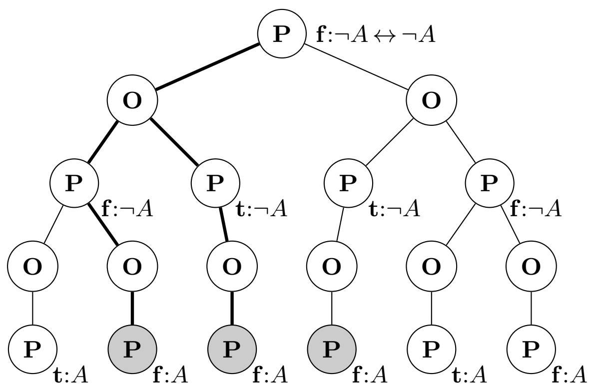
- *If F is an atomic formula the game consists of a single* **P***-node.*
  - *If $F\in\mathrm{PV}$ then* **P** *wins if $\alpha(F)=w$, otherwise* **O** *wins.*
  - *If $F\in\mathrm{PC}_{\mathcal{L}}$ then* **P** *wins if $w\in\widetilde{F}$, otherwise* **O** *wins.*
- *If $F = w{:}\diamond(F_1,\ldots,F_n)$ then, just like for $\mathcal{M}$-games, the game begins with a corresponding instance of the tree in Figure 1. The only difference to $\mathcal{M}$-games is that the labels at the leaves of the rule tree in Figure 1 are now obtained according to the complete disjunctive truth condition $\bar{\Delta}^w_{\diamond(A_1,\ldots,A_n)}$. Each of the lower* **P***-nodes labeled with $w^i_j{:}F_j$, obtained by instantiating F_j for A_j is the root of an unrestricted $\mathcal{N}$-game for $w^i_j{:}F_j$ under α. If the complete disjunctive truth condition is empty, then the* **P***-node labeled by $w{:}F$ is a game at which* **O** *wins and* **P** *loses.*

The notion of a (winning) strategy for **P** *in $\mathcal{N}$-games remains exactly as in Definition 7. I.e., strategies arise from game trees by deleting all but one of the successor* **P***-nodes to every* **O***-node.*

Remark 13 Unrestricted $\mathcal{N}$-games cannot be distinguished *locally* from $\mathcal{M}$-games. More precisely, any group of nodes that arises by instantiating a single rule tree, i.e. some **P**-node, its succeeding **O**-nodes and the immediate successors (**P**-nodes) of the latter, could occur in an $\mathcal{N}$-tree as well as in an $\mathcal{M}$-tree. The only difference is as follows: in a rule tree based on a complete disjunctive truth condition for $w{:}\diamond(A_1, \dots, A_n)$ with respect to a Nmatrix $\mathcal{N}$ there may occur an **O**-node that has exactly the same sequence of successor **P**-nodes as an **O**-node that occurs in a rule tree for $v{:}\diamond(A_1, \dots, A_n)$, where $v \neq w$ (but where the truth condition refers to the same Nmatrix $\mathcal{N}$). This cannot happen for rules based on complete disjunctive normal forms with respect to a matrix $\mathcal{M}$. With respect to given matrix, determinism implies that each conjunction $\bigwedge\!\!\!\bigwedge_{1 \leq i \leq n} w_i{:}A_i$, where $w_1, \dots, w_n \in \mathcal{V}$ occurs as disjunct of exactly one complete disjunctive normal form for a signed formula of type $w{:}\diamond(A_1, \dots, A_n)$, where $w \in \mathcal{V}$. This observation will become important below, for describing a certain 'pruning mechanism' for unrestricted $\mathcal{N}$-games.

Unrestricted $\mathcal{N}$-games characterize neither dynamic nor static valuations as illustrated be the following example.

Example 3 *Let us consider an unrestricted $\mathcal{N}$-game for a language with the classical equivalence connective $\leftrightarrow$ (for which we have already presented adequate rules in Section 3) and the 'liberal negation' $\neg$ considered in Example 1, at the end of Section 2), where $\tilde{\neg}(\mathbf{t}) = \{\mathbf{f}\}$, $\tilde{\neg}(\mathbf{f}) = \{\mathbf{f}, \mathbf{t}\}$. According to Definition 12 the game for $\mathbf{f} : \neg A \leftrightarrow \neg A$ under the assignment $\alpha(A) = \mathbf{f}$ looks as follows:*



where the gray leaf nodes represent those final states at which **P** *wins. The thick lines single out a sub-tree that constitutes a winning strategy for* **P** *for* $\mathbf{f}{:}\neg A \leftrightarrow \neg A$ *under α. This should be contrasted with the fact that there exists neither a dynamic nor a static valuation in the indicated Nmatrix semantics that returns* $\mathbf{f}$ *for* $\neg A \leftrightarrow \neg A$ *under any assignment.*

The key to defining a form of valuation that remains close to Definition 8, but matches unrestricted $\mathcal{N}$-games, is to distinguish *occurrences* of formulas from the formulas themselves.

Definition 13 *An* occurrence $\overline{G}$ in F *of a subformula* G *of* F *is defined as a tuple* $\overline{G} = (G, p, F)$*, where* p *is some marker that uniquely identifies a specific position within* F *at which* G *occurs. The set of all occurrences of subformulas of* F *in* F *is denoted by* $osf(F)$.

Remark 14 We have deliberately refrained from specifying *how* positions within formulas are represented formally, since there is no need to worry about details of such a representation here. Whenever it is necessary to distinguish between occurrences of some formula G and the formula itself we will use $\overline{G}$ to denote the relevant occurrence. (The context will always fix the formula F of whose subformulas we speak.)

Definition 14 (Liberal valuation) *Given an Nmatrix* $\mathcal{N}$ *for a language* $\mathcal{L}$*, a* liberal valuation in $\mathcal{N}$ on $F \in \mathrm{FORM}_{\mathcal{L}}$ over an assignment $\alpha : \mathrm{PV} \to \mathcal{V}$ *is a function* $lv^{\alpha}_{\mathcal{N}} : osf(F) \to \mathcal{V}$ *that satisfies the following conditions.*

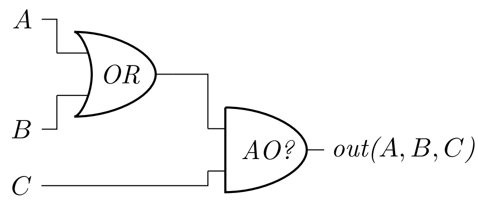
(a) $lv^{\alpha}_{\mathcal{N}}(\overline{G}) = \alpha(G)$ *if* $G \in \mathrm{PV}$,

(b) $lv^{\alpha}_{\mathcal{N}}(\overline{G}) \in \widetilde{G}$ *if* $G \in \mathrm{PC}_{\mathcal{L}}$,

(c) $lv^{\alpha}_{\mathcal{N}}(\overline{\diamond(\overline{G_1}, \ldots, \overline{G_n})}) \in \widetilde{\diamond}(lv^{\alpha}_{\mathcal{N}}(\overline{G_1}), \ldots, lv^{\alpha}_{\mathcal{N}}(\overline{G_n}))$ *if* $\diamond \in \mathrm{OP}_{\mathcal{L}}$ *with* $ar(\diamond) = n$.

The concept of valuations defined on occurrences of (sub)formulas may seem odd at first sight. But we argue that it amounts to a consequent continuation of the path opened up by the move from static to dynamic valuations. To see this, we take up an example from the introduction of the handbook article [5] by Arnon Avron and Anna Zamansky. This example is particularly nice, since it supports the illustration of all three forms of non-deterministic valuations in reference to the same scenario.

Example 4 *In [5] the authors invite us to consider "inherent non-deterministic behavior" of a circuit like the following:*



If the gate marked by 'AO?' is either an OR-gate or an AND-gate, but we *do not know which of the two, we can use the non-deterministic truth table*

		AO?
t	**t**	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{f},\mathbf{t}\}$
f	**t**	$\{\mathbf{f},\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

and model our knowledge about the behavior of the gate by static valuation *of the formula* $out(A,B,C) = AO?(A \vee B, C)$. *Moreover it is argued in [5] that* dynamic valuation *allows one to approximate the behavior of a faulty AND-gate that responds correctly when its inputs are equivalent, but responds unpredictably otherwise. But note that there is no difference at all between static, dynamic, or liberal valuation if there is only a single occurrence of a non-deterministic connective in a formula. The difference between the different types of valuation emerges only if we consider a circuit where there is more than one occurrence of, say, an AO?-gate. If these occurrences, unknown to us, refer either all to an OR-gate or all to AND-gate then static valuation is adequate to model the expected behavior of the circuit. If, instead, each occurrence of an AO?-gate might behave like either classical gate individually, then we need dynamic or liberal valuation. Dynamic valuation is adequate if we know that all AO?-gates behave identical whenever identical sub-circuits are connected to the first input as well as to the second input, respectively. If however, the behavior of AO?-gates is truly unpredictable, i.e., if no constraint other than the one specified in the truth table for 'AO?' restricts the non-deterministic behavior of corresponding gates, then clearly liberal valuation has to be used to model the behavior of the whole circuit.*

We can think of many other examples where liberal valuations provide an appropriate modeling tool; however, we will leave the investigation of this (seemingly as yet unexplored) type of Nmatrix based valuation to future work.

From now on, we will assume that in an unrestricted $\mathcal{N}$-game for $v{:}F$, the signed formulas labeling the **P**-nodes actually refer to *occurrences* of the corresponding subformulas of F. This in particular implies that in any *strategy* for **P** in such a game, each occurrence of a subformula of F occurs at most once (together with a truth value) as a label of a **P**-node.

Theorem 3 *Given a language* $\mathcal{L}$ *and an Nmatrix* $\mathcal{N}$*, the following are equivalent for every formula* F*, truth value* $v \in \mathcal{V}$ *and assignment* α*.*

(1) **P** *has a winning strategy for the unrestricted* $\mathcal{N}$*-game for* $v{:}F$ *under* α*.*

(2) $lv^{\alpha}_{\mathcal{N}}(F) = v$ *in* $\mathcal{N}$ *for some liberal valuation* $lv^{\alpha}_{\mathcal{N}}$ *on* F *over* α*.*

Proof. In complete analogy to the proof of Theorem 2, every winning strategy induces a suitable liberal valuation, and every liberal valuation induces a corresponding winning strategy. The only difference is that we now assign truth values to individual *occurrences* of subformulas of F and not to the subformulas themselves. □

Remark 15 Since liberal valuations may treat different occurrences of syntactically identical subformulas differently, such valuations are not stable under substitution. Therefore one does not obtain a logic (at least not in the usual sense) by singling out the set of formulas that evaluate to a designated truth value under all liberal valuations. Alternative ways of defining validity or a consequence relation are conceivable. (E.g., one could involve a probability distribution over all possible liberal valuations in various ways). Here, however, we only claim that liberal valuations are meaningful from a purely semantic perspective, as indicated not only by their relation to $\mathcal{N}$-games, but also in Example 4, above.

The concept of unrestricted semantic games provides an alternative route towards a characterization of dynamic valuations. that is based on the following simple observation.

Proposition 4 *A liberal valuation $lv^{\alpha}_{\mathcal{N}}$ in $\mathcal{N}$ on F over α, where $lv^{\alpha}_{\mathcal{N}}(\overline{G}) = lv^{\alpha}_{\mathcal{N}}(\overline{G}')$ whenever $\overline{G}$ and $\overline{G}'$ are occurrences of the same subformula $G \in sf(F)$, is a partial dynamic valuation in $\mathcal{N}$ on $sf(F)$ over α.*

Proposition 4 suggests that, in order to arrive at a game that matches a dynamic valuation, one should prune a given unrestricted semantic game in such a manner that a unique truth value is assigned to each subformula in any **P**-winning strategy in the resulting game tree.

In Remark 13 we have hinted at the difference between unrestricted $\mathcal{N}$-games and $\mathcal{M}$-games. In the pruning process for $\mathcal{N}$-games we have to focus on the same type of configurations of nodes as described in Remark 13. To support the identification of such configurations of nodes we provide the following definitions.

Definition 15 *Let σ be an unrestricted semantic game. An* **O***-node n in σ is of* type $(w_1, \ldots, w_n)$*, where $w_1, \ldots w_n \in \mathcal{V}$, if the successor nodes of n (scanned from left to right*[5]*) are labeled by $w_1{:}F_1, \ldots, w_n{:}F_n$, respectively. Let n_1 and n_2 be* **O***-nodes of the same type, where the label of the* **P***-node preceding n_1 in σ is $u{:}\diamond(F_1, \ldots, F_n)$ and the label of the* **P***-node preceding n_2 in σ is $v{:}\diamond(G_1, \ldots, G_n)$. (Note that the exhibited formulas share their outermost connective.) If $u \neq v$ then (n_1, n_2) is called an* unresolved static pair*. If, in addition, $F_i = G_i$ for all $i = 1, \ldots, n$ then (n_1, n_2) is also called an* unresolved dynamic pair *if $u \neq v$.*

Two **O***-nodes (not necessarily of the same type and not necessarily distinct) also form an* unresolved, dynamic as well as static, pair *in σ if among their successor nodes there are two* **P***-nodes labeled by $u{:}C$ and $v{:}C$, where $u \neq v$ and C is a propositional constant. We call such a pair also an* unresolved end pair.

We stipulate that the above notions of coordination and of unresolved pairs transfer from σ to its sub-trees.

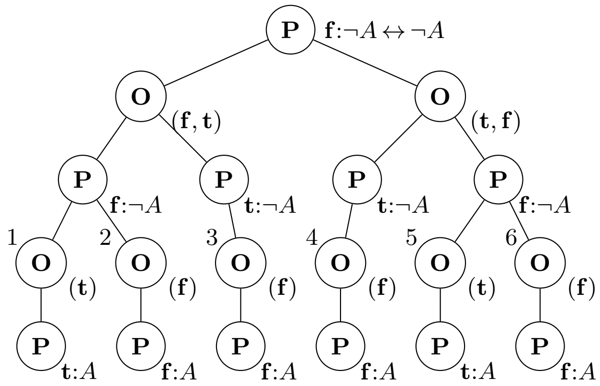
Remark 16 Remember that each **O**-node n refers to a particular disjunct in the complete disjunctive truth condition for the signed formula $u{:}\diamond(F_1, \ldots, F_n)$

[5]Remember that all game trees are ordered by definition.

that labels the **P**-node preceding n. Unresolved pairs locate **O**-nodes in a game that refer (via the corresponding rule tree) to identical disjuncts in the complete disjunctive truth condition for the signed formulas $u{:}\diamond(A_1, \ldots, A_n)$ and $v{:}\diamond(A_1, \ldots, A_n)$ that are instantiated to $u{:}\diamond(F_1, \ldots, F_n)$ and to $v{:}\diamond(G_1, \ldots, G_n)$, respectively, in the game.

Remark 17 If (n_1, n_2) is a static or dynamic unresolved pair then so is (n_2, n_1). Consequently we will not distinguish (n_1, n_2) from (n_2, n_1) when talking about unresolved pairs.

Example 5 *To illustrate the terminology introduced in Definition 15, we take up Example 3. For easier reference, we re-draw the unrestricted game tree, but now label the* **O***-nodes by their respective types. Moreover, we attach names (numbers* 1 *to* 6*, at the upper left hand side) to the lower* **O***-nodes.*



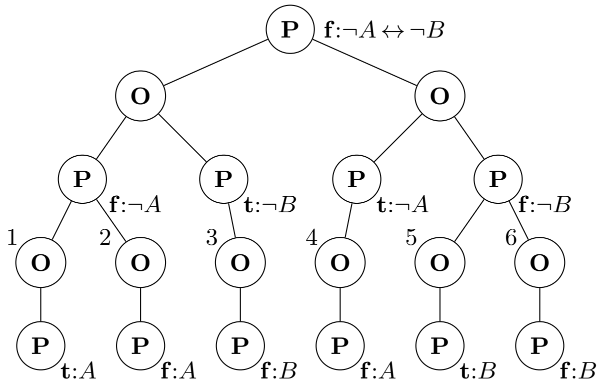
In this game tree $(2, 3)$*,* $(2, 4)$*,* $(3, 6)$*, and* $(4, 6)$ *are static as well as dynamic unresolved pairs.*

To exemplify the difference between static and dynamic pairs we replace the formula at the root of the by $\neg A \leftrightarrow \neg B$*, where* A *and* B *are different atomic formulas. We will re-use that game in Example 6 and therefore exhibit it in Figure 2.* $(2, 3)$*,* $(2, 4)$*,* $(3, 6)$*, and* $(4, 6)$ *remain unresolved static pairs in the tree for* $\mathbf{f}{:}\neg A \leftrightarrow \neg B$*, but only* $(2, 4)$ *and* $(3, 6)$ *are also unresolved dynamic pairs now.*

We could specify an appropriately restricted $\mathcal{N}$-game simply as a maximal sub-tree of an unrestricted semantic game σ, in which no unresolved (dynamic or static) pair exists. However, in the spirit of game semantics, we will instead describe a stepwise pruning process in which the players **P** and **O** interact repeatedly.

Definition 16 (Pruning) *Starting from an unrestricted* $\mathcal{N}$*-game* σ*,* dynamic (static) pruning *reduces the current* $\mathcal{N}$*-game tree to a sub-tree of* σ *by repeating the following round of interaction between* **P** *and* **O** *until no dynamic (static) pair remains.*

Figure 2: Unrestricted game tree for $\mathbf{f}{:}\neg A \leftrightarrow \neg B$.



1. **O** *picks an unresolved dynamic (static) pair* (n_1, n_2) *in the current game.*

2. *In reply,* **P** *removes either the sub-tree rooted* n_1 *or the one rooted in* n_2 *from the current game.*

The resulting tree is called a restricted dynamic (static) $\mathcal{N}$-game.

The winning conditions for restricted (dynamic or static) $\mathcal{N}$*-games remain like in the corresponding unrestricted game. Let* α *be the relevant assignment and let* n *be a leaf node labeled by* $v{:}F$*. Then* **P** *wins at* n *if* F *is a propositional variable such that* $\alpha(F) = v$ *or if* $F \in \mathrm{PC}_{\mathcal{L}}$ *and* $v \in \widetilde{F}$*. If* F *is not atomic then* **P** *looses at* n*.*

Remark 18 Note that new leaf nodes may arise by pruning. Such new leaf nodes are always **P**-nodes labeled by a non-atomic signed formula. By definition, **P** looses the game at such nodes.

Remark 19 Although **P** and **O** interact in rounds that admit choices by the two players, pruning actually is *not a game* in the sense of game theory. There are no winning conditions. The interaction is just intended to lead to some sub-tree of the unrestricted game in which there are no unresolved pairs.

Example 6 *We continue Example 5 to illustrate the pruning process.*

In the first step of static pruning applied to the $\mathcal{N}$*-game depicted in Figure 2* **O** *can pick one of the following unresolved static pairs of* **O***-nodes:* $(2,3)$, $(2,4)$, $(3,6)$, $(4,6)$. *Suppose* **O** *picks* $(2,3)$ *and* **P** *replies by picking node* 3 *and removing it together with the* **P***-node below* 3*. In the next round of pruning* **P** *picks* $(2,4)$ *and* **P** *replies by removing node* 4 *together with the* **P***-node below* 4*. At this stage no further unresolved static pairs are left. The resulting restricted static* $\mathcal{N}$*-game is depicted in Figure 3. Note that this tree is also an* $\mathcal{M}$*-game, where the matrix* $\mathcal{M}$ *contains the truth table* $\widehat{\neg}(\mathbf{f}) = \mathbf{f}$, $\widehat{\neg}(\mathbf{t}) = \mathbf{f}$*. Under this semantics all negated formulas are false and hence there is no winning strategy for* **P** *for* $\mathbf{f} : \neg A \leftrightarrow \neg B$*.*

Figure 3: Restricted $\mathcal{N}$-game tree for $\mathbf{f}{:}\neg A \leftrightarrow \neg B$ that coincides with a particular $\mathcal{M}$-game.

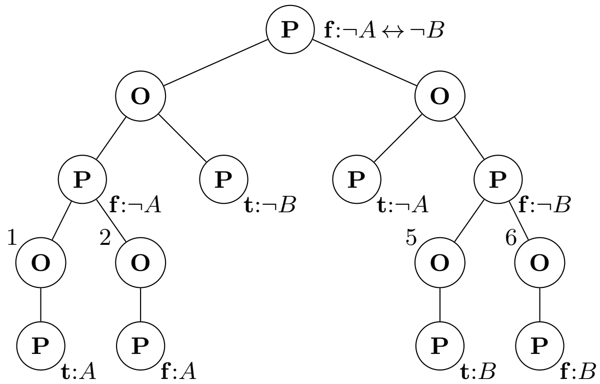


Figure 4: Restricted $\mathcal{N}$-game tree for $\mathbf{f}{:}\neg A \leftrightarrow \neg B$ that does not correspond to any $\mathcal{M}$-game.

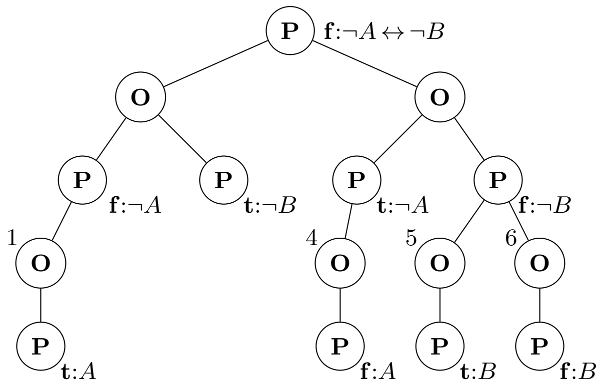
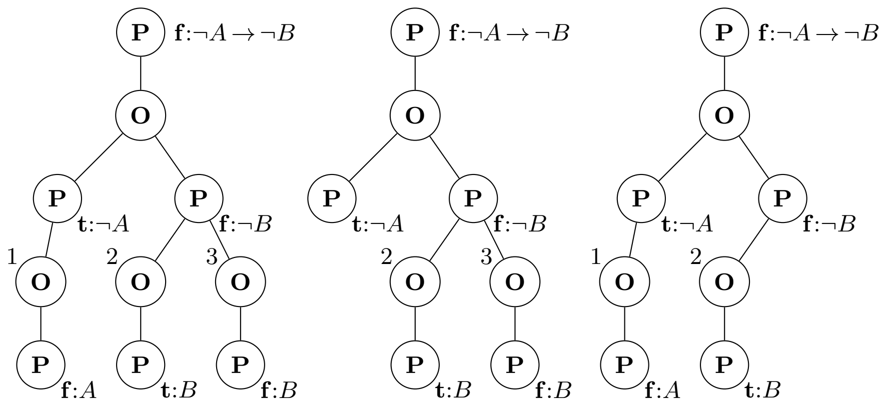


Figure 5: Unrestricted $\mathcal{N}$-game and restricted static $\mathcal{N}$-games for $\mathbf{f}{:}\neg A \to \neg B$.



In contrast, the classical $\mathcal{M}$-game (where $\widehat{\neg}(\mathbf{f}) = \mathbf{t}$) emerges if the the original tree (Figure 2) is pruned at nodes 2 *and* 6 *instead. However, we emphasize that* not *all restricted static $\mathcal{N}$-games are $\mathcal{M}$-games for some deterministic matrix $\mathcal{M}$. In particular the restricted static $\mathcal{N}$-game depicted in Figure 4, which results from removing nodes* 2 *and* 3 *through pruning, does not correspond to any matrix.*

Although we have referred to static pruning in describing the restricted games in Figures 2 and 4, the same trees may be obtained from dynamic pruning as well. However this is a special feature of this particular example. For an example that shows that restricted static trees are different from restricted dynamic trees in general, consider once more the 'liberal' non-deterministic negation; this time combined with classic implication ($\to$). The leftmost tree in Figure 5 is the unrestricted $\mathcal{N}$-game for $\mathbf{f}{:}\neg A \to \neg B$. There is only one unresolved static pair in this tree, namely $(1, 3)$. The tree in the middle of Figure 5 depicts the restricted static $\mathcal{N}$-game resulting from removing node 1*. The result of the other possibility for static pruning, namely to remove node* 3*, is depicted in the tree at the right hand side. Since $(1, 3)$ is not a unresolved dynamic pair, the unrestricted $\mathcal{N}$-game is also the only restricted dynamic $\mathcal{N}$-game in this case.*

Theorem 4 *Given a language $\mathcal{L}$ and an Nmatrix $\mathcal{N}$, the following holds for every formula F, truth value $v \in \mathcal{V}$, and assignment α.*

(1) *For every winning strategy for* **P** *in a restricted dynamic $\mathcal{N}$-game for $v{:}F$ under α there is a dynamic valuation $dv^{\alpha}_{\mathcal{N}}$ over α such that $dv^{\alpha}_{\mathcal{N}}(F) = v$.*

(2) *If $dv^{\alpha}_{\mathcal{N}}(F) = v$ for some dynamic valuation $dv^{\alpha}_{\mathcal{N}}$ over α then* **P** *has winning strategy in some restricted dynamic $\mathcal{N}$-game for $v{:}F$ under α.*

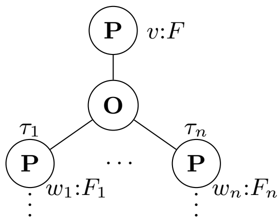
Both statements remain true when we replace 'dynamic' by 'static' everywhere.

Proof. We show the case for dynamic valuation. The case for static valuation is completely analogous.

(1): Let τ be a winning strategy for **P** in a restricted dynamic $\mathcal{N}$-game for $v{:}F$ under α. A valuation dv_τ is called *induced by* τ if $dv_\tau(G) = w$ for every signed formula $w{:}G$ that labels some **P**-node in τ. We show that τ induces an appropriate dynamic valuation dv_τ by induction on its depth $d(\tau)$.

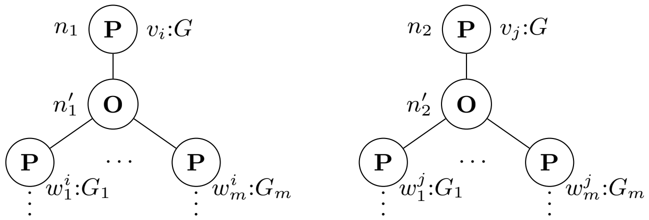
- $d(\tau) = 0$: In this case τ consists of a single **P**-node labeled by $v{:}F$. Since τ is a winning strategy, we have either $F \in \mathrm{PV}$ and $dv_\tau(F) = \alpha(F) = v$, or else $F \in \mathrm{PC}_\mathcal{L}$ and $dv_\tau(F) = v \in \widetilde{F}$, as required. (Note that τ cannot consist in a leaf node labeled by a compound formula, since **P** loses at all such nodes.)

- $d(\tau) > 0$: By definition, τ begins as follows:



where $F = \diamond(F_1, \ldots, F_n)$ and where $v \in \widetilde{\diamond}(w_1, \ldots, w_n)$, since the label of the lower **P**-nodes are the conjuncts of some disjunct in the complete disjunctive truth condition for $w{:}\diamond(F_1, \ldots, F_n)$. By the induction hypothesis we obtain valuations $dv_{\tau_1}, \ldots, dv_{\tau_n}$ induced by the sub-trees $\tau_1, \ldots, \tau_n$ of τ that are rooted in the lower **P**-nodes labeled by $w_1{:}F_1, \ldots, w_n{:}F_n$, respectively. We know that $dv_{\tau_i}(F_i) = w_i$, where dv_{τ_i} is a dynamic valuation over α for $i = 1, \ldots, n$. It remains to show that $dv_{\tau_i}(G) = dv_{\tau_j}(G)$ whenever $G \in \mathrm{sf}(F_i) \cap \mathrm{sf}(F_j)$ $(1 \leq i, j \leq n)$, and therefore the valuations jointly extend to dv_τ as required.

  Suppose, to the contrary, that $dv_{\tau_i}(G) \neq dv_{\tau_j}(G)$ for some $i, j \in \{1, \ldots, n\}$. Moreover assume without loss of generality that G is a smallest subformula of F with this property. G cannot be a propositional variable, since dv_{τ_i} and dv_{τ_j} are defined over the same assignment α. G also cannot be a propositional constant, since then there were an unresolved end pair in τ. But all such unresolved pairs have been removed during pruning from the game tree and therefore also cannot appear in a strategy. The remaining possibility is that $G = \diamond'(G_1, \ldots, G_m)$ for some m-ary $\diamond' \in \mathrm{OP}_\mathcal{L}$. Thus there are two **P** nodes n_1 and n_2 in τ that are labeled by $v_i{:}G$ and $v_j{:}G$, respectively, where $v_i \neq v_j$. Since τ is a winning strategy for **P**, neither n_1 nor n_2 can be a root node. The corresponding sub-trees therefore look as follows:



By the assumption that G is the smallest formula where $dv_{\tau_i}(G) \neq dv_{\tau_j}(G)$, we obtain $w^i_\ell = w^j_\ell$ for $1 \leq \ell \leq m$. But this implies that (n'_1, n'_2) is an unresolved dynamic pair, which contradicts the fact that τ and thus also τ_i and τ_j are sub-trees of a restricted $\mathcal{N}$-game, where all unresolved dynamic pairs have been removed through pruning.

(2): Let $dv^\alpha_\mathcal{N}(F) = v$ for some dynamic valuation $dv^\alpha_\mathcal{N}$ over α. We show by induction on F that a winning strategy τ for a restricted dynamic $\mathcal{N}$-game rooted in $v{:}F$ can be constructed, where $dv^\alpha_\mathcal{N}(G) = w$ whenever a **P**-node labeled by $w{:}G$ appears in τ.

- If F is atomic, then τ consists just in a single **P**-node labeled by $v{:}F$.
- Suppose $F = \diamond(F_1, \ldots, F_n)$ and $dv^\alpha_\mathcal{N}(F_i) = w_i$: By the induction hypothesis there is a winning strategy τ_i for **P** in some restricted $\mathcal{N}$-game σ_i for $w_i{:}F_i$ under α for every $i = 1, \ldots, n$. Moreover, for every signed formula $w{:}G$ that labels a **P**-node in τ_i we have $dv^\alpha_\mathcal{N}(G) = w$. $dv^\alpha_\mathcal{N}(F) = v$ implies $v \in \widetilde{\diamond}(w_1, \ldots, w_n)$. Consequently there is a disjunct $\bigwedge\!\!\!\bigwedge_{1\leq i\leq n} w_i{:}F_i$ in the complete disjunctive truth condition for $v{:}\diamond(F_1, \ldots, F_n)$. Since the rules of the game correspond to complete disjunctive truth conditions, this means that we obtain the required winning strategy τ by joining the root (**P**-)nodes of $\tau_1, \ldots, \tau_n$ by a common parent **O**-node, and add a **P**-node labeled by $v{:}F$ as a parent node to this **O**-node.

Remember that we have $dv^\alpha_\mathcal{N}(G) = w$ for every signed formula $w{:}G$ occurring as a label of some **P**-node in τ. Since every dynamic valuation assigns a single truth value to any given formula, τ does not contain any unresolved dynamic pair. Moreover, $dv^\alpha_\mathcal{N}$ constrains the choices of **P** in the pruning process for the removal of sub-trees of the original $\mathcal{N}$-game σ: **P** should not remove any **O**-node that is the successor node of a **P**-node labeled by $w{:}G$, where $w = dv^\alpha_\mathcal{N}(G)$. Any dynamic pruning process that obeys this constraint results in a restricted game tree for which the sub-tree τ, specified above, is a winning strategy for **P**. □

Remark 20 As already indicated in Example 6, there is no direct correspondence between restricted static or dynamic $\mathcal{N}$-games and static or dynamic valuations. Theorem 4 rather expresses a correspondence between *winning strategies* in such games and valuations. Without going into any details we remark that by defining a somewhat more involved pruning procedure one can in fact extract sub-trees from unrestricted $\mathcal{N}$-games that directly correspond to arbitrary static or dynamic valuations. This should not be surprising, since for every given signed formula and assignment the corresponding $\mathcal{M}$-game is just a sub-tree of an unrestricted $\mathcal{N}$-game, whenever the matrix $\mathcal{M}$ is a determination of the Nmatrix $\mathcal{N}$. This observation directly covers all static valuations; but a similar statement holds for dynamic valuations as well.

5 Conclusion

Inspired by a fresh look at Hintikka's semantic game for classical logic, we have introduced the concept of $\mathcal{M}$-games, matching the valuation of formulas with respect to a matrix $\mathcal{M}$, i.e., a collection of arbitrary finite truth tables. An adequate game rule for a particular connective and truth value can be directly read off from the corresponding truth table. This still holds when we generalize $\mathcal{M}$-games to unrestricted $\mathcal{N}$-games, for arbitrary collections (Nmatrices) $\mathcal{N}$ of finite non-deterministic truth tables. As we have seen in Section 4, **P**'s winning strategies in unrestricted $\mathcal{N}$-games correspond neither to static nor to dynamic valuations, but rather to the new concept of 'liberal valuations', which arises naturally not only from a game semantic perspective, but arguably generalizes dynamic valuations in a manner that is independently suggested by certain application scenarios. In any case, the concept allows to characterize dynamic and static valuations in a new manner; namely, as arising from removing certain options for player **P** from the game tree. The corresponding restrictions of $\mathcal{N}$-games can be obtained as results of a pruning process that proceeds by stepwise interaction of the two players and thus fits the context of game semantics.

We consider this work as a starting point, bringing to light a number of directions for further investigation. We conclude by highlighting four of these further topics.

- We have only studied propositional valuations here. However, matrix semantics as well as Nmatrix semantics can be extended to a wide class of quantifiers, in particular so-called distribution quantifiers (see, e.g., [6]) and their non-deterministic variants (see, e.g., [4]). We conjecture that the game semantics developed in this paper can be straightforwardly lifted to the first order level. This in turn will provide a basis for investigating many-valued as well as non-deterministic variants of Independence-Friendly Logic (IF-Logic, see [19, 15]).

- We have seen how liberal valuations arise as counterparts of **P**'s winning strategies in unrestricted $\mathcal{N}$-games. Dynamic and static semantics require an additional game-like preprocessing procedure ('pruning') that properly restricts $\mathcal{N}$-games. This triggers the question, whether further (new) types of non-deterministic valuations can be characterized by corresponding variants of pruning. For example, preference relations on truth values may be introduced that guide **P**'s choices in resolving unresolved pairs. Moreover, one may also investigate whether, instead of restricting **P**'s options in the game, it makes sense to restrict also **O**'s option in some (systematic) way.

- Related to the previous item, one may consider various combinations of liberal, dynamic, and static semantics. Indeed, from a game semantic point of view, such combinations naturally arise from subjecting a corresponding mixture of static and dynamic unresolved pairs to pruning.

- Our presentation of $\mathcal{N}$-games relies on the finiteness of the underlying Nmatrices. However Nmatrices based on infinite sets of truth values have been found useful as well (see, e.g., [5]). It is not clear to what extent and how one can generalize semantic games to cover also valuations over infinitely many values in a finitary manner.

Once more we want to emphasize that the investigation of these topics should not be seen as purely theoretical exercise, undertaken for the sake of generalization itself. We rather expect that appropriate forms of generalization deepen the understanding of the underlying classical concepts. Moreover, in the spirit of 'logic engineering' [1] or of 'universal logic' [7], one should be prepared to assemble logics for different application scenarios by instantiating general tools and concepts that extend and variate classical logic in various directions. Nmatrix semantics certainly has a lot to offer in that vain. Corresponding games may well prove to be a useful addendum to that logical toolbox.

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