Summer School for Proof Theory in First-Order Logic
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Games and Analytic Proof Systems
Part 2

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Overview

Part 1 (yesterday)

- the most basic logic game: Hintikka's game for classical logic
- from Hintikka's game to sequent calculi via disjunctive states
- Hintikka's game and many truth values:
  - many-valued truth tables, Nmatrices
  - Giles's game for Łukasiewicz logic
- analyzing a hypersequent calculus using games

Part 2 (today)

- Lorenzen's dialogue game for intuitionistic logic
- parallel dialogue games and hypersequent systems
- A brief interlude: alternative forms of game semantics
- Substructural logics: Paoli's system **LL**
- Lorenzen-style rules for **LL** and other substructural logics
- Conclusion & further topics

Dialogues as logical foundations:

Remember:

"logic, like sex, works better when another person is involved"

Imagine a dialogue, where a Proponent **P** tries to defend a logically complex statement against attacks by an Opponent **O**. The dialogue stepwise reduces complex assertions to their components.

Lorenzen's central idea (*'Logik und Agon'*, late 1950s):

G logically follows from $F_1, \ldots, F_n$ means:

P *can always win an antagonistic, rational dialogue starting with her assertion of* G*, if* **O** *has granted* $F_1, \ldots, F_n$

Some basic features of Lorenzen style dialogues:

- attack moves and corresponding defense moves refer to outermost connectives and quantifiers of assertions
- both, **P** and **O**, may launch attacks and defend against attacks during the course of a dialogue
- moves alternate strictly between **P** and **O**

Logical dialogue rules:

X/**Y** stands for **P**/**O** or **O**/**P**

statement by **X**	attack by **Y**	defense by **X**
$A \wedge B$	l? or r? (**Y** chooses)	A or B, accordingly
$A \vee B$	?	A or B (**X** chooses)
$A \supset B$	A	B
$\neg A$	A	(none)
$\forall x A(x)$	?c (**Y** chooses)	$A(c)$
$\exists x A(x)$	?	A(c) (**Y** chooses c)

Winning conditions for **P**:

W: **O** has already granted **P**'s active formula

W$\bot$: **O** has granted $\bot$

active formula . . . last[†] formula asserted by **P**, either attacked or to be attacked next by **O**, but not yet defended

[†] we will drop 'last' later $\Rightarrow$ more than one active formula possible

Structural rules:

Start:	**O** starts by attacking **P**'s initial assertion (formula)
Alternate:	moves strictly alternate between **O** and **P**
Atom:	atomic formulas (including $\bot$) can neither be attacked nor defended by **P**
'E-rule':	each (but the first) move of **O** reacts directly to the immediately preceding move by **P**
'F-rule':	**P** defends only active formulas

NB:

Lorenzen-style games are quite different from semantic games:

- Hintikka- and Giles-style games are about taking a certain truth value in a given interpretation, not about validity
- the provability games resulting from the 'states-to-disjunctive states' translation are also different from Lorenzen-style games

Analyzing winning strategies for Lorenzen's game

Definition:

A **winning strategy** (for **P**) is a finite tree, whose branches are dialogues that end in winning states for **P**, s.t.

– **P**-nodes have at most one successor;

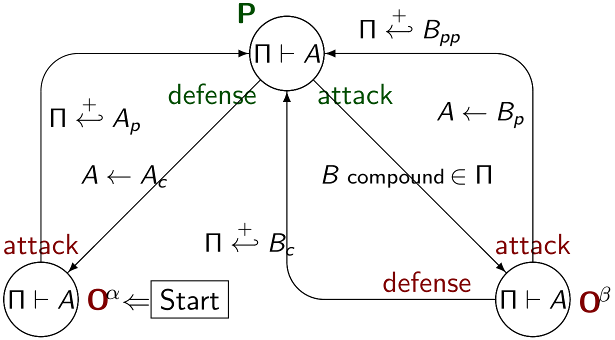
– **O**-nodes have successors for each possible next move by **O**.

Note:

Dialogues are traces in the corresponding state transition system.

Winning strategies arise by 'unwinding' the state transition system.

Dialogues as state transitions (implicational fragment):



$A_c = G$ if $A = (F \supset G)$, empty otherwise
$A_p = F$ if $A = (F \supset G)$, empty otherwise
$B_{pp} = F$ if $B = ((F \supset G) \supset H))$, empty otherwise

Adequacy for intuitionistic logic

Theorem (Lorenzen, Lorenz, Felscher, . . .):
P has a winning strategy when initially asserting F

if and only if

F is valid according to intuionistic logic (**I**).

Our version of the adequacy theorem:

Theorem:

Winning strategies correspond to cut-free $\mathbf{LI}'$-proofs.

Remark on adequacy proofs:
Lorenzen and Lorenz never succeeded completely.
First full proof for by Felscher (*APAL, 1985*).
Many proofs (some 'gappy') have appeared since: Krabbe, Rahman, Keiff, Sorensen, Clerbout, Alama/Konks/Uckelman, F,. . .

LI$'$: the proof search friendly version of LI (LJ?)

Axioms:

'confine weakening to axioms':

$\bot, \Pi \longrightarrow C$ and $A, \Pi \longrightarrow A$

Logical rules:

'keep a copy of the main (i.e. reduced) formula around'
(by melting the logical rule with contraction):

$$\frac{A \supset B, \Pi \longrightarrow A \qquad B, A \supset B, \Pi \longrightarrow C}{A \supset B, \Pi \longrightarrow C}\ (\supset, l)$$

$$\frac{A, \Pi \longrightarrow B}{A, \Pi \longrightarrow A \supset B}\ (\supset, r)$$

From winning strategies to $\mathbf{LI}'$-derivations

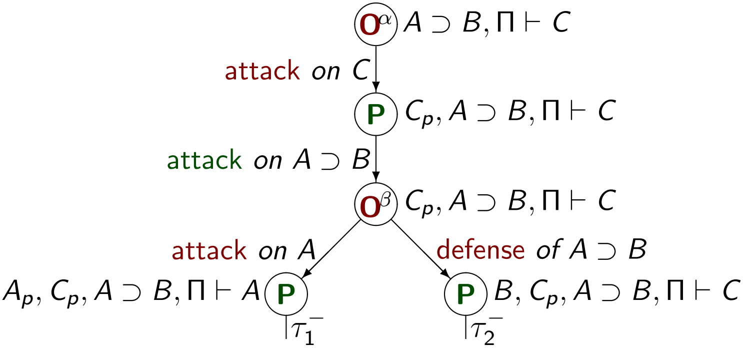
Theorem ('Soundness of the game')
Every winning strategy τ for $\Pi \vdash C$ can be transformed into an $\mathbf{LI}'$-proof of $\Pi \longrightarrow C$.

Proof idea:

- induction on the depth of τ
- induction step:
  each **P**–**O**–**P** cycle of moves translates
  into one (branch of) an $\mathbf{LI}'$-inference step

From LI′-derivations to winning strategies
('Completeness of the game')

The case for $A \supset B, \Pi \longrightarrow C$:



τ_1^- ... winning strategy from **LI′**-proof of $A \supset B, \Pi \longrightarrow A$
τ_2^- ... winning strategy from **LI′**-proof of $B, A \supset B, \Pi \longrightarrow C$

Lorenzen-style games: some other logics

- Already Lorenzen realized: If **P** may defend not just a single 'active formula', but also previously challenged formulas instead, the game characterizes classical logic
- dialogue games for modal logics (Rahman, Rückert, Blackburn, Keif, Sticht, ...):
  e.g., modeling possible worlds by 'dialogical contexts'
- Rahman/Rückert (*Synthese 2001*): 'dialogical connexive logic'
  Winning strategies for $\neg(A \supset \neg A)$ and $\neg(\neg A \supset A)$ via rules for new operators modeling 'defensibility'/'attackability'

Note:
In all these cases relations between **P**'s winning strategies and analytic proofs (usually tableau-style) can be established

$\mathbf{HLI}'$: A hypersequent calculus for intuitionistic logic

Exactly as $\mathbf{LI}'$ except for the presence of side hypersequents:

Axioms:

$$\bot, \Pi \longrightarrow C \mid \mathcal{H} \quad \text{and} \quad A, \Pi \longrightarrow A \mid \mathcal{H}$$

Logical rules:

$$\frac{A \supset B, \Pi \longrightarrow A \mid \mathcal{H} \qquad B, A \supset B, \Pi \longrightarrow C \mid \mathcal{H}}{A \supset B, \Pi \longrightarrow C \mid \mathcal{H}} \ (\supset, l)$$

$$\frac{A, \Pi \longrightarrow B \mid \mathcal{H}}{A, \Pi \longrightarrow A \supset B \mid \mathcal{H}} \ (\supset, r)$$

Note:

The side hypersequents are clearly redundant here, but may be useful in representing choices in proof search (once the 'obvious' external structural rules are in place ...)

Internal structural rules:

$$\frac{A, A, \Pi \longrightarrow C \mid \mathcal{H}}{A, \Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{I-contr.}) \qquad \frac{\Pi \longrightarrow C \mid \mathcal{H}}{A, \Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{I-weakening})$$

$$\frac{\Pi \longrightarrow A \mid \mathcal{H} \quad A, \Pi \longrightarrow C \mid \mathcal{H}'}{\Pi \longrightarrow C \mid \mathcal{H} \mid \mathcal{H}'} \ (\textit{cut})$$

Remember: cut and internal weakening are redundant!

External structural rules:

$$\frac{\mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{E-weakening}) \qquad \frac{\Pi \longrightarrow C \mid \Pi \longrightarrow C \mid \mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{E-contr.})$$

Note:

E-weakening records the dismissal of an alternative in proof search.
E-contraction records a 'backtracking point' for such an alternative.

Parallel dialogue games

General features of our form of parallelization:

- Ordinary dialogues (**I**-dialogues) appear as subcases of the more general parallel framework.
- **P** may initiate additional dialogues by ‘cloning’.
- To win a set of parallel dialogues, **P** has to win at least one of the component **I**-dialogues.
- Synchronization between parallel **I**-dialogues is invoked by **P**’s decision to merge some **I**-dialogues (‘component dialogues’) into one. **O** may react to this in different ways.

Notions for parallel dialogue games

A parallel **I**-dialogue (*P*-**I-dialogue**) is a sequence of global states connected by internal or external moves.

Global state:
$\{\Pi_1 \vdash_{\iota 1} C_1,\ \ldots,\ \Pi_n \vdash_{\iota n} C_n\}$
(Set of uniquely indexed component **I**-dialogue sequents.)

Internal move:
Set of **I**-dialogue moves: at most one for each component.

External move:
May add or remove components, but does not change the status — **P**'s or **O**'s turn to move — of existing components.

Basic external moves:

fork: **P** duplicates a **P**-component of the current global state.

cancel: **P** removes an arbitrary **P**-component (if the global state contains another **P**-component).

Towards proving adequacy: Sequentialized and normal P-**I**-dialogues

Sequentiality: internal moves are singletons.

Normality:
- **P**-moves are immediately followed by **O**-moves referring to the same component(s)
- external moves (possibly consisting of a **P**-**O**-round) are followed by **P**-moves

Lemma:
Every finite P-**I**-dialogue can be translated into an equivalent sequentialized and normal P-**I**-dialogue.

Theorem:

Winning strategies for sequentialized and normal P-**I**-dialogues correspond to $\mathbf{HLI}'$-proofs.

Example: Characterizing Gödel-Dummett logic

HLC′ is obtained from **HLI**′ by adding:

$$\frac{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H} \qquad \Pi_1, \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}{\Pi_1 \longrightarrow C_1 \mid \Pi_2 \longrightarrow C_2 \mid \mathcal{H}} \ (com')$$

This correponds to the following ‘synchronisation rule’:

lc-merge:
1. **P** picks two **P**-components $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$.
2. **O** chooses either C_1 or C_2 as the current formula of the merged component with granted formulas $\Pi_1 \cup \Pi_2$.

Theorem:

Winning strategies for P-**I**-dialogues with lc-merge can be translated into cut-free **HLC**′-proofs, and vice versa.

Other forms of synchronization:

System	rule	external move(s)
P-**Cl**	class	**P** merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$
P-**LQ**	lq	**P** merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} \bot$ into $\Pi \cup \Gamma \vdash_{\iota 2} \bot$
P-**LC**	lc	**P** picks $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$
		O chooses $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \cup \Pi_2 \vdash_{\iota 2} C_2$
P-**sLC**	lc0	**P** picks $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$
		O chooses $\Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \vdash_{\iota 2} C_2$
	sp	**P** merges $\Pi \vdash_{\iota 1} C$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$
P-$\mathbf{G_n}$	g_n	**P** picks the components
		$\Pi_1 \vdash_{\iota 1} C_1$, and $\ldots \Pi_{n-1} \vdash_{\iota[n-1]} C_{n-1}$, and $\Pi_n \vdash_{\iota n}$
		O chooses one of
		$\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$, $\Pi_2 \cup \Pi_3 \vdash_{\iota 2} C_2$, ..., or
		$\Pi_{n-1} \cup \Pi_n \vdash_{\iota[n-1]} C_{n-1}$

Interlude: Alternative forms of game semantics

- Blass (APAL 1992): game semantics for affine linear logic
  – new paradigm: 'logical connectives as game operators'
  – only additive connectives, otherwise 'counter examples'
  – negation as role switch
- Abramsky/Jagadeesan (JSL 1994): full completeness
  – paradigm: formulas = games, strategies = proofs
  – multiplicative connectives are covered
  – high level of abstraction
- Japaridze's computability logic CL (since 2003)
  – games as a general model of interactive computation
  – computational constructions induce (many) connectives
  – certain principles of linear logic get invalidated
- Girard's Locus Solum ('ludics') (2001):
  'loci': pointers to subformulas, 'designs': corresponding proofs
  attempts to provide a logic of inference rules as interactions

Back to Lorenzen-style games: some other logics

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- dialogue games for modal logics (Rahman, Rückert, Blackburn, Keif, ...): modeling possible worlds by 'dialogical contexts'
- Rahman/Rückert (*Synthese 2001*): 'dialogical connexive logic' Winning strategies for $\neg(A \supset \neg A)$ and $\neg(\neg A \supset A)$ via rules for new operators modeling 'defensibility'/'attackability'

Note:

In all these cases relations between **P**'s winning strategies and analytic proofs (usually tableau-style) are readily established

Substructurual logics: Paoli's system LL

Axioms: $A \longrightarrow A \qquad \longrightarrow 1 \qquad 0 \longrightarrow$

Logical rules (without negation):

$$\frac{A, B, \Gamma \longrightarrow \Delta}{A \otimes B, \Gamma \longrightarrow \Delta}\ (\otimes, l) \qquad \frac{\Gamma \longrightarrow \Delta, A \quad \Pi \longrightarrow \Sigma, B}{\Gamma, \Pi \longrightarrow \Delta, \Sigma, A \otimes B}\ (\otimes, r)$$

$$\frac{A, \Gamma \longrightarrow \Delta}{A \wedge B, \Gamma \longrightarrow \Delta} \Big/ \frac{B, \Gamma \longrightarrow \Delta}{A \wedge B, \Gamma \longrightarrow \Delta}\ (\wedge, l) \qquad \frac{\Gamma \longrightarrow \Delta, A \quad \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \wedge B}\ (\wedge, r)$$

$$\frac{A, \Gamma \longrightarrow \Delta \quad B, \Pi \longrightarrow \Sigma}{A \oplus B, \Gamma, \Pi \longrightarrow \Delta, \Sigma}\ (\oplus, l) \qquad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, A \oplus B}\ (\oplus, r)$$

$$\frac{A, \Gamma \longrightarrow \Delta \quad B, \Gamma \longrightarrow \Delta}{A \vee B, \Gamma \longrightarrow \Delta}\ (\vee, l) \qquad \frac{\Gamma \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, A \vee B} \Big/ \frac{\Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \vee B}\ (\vee, r)$$

$$\frac{\Gamma \longrightarrow \Delta, A \quad B, \Pi \longrightarrow \Sigma}{A \supset B, \Gamma, \Pi \longrightarrow \Delta, \Sigma}\ (\supset, l) \qquad \frac{A, \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \supset B}\ (\supset, r)$$

$$\frac{\Gamma \longrightarrow \Delta}{1, \Gamma \longrightarrow \Delta}\ (1, l) \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, 0}\ (0, r)$$

NB: no structural rules (except cut)

Reading LL-rules as Lorenzen-style game rules

Note: like in Lorenzen's game for classical logic, there is a multiset of 'active formulas': (to be) attacked by **O**, but not yet defended

(1) 'weakening-free' axioms: $\Rightarrow$ winning conditions:
$A \longrightarrow A$: **O** has already granted **P**'s only active formula A
$\longrightarrow 1$: **P**'s only active formula is 1
$0 \longrightarrow$: **O** grants 0; no assertion of **P** is left undefended
moreover, in each case: all other assertions of **O** have already been marked as attacked as well as defended

(2) rules for the logical constants 0 and 1:
$(0, r)$: when **O** attacks **P**'s assertion of 0, it gets removed
$(1, l)$: when **O** grants 1, **P** may ask for its removal
removal from the active dialogue state means:
marked as already attacked as well as defended

(3) no (built in or explicit) contraction in additive rules:
Each formula granted by **O** is attacked at most once;
this renders Lorenzen's $\wedge$- and $\vee$-rules adequate for **LL**

Lorenzen-style rules for LL (ctd.)

(4) (multiplicative) implication:

- **P** attacks **O**'s assertion of $A \supset B$ by partitioning **O**'s unattacked assertions Γ into Γ_1 and Γ_2 and **P**'s active formulas Δ into Δ_1 and Δ_2 and lets **O** choose between:
  (1) **P** defends A and Δ_1 if **O** only grants Γ_1
  (2) **O** grants B in addition to Γ_2 and **P** defends Δ_2
- **O** attacks on **P**'s $A \supset B$: Lorenzen's original rule applies

(5) multiplicative conjunction:

- **O** attacks **P**'s assertion of $A \otimes B$:
  P partitions as in (4) above and lets **O** choose between
  (1) **P** defends A and Δ_1 if **O** grants Γ_1
  (2) **P** defends B and Δ_2 if **O** grants Γ_2
- **P** attacks **O**'s $A \otimes B$: **O** has to grant A as well as B

(6) multiplicative disjunction: analogous to conjunction

Lorenzen-style rules for other substructural logics

- Dialethic (paraconsistent) LL^{Λ}:
  P also wins if nothing is granted by **O** and **P**
- Adding $\bot$ and $\top$ – (bounded lattice-theoretic) LL^{B}:
  P also wins if **O** grants $\bot$ or attacks **P**'s assertion of $\top$
- Adding contraction – (relevant) LR^{ND}:
  P can ask for an additional copy of any formula granted by **O**
  P can add a copy of any active formula
- Adding weakening – (affine) LL^{A}:
  P may remove any formula granted by **O** as well as
  any of her active formulas

Note: various combinations and variants of these modifications lead to characterizations of well known substructural logics

Conclusions

Regarding Part 2 (today)

When freed from Lorenzen's commitment on intuitionistic logic, dialogue games provide a versatile frame for characterizing many different logics, relating to variants of (hyper)sequent systems.

Regarding Part 1 (yesterday)

Semantic games can be translated systematically into analytic proof systems via lifting from ordinary game states to disjunctive states.

Further topics (not treated in this course):

- Blass/Abramsky-style game semantics and sequent systems
- Client/Server-games and sequent systems
- game interpretation of admissible rules (in particular cut)
- semantic game rules for generalized quantifiers
- dialogue rules for linear logic exponentials '!' and "?
- models of proof search: **P**-**O** as 'Client-Server' (Blass) induces models of different proof search strategies
- there are many other types of games in logic: can we find interesting connections to proof theory?