Summer School for Proof Theory in First-Order Logic
Funchal, Madeira, August 2017

Games and Analytic Proof Systems
Part 1

Chris Fermüller

Vienna University of Technology
Theory and Logic Group
www.logic.at/people/chrisf/

Motivation

From a recent TLS reviewof

Hugo Mercier and Dan Sperber:
THE ENIGMA OF REASON –
A new theory of human understanding

"Reasoning is not meant to be done alone in a room. From Mercier and Sperber's evolutionary perspective, reasoning, like sex, works better when another person is involved."

Cecilia Heyes, TLS, July 28, 2017

Overview

Part 1 (today)

- the most basic logic game:
  Hintikka's game for classical logic
- from Hintikka's game to sequent calculus via disjunctive states
- Hintikka's game and many truth values:
  - many-valued truth tables, Nmatrices
  - Giles's game for Łukasiewicz logic
- analyzing a hypersequent calculus using games

Part 2 (tomorrow)

- Lorenzen's dialogue game for intuitionistic logic
- parallel dialogue games and hypersequent systems
- A brief interlude: alternative forms of game semantics
- Substructural logics: Paoli's system **LL**
- Lorenzen-style rules for **LL** and other substructural logics
- Conclusion & further topics

The most basic logic game: Hintikka's game for classical propositional logic

Idea:

The meaning of connectives is encoded in a game:

– players I and You, acting in role **P** (proponent) or **O** (opponent)

– **P** (initially I) asserts that F is true (**t**) under a given interpretation $\mathcal{I}$, while **O** seeks to establish that F is false (**f**)

Rules of the game refer to the form of the current formula:

$F \wedge G \Rightarrow$ **O** chooses F or G, **P** asserts F or G, accordingly

$F \vee G \Rightarrow$ **P** asserts F or G, according to her own choice

$\neg F \Rightarrow$ after switching roles **P** (the other player) asserts F

Winning condition:

If an atom A is reached, **P** wins if A is true in $\mathcal{I}$, otherwise **O** wins

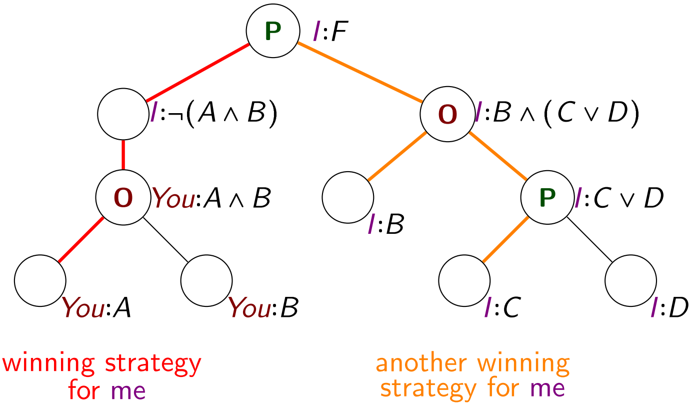
Central Fact: (characterization of Tarski's "truth in a model")

I have a winning strategy iff F is true in $\mathcal{I}$

Hintikka's game: an example

$F = \neg(A \wedge B) \vee (B \wedge (C \vee D))$

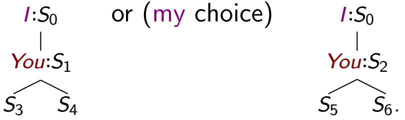
$\mathcal{I}$: $v_{\mathcal{I}}(B) = v_{\mathcal{I}}(C) = \mathbf{t}$, $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(D) = \mathbf{f}$



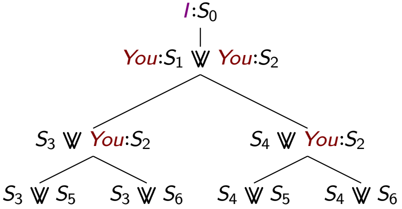
Extracting a classical sequent calculus in 3 steps:

Step 1: from strategies to disjunctive strategies

Suppose players I and You have the following choices:



corresponding disjunctive strategy:



Example (ctd.) – a disjunctive strategy for player I

$$I{:}\neg(A \wedge B) \vee (B \wedge (C \vee D))$$

$$I{:}\neg(A \wedge B) \Vvee I{:}(B \wedge (C \vee D))$$

$$You{:}A \wedge B \Vvee I{:}(B \wedge (C \vee D))$$

$$You{:}A \Vvee You{:}B \Vvee I{:}(B \wedge (C \vee D))$$

$$You{:}A \Vvee You{:}B \Vvee I{:}B \qquad You{:}A \Vvee You{:}B \Vvee I{:}C \vee D$$

$$You{:}A \Vvee You{:}B \Vvee I{:}C \Vvee I{:}D$$

NB:

– if we replace, e.g., C by A, then the formula becomes tautological

– correspondingly, every final disjunctive state contains some atom with both labels (‘You’ as well as ‘I’)

$\Longrightarrow$ we can find a winning strategy for any interpretation!

Extracting a classical sequent calculus (ctd.):

Step 2: Formulate the game rules from *I*/*You* perspective

Remember that there 2 kinds of choices involved:

– use meta-level disjunction ($\bigvee\!\!\!\!\bigvee$) for I-choices
– use branching for You-choices

$$\frac{You{:}A_1 \wedge A_2}{You{:}A_1 \;\bigvee\!\!\!\!\bigvee\; You{:}A_2}\ \wedge\text{-You} \qquad \frac{I{:}A_1 \wedge A_2}{I{:}A_1 \qquad I{:}A_2}\ \wedge\text{-I}$$

$$\frac{I{:}A_1 \vee A_2}{I{:}A_1 \;\bigvee\!\!\!\!\bigvee\; I{:}A_2}\ \vee\text{-I} \qquad \frac{You{:}A_1 \vee A_2}{You{:}A_1 \qquad You{:}A_2}\ \vee\text{-You}$$

$$\frac{I{:}\neg A}{You{:}A}\ \neg\text{-I} \qquad \frac{You{:}\neg A}{I{:}A}\ \neg\text{-You}$$

Extracting a classical sequent calculus (ctd.):

Step 3: sequents as meta-disjunctions of *I*/*You*-signed formulas

- Put *I*-signed formulas to the right of the sequent arrow '$\vdash$' and put *You*-signed formulas to the left of '$\vdash$'
- add 'side formulas' (Γ, Δ) to the main (exhibited) ones
- write the rules upside down

$$\frac{A_1, A_2, \Gamma \vdash \Delta}{A_1 \wedge A_2, \Gamma \vdash \Delta}\ \wedge\text{-l} \qquad \frac{\Gamma \vdash \Delta, A_1 \qquad \Gamma \vdash \Delta, A_2}{\Gamma \vdash \Delta, A_1 \wedge A_2}\ \wedge\text{-r}$$

$$\frac{A_1, \Gamma \vdash \Delta \qquad A_2, \Gamma \vdash \Delta}{A_1 \vee A_2, \Gamma \vdash \Delta}\ \vee\text{-l} \qquad \frac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A_1 \vee A_2}\ \vee\text{-r}$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg\text{-l} \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg\text{-r}$$

NB: These are the logical rules for $\wedge/\vee/\neg$ for **LK** (additively)!

Extracting a classical sequent calculus (ctd.):

What about structural rules and initial sequents?

- permutation corresponds to commutativity of $\bigvee\!\!\!\!\bigvee$
- contraction corresponds to idempotency of $\bigvee\!\!\!\!\bigvee$
- weakening is moved to initial sequents
  which are now those containing an atomic formula appearing with both labels (I and You = 'left' and 'right')

What about implication?

- remember: $A \to B$ can be defined as $\neg A \vee B$
- a corresponding rule arises by combining **P**'s choice with a role switch, when **P** chooses $\neg A$
- further connectives call for combining **P** and **O** choices (two-level rules, instead of a single choice or role switch)

What about quantifiers?

NB: Hintikka's covers classical first-order logic, i.e. also $\forall$ and $\exists$

Quantifier rules (assuming a constant for each domain element):

$\forall x F(x) \Rightarrow$ **O** chooses a constant c, **P** asserts $F(c)$

$\exists x F(x) \Rightarrow$ **P** chooses a constant c and asserts $F(c)$

- the assumption regarding constants is inessential: Hintikka actually referred to formulas + variable assignments
- if I am **P** then the $\exists$-rule leads directly to the **LK**-rule $\exists$-r similarly for the **LK**-rule $\forall$-l, if I am in role **O**
- for the **LK**-rules with eigenvariable condition the lifting from ordinary game states to disjunctive states has to be 'compactified' using a unique (new) placeholder variable to represent the infinite branching

Hintikka's game and (many) truth values

Observation: Hintikka's game looks tightly classical (bivalent):

- winning/loosing corresponds to atomic truth/falsehood
- there are just three types of moves:
  — **P**'s choice
  — **O**'s choice
  — role switch
  NB: combinations of such moves (as, e.g., in →-rule) don't lead beyond classical logic

Two ways to generalize to many-valued logics:

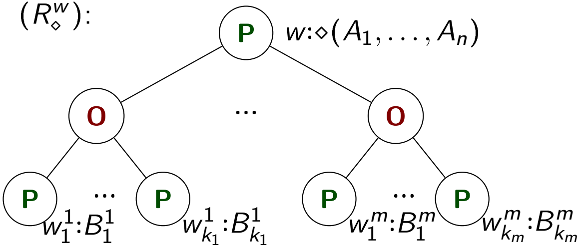
(1) Many-valued payoffs: leads to $\wedge / \vee / \neg$ as $\min / \max / 1 - x$ and calls for further generalizations (⇒ Giles's game)

(2) Taking role switch as the clue:
The "role assignment" can be seen as truth value
P asserts "**t**:*F*" before, but "**f**:*F*" after role switch.

Corresponding generalization:
P always asserts (and **O** always denies) a statement of the form "*F* takes value *w*", denoted as $w{:}F$

General format of a game rule for connective $\diamond$:

$(R^w_\diamond)$:

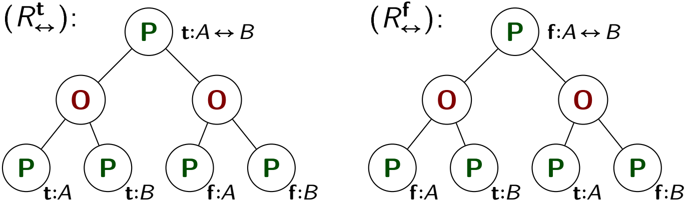


where $w^i_j \in TV$ and $B^i_j \in \{A_1,\ldots,A_n\}$ for $1 \le i \le m$, $1 \le j \le k_i$.

Remark:

A dual form, where **O** chooses first, leads to equivalent results

Concrete rule instances (classical equivalence):



Note: The rules correspond to external disjunctive normal forms
(the dual form corresponds to conjuctive normal forms)

$$\mathbf{t}{:}A \leftrightarrow B \equiv ((\mathbf{t}{:}A \wedge\!\!\!\wedge \mathbf{t}{:}B) \vee\!\!\!\vee (\mathbf{f}{:}A \wedge\!\!\!\wedge \mathbf{f}{:}B))$$

$$\mathbf{f}{:}A \leftrightarrow B \equiv ((\mathbf{f}{:}A \wedge\!\!\!\wedge \mathbf{t}{:}B) \vee\!\!\!\vee (\mathbf{t}{:}A \wedge\!\!\!\wedge \mathbf{f}{:}B))$$

Normal forms can be directly read off from arbitrary truth tables!

From truth tables to semantic games

Definition:

A matrix semantics $\mathcal{M}$ for a propositional language $\mathcal{L}$ specifies a truth table $\widehat{\diamond}$ over truth values $\mathcal{V}$ for each connective $\diamond$ in $\mathcal{L}$.
This induces a (deterministic) valuation $v^{\alpha}_{\mathcal{M}} : \mathrm{FORM}_{\mathcal{L}} \to \mathcal{V}$ over an assignment $\alpha : \mathrm{PV} \to \mathcal{V}$, as usual.

Definition:

Given a matrix semantics $\mathcal{M}$, a corresponding $\mathcal{M}$-game (played under an assignment α) is obtained from external disjunctive normal forms for every pair $w{:}\diamond(F_1, \ldots, F_n)$, as outlined before.
Ending in $w{:}A$, **P** wins if $v^{\alpha}_{\mathcal{M}}(A) = w$, otherwise **O** wins.

Theorem:

For every matrix semantics $\mathcal{M}$ and assignment α t.f.a.e.:

(1) **P** has a winning strategy for the $\mathcal{M}$-game starting with $w{:}F$.

(2) $v^{\alpha}_{\mathcal{M}}(F) = w$, where $v^{\alpha}_{\mathcal{M}}$ is the valuation over α.

Generalization to Avron's Nmatrices

Claim:

Starting with semantic games to obtain truth-functional semantics straightforwardly leads to Nmatrices!

Definition:

An Nmatrix semantics $\mathcal{N}$ specifies a nondeterministic truth table $\widehat{\diamond} : \mathcal{V}^n \mapsto (2^{\mathcal{V}} - \emptyset)$ for each n-ary connective $\diamond$.

At least two possible forms nondeterministic valuations arise:

Dynamic valuation:

– $\vec{v}^{\alpha}_{\mathcal{N}}(F) = \alpha(F)$ if $F \in \mathrm{PV}$

– $\vec{v}^{\alpha}_{\mathcal{N}}(\diamond(F_1, \ldots, F_n)) \in \widetilde{\diamond}(\vec{v}^{\alpha}_{\mathcal{N}}(F_1), \ldots, \vec{v}^{\alpha}_{\mathcal{N}}(F_n))$ for n-ary $\diamond$

Static valuation:

A static valuation $\check{v}^{\alpha}_{\mathcal{N}}$ is a dynamic valuation satisfying $\check{v}^{\alpha}_{\mathcal{N}}(\diamond(G_1, \ldots, G_n) = \check{v}^{\alpha}_{\mathcal{N}}(\diamond(F_1, \ldots, F_n))$ if $\check{v}^{\alpha}_{\mathcal{N}}(G_i) = \check{v}^{\alpha}_{\mathcal{N}}(F_i)$

Caveat: While static and dynamic valuations can modeled, a new type of valuation ('liberal valuation') is more natural

Game rules and Nmatrices

Observation:

Arbitrary collections of rules $R_{\diamond}^{w}$ (one for each pair $\langle w, \diamond \rangle$) do not correspond to truth functional finite valued logics. (Just consider identical external normal forms for different truth values.)

However:

Every collection of rules $R_{\diamond}^{w}$ does determine a particular form of valuation over some finite-valued Nmatrix semantics $\mathcal{N}$: namely one, where different occurrences of the same subformula might be evaluated differently. (Otherwise like dynamic valuation.)

We call such valuations (over an $\mathcal{N}$) liberal valuations over $\mathcal{N}$ and the corresponding games $\mathcal{N}$-games.

Theorem:

For every Nmatrix semantics $\mathcal{N}$ and assignment α t.f.a.e.:

(1) **P** has a winning strategy for the $\mathcal{N}$-game starting with $w{:}F$.

(2) $\tilde{v}_{\mathcal{N}}^{\alpha}(F) = w$, where $\tilde{v}_{\mathcal{N}}^{\alpha}$ is some liberal valuation over α.

Connection to analytic proof systems

- The translation from ordinary game states to disjunctive states can be applied to applied to $\mathcal{M}$-games just as well.

  This results in signed sequent calculi (also known as 'many-sided sequent systems').

- Nmatrices were invented for the analysis of sequent systems!

  There is tight connection between Nmatrix-based semantics and so-called canonical proof systems.

  General criteria for cut-eliminability arise in this manner.

- Nmatrices naturally arise if certain logical logical rules are missing in canonical proof systems.

  (With Esther Corsi, we have recently applied this concept to 'logics of argumentation' arising from Dung-style abstract argumentation frames.)

Robin Giles about reasoning in theories of physics

Robin Giles 1974/77: 'A non-classical logics for physics'

Principles of Giles's analysis of reasoning:

- All assertions have to be tested with respect to concrete (instances of) binary experiments:
  Each atomic assertion $P(t_1, \ldots, t_n)$ is connected to a parameterized experiment $E_P^{t_1,..,t_n}$ that may fail or succeed.
- Experiments may show dispersion: different instances of the same experiment may yield different results.
- To provide a tangible meaning to sentences one imagines a dialogue between me and you, where we are willing to pay 1€ to the opponent for each false atomic assertion, i.e., one where the corresponding instance of the experiment fails. Note: since experiments are dispersive, assertions are risky!
- A tenet collects all assertions of a player (me or you).
  Repetita juvant: Tenets are multisets of interpreted sentences.

Important observations:

- I can quantify the expected loss for my tenet $\{q_1, \ldots, q_n\}$ of atomic assertions by assigning a subjective failure probability $\langle q_i \rangle$ to the experiment E_{q_i}.
- While these probabilities may have some objective grounds they are still subjective in the sense that I don't care which values you associate with the same experiments.
- Events are (unrepeatable) instances of (repeatable) elementary experiments. In other words:
experiments are event types, such that the same probabilities are assigned to events of the same type.
- Final (or: atomic) game states of are denoted by $[p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m]$, where $\{p_1, \ldots, p_n\}$ is your tenet and $\{q_1, \ldots, q_m\}$ is my tenet of assertions.
My corresponding risk, i.e., my expected loss of money is

$$\sum_{1 \leq i \leq m} \langle q_i \rangle € - \sum_{1 \leq j \leq n} \langle p_j \rangle €$$

What about logically complex statements?

NB: So far, no logic has been involved!

For the reduction of logically complex assertions to atomic ones, Giles suggests a game referring to Lorenzen's dialogue game, rather than to Hintikka. (Giles seemingly didn't know about Hintikka's semantic game, introduced roughly at the same time.)

Giles states the rules in the following — old fashioned — way:

- *He who asserts $A \to B$ agrees to assert B if his opponent will assert A.*
- *He who asserts $A \vee B$ undertakes to assert either A or B at his own choice.*
- *He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.*

Defining $\neg A = A \to \bot$ leads to

- *He who asserts $\neg A$ agrees to pay 1\$ to his opponent if he will assert A.*

Observations about the dialogue part of Giles's game

(1) Assertions are attacked at most once: *'repetita juvant'*.

(2) Principle of limited liability for attacks:
The players may explicitly choose not to attack an assertion.

(3) In contrast to Lorenzen:

- no regulations on the succession of moves!
- no restrictions on who can attack what!

(4) Giles defends the $\wedge$-rule by reference to the above principle of limited liability: each assertion carries a maximal risk of 1\$.

Giles has no rule for strong conjunction (&)!
By extending the principle to defense move we obtain:

- *If a player asserts $A \& B$ she has to assert either both, A and B, or else has to assert $\perp$ (i.e., to pay 1€).*

Adequateness of Giles's game for propositional Ł

Theorem (coarse version):
I always have a strategy for avoiding expected loss for precisely those initial statements that are valid in Łukasiewicz logic.

Theorem (refined version):
Suppose we play the game starting with my assertion of F with respect to given assignment $\langle\cdot\rangle$ of risk values to atomic assertions.

The following are equivalent:

- F evaluates to $1-x$ in Łukasiewicz logic under the interpretation that assigns $1 - \langle p \rangle$ to each atom p.
- My best strategy guarantees that the play ends in a state, where my risk is at most x€, while You have a strategy enforcing that my risk is at least x€.

Can games help to analyze analytic proof systems?

Suppose I'd announce that I want to talk about the the logic given by the following Hilbert-style system:

1. $(A \to B) \to C) \to ((((B \to A) \to C) \to C)$
2. $(A \to B) \to ((B \to C) \to (A \to C))$
3. $\bot \to A$
4. $((A \to \bot) \to \bot)) \to A$
5. $(A \& B) \to B$
6. $(A \& B) \to (B \& A)$
7. $(A \& (A \to B)) \to (B \& (B \to A))$
8. $((A \& B) \to C) \to (A \to (B \to C))$
9. $(A \to (B \to C)) \to ((A \& B) \to C)$

Modus Ponens is the only inference rule

You were justified to loose interest in my presentation, because of this obviously(?) inadequate presentation of a logic!

An improvement?

Suppose I replace the above list of axioms by

$$
\begin{array}{ll}
1 & A \to (B \to A) \\
2 & (A \to B) \to ((B \to C) \to (A \to C)) \\
3 & (\neg A \to \neg B) \to (B \to A) \\
4 & ((A \to B) \to B) \to ((B \to A) \to A)
\end{array}
$$

A much improved start:

I want to talk about

- one of three fundamental fuzzy logics
- the logic based on the t-norm $\max(0, x + y - 1)$
- the logic of all MV-algebras
- the logic where formulas represent McNaughton functions
- the logic of Mundici's Ulam-Renyi game semantics
- the only fuzzy logic where *all* truth functions are continuous
- ...

In other words: Łukasiewicz logic **Ł**!

Formal reasoning

The above remarks seem to suggest:

- proof theoretic (syntactic) presentations are uninformative
- algebraic (semantic) characterizations are needed

But what if we focus on formal reasoning (within the logic)?!

Hilbert style systems are problematic (also) for this purpose!
But:
think of Gentzen's characterization of classic vs. intuitionistic inference in terms of the cut-free sequent calculus!

NB. The following systems are related in this respect:

- analytic tableaux
- natural deduction
- calculus of structures
- ...

HŁ – A hypersequent system for Łukasiewicz logic:

Initial sequents:

$$A \vdash A \ (ID) \qquad \vdash \ (EMPTY) \qquad \bot, \Gamma \vdash A \ (\bot, l)$$

Logical rules:

$$\frac{B, \Gamma \vdash \Delta, A \mid \Gamma \vdash \Delta \mid \mathcal{H}}{A \to B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\to, l) \qquad \frac{A, \Gamma \vdash \Delta, B \mid \mathcal{H} \qquad \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{H}} \ (\to, r)$$

$$\frac{A, \Gamma \vdash \Delta \mid B, \Gamma \vdash \Delta \mid \mathcal{H}}{A \wedge B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\wedge, l) \qquad \frac{\Gamma \vdash \Delta, A \mid \mathcal{H} \qquad \Gamma \vdash \Delta, B \mid \mathcal{H}}{\Gamma \vdash \Delta, A \wedge B \mid \mathcal{H}} \ (\wedge, r)$$

$$\frac{A, B, \Gamma \vdash \Delta \mid \mathcal{H} \qquad \bot, \Gamma \vdash \Delta \mid \mathcal{H}}{A \,\&\, B, \Gamma \vdash \Delta \mid \mathcal{H}} \ (\&, l) \qquad \frac{\Gamma \vdash \Delta, A, B \mid \Gamma \vdash \Delta, \bot \mid \mathcal{H}}{\Gamma \vdash \Delta, A \,\&\, B \mid \mathcal{H}} \ (\&, r)$$

Structural rules:

$$\frac{\mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}} \ (EW) \qquad \frac{\Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}} \ (EC) \qquad \frac{\Gamma \vdash \Delta \mid \mathcal{H}}{A, \Gamma \vdash \Delta \mid \mathcal{H}} \ (IW)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}}{\Gamma_1 \vdash \Delta_2 \mid \Gamma_2 \vdash \Delta_1 \mid \mathcal{H}} \ (SPLIT) \qquad \frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{H} \qquad \Gamma_2 \vdash \Delta_2 \mid \mathcal{H}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}} \ (MIX)$$

Although unusual, **HŁ** has nice properties:

- sound and complete for **Ł**
- (potentially much shorter, but hard to find) proofs using

$$\frac{\mathcal{H} \mid \Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2 \mid \mathcal{H}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}} \; (CUT)$$

  can be stepwise transformed into cut-free proofs [CM03]
- applications of structural rules can be limited to atomic hypersequents (except (EW) for trivial reasons)
- the 'purely logical' version of **HŁ** reduces all complex hypersequents to atomic hypersequents, for which validity can be checked in PTIME

Nevertheless:
is **HŁ** a really convincing analysis of actual reasoning?!

Dialogue games and the meaning of connectives

Lorenzen/Giles Idea (similar to Hintikka):
The meaning of a logical connective is given by dialogue game rules, like the following:

Let **P**/**O** stand for me/you or for you/me

P asserts	'attack' by **O**	answer by **P**
$A \to B$	A	B
$A \vee B$	'?'	A or B (**P** chooses)
$A \wedge B$	'l?' or 'r?' (**O** chooses)	A or B (accordingly)
$A \& B$	'?'	A and B
$\forall x\, A(x)$	't?' (**O** chooses)	$A(t)$
$\exists x\, A(x)$	'?'	$A(t)$ (**P** chooses)

Note: $\neg A$ abbreviates $A \to \bot$.
The assertion '$\bot$' is always false.

The rules in sequent-style format

State of the game: $[A_1, \ldots, A_n \parallel B_1, \ldots, B_m]$

I assert $B_1, \ldots, B_m$, while you assert $A_1, \ldots, A_n$

The rules from my point of view (for brevity, only $\rightarrow$ and $\&$):

$$\frac{[B, \Gamma \parallel \Delta, A]}{[A \rightarrow B, \Gamma \parallel \Delta]} \ (\rightarrow, me) \qquad \frac{[A, \Gamma \parallel \Delta, B]}{[\Gamma \parallel \Delta, A \rightarrow B]} \ (\rightarrow, you)$$

$$\frac{[A, B, \Gamma \parallel \Delta]}{[A \& B, \Gamma \parallel \Delta]} \ (\&, me) \qquad \frac{[\Gamma \parallel \Delta, A, B]}{[\Gamma \parallel \Delta, A \& B]} \ (\&, you)$$

Note: the labels refer to the attacking player

- complex statements are decomposed exactly once
- no ‘hedging’ or ‘refuse to attack’ is allowed
- arbitrary states are reduced to atomic states
- no winning conditions formulated yet!

Dialogues as evaluation games

NB: If we add an evaluation function – assigning real numbers to atomic states – to the dialogue rules we obtain an evaluation game

A simple, but interesting example:

1. assign an arbitrary pay-off value $v(p) \in \mathbb{R}$ to each atom p
2. define $v([p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m]) = \sum_i v(q_i) \; - \sum_j v(p_j)$
3. $\Longrightarrow$ finite 2-person game with perfect information: guaranteed pay-off for me can be calculated using induction following the max-min strategy for finite game trees

The resulting logic is Slaney's Abelian logic (which coincides with one of Casari's logic of comparison):

- 'truth value set' is $\mathbb{R}$
- truth function for conjunction: addition
- truth function for implication: subtraction
- validity: value ≥ 0 under all assignments

Dialogues as evaluation games (ctd.)

To obtain Łukasiewicz logic we have to do three things:

(1) restrict to $v(p) \in [0,1]$ for atoms p; $v(\bot)=0$

(2) allow refusion to attack (no player is forced to attack)

(3) allow hegding of maximal loss: instead of defending my(your) assertion I(you) can replace it by asserting $\bot$

A simplification:

(2) is only relevant for implication ($\to$).

(3) is only relevant for strong conjunction ($\&$).

The resulting rules are:

$$\frac{[B, \Gamma \parallel \Delta, A]}{[A \to B, \Gamma \parallel \Delta]}\ (\to, \mathit{me}) \quad \frac{[\Gamma \parallel \Delta]}{[A \to B, \Gamma \parallel \Delta]}\ (\to, \mathit{me}) \quad \frac{[A, \Gamma \parallel \Delta, B] \qquad [\Gamma \parallel \Delta]}{[\Gamma \parallel \Delta, A \to B]}\ (\to, \mathit{you})$$

$$\frac{[A, B, \Gamma \parallel \Delta] \quad [\bot, \Gamma \parallel \Delta]}{[A \& B, \Gamma \parallel \Delta]}\ (\&, \mathit{me}) \quad \frac{[A, \Gamma \parallel \Delta, B]}{[\Gamma \parallel \Delta, A \& B]}\ (\&, \mathit{you}) \quad \frac{[\Gamma \parallel \Delta, \bot]}{[\Gamma \parallel \Delta, A \& B]}\ (\&, \mathit{you})$$

What is the relation to Giles's game?

Remember: Giles talks about:

- payments to the opponent for each false assertion
- dispersive experiments that decide about the truth/falsity of atomic assertions
- probabilities associated with experiments
- minimizing risk (expected amount of payments)

Have we lost the connection to Giles's approach?!

No!

Giles's story about dispersive experiments etc. is only a proposal to attach tangible meaning to $v(p)$ and to $v([p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m])$

My expected loss in such a final state can be calculated to be $\sum_i \langle q_i \rangle - \sum_j \langle p_j \rangle$€, where $\langle p \rangle$ is short for the risk associated with the corresponding experiment E_p: $\langle p \rangle = 1 - \pi(E_p)$

Minimizing my expected payment to You amounts to maximizing v

From evaluation games to hypersequent systems

Giles's game – and its variants – are semantic games, i.e., interactive forms of determining truth values (Giles: risk values), given particular assignments.

While the rules can be presented in sequent format we still seem to be far from a hypersequent calculus like **HŁ** for checking validity.

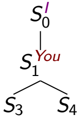
However, we can use the same generic way as for Hintikka's games to turn the semantic game into a provability game:
Keep all choices available: states $\longrightarrow$ disjunctive states

Resulting disjunctive strategies can be seen as
– either referring to a generalized, parallel version of the game
– or simply a bookkeeping device that collects all relevant ordinary strategies into one combined structure (tree)

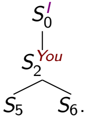
Evaluation of atomic disjunctive states:
winning means: at least one component state is winning (for me)

From strategies to disjunctive strategies

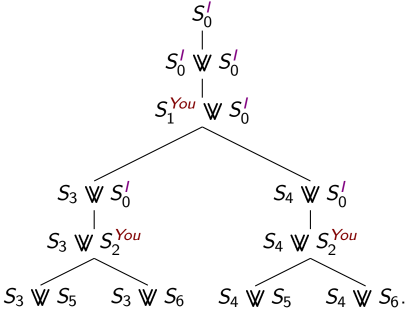
Suppose players me and you have the following choices:



or (my choice)



corresponding disjunctive strategy:



Disjunctive winning strategy for $(p \to q) \vee (q \to p)$

$$[\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel (p \to q) \vee (q \to p)]^{You} \mathbin{\mathbb{W}} [\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel \underline{(p \to q) \vee (q \to p)}]^{I} \mathbin{\mathbb{W}} [\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel p \to q]^{You} \mathbin{\mathbb{W}} [\ \parallel (p \to q) \vee (q \to p)]^{You}$$

$$[\ \parallel p \to q]^{You} \mathbin{\mathbb{W}} [\ \parallel \underline{(p \to q) \vee (q \to p)}]^{I}$$

$$[\ \parallel p \to q]^{You} \mathbin{\mathbb{W}} [\ \parallel q \to p]^{You}$$

$$[\ \parallel \underline{p \to q}]^{You} \mathbin{\mathbb{W}} [\ \parallel q \to p]^{You}$$

$$[p \parallel q] \mathbin{\mathbb{W}} [\ \parallel q \to p]^{You} \qquad [\ \parallel\ ] \mathbin{\mathbb{W}} [\ \parallel q \to p]^{You}$$

$$[p \parallel q] \mathbin{\mathbb{W}} [\ \parallel \underline{q \to p}]^{You} \qquad [\ \parallel\ ] \mathbin{\mathbb{W}} [\ \parallel \underline{q \to p}]^{You}$$

$$[p \parallel q] \mathbin{\mathbb{W}} [q \parallel p] \qquad [p \parallel q] \mathbin{\mathbb{W}} [\ \parallel\ ] \qquad [\ \parallel\ ] \mathbin{\mathbb{W}} [q \parallel p] \qquad [\ \parallel\ ] \mathbin{\mathbb{W}} [\ \parallel\ ].$$

Disjunctive game rules are hypersequent rules!

Rules of the disjunctive game:

$$\frac{[B,\Gamma \parallel \Delta, A] \Cup [\Gamma \parallel \Delta] \Cup \mathcal{H}}{[A \to B,\Gamma \parallel \Delta] \Cup \mathcal{H}}\ (\to, l) \qquad \frac{[A,\Gamma \parallel \Delta, B] \Cup \mathcal{H} \quad [\Gamma \parallel \Delta] \Cup \mathcal{H}}{[\Gamma \parallel \Delta, A \to B] \Cup \mathcal{H}}\ (\to, r)$$

$$\frac{[A,\Gamma \parallel \Delta] \Cup [B,\Gamma \parallel \Delta] \Cup \mathcal{H}}{[A \wedge B,\Gamma \parallel \Delta] \Cup \mathcal{H}}\ (\wedge, l) \qquad \frac{[\Gamma \parallel \Delta, A] \Cup \mathcal{H} \quad [\Gamma \parallel \Delta, B] \Cup \mathcal{H}}{[\Gamma \parallel \Delta, A \wedge B] \Cup \mathcal{H}}\ (\wedge, r)$$

$$\frac{[A, B,\Gamma \parallel \Delta] \Cup \mathcal{H} \quad [\bot,\Gamma \parallel \Delta] \Cup \mathcal{H}}{[A \,\&\, B,\Gamma \parallel \Delta] \Cup \mathcal{H}}\ (\&, l) \qquad \frac{[\Gamma \parallel \Delta, A, B] \Cup [\Gamma \parallel \Delta, \bot] \Cup \mathcal{H}}{[\Gamma \parallel \Delta, A \,\&\, B] \Cup \mathcal{H}}\ (\&, r)$$

Disjunctive game rules are hypersequent rules!

Logical rules of **HŁ**:

$$\frac{B, \Gamma \vdash \Delta, A \mid \Gamma \vdash \Delta \mid \mathcal{H}}{A \to B, \Gamma \vdash \Delta \mid \mathcal{H}}\ (\to, l) \qquad \frac{A, \Gamma \vdash \Delta, B \mid \mathcal{H} \qquad \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{H}}\ (\to, r)$$

$$\frac{A, \Gamma \vdash \Delta \mid B, \Gamma \vdash \Delta \mid \mathcal{H}}{A \wedge B, \Gamma \vdash \Delta \mid \mathcal{H}}\ (\wedge, l) \qquad \frac{\Gamma \vdash \Delta, A \mid \mathcal{H} \qquad \Gamma \vdash \Delta, B \mid \mathcal{H}}{\Gamma \vdash \Delta, A \wedge B \mid \mathcal{H}}\ (\wedge, r)$$

$$\frac{A, B, \Gamma \vdash \Delta \mid \mathcal{H} \qquad \bot, \Gamma \vdash \Delta \mid \mathcal{H}}{A \,\&\, B, \Gamma \vdash \Delta \mid \mathcal{H}}\ (\,\&\,, l) \qquad \frac{\Gamma \vdash \Delta, A, B \mid \Gamma \vdash \Delta, \bot \mid \mathcal{H}}{\Gamma \vdash \Delta, A \,\&\, B \mid \mathcal{H}}\ (\,\&\,, r)$$

What happened to structural rules and initial sequents?

Initial sequents: $A \vdash A$ (ID) $\qquad \vdash$ $(EMPTY)$ $\qquad \bot, \Gamma \vdash A$ $(\bot, l)$

Structural rules:

$$\frac{\mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}}\ (EW) \qquad \frac{\Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta \mid \mathcal{H}}\ (EC) \qquad \frac{\Gamma \vdash \Delta \mid \mathcal{H}}{A, \Gamma \vdash \Delta \mid \mathcal{H}}\ (IW)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}}{\Gamma_1 \vdash \Delta_2 \mid \Gamma_2 \vdash \Delta_1 \mid \mathcal{H}}\ (SPLIT) \qquad \frac{\Gamma_1 \vdash \Delta_1 \mid \mathcal{H} \qquad \Gamma_2 \vdash \Delta_2 \mid \mathcal{H}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid \mathcal{H}}\ (MIX)$$

Remember: the structural rules of **HŁ** are only needed at the atomic level. (For proving sequents (EW) is redundant.)

If we are satisfied with more complex initial sequents then the structural rules are redundant!

Should we be satisfied with complex initial sequents?

In this case: yes!
Reason: it can be checked in PTIME whether a given atomic hypersequent is valid or not.

Other t-norm based fuzzy logics

t-norms are binary operations on $[0,1]$ that are associative, commutative, non-decreasing (in both arguments) with 1 as unit.

Three fundamental logics based on t-norms $\circ$ and their residua:

Logic	$x \circ y$	$x \Rightarrow y$, for $x > y$
Łukasiewicz:	$\max(0, x + y - 1)$	$1 - x + y$
Gödel:	$\min(x, y)$	y
Product:	$x \cdot y$	y/x

Gödel logic G: hypersequents [Avron 91], sequents-of-relation [Baaz/F 99], parallellized dialogue games [F 02], ...

Łukasiewizc logic Ł: hypersequents [Metcalfe et.al. 02]

Product logic P: hypersequent calculus [Metcalfe et.al. 03]

Other ways of combining elementary claims

(We translate “loosing when false” into “winning when true”)

Basic idea: $[p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m]$ denotes my expect gain when betting for positive results of the q_i's against your bet for positive results of the p_i's.

This is ambigous!

“Beting for $B_1, \ldots, B_m$” can mean (at least) one of the following

- betting separately: $\langle B_1, \ldots, B_m \rangle =_{df.} \sum_i \langle B_i \rangle$ ($\Rightarrow$ logic **Ł**)
- betting jointly: $\langle B_1, \ldots, B_m \rangle =_{df.} \prod_i \langle B_i \rangle$ ($\Rightarrow$ logic **P** [**CHL**])
- worst case bet: $\langle B_1, \ldots, B_m \rangle =_{df.} \min_i \langle B_i \rangle$ ($\Rightarrow$ logic **G**)

Underlying dialogue rules (i.e., ‘meaning postulates’ for connectives) remain unchanged! Only the axioms change!

However:
The rules for constructing strategies must be made more explicit.

Other logics:

"Making the rules for constructing strategies more explicit" means: making the (implicit) case distinction $A \leq B / B < A$ explicit, at least in the rules for implication.

This obviously requires to consider "<" in addition to "≤" in denoting (disjunctive) states and corresponding (hyper)sequents.

With hindsight, "<" should have been there from the beginning!

Observe:

With ≤ the game is not zero-sum: both players can 'win' (or possibly none, if we require a positive expected gain).

A case where we don't have to change anything except axioms:

Cancellative hoop logic CHL: like product, but over $(0, 1]$

(Another case is classical logic!)

Summary and Conclusion

- Analytic ('Gentzen style') proof systems are needed for effective proof search, but also for analyzing reasoning within a logic like Łukasiewicz logic **Ł**.
- Hypersequents enable useful analytic systems, but seem problematic as formal models of reasoning.
- Dialogue games, like Giles's for **Ł**, model reasoning from first principles, but seem only to refer to truth evaluation.
- We have shown:

  Constructing disjunctive strategies for Giles-style games corresponds directly to logical hypersequent rules.

  Structural rules are only needed to reduce valid atomic hypersequents into simple sequents.

  This principle generalizes to other fuzzy logics.

Selected References

- C.F., G. Metcalfe: *Giles's Game and the Proof Theory of Lukasiewicz Logic*. *Studia Logica* 92(1): 27-61 (2009).
- R. Giles: *A non-classical logic for physics.*
  In: R. Wojcicki, G. Malinkowski (eds.) *Selected Papers on Łukasiewicz Sentential Calculi*. 1977, 13-51
  Short version in: *Studia Logica* 4(33): 399-417 (1974).
- A. Ciabattoni, C.F., G. Metcalfe: *Uniform Rules and Dialogue Games for Fuzzy Logics*.
  LPAR 2004, Springer LNAI 3452 (2005), 496-510.
- C.G. Fermüller, O. Majer: *Equilibrium Semantics for IF Logic and Many-Valued Connectives*.
  TbiLLC 2015, Springer LNCS 10148 (2016), 290-312
- C.F.: *Dialogue Games for Many-Valued Logics – an Overview.*
  Studia Logica 90(1): 43-68 (2008).

check https://www.logic.at/staff/chrisf/selected.html