Power and Limits of Structural Display Rules

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What can (and what cannot) be expressed by structural display rules? Given a display calculus, we present a systematic procedure for transforming axioms into structural rules. The conditions for the procedure are given in terms of (purely syntactic) abstract properties of the base calculus and thus the method applies to large classes of calculi and logics. If the calculus satisfies certain additional properties we prove the converse direction thus characterising the class of axioms that can be captured by structural display rules. Determining if an axiom belongs to this class or not is shown to be decidable. Applied to the display calculus for tense logic, we obtain a new proof of Kracht's Display Theorem I.

Additional Key Words and Phrases: proof theory, display calculus, structural rules, display theorem

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1. INTRODUCTION

[Gentzen 1935] introduced a proof system called the sequent calculus as a tool for studying the structure of proofs in classical and intuitionistic logic. The main result is the cut-elimination theorem which shows how to eliminate the cut-rule from derivations (i.e. proofs) in a specific calculus, leading to an *analytic* calculus for the logic. The feature of an analytic calculus is the *subformula property* which states that every formula that occurs in a derivation is a subformula of the formula to be proved; this allows us to prove important results about the formalized logic and is key for developing automated reasoning methods. Despite the successful formalisation of many important logics, certain interesting logics do not fit into the sequent calculus framework. Moreover, cut-free sequent calculi suffer from a lack of modularity: even when such a calculus is known for a logic, it is often not clear how to define cut-free sequent calculi for the extensions of the logic that are obtained by the addition of further properties (e.g. as new axioms to its Hilbert calculus).

A large range of formalisms extending the sequent calculus have been introduced in the last few decades to define analytic calculi for logics apparently lacking a cut-free sequent formalisation and to alleviate the problem of modularity while still retaining cut-elimination. Prominent examples include the hypersequent calculus [Avron 1987], the display calculus [Belnap 1982], labelled deductive systems [Fitting 1983; Negri 2005], nested sequent systems [Kashima 1994; Brünnler 2006] and the calculus of structures [Guglielmi 2007]. And yet, despite a large number of papers in the literature dealing with this topic some logics still lack an analytic calculus (e.g. the logic of cancellative residuated lattices [Bahls et al. 2003]). It is not known if this is due to the lack of the "correct" inference rule(s) and/or cut-elimination proof, or the lack of an appropriate formalism, or if there is some fundamental obstacle preventing these logics from having an analytic calculus.

Systematic procedures to automate the introduction of analytic calculi from (axiomatic or semantic) specifications of logics are therefore highly sought-after and very useful to deal with the new

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logics that emerge on a regular basis. Results in this area also yield deeper insights into the expressive power and fundamental properties of the different proof theoretic formalisms. This paper tackles exactly this challenge, focusing on the *display calculus*. Introduced under the name Display Logic, the display calculus [Belnap 1982] is a powerful and semantic-independent formalism that has been used to formalise a variety of different logics ranging from resource-oriented logics [Goré 1998b; 1998a; Brotherston 2012] to temporal logics [Kracht 1996]. The display calculus extends Gentzen's language of sequent, comprising of the structural connectives comma and $\vdash$, with new n-ary connectives. While the comma is usually assumed to be associative (and often commutative), no implicit assumptions are made about the n-ary structural connectives in display calculi and properties such as associativity are stated explicitly, using rules built exclusively from structural connectives and variables (*structural rules*). An attractive feature of the display calculus is the general cut-elimination theorem which leads to analyticity and applies to all display calculi obeying eight syntactic conditions C1–C8; only one of them (C8) is non-trivial to verify, and that condition is not relevant for structural rules. These features make the display calculus an ideal candidate for capturing large classes of logics in a unified way, irrespective of their semantics or connectives, and motivates our interest in *analytic (structural) rules* i.e. structural rules satisfying C1–C7.

Various algorithms have been proposed to define analytic calculi for extensions of logics in a uniform and modular manner, e.g. [Kracht 1996; Negri 2005; Ciabattoni et al. 2008; Ciabattoni et al. 2009; Goré et al. 2011; Ciabattoni et al. 2012; Lellmann and Pattinson 2013; Lahav 2013; Marin and Straßburger 2014; Lellmann 2014]. Yet, they all start with a specific calculus in some proof-theoretic framework and transform Hilbert axioms or semantic conditions into suitable rules. Moreover, excepting [Ciabattoni et al. 2012] (sequent structural rules over intuitionistic Lambek logic), [Kracht 1996] (display structural rules over tense logic Kt) and [Lellmann 2014] (hypersequent logical rules over classical logic), these algorithms work in one direction only and do not tell us if a different procedure could capture a larger class of logics.

In contrast with the existing results, the emphasis in this paper is on providing a *methodology* to construct uniform and modular calculi for different classes of logics and on *understanding* how far the modular construction can be developed using the display calculus. Instead of starting from a display calculus for a specific logic, our transformation from Hilbert axioms into structural display rules applies to *any* display calculus satisfying natural properties (the *amenability* conditions). We identify a hierarchy of axiom classes—computed as a function of the invertible logical rules of the chosen base calculus—and show how to translate axioms from suitable classes (*acyclic $\mathcal{I}_2$ axioms*) into equivalent structural display rules satisfying Belnap's conditions C1-C8. More invertible rules in the base calculus lead to larger sets of axioms in each suitable class and hence to the construction of analytic calculi for more logics. The crucial point is that the amenability conditions are purely syntactic abstract conditions on the display calculus. Furthermore we prove the converse direction, namely that under few additional conditions on the chosen base calculus, every structural display rule satisfying C1-C8 actually corresponds to an acyclic $\mathcal{I}_2$ axiom. In other words, the analytic structural rule extensions of a calculus are *characterised* by its acyclic $\mathcal{I}_2$ axioms. Determining if an axiom is acyclic $\mathcal{I}_2$ or not is shown to be decidable.

Our result applies to many (base) calculi, including the calculi for non-associative Bi-Lambek logic [Goré 1998a], Bi-Intuitionistic Logic HB [Wolter 1998], bunched logics [Brotherston 2012] and tense logic Kt [Kracht 1996], and sheds light on the expressive power of analytic structural rules. As a corollary we provide an alternative—and fully checkable—proof of [Kracht 1996] characterisation of analytic structural rule extensions of the display calculus δKt for Kt.

The paper is an extended version of [Ciabattoni and Ramanayake 2013], where we gave the algorithm for transforming axioms into analytic structural rules. The proof of the reverse direction, which leads to a characterisation of analytic structural rule extensions, does not appear in that work.

The paper is organized as follows: Section 2 provides a short introduction to the display calculus (see, e.g., [Wansing 1998; Restall 1998; Ciabattoni et al. 2014] for more details). The algorithm for transforming axioms into structural display rules is described in Section 3, and compared in 3.3 with the seminal algorithm in [Ciabattoni et al. 2008]. The converse direction is contained in Section 4. The case study of tense logics is discussed in Section 5, where our method is compared with Kracht's method and a new proof of his Display Theorem I is presented.

2. DISPLAY CALCULI IN A NUTSHELL

Since this work establishes a general result on display calculi, we provide an abstract introduction of display calculi (independent of any particular calculus or logic).

Definition 2.1. An *a-structure* (resp. *s-structure*) is built from logical formulae and structure constants using structural connectives. A *display sequent* $X \vdash Y$ is a tuple (X, Y) where X (*antecedent*) is an *a-structure* and Y (*succedent*) is an *s-structure*.

We use the term structure to mean an a-structure or an s-structure. To see a concrete example, the reader may find it helpful to look ahead to Example 2.10, where the a-structures and s-structures of the display calculus for Bi-Lambek logic are explicitly defined. A structure Z is a *substructure* of X (denoted $X[Z]$) if Z occurs in X. Trivially every structure is a substructure of itself.

An N-premise *rule* ($N \geq 0$) is a sequence $(s_1, \dots, s_N, s_{N+1})$ of display sequents, written:

$$\frac{s_1 \qquad \cdots \qquad s_N}{s_{N+1}}$$

The sequent s_{N+1} is called the *conclusion* of the rule and the remaining sequents are called the *premises* of the rule. In the case of a 0-premise rule (also called an *initial sequent*), for brevity we simply write the conclusion, omitting the horizontal line. A *calculus* is a set of rules, typically including initial sequents and the *cut-rule*. The rules of the calculus are usually presented as rule schemata. By this we mean that the rule is built from *schematic sequents* each of the form $X \vdash Y$ where X and Y are *schematic structures*, built from *schematic* (structure and formula) *variables* using the structural and the logical connectives and constants. A concrete sequent is obtained from a schematic sequent by substituting a formula (resp. structure) for each schematic formula (structure) variable. A concrete instance of a rule is obtained by replacing the premises and conclusion by concrete sequents. The use of rule schemata in presenting the rules of a calculus is standard. In particular, following standard practice, we do not always distinguish explicitly between a rule instance and a rule schema.

Given a calculus $\mathcal{C}$ and sequent s, we assume that the set of concrete rule instances in $\mathcal{C}$ with conclusion s is finite and computable. For a concrete calculus this can be vertified by inspection. In the abstract case we need to explicitly demand this property from $\mathcal{C}$.

Definition 2.2 (*derivation from assumptions*). Let $\mathcal{C}$ be a calculus and $\mathcal{S}$ a set of sequents. A *derivation (assuming $\mathcal{S}$)* of a sequent s is a directed tree rooted at s where the nodes are display sequents, the leaves are initial sequents or belong to $\mathcal{S}$, and the edges are defined according to the rules of $\mathcal{C}$ (from premises to conclusion).

Notation: We write For$\mathcal{L}$ to denote the formulae of a language $\mathcal{L}$. We use $p, q, \dots$ for propositional variables; $A, B, \dots$ both for formulae and for schematic formulae; L, M, N for structure variables; and $X, Y, U, V, \dots$ to denote either (concrete) structures or schematic structures. This will not cause confusion in practice.

Example 2.3 (*difference between a structure variable and a schematic structure*).
A structure variable can be instantiated to obtain any structure while a schematic structure cannot, in general, be instantiated to obtain an arbitrary structure. Looking ahead to Example 2.10, $(L > M), N$ is an example of a schematic structure constructed from three structure variables L, M and N. A structure variable can be instantiated by, for example, p or $p > q$ or $(p > q), r$. Of these three (concrete) structures, only the latter can be obtained via instantiation of the schematic structure $(L > M), N$. Thus a schematic structure may be viewed as possessing an underlying 'shape'.

Certain types of rules in the calculus will be of special interest to us. A *structural* rule is constructed from structure variables using structural connectives and structure constants (so no schematic formula variables, logical connectives or constants are present). The display rules (see Definition 2.7) and the cut-rule (below) are important examples of structural rules:

$$\frac{L \vdash A \qquad A \vdash M}{L \vdash M}\, cut$$

(where L and M are structure variables and A is a schematic formula variable). The *logical* rules introduce logical connectives into the conclusion. Clearly a rule cannot be both a logical rule and a structural rule.

The calculus $\mathcal{C} + \{\rho_i\}_{i \in I}$ obtained by the addition (strictly speaking, set union) of structural rules $\{\rho_i\}_{i \in I}$ to a calculus $\mathcal{C}$ is called a *structural rule extension* (of $\mathcal{C}$).

A rule is *derivable* in a calculus $\mathcal{C}$ if there is a derivation of every concrete conclusion *assuming* the corresponding premises.

Definition 2.4 (*invertible*). A rule is *invertible* if there is a derivation of each concrete premise assuming the conclusion.

In this paper we will only consider logical rules containing more logical connective occurrences in the conclusion than in any premise of the rule (see Remark 3.8). Furthermore, we assume[1] that the conclusion of invertible logical rules has a schematic formula on one side of the sequent and the other side consists of a structure variable (used in the proof of Lemma 3.27).

Definition 2.5 (*equivalent rules*). Let $\mathcal{R}_0$ and $\mathcal{R}_1$ be sets of rules. We say that $\mathcal{R}_0$ and $\mathcal{R}_1$ are *equivalent* in $\mathcal{C}$ if each rule in $\mathcal{R}_i$ is derivable in $\mathcal{C} + \mathcal{R}_{1-i}$ for $i = 0, 1$.

Remark 2.6. Viewing a sequent $X \vdash Y$ as the 0-premise rule with conclusion $X \vdash Y$, we can define in the obvious way what it means for two *sets of sequents* to be equivalent, and for a sequent to be equivalent to a rule.

Now we define the crucial display property whence display calculi get their name. The abstract definition here is slightly more involved than what is encountered for concrete calculi. This is because we need to demand properties here that can simply be verified by inspection in a concrete case.

Definition 2.7 (*display property and display rules*). A calculus $\mathcal{C}$ is said to have the *display property* if it contains a set of single-premise structural rules (the *display rules*) such that

(i) The rule upwards from conclusion to premise of a display rule is also a display rule.
(ii) Suppose that Z occurs in $X \vdash Y$. Then $Z \vdash U$ or $U \vdash Z$ (but not both) are effectively derivable from $X \vdash Y$, for some U, using the display rules.

A structure Z is *displayed* (in a sequent) if the sequent has the form $Z \vdash U$ or $U \vdash Z$. In the former (resp. latter) case, the occurrence Z is said to be *a-part* (*s-part*) in the sequent. If Z is a structure/formula then the sequent is said to display Z as an a-part (resp s-part) structure/formula. Note that we do not exclude the possibility that a substructure can be displayed in more that one way or using arbitrarily long sequences of display rule applications.

A *display calculus* is a calculus with the display property.

Since a formula is itself a structure, the display property applies to a formula occurring in a sequent but not to its proper subformulae. The motivation of the display property is that it permits a finer manipulation of a sequent that is not possible with the usual Gentzen sequent. This finer control permits, for example, a straightforward proof of the general cut-elimination theorem [Belnap 1982]. Contrast, for example, with the typically intricate and delicate proofs of cut-elimination for nested sequent calculi e.g. [Marin and Straßburger 2014].

[1]This is a very natural requirement if we regard the invertible logical rules as *rewrite rules* in the sense of [Goré 1998b].

Definition 2.8 (*logic of $\mathcal{C}$*). Let $\mathcal{C}$ be a display calculus. For an a-structure constant $\mathbf{I}$, the set $L_{\mathbf{I}}(\mathcal{C}) = \{A \text{ is a formula} \mid \mathbf{I} \vdash A \text{ is derivable in } \mathcal{C}\}$ is called the *logic of $\mathcal{C}$ w.r.t* $\mathbf{I}$.

Remark 2.9. Clearly $L_{\mathbf{I}}(\mathcal{C})$ is parametrised by the structural constant $\mathbf{I}$. In the case of a concrete calculus, the appropriate structural constant needs to be chosen. To understand the role of $\mathbf{I}$, recall that Gentzen's LK [Gentzen 1935] is a sequent calculus for classical logic in the sense that $\Rightarrow A$ (note: empty antecedent) is derivable in LK iff A is a theorem of classical logic. Roughly speaking, the empty antecedent in LK is abstracted by the a-structure constant $\mathbf{I}$.

Since this paper is concerned with axiomatic extensions of a logic we will define a logic as the set of derivable formulae in a Hilbert calculus. Recall that a *Hilbert calculus* consists of a set of rule schemata (including zero-premise rules, i.e. axioms) built from propositional variables and logical connectives, and contains the rule of *modus ponens* and the rule of *uniform substitution*. The latter permits the uniform substitution of a propositional variable with an arbitrary formula. A *derivation* of a formula A assuming $A_1, \ldots, A_N$ in the Hilbert calculus $\mathcal{H}$ is a sequence of formulae such that each is an instance of an axiom, or of A_i, or follows from the previous formulae using the rules of $\mathcal{H}$. Define $L(\mathcal{H}) = \{\, B \mid \text{there is a derivation in } \mathcal{H} \text{ of } B \text{ from no assumptions} \,\}$.

Note: alternative presentations of Hilbert calculi use schematic variables rather than propositional variables and the rule of uniform substitution. Schematic variables can be uniformly substituted with any formula so the rule of uniform substitution is not required. Here we use the propositional variables/rule of uniform substitution presentation so that the Hilbert axioms are clearly contrasted with sequents in the display calculus. Nevertheless, at certain points in the text (e.g. the proof of Lemma 3.15) we will move between these presentations and treat the propositional variables as schematic variables for formulae.

Example 2.10 (*Bi-Lambek logic*). (Non-associative) Bi-Lambek logic Bi-FL [Lambek 1993] is obtained by augmenting the language of Lambek calculus with the right $\rightarrow_d$ and left $\leftarrow_d$ coimplication connectives. The set For$\mathcal{L}_{\text{Bi-FL}}$ of formulae are given as follows:

$$\mathcal{F} ::= \text{prop. variable } p \mid \top \mid \bot \mid 1 \mid 0 \mid \mathcal{F} \cdot \mathcal{F} \mid \mathcal{F} + \mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \rightarrow \mathcal{F} \mid \mathcal{F} \leftarrow \mathcal{F} \mid \mathcal{F} \rightarrow_d \mathcal{F} \mid \mathcal{F} \leftarrow_d \mathcal{F}$$

The display calculus δBi-FL [Goré 1998b] is built from sequents $X \vdash Y$ where $X \in \mathfrak{S}_{\text{ant}}$ (a-structures) and $Y \in \mathfrak{S}_{\text{suc}}$ (s-structures):

$$\mathfrak{S}_{\text{ant}} ::= A \in \text{For}\mathcal{L}_{\text{Bi-FL}} \mid \mathbf{I} \mid \Phi \mid \mathfrak{S}_{\text{ant}}, \mathfrak{S}_{\text{ant}} \mid \mathfrak{S}_{\text{ant}} > \mathfrak{S}_{\text{suc}} \mid \mathfrak{S}_{\text{ant}} < \mathfrak{S}_{\text{suc}}$$
$$\mathfrak{S}_{\text{suc}} ::= A \in \text{For}\mathcal{L}_{\text{Bi-FL}} \mid \mathbf{I} \mid \Phi \mid \mathfrak{S}_{\text{suc}}, \mathfrak{S}_{\text{suc}} \mid \mathfrak{S}_{\text{ant}} > \mathfrak{S}_{\text{suc}} \mid \mathfrak{S}_{\text{ant}} < \mathfrak{S}_{\text{suc}}$$

In the following, the double line is used as notation to indicate two rules (read in the downward direction to see one rule and upwards for the other). Each rule of the pair denoted in this way is necessarily invertible. To save space, we also bundle two double line rules together and write these as a single 'object' of three lines. Thus each such object describes four rules in all. The display and structural rules are:

display rules

$$\dfrac{\dfrac{M \vdash L > N}{L, M \vdash N}}{L \vdash M < N} \qquad \dfrac{\dfrac{M > L \vdash N}{L \vdash M, N}}{L < N \vdash M} \qquad \dfrac{\dfrac{\mathbf{I}, L \vdash M}{L \vdash M}}{L, \mathbf{I} \vdash M} \qquad \dfrac{\dfrac{L \vdash M, \mathbf{I}}{L \vdash M}}{L \vdash \mathbf{I}, M}$$

$$\dfrac{\mathbf{I} \vdash M}{L \vdash M} \qquad \dfrac{L \vdash \mathbf{I}}{L \vdash M} \qquad \dfrac{\dfrac{L, \Phi \vdash M}{L \vdash M}}{\Phi, L \vdash M} \qquad \dfrac{\dfrac{L \vdash M, \Phi}{L \vdash M}}{L \vdash \Phi, M}$$

It is easy to check that the display property holds. For every structural connective that may occur as head symbol in the antecedent or succedent, there is a display rule that can be used to 'peel-away' that connective thus revealing its nested substructure. Displaying a substructure is thus computable so Definition 2.7(ii) is satisfied. Moreover, Definition 2.7(iii) holds because the set of sequents display equivalent to a given sequent is finite and computable. This follows from the observation that each display rule preserves the total number of structural connectives in the sequent.

The calculus also contains the cut rule and the following *initial sequents* and *logical rules*:

$$\bot \vdash \mathbf{I} \qquad p \vdash p \qquad \mathbf{I} \vdash \top \qquad \frac{\mathbf{I} \vdash L}{\top \vdash L}\ \top l$$

$$\frac{L \vdash \mathbf{I}}{L \vdash \bot}\ \bot r \qquad 0 \vdash \Phi \qquad \Phi \vdash 1$$

$$\frac{\Phi \vdash M}{1 \vdash M} \qquad \frac{L \vdash \Phi}{L \vdash 0} \qquad \frac{A_i \vdash L}{A_1 \wedge A_2 \vdash L}\ \wedge l; i \in \{1,2\}$$

$$\frac{L \vdash A \qquad L \vdash B}{L \vdash A \wedge B}\ \wedge r \qquad \frac{A \vdash L \qquad B \vdash L}{A \vee B \vdash L}\ \vee l \qquad \frac{L \vdash A_i}{L \vdash A_1 \vee A_2}\ \vee r; i \in \{1,2\}$$

$$\frac{L \vdash A \qquad B \vdash M}{A \to B \vdash L > M} \to l \qquad \frac{L \vdash A > B}{L \vdash A \to B} \to r \qquad \frac{B < A \vdash L}{B \leftarrow_d A \vdash L} \leftarrow_d l$$

$$\frac{L \vdash A \qquad B \vdash M}{L < M \vdash A \leftarrow_d B} \leftarrow_d r \qquad \frac{L \vdash A < B}{L \vdash A \leftarrow B} \leftarrow\mathrm{r} \qquad \frac{A > B \vdash M}{A \to_d B \vdash M} \to\mathrm{l}$$

$$\frac{A \vdash L \qquad M \vdash B}{L > M \vdash A \to_d B} \to_d\mathrm{r} \qquad \frac{A, B \vdash M}{A \cdot B \vdash M}\ \cdot\mathrm{l} \qquad \frac{L \vdash A \qquad M \vdash B}{L, M \vdash A \cdot B}\ \cdot\mathrm{r}$$

$$\frac{A \vdash L \qquad B \vdash M}{A + B \vdash L, M}\ +\mathrm{l} \qquad \frac{L \vdash A, B}{L \vdash A + B}\ +\mathrm{r} \qquad \frac{A \vdash L \qquad M \vdash B}{A \leftarrow B \vdash L < M} \leftarrow\mathrm{l}$$

It can be proved that *Bi-FL is the logic of δBi-FL* (we also say that δBi-FL is a calculus for Bi-FL) in the sense that $A \in$ Bi-FL iff $\mathbf{I} \vdash A$ is derivable in δBi-FL.

A calculus is said to be *cut-eliminable* if it is possible to eliminate all occurrences of the cut-rule from a given derivation in order to obtain a *cut-free* derivation of the same sequent. A display calculus has the *subformula property* if every formula that occurs in a cut-free derivation appears as a subformula of the final sequent. An important feature of the display calculus is Belnap's conditions C1–C8. In the following we write formula (resp. structure) variable to mean a schematic formula (structure) variable.

(C1) Each formula occurring in a premise of a rule instance is a subformula of some formula in the conclusion.
(C2) Occurrences of the identical structure variable in a rule are said to be *congruent* to each other.
(C3) Each structure variable in the premise is congruent to at most one structure variable in the conclusion. I.e. no two structure variables in the conclusion are congruent to each other.
(C4) Congruent structure variables are all either a-part or s-part structures.
(C5) A schematic formula variable in the conclusion of a rule ρ is either the entire antecedent or the entire succedent. This formula is called a *principal formula* of ρ.
(C6/7) Each rule is closed under uniform substitution of arbitrary structures for congruent variables.
(C8) If there are rules ρ and σ with respective conclusions $L \vdash A$ and $A \vdash M$ with formula A principal in both inferences (see C5) and if *cut* is applied to yield $L \vdash M$, then either $L \vdash M$ is identical to either $L \vdash A$ or $A \vdash M$; or it is possible to pass from the premises of ρ and σ to $L \vdash M$ by means of inferences falling under cut where the cut-formula always is a proper subformula of A.

The condition C8 is on the *set* of rules of the calculus. A display calculus satisfies one of the rules in C1–C7 if each rule in the calculus satisfies that condition, and the display calculus satisfies C8 if the set of all rules satisfy C8. Belnap's general cut-elimination theorem states that C2–C8 constitute sufficient conditions for a calculus to be cut-eliminable. Meanwhile C1 is the subformula property. Of the conditions, only C8 is non-trivial to check. Since C8 is pertinent only to logical rules, structural rule extensions of a calculus satisfying C8 preserve this property.

Example 2.11. It is easy to check that δBi-FL satisfies conditions C1-C8, hence it is a cut-eliminable calculus with the subformula property.

Remark 2.12. Suppose that below left is an *instance* of a structural rule r satisfying C1-C7. Then below right is also an *instance* of r when σ is the substitution $A \mapsto X$ where A is a formula and X is an arbitrary concrete structure.

$$\frac{s_1 \qquad \cdots \qquad s_N}{s_{N+1}}\ \mathrm{r} \qquad\qquad \frac{s_1\sigma \qquad \cdots \qquad s_N\sigma}{s_{N+1}\sigma}\ \mathrm{r}$$

LEMMA 2.13. *Let s_1 be the premise of a display rule satisfying C1-C7 and s_2 its conclusion. Then s_1 and s_2 contain exactly the same structure variables, each with multiplicity* 1.

PROOF. From C3, the multiplicity of a structure variable in s_2 must be 1. Also s_1 cannot contain a structure variable that does not appear in s_2 as this would violate C1. The claim follows by now reasoning on the rule with premise s_2 and conclusion s_1, which is a display rule by Def. 2.6(i). □

3. POWER OF STRUCTURAL DISPLAY RULES

We present an algorithm to transform a large class of Hilbert axioms into equivalent structural display rules that preserve cut-elimination and the subformula property when added to a suitable base calculus. The conditions for the procedure are given in terms of purely syntactic abstract properties of the base calculus and thus the method applies to large classes of calculi and logics. This permits the automated construction of (infinitely) many display calculi in a uniform and modular way.

More precisely, given a Hilbert calculus $\mathcal{H}$ and a display calculus $\mathcal{C}$ for $L(\mathcal{H})$ such that $\mathcal{C}$ and $\mathcal{H}$ 'simulate' each other (see Definition 3.33), we show how to obtain structural rules $r_1, \ldots, r_m$ so that $\mathcal{C} + \{r_1, \ldots, r_m\}$ is a cut-eliminable calculus with subformula property for the axiomatic extension $\mathcal{H} + A_1 + \ldots + A_n$. Our method is constructive and works whenever the base calculus $\mathcal{C}$ is 'expressive enough' (i.e., it is *amenable*, Def. 3.1 below) and each formula A_i is of a specific syntactic form that is determined by the logical rules invertible in $\mathcal{C}$.

3.1. From $\mathbf{I} \vdash A$ to equivalent structural rules

Definition 3.1 (*amenable calculus*). Suppose that $\mathcal{C}$ is a display calculus which contains an a-structure constant and an s-structure constant—for brevity, use $\mathbf{I}$ to denote both constants—and satisfies C1–C8. Let $\mathfrak{S}_{\text{ant}}$ and $\mathfrak{S}_{\text{suc}}$ denote the class of a- and s-structures of $\mathcal{C}$, and let $\mathcal{L}$ be the language of $L_{\mathbf{I}}(\mathcal{C})$. A display calculus satisfying the following conditions is said to be *amenable*.

1 (*interpretation functions*) There are functions $\mathrm{l} : \mathfrak{S}_{\text{ant}} \mapsto \mathrm{For}\mathcal{L}$ and $\mathrm{r} : \mathfrak{S}_{\text{suc}} \mapsto \mathrm{For}\mathcal{L}$ such that $\mathrm{l}(A) = A = \mathrm{r}(A)$ for $A \in \mathrm{For}\mathcal{L}$, and for arbitrary $X \in \mathfrak{S}_{\text{ant}}$ and $Y \in \mathfrak{S}_{\text{suc}}$:
  (i) $X \vdash \mathrm{l}(X)$ and $\mathrm{r}(Y) \vdash Y$ are derivable in $\mathcal{C}$.
  (ii) if $X \vdash Y$ is derivable in $\mathcal{C}$ then so is $\mathrm{l}(X) \vdash \mathrm{r}(Y)$.

2 (*logical constants*) There are logical constants $c_a, c_b \in \mathrm{For}\mathcal{L}$ such that the following sequents are derivable for arbitrary $X \in \mathfrak{S}_{\text{ant}}$ and $Y \in \mathfrak{S}_{\text{suc}}$:

$$c_a \vdash Y \qquad\qquad X \vdash c_s$$

3 (*logical connectives*) There are binary connectives $\vee, \wedge \in \mathcal{L}$ and the following are derivable:
  (i) commutativity: $A \star B \vdash B \star A$ where $\star \in \{\vee, \wedge\}$
  (ii) associativity: $A \star (B \star C) \vdash (A \star B) \star C$ and $(A \star B) \star C \vdash A \star (B \star C)$
  Also, for $A, B \in \mathrm{For}\mathcal{L}$, $X \in \mathfrak{S}_{\text{ant}}$ and $Y \in \mathfrak{S}_{\text{suc}}$:
  (a)$_\vee$ $A \vdash Y$ and $B \vdash Y$ implies $A \vee B \vdash Y$
  (b)$_\vee$ $X \vdash A$ implies $X \vdash A \vee B$ for any formula B.
  (a)$_\wedge$ $X \vdash A$ and $X \vdash B$ implies $X \vdash A \wedge B$
  (b)$_\wedge$ $A \vdash Y$ implies $A \wedge B \vdash Y$ for any formula B.

Remark 3.2. In the above definition:

— the function l (resp. r) 'interprets' the structural connectives in the antecedent (resp. succedent);
— we use the notation $\wedge$ and $\vee$ to reflect that in a calculus for intuitionistic or classical logic, the standard connectives of conjunction and disjunction satisfy the properties in Definition 3.1.3.

Example 3.3 (δ*Bi-FL*). The calculus δBi-FL (see Example 2.10) is amenable. Indeed define the functions $l : \mathfrak{S}_{\text{ant}} \mapsto \text{For}\mathcal{L}_{\text{Bi-FL}}$ and $r : \mathfrak{S}_{\text{suc}} \mapsto \text{For}\mathcal{L}_{\text{Bi-FL}}$:

$$
\begin{array}{rlrl}
l(A) &= A & r(A) &= A \\
l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\
l(\Phi) &= 1 & r(\Phi) &= 0 \\
l(X,Y) &= l(X) \cdot l(Y) & r(X,Y) &= r(X) + r(Y) \\
l(X < Y) &= l(X) \leftarrow_d r(Y) & r(X < Y) &= l(X) \leftarrow r(Y) \\
l(X > Y) &= l(X) \rightarrow_d r(Y) & r(X > Y) &= l(X) \rightarrow r(Y)
\end{array}
$$

We prove $X \vdash \mathrm{l}(X)$ and $\mathrm{r}(Y) \vdash Y$ (Definition 3.1.1) simultaneously by induction on the size of X and Y. The base cases are:

$$A \vdash A \qquad \mathbf{I} \vdash \mathrm{l}(\mathbf{I}) \qquad \Phi \vdash \mathrm{l}(\Phi) \qquad \mathrm{r}(\mathbf{I}) \vdash \mathbf{I} \qquad \mathrm{r}(\Phi) \vdash \Phi$$

Each of these is derivable in δBi-FL. Inductive case: we must prove $X \vdash \mathrm{l}(X)$ and $\mathrm{r}(Y) \vdash Y$ for each of the following:

$$
\begin{array}{ccc}
X = U, V & X = U < V & X = U > V \\
Y = U, V & Y = U < V & Y = U > V
\end{array}
$$

We give the proof for $Y = U > V$ (the other cases are similar). We need to obtain a derivation of $\mathrm{r}(U > V) \vdash U > V$ i.e. $\mathrm{l}(U) \rightarrow \mathrm{r}(V) \vdash U > V$. The following suffices—the derivations of $U \vdash \mathrm{l}(U)$ and $\mathrm{r}(V) \vdash V$ are obtained from the induction hypothesis:

$$\frac{U \vdash \mathrm{l}(U) \qquad \mathrm{r}(V) \vdash V}{\mathrm{l}(U) \rightarrow \mathrm{r}(V) \vdash U > V} \rightarrow\mathrm{l}$$

That $X \vdash Y$ implies $\mathrm{l}(X) \vdash \mathrm{r}(Y)$ is shown by induction on the size of X and Y. Definition 3.1.2 holds due to the following derivations (here $c_a := \bot$ and $c_s := \top$):

$$\frac{\bot \vdash \mathbf{I}}{\bot \vdash Y} \qquad \frac{\mathbf{I} \vdash \top}{X \vdash \top}$$

Finally, Definition 3.1.3 can be verified by inspection of the rules for $\vee$ and $\wedge$.

Remark 3.4. Note that Condition 2 in the original definition of amenability in [Ciabattoni and Ramanayake 2013] required the presence in $\mathcal{C}$ of the following rules

$$\frac{\mathbf{I} \vdash L}{Y \vdash L}\, l\mathbf{I} \qquad \frac{L \vdash \mathbf{I}}{L \vdash Y}\, r\mathbf{I}$$

The present condition only specifies the sequents that are derivable (and not the specific form of the rule that should derive it).

Example 3.5 (*Bunched logics*). The bunched logics $\{\mathrm{BI}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$ are obtained as the free combination of the intuitionistic and classical logic with multiplicative intuitionistic and classical linear logic. A display calculus [Brotherston 2012] has been given for each logic in $\{\mathrm{BI}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$. By inspection each calculus is amenable.

Our algorithm abstracts and reformulates for display calculi the procedure in [Ciabattoni et al. 2008; Ciabattoni et al. 2009] for (hyper)sequent calculi and substructural logics. To transform axioms into structural rules we use: (1) the invertible logical rules of $\mathcal{C}$ and (2) the display calculus formulation, below, of the so-called Ackermann's lemma [Ciabattoni et al. 2008; Conradie and Palmigiano 2012] allowing a formula in a rule to switch sides of the sequent moving from conclusion to premises.

LEMMA 3.6 (ACKERMANN'S LEMMA). *Each of the following pairs of rules is pairwise equivalent in an amenable calculus where $A \in For\mathcal{L}$, $\mathcal{S}$ is a set of sequents and M is a structure variable not in $\mathcal{S}$ or X.*

$$\frac{\mathcal{S}}{X \vdash A}\rho_1 \qquad \frac{\mathcal{S} \quad A \vdash M}{X \vdash M}\rho_2 \qquad\qquad \frac{\mathcal{S}}{A \vdash X}\delta_1 \qquad \frac{\mathcal{S} \quad M \vdash A}{M \vdash X}\delta_2$$

PROOF. ($\rho_1 \Rightarrow \rho_2$) Suppose that we have concrete instances $\mathcal{S} \cup \{A \vdash Y\}$ of the premises of ρ_2. Applying ρ_1 to $\mathcal{S}$ we get $X \vdash A$. Applying cut with $A \vdash Y$ we get $X \vdash Y$ and thus it follows that ρ_2 is derivable in a calculus containing ρ_1.

($\rho_2 \Rightarrow \rho_1$) Given concrete instances of the premises $\mathcal{S}$ of ρ_1. Observe that $A \vdash A$ is derivable. Applying ρ_2 to $\mathcal{S} \cup \{A \vdash A\}$ we get $X \vdash A$ as required.

The proof that δ_1 and δ_2 are equivalent is analogous. □

We now give an abstract description of the axioms that we can handle. The description is based on the invertible logical rules of the chosen display calculus $\mathcal{C}$ and is inspired by the classification in [Ciabattoni et al. 2008] for formulae of intuitionistic Lambek logic with exchange FLe. We identify three classes of axioms in the language of $\mathcal{L}$ from which the logical connectives can be eliminated using the invertible logical rules of $\mathcal{C}$ (modulo the display rules) at various levels. The intuition behind the three classes is the following (see Definition 3.11 for the formal definition):

0-inverted axioms $\mathcal{I}_0(\mathcal{C})$. Propositional variables.

1-inverted axioms $\mathcal{I}_1(\mathcal{C})$. Formulae A whose logical connectives can be eliminated by repeatedly applying the *invertible logical rules* backwards starting with $\mathbf{I} \vdash A$ (thus obtaining sets of sequents built from propositional variables and structure constants using the structural connectives of $\mathcal{C}$).

2-inverted axioms $\mathcal{I}_2(\mathcal{C})$. Formulae A whose logical connectives can be eliminated by applying the *invertible logical rules* to the premises of those rules obtained by applying some invertible rules to $\mathbf{I} \vdash A$ followed by Lemma 3.6.

Definition 3.7 (*inv*). Given a display calculus $\mathcal{C}$ and sequent $X \vdash Y$: the set inv($X \vdash Y$) consists of all the sets of sequents obtained by applying upwards (i.e. from conclusion to premise) some sequence of invertible logical rules in $\mathcal{C}$ (and display rules in order to display the formula occurring in the sequent) starting from $X \vdash Y$.

So inv($X \vdash Y$) has the form $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ where each $\mathcal{S}_j$ is a set of sequents.

Let us identify a distinguished subset inv$^{\text{all}}$($X \vdash Y$) $\subseteq$ inv($X \vdash Y$) consisting of those sets of sequents that are obtained by applying invertible logical rules (and display rules where required) *as much as possible*. If all maximal sequences of invertible logical rules applied upwards yield the same set of sequents up to display equivalence, then inv$^{\text{all}}$($X \vdash Y$) is a singleton set.

Remark 3.8. Notice that inv($X \vdash Y$) is computable since displaying a substructure is computable—Definition 2.7(ii)—and each application of a logical rule reduces the number of logical connectives in each premise (see below Definition 2.4). Similarly, inv$^{\text{all}}$($X \vdash Y$) is computable.

Example 3.9. Let A be the axiom $(p \to 0) + ((p \to 0) \to 0)$ for the weak excluded middle. With respect to the calculus δBi-FL (Example 2.10), the set inv($\mathbf{I} \vdash A$) consists of those sets of sequents obtained by applying some number of invertible rules. There is only a single invertible rule that can be applied to $\mathbf{I} \vdash A$ and then $\{\mathbf{I} \vdash (p \to 0), ((p \to 0) \to 0)\} \in$ inv($\mathbf{I} \vdash A$). Applying two invertible rules to $\mathbf{I} \vdash A$ (and display rules where required, of course) yields that the following sets belong to inv($\mathbf{I} \vdash A$):

$$\{\mathbf{I} < (p \to 0) \to 0 \vdash p > 0\} \qquad \{p \to 0 > \mathbf{I} \vdash (p \to 0) > 0\}$$

For the sake of clarity we apply further display rules to present the sequents with $\mathbf{I}$ displayed (we may do this because the display rules hold in both directions so logical equivalence is preserved):

$$\{\mathbf{I} \vdash (p > 0), ((p \to 0) \to 0)\} \qquad \{\mathbf{I} \vdash (p \to 0), ((p \to 0) > 0)\}$$

Applying three invertible rules to $\mathbf{I} \vdash A$ leads to the following, each thus a member of inv($\mathbf{I} \vdash A$):

$$\{\mathbf{I} \vdash (p > \Phi), ((p \to 0) \to 0)\} \quad \{\mathbf{I} \vdash (p \to 0), ((p \to 0) > \Phi)\} \quad \{\mathbf{I} \vdash (p > 0), ((p \to 0) > 0)\}$$

Continuing in this way until all possible invertible rules have been applied leads to the following set which is thus a member of inv$^{\text{all}}$($\mathbf{I} \vdash A$) $\subseteq$ inv($\mathbf{I} \vdash A$):

$$\{\mathbf{I} \vdash (p > \Phi), ((p \to 0) > \Phi)\}$$

The reason that the above set is the only element of inv$^{\text{all}}$($\mathbf{I} \vdash A$) is because every maximal sequence of invertible rules in δBi-FL applied to $\mathbf{I} \vdash A$ leads to this set (up to display equivalence).

A set $\{U_i \vdash V_i\}_{i\in\Omega}$ of sequents is said to contain no logical connectives if all $\{U_i\}_{i\in\Omega}$ and $\{V_i\}_{i\in\Omega}$ are free of logical connectives.

Definition 3.10 (*soluble*). A formula $A \in$ For$\mathcal{L}$ is *a-soluble* (resp. *s-soluble*) if there is some $\{U_i \vdash V_i\}_{i\in\Omega} \in$ inv($A \vdash \mathbf{I}$) (resp. $\in$ inv($\mathbf{I} \vdash A$)) containing no logical connectives.

The most external connective of an a-soluble formula has left introduction rules that are invertible (i.e. it is a *positive* connective [Andreoli 1992]), while the most external connective of an s-soluble formula has invertible right introduction rules (i.e. it is a *negative* connective).

Definition 3.11 ($\mathcal{I}_0(\mathcal{C})$, $\mathcal{I}_1(\mathcal{C})$, $\mathcal{I}_2(\mathcal{C})$). Let $\mathcal{C}$ be an amenable calculus and let $\mathcal{L}$ denote the language of $L_{\mathbf{I}}(\mathcal{C})$. The classes $\mathcal{I}_j(\mathcal{C}) \subseteq$ For$\mathcal{L}$ for $j \in \{0,1,2\}$ are defined in the following way.

- $\mathcal{I}_0(\mathcal{C})$: $A \in$ For$\mathcal{L}$ belongs to the class $\mathcal{I}_0(\mathcal{C})$ if A is a propositional variable.

$A \in$ For$\mathcal{L}$ belongs to the following classes if there is some $\{U_i \vdash V_i\}_{i\in\Omega} \in$ inv($\mathbf{I} \vdash A$) such that

- $\mathcal{I}_1(\mathcal{C})$: $\{U_i \vdash V_i\}_{i\in\Omega}$ contain no logical connectives
- $\mathcal{I}_2(\mathcal{C})$: each a-part formula in $U_i \vdash V_i$ is s-soluble and each s-part formula in $U_i \vdash V_i$ is a-soluble, for each $i \in \Omega$.

We say that $\{U_i \vdash V_i\}_{i\in\Omega}$ *witnesses* $A \in \mathcal{I}_j(\mathcal{C})$.

We will often write $\mathcal{I}_j$ for $\mathcal{I}_j(\mathcal{C})$ when the discussion applies to a generic amenable calculus.

As each propositional variable is both a-soluble and s-soluble it follows that:

$$\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2$$

Remark 3.12. If $A \in \mathcal{I}_j(\mathcal{C})$ then it must be the case that every element in inv$^{\text{all}}$($\mathbf{I} \vdash A$) witnesses it. For this reason all the results in this section hold if we use an arbitrary (fixed) element of $\{$inv$^{\text{all}}(X \vdash Y)\}$ instead of inv($X \vdash Y$). However, the more general Definition 3.7 permits the proof of Lemma 4.4 in the following section.

Example 3.13. Let A be the axiom $(p \to 0) + ((p \to 0) \to 0)$. Let us verify that $A \in \mathcal{I}_2(\delta\text{Bi-FL})$. In Example 3.9 we saw that applying all possible invertible rules upwards starting with $\mathbf{I} \vdash A$ yields the set inv$^{\text{all}}$($\mathbf{I} \vdash A$) consisting of a single element $\{\mathbf{I} \vdash (p > \Phi), ((p \to 0) > \Phi)\}$. In the sequent $\mathbf{I} \vdash (p > \Phi), ((p \to 0) > \Phi)$, the occurrence $p \to 0$ is a-part. It remains to check that this formula is s-soluble (the other formula occurrence is a propositional variable so it is soluble). Since inv($\mathbf{I} \vdash p \to 0$) contains the singleton set $\{\mathbf{I} \vdash p > \Phi\}$ this is indeed the case.

Example 3.14. Consider the display calculus δBi-FL and let A_1 be the axiom $(p \to q) + (q \to p)$. Then $\{\mathbf{I} \vdash (p > q), (q > p)\} \in$ inv($\mathbf{I} \vdash A_1$) and hence $A_1 \in \mathcal{I}_1(\delta\text{Bi-FL})$.

Henceforth a rule whose conclusion is constructed from structure variables and structure constants using structural connectives, and whose premises might additionally contain propostional variables will be called a *semi-structural* rule.

Given any axiom within the class $\mathcal{I}_2(\mathcal{C})$, the proof of the following proposition contains an algorithm to extract equivalent semi-structural rules satisfying conditions C2-C7 such that the calculi obtained from $\mathcal{C}$ by the addition of these rules preserve C8[2] and hence have cut-elimination, notwithstanding the presence of propositional variables in the premises of the semi-structural rules. If the axioms satisfy the additional condition of acyclicity[3] (Definition 3.23) then the semi-structural rules can be transformed into equivalent structural rules satisfying C1–C7. The steps of the algorithm which lead to a cut-eliminable calculus with the subformula property are summarized in Figure 1.

PROPOSITION 3.15. *Let $\mathcal{C}$ be an amenable calculus and $A \in \mathcal{I}_2(\mathcal{C})$ (witnessed by some $\{U_i \vdash V_i\}_{i\in\Omega} \in inv^{all}(\mathbf{I} \vdash A)$). There are computable semi-structural rules $\{\rho_i\}_{i\in\Omega}$ equivalent to $\mathbf{I} \vdash A$ in $\mathcal{C}$ such that $\mathcal{C} + \{\rho_i\}_{i\in\Omega}$ is a cut-eliminable calculus satisfying C2–C8.*

PROOF. The conclusion of the semi-structural rules are built from structure variables and constants using structural connectives while the formula A in the sequent $\mathbf{I} \vdash A$ is built from propositional variables using the logical connectives. In order to meaningfully discuss the equivalence of the semi-structural rules (which permit uniform substitution of concrete structures for structure variables) and the sequent $\mathbf{I} \vdash A$, concrete instances of the latter are obtained via uniform substitution of formulae for propositional variables (see the note above Example 2.10).

First note that $\mathbf{I} \vdash A$ is equivalent to $\{U_i \vdash V_i\}_{i\in\Omega}$ in $\mathcal{C}$. We have noted that the set $\text{inv}^{\text{all}}(\mathbf{I} \vdash A)$ is computable (Remark 3.8). This together with the construction below yields computability of the semi-structural rules.

Let us construct a semi-structural rule equivalent to each $U_i \vdash V_i$. Suppose that $U_i \vdash V_i$ consists of a-part formulae $C_1, \ldots, C_n$ and s-part formulae $D_1, \ldots, D_m$. First display C_1 in $U_i \vdash V_i$ as $C_1 \vdash W_1$ (for some structure W_1) and then apply Lemma 3.6 to obtain the equivalent rule below left. Note that the M_1 in the rule is a new structure variable. Next display C_2 in the conclusion of the rule below left as $C_2 \vdash W_2$ (for some structure W_2) and apply Lemma 3.6 to obtain the equivalent rule below right (recall M_i is a structure variable, W_i is a structure and C_i is a formula):

$$\frac{M_1 \vdash C_1}{M_1 \vdash W_1} \qquad\qquad \frac{M_1 \vdash C_1 \qquad M_2 \vdash C_2}{M_2 \vdash W_2}$$

Repeat in this way until Lemma 3.6 has been applied to every C_i. Next, in the conclusion of the rule obtained in the previous step display D_1 as $W_{n+1} \vdash D_1$ (for some structure W_{n+1}) and apply Lemma 3.6 (replace D_1 with the new structure variable M_{n+1}). Repeat in this way until Lemma 3.6 has been applied to every D_i. In this way we ultimately obtain the following rule.

$$\frac{M_1 \vdash C_1 \ \ldots \ M_n \vdash C_n \quad D_1 \vdash M_{n+1} \ \ldots \ D_m \vdash M_{n+m}}{W_{n+m} \vdash M_{n+m}}$$

Here W_{n+m} is constructed only from structure variables $M_1, \ldots, M_{n+m-1}$ (each of which occurs exactly once) and structure constants using structural connectives. Since $A \in \mathcal{I}_2$, every C_i (resp. D_i) formula is s-soluble (a-soluble) and so the following is a semi-structural rule equivalent to $U_i \vdash V_i$:

$$\frac{\mathcal{S}_1 \ \ldots \ \mathcal{S}_n \quad \mathcal{S}_{n+1} \ \ldots \ \mathcal{S}_{n+m}}{W_{n+m} \vdash M_{n+m}} \rho_i$$

Here $\mathcal{S}_j$ is an element of $\text{inv}^{\text{all}}(M_j \vdash C_j)$ $(1 \leq j \leq n)$ and $\text{inv}^{\text{all}}(D_j \vdash M_{n+j})$ $(n+1 \leq j \leq n+m)$. As the conclusion of ρ_i does not contain multiple occurrences of structure variables C2 and C3 hold.

[2] Recall that C8 only applies to logical rules and hence is preserved under the addition of structural rules.

[3] An analogous condition is used to adapt the original algorithm in [Ciabattoni et al. 2008] to non-commutative sequent calculi [Ciabattoni et al. 2012] and to multiple-conclusion (hyper)sequent calculi [Ciabattoni et al. 2009].

C4 holds for all structure variables in the rule (but possibly not for the propositional variables) and C5–C8 are easily satisfied. Though the premises of ρ_i might contain propositional variables that do not occur in the conclusion (and hence C1 may not hold) cut-elimination for $\mathcal{C} + \{\rho_i\}_{i\in\Omega}$ proceeds without difficulty as no propositional variable occurs in the conclusion of any ρ_i. □

Example 3.16. We saw in Example 3.13 that inv($\mathbf{I} \vdash A$) consists of the set $\{\mathbf{I} \vdash (p > \Phi), ((p \to 0) > \Phi)\}$. Display the occurrence of $p \to 0$ in $\mathbf{I} \vdash (p > \Phi), ((p \to 0) > \Phi)$ to obtain the sequent $p \to 0 \vdash \Phi < ((p > \Phi) > \mathbf{I})$. Now applying Lemma 3.6 we get the equivalent rule below left. Starting with the conclusion of the rule below left, display the remaining occurrence of p to get the sequent $p \vdash \mathbf{I} < (\Phi < (M_1 > \Phi))$. Apply Lemma 3.6 again to get the semi-structural rule below centre. This is not yet a semi-structural rule because it contains a logical connective. Applying the invertible rules to the left premise we finally get the semi-structural rule below right.

$$\frac{M_1 \vdash p \to 0}{M_1 \vdash \Phi < ((p > \Phi) > \mathbf{I})} \qquad \frac{M_1 \vdash p \to 0 \quad M_2 \vdash p}{M_2 \vdash \Phi < (\mathbf{I} < (M_1 > \Phi))} \qquad \frac{M_1 \vdash p > \mathbf{I} \quad M_2 \vdash p}{M_2 \vdash \Phi < (\mathbf{I} < (M_1 > \Phi))}$$

Definition 3.17 (*analytic structural rules*). An *analytic structural rule* is a structural rule that satisfies C1–C7.

Notice that if a display calculus satisfies C1–C8 then any extension of that calculus by analytic structural rules (*analytic structural rule extension*) also satisfies C1–C8.

Remark 3.18. [Kracht 1996] refers to analytic structural rule (extensions) as *proper* structural rule (extensions).

Restricting our attention to a subclass of $\mathcal{I}_2$ axioms satisfying the additional condition of *acyclicity* (Definition 3.24 below), we transform the semi-structural rules in the above proposition into equivalent analytic structural rules. The transformation given below mirrors the 'completion' procedure in [Ciabattoni et al. 2009] and amounts to applying the cut-rule to the premises of the semi-structural rules. We formalise this by defining an operation that takes a set $\mathcal{S}$ of sequents (containing the propositional variable p, say) and returns a set $\mathcal{S}_p$ of sequents that does not contain p (think of this as applying the cut-rule in 'all possible ways' to all the occurrences of p). Sometimes it is not possible to remove all occurrences of p. Indeed, this operation is successful if $\mathcal{S}$ satisfies certain conditions—in our terminology: $\mathcal{S}$ respects multiplicities wrt p. A set $\mathcal{S}$ is acyclic if this operation can be repeated to obtain ultimately a set of sequents not containing any propositional variables.

Let $\mathbb{V}(\mathcal{S})$ denote the set of propositional variables occurring in a set $\mathcal{S}$ of sequents.

Definition 3.19 (*respect multiplicities*). A nonempty set $\mathcal{S}$ of sequents is said to *respect multiplicities wrt* a propositional variable $p \in \mathbb{V}(\mathcal{S})$ if $\mathcal{S}$ can be partioned into one of the forms below using the display rules for fixed p:

$$\{p \vdash U \,|\, p \notin U\} \cup \{V \vdash p \,|\, \text{every } p \text{ in } V \vdash p \text{ is s-part}\} \cup \{S \,|\, p \notin S\} \quad (1)$$
$$\{U \vdash p \,|\, p \notin U\} \cup \{p \vdash V \,|\, \text{every } p \text{ in } p \vdash V \text{ is a-part}\} \cup \{S \,|\, p \notin S\} \quad (2)$$

An alternative definition is that (i) no $S \in \mathcal{S}$ contains both an a-part and s-part occurrence of p (eg. $p \vdash p$ cannot be in $\mathcal{S}$), and (ii) there do not exist $S_1, S_2 \in \mathcal{S}$ such that S_1 contains multiple (ie.>1) a-part occurrences of p and S_2 contains multiple s-part occurrences of p.

Example 3.20. Consider a display calculus containing a structural connective $\otimes$ such that both occurrences of p in $p \otimes p \vdash X$ (resp. $Y \vdash p \otimes p$) are a-part (s-part). If $(p \otimes p \vdash X) \in \mathcal{S}$ and $(Y \vdash p \otimes p) \in \mathcal{S}$, then $\mathcal{S}$ does not respect multiplicities wrt p because it contains sequents with multiple a-part occurrences of p and multiple s-part occurrences of p.

Given a set $\mathcal{S}$ of sequents respecting multiplicities wrt p. An equivalent set (cf. Remark 2.6) not containing p can be constructed as indicated in the following definition.

Definition 3.21 ($\mathcal{S}_p$). Let $\mathcal{S}$ be a set of sequents respecting multiplicities wrt p. If it is not the case that $(p \vdash U) \in \mathcal{S}$ *and* $(V \vdash p) \in \mathcal{S}$ (up to display equivalence) then define $\mathcal{S}_p$ as

$\{S \in \mathcal{S} \,|\, p \notin S\}$. Otherwise, define $\mathcal{S}_p$ as the union of $\{S \,|\, p \notin S\}$ and one of the following, depending on the display equivalent form of $\mathcal{S}$ as (1) or (2), respectively.

$$\{S \,|\, \text{obtain } S \text{ from } (V \vdash p) \in \mathcal{S} \text{ by substituting each occurrence of } p \text{ with a } U \text{ s.t. } p \vdash U \in \mathcal{S}\ \}$$
$$\{S \,|\, \text{obtain } S \text{ from } (p \vdash V) \in \mathcal{S} \text{ by substituting each occurrence of } p \text{ with a } U \text{ s.t. } U \vdash p \in \mathcal{S}\ \}$$

LEMMA 3.22. *If $\mathcal{S}$ respects multiplicities wrt p, then p does not occur in $\mathcal{S}_p$.*

PROOF. Follows immediately from the form of $\mathcal{S}$ and the definition of $\mathcal{S}_p$. □

Definition 3.23 (*acyclic set*). Let $\mathcal{C}$ be a display calculus. A finite set $\mathcal{S}$ of sequents built from structure variables, structure constants and propositional variables using structural connectives is *acyclic* if (i) $\mathbb{V}(\mathcal{S}) = \emptyset$ or (ii) $\exists p \in \mathbb{V}(\mathcal{S})$ such that $\mathcal{S}$ respects multiplicities wrt p and $\mathcal{S}_p$ is acyclic.

Definition 3.24 (*acyclic formula*). Let $A \in \mathcal{I}_2(\mathcal{C})$. If there is some set $\{\rho_i\}_{i\in\Omega}$ of semi-structural rules equivalent to A obtained according to Proposition 3.15 such that the premises of each ρ_i ($i \in \Omega$) are acyclic, then A is called an acyclic formula.

Example 3.25. In Example 3.16 we computed the semi-structural rule equivalent to A: $(p \to 0) + ((p \to 0) \to 0)$. To check if A is acyclic we need to check if the set $\mathcal{S} = \{M_1 \vdash p > \Phi, M_2 \vdash p\}$ of premises of the equivalent semi-structural rule is acyclic. Let us unfold the recursive definition of acyclicity. Certainly $\mathbb{V}(\mathcal{S}) = \{p\} \neq \emptyset$. Noting that $M_1 \vdash p > \Phi$ is display equivalent to $p \vdash \Phi < M_1$ we see that $\mathcal{S}$ can be written as $\{p \vdash \Phi < M_1, M_2 \vdash p\}$ using only the display rules. This set respects multiplicities wrt p—it is the form (1) in Definition 3.19. We now compute

$$\mathcal{S}_p = \{S \mid \text{obtain } S \text{ from } M_2 \vdash p \text{ by substituting } p \text{ with } \Phi < M_1\} = \{M_2 \vdash \Phi < M_1\}$$

The construction of $\mathcal{S}_p$ from $\mathcal{S}$ can be read as applying 'all possible cuts' on p in $\mathcal{S}$. Since $\{M_2 \vdash \Phi < M_1\}$ contains no propositional variables, it is acyclic. We conclude that A is acyclic.

Remark 3.26. The abstract definition of acyclic formula above is procedural. In the case of a concrete calculus, a declarative definition might be obtained.

The following shows that checking acyclicity is decidable.

LEMMA 3.27. *Determining if a given formula is acyclic is decidable.*

PROOF. First we need to decide if the given formula $A \in \mathcal{I}_2(\mathcal{C})$, i.e. is there some $\{U_i \vdash V_i\}_{i\in\Omega} \in \text{inv}(\mathbf{I} \vdash A)$ such that each a-part formula in $U_i \vdash V_i$ is s-soluble and each s-part formula in $U_i \vdash V_i$ is a-soluble for some $i \in \Omega$. Computing the function $\text{inv}(\mathbf{I} \vdash A)$ and checking a/s-solubility rely on displaying formulae, which is effective by Definition 2.7(ii), and on checking if an invertible logical rule can be applied upwards to that formula. The latter depends only on the head-connective of the formula and on whether the formula is a-part or s-part.[4] As we have allowed for the possibility that a substructure can be displayed in more than one way, in order to decide if $A \in \mathcal{I}_2(\mathcal{C})$ it remains to show: if one way of displaying a substructure yields that $A \notin \mathcal{I}_2(\mathcal{C})$ then $A \notin \mathcal{I}_2(\mathcal{C})$ irrespective of how we choose to display the substructures. By Lemma 2.13, a formula occurrence cannot 'disappear' under any sequence of display rules. Moreover by C4, an a-part (s-part) formula remains a-part (resp. s-part) whichever display rules are used. So if $A \notin \mathcal{I}_2(\mathcal{C})$ this can only be due to the unavailability of a suitable invertible logical rule to apply upwards to some a-part or s-part subformula of A and this problem persists irrespective of the display strategy.

We have already seen that obtaining the (finite) set of sets of semi-structural rules equivalent to $\mathbf{I} \vdash A$ ($A \in \mathcal{I}_2(\mathcal{C})$) is effective (Lemma 3.15). If there is some set $\{\rho_i\}_{i\in\Omega}$ of semi-structural rules such that the set of premises of each ρ_i is acyclic, then A is acyclic. Otherwise it is not.

Thus, to complete the proof, we must show how to decide if a finite set $\mathcal{S}$ of premises of semi-structural rules is acyclic or not. The proof proceeds by induction on $|\mathbb{V}(\mathcal{S})|$. The base case is trivial.

[4]Note that the invertible logical rules can be applied upwards irrespective of the structure on the other side of the sequenta—see the text following Definition 2.4.

Suppose $|\mathbb{V}(\mathcal{S})| = n + 1$. Following Definition 3.23, check if $\mathcal{S}$ respects multiplicities wrt $p \in \mathbb{V}(\mathcal{S})$. We can check the latter by effectively displaying the substructures via Definition 2.7(ii). As in the paragraph above, we need to show that if one way of displaying does not yield that the set respects multiplicities, then no other way of displaying can yield that the set respects multiplicities. To see this, suppose that $\mathcal{S}$ does not respect multiplicities wrt p using the effective way of displaying a substructure. This means that either (i) there is an a-part and s-part occurrence of p in some sequent in $\mathcal{S}$ or (ii) there is in $\mathcal{S}$, a sequent containing multiple s-part occurrences of p and a sequent containing multiple a-part occurrences of p. In either case, Lemma 2.13 and C4 assures us that any other way of displaying the structures will lead to the same result, and hence $\mathcal{S}$ does not respect multiplicities wrt p irrespective of how the substructures are displayed.

We can check the latter by effectively displaying the substructures. As in the paragraph above, we need to show that if one way of displaying does not yield that the set respects multiplicities, then no other way of displaying can yield it. To see this, suppose that $\mathcal{S}$ does not respect multiplicities wrt p. This means that $\mathcal{S}$ contains either (i) a sequent having an a-part and s-part occurrence of p or (ii) a sequent containing multiple s-part occurrences of p and a sequent containing multiple a-part occurrences of p. In either case, Lemma 2.13 and C4 assures us that any other way of displaying the structures will lead to the same result. If $\mathcal{S}$ does not respect multiplicities for any $p \in \mathbb{V}(\mathcal{S})$ then the set is not acyclic. Otherwise, for each p such that $\mathcal{S}$ respects multiplicities wrt p, check if $\mathcal{S}_p$ is acyclic; if it is then $\mathcal{S}$ is acyclic. Since $\mathbb{V}(\mathcal{S}_p) < n+1$ we can use the induction hypothesis to decide this. If $\mathcal{S}_p$ is not acyclic for any such p then $\mathcal{S}$ is not acyclic. □

Remark 3.28. Every formula $A \in \mathcal{I}_1$ is acyclic. To see this, follow the above definition. Since $A \in \mathcal{I}_1$, there is some $\mathcal{S} = \{S^1, \ldots, S^N\} \in \text{inv}^{\text{all}}(\mathbf{I} \vdash A)$ that witnesses this i.e. every a-part formula A^j_k and s-part formula B^j_l in S^j ($1 \leq j \leq N$) is a propositional variable. Since every union of singleton sets of the form $\{L \vdash p\}$ and $\{q \vdash M\}$ is acylic the result follows.

We are ready to show that every acyclic $\mathcal{I}_2$ axiom has equivalent analytic structural rules. Recall that the $\mathcal{I}_2$ axiom A is acyclic if there is an equivalent set of semi-structural rules whose set of premises are each acyclic. The base case of the definition of acyclic set is a set containing no propositional variables. Then the corresponding rule is already an analytic structural rule. In the following proposition we show that the inductive part of the definition (which amounts to deleting a propositional variable p from the set) preserves equivalence.

Informally speaking, the proposition states that if the set $\mathcal{S}$ of premises of a semi-structural rule ρ is acyclic, then ρ is equivalent to the rule ρ_p with premises $\mathcal{S}_p$, where $\mathcal{S}_p$ is obtained from $\mathcal{S}$ by applying cut in 'all possible ways' with cut-formula p. Before we state and prove the proposition, we illustrate the proof with an example.

Example 3.29. Let $\mathcal{C}$ be an amenable calculus and let ρ be the semi-structural rule below left (so the conclusion s does not contain any propositional variables). We claim that ρ is equivalent in $\mathcal{C}$ to the rule ρ_p below right.

$$\frac{X_1 \vdash p \qquad p \vdash Y_1 \qquad p \vdash Y_2}{s}\,\rho \qquad\qquad \frac{X \vdash Y_1 \qquad X \vdash Y_2}{s}\,\rho_p$$

Indeed, given concrete premises of ρ, we can obtain s by applying the cut-rule to the formula instantiating the propositional variable p and then applying ρ_p. For the other direction, given concrete premises of ρ_p, we need to find a formula instantiating p in order to construct concrete premises for ρ. The required formula is $\text{r}(Y_1) \wedge \text{r}(Y_2)$. Indeed, making use of Definition 3.1 (the dashed lines indicate some number of rules in $\mathcal{C}$) here is the derivation of s using ρ:

$$\dfrac{\dfrac{\dfrac{X \vdash Y_1}{X \vdash \text{r}(Y_1)} \qquad \dfrac{X \vdash Y_2}{X \vdash \text{r}(Y_2)}}{X \vdash \text{r}(Y_1) \wedge \text{r}(Y_2)} \qquad \dfrac{\text{r}(Y_1) \vdash Y_1}{\text{r}(Y_1) \wedge \text{r}(Y_2) \vdash Y_i} \qquad \dfrac{\text{r}(Y_2) \vdash Y_2}{\text{r}(Y_1) \wedge \text{r}(Y_2) \vdash Y_i}}{s}\,\rho$$

PROPOSITION 3.30. *Let $\mathcal{C}$ be an amenable calculus, $\mathcal{S}$ an acyclic set of sequents and $p \in \mathbb{V}(\mathcal{S})$. Then the semi-structural rule ρ with premises $\mathcal{S}$ and the semi-structural rule ρ_p with premises $\mathcal{S}_p$ are equivalent in $\mathcal{C}$.*

PROOF. Let $\mathcal{S}$ be any acyclic set of sequents. There are two cases to consider.
(i) Suppose that $\mathcal{S}$ does not contain sequents of the form $p \vdash U$ *and* $V \vdash p$. Then $\mathcal{S}$ has one of the following forms

$$\{V_1 \vdash p, \ldots, V_{n+1} \vdash p\} \cup \{S \,|\, p \notin S\} \qquad \{p \vdash V_1, \ldots, p \vdash V_{n+1}\} \cup \{S \,|\, p \notin S\}$$

and $\mathcal{S}_p$ is $\{S \in \mathcal{S} \,|\, p \notin S\}$. Suppose the case above right (the other case is similar). One direction is immediate, and to show that ρ_p is derivable in $\mathcal{C} + \rho$ it is enough to apply ρ using the sequents $\{c_a \vdash V_i[p \mapsto c_a]\}_{1 \leq i \leq n+1}$ for the missing premises. These sequents are derivable due to Definition 3.1.2.

(ii) Suppose that $\mathcal{S}$ contains sequents of the form $p \vdash U$ *and* $V \vdash p$. Clearly ρ is derivable in $\mathcal{C} + \rho_p$—it suffices to apply the cut-rule (and display rules) to concrete premises of ρ and then apply ρ_p. For the other direction, assume, to fix ideas that the premises $\mathcal{S}$ of ρ have the form (1) in Definition 3.19 (the other case is similar, use (a)$_\vee$ and (b)$_\vee$ from Def. 3.1(3) instead of (a)$_\wedge$ and (b)$_\wedge$), i.e.,

$$\{p \vdash U_i \,|\, p \notin U_i; 1 \leq i \leq n\} \cup \{V \vdash p \,|\, \text{every } p \text{ in } V \vdash p \text{ is s-part}\} \cup \{S \,|\, p \notin S\}$$

Then the premises $\mathcal{S}_p$ of ρ_p have the following form:

$$\{S \,|\, S \text{ is a subst. instance of } V \vdash p \in \mathcal{S} \text{ s.t. each occ. } p \mapsto U_i \text{ for some } 1 \leq i \leq n)\ \} \cup \{S \,|\, p \notin S\}$$

We now want to use $\mathcal{S}_p$ to 'reconstruct' a concrete instance of $\mathcal{S}$ by instantiating p with a suitable formula. It may be helpful for the reader to read the following steps in parallel with Example 3.29. For each sequent in the above set, display each occurrence of U_i (necessarily in the succedent since U_i is s-part) and apply the function r to get the set

$$\{S \,|\, S \text{ is a subst. instance of } V \vdash p \in \mathcal{S} \text{ s.t. each occ. } p \mapsto \mathrm{r}(U_i) \text{ for some } 1 \leq i \leq n)\ \} \cup \{S \,|\, p \notin S\}$$

Suppose that we are given concrete instances of the premises of ρ_p. Repeatedly using (a)$_\wedge$, Definition 3.1.1(ii) and the display rules, obtain the set $\mathcal{S}_p^*$

$$\{S \,|\, S \text{ is a subst. instance of } V \vdash p \in \mathcal{S} \text{ s.t. each occ. } p \mapsto \wedge_{1 \leq i \leq n}\mathrm{r}(U_i)\ \} \cup \{S \,|\, p \notin S\}$$

Making use of (b)$_\wedge$ and Definition 3.1.1(i), derive the set $\{\wedge_{1 \leq j \leq n}\mathrm{r}(U_j) \vdash U_i\}_{1 \leq i \leq n}$ of sequents. By inspection, this set together with $\mathcal{S}_p^*$ yield concrete instances of the premises of ρ (in particular, p has been instantiated with $\wedge_{1 \leq i \leq n}\mathrm{r}(U_i)$). Applying ρ to these and noting that ρ and ρ_p have the same conclusion, we have that ρ_p is derivable in $\mathcal{C} + \rho$. □

THEOREM 3.31. *Let $\mathcal{C}$ be an amenable calculus. If $A \in \mathcal{I}_2(\mathcal{C})$ is acyclic, then there are analytic structural rules $\{\rho'_i\}_{i \in \Omega}$ equivalent to $\mathbf{I} \vdash A$ such that $\mathcal{C} + \{\rho'_i\}_{i \in \Omega}$ is a cut-eliminable calculus satisfying C1–C8 (i.e. an analytic structural rule extension of $\mathcal{C}$).*

PROOF. Let $\{\rho_i\}_{i \in \Omega}$ be the semi-structural rules equivalent to $\mathbf{I} \vdash A$ in $\mathcal{C}$ obtained in Proposition 3.15. Notice that each ρ_i might violate (only) Belnap's condition C1 due to the presence of propositional variables in the set $\mathcal{S}^i$ of sequents that are its premises.
Since A is acyclic, by Definition 3.24 we have that $\mathcal{S}^i$ is acyclic. Let $\mathbb{V}(\mathcal{S}^i) = \{p_1, p_2, \ldots, p_n\}$. By (repeatedly applying) Proposition 3.30 the rule ρ'_i obtained from ρ_i by replacing the premises $\mathcal{S}^i$ with $((\ldots(\mathcal{S}^i_{p_1})_{p_2}\ldots)_{p_{n-1}})_{p_n}$ is an equivalent analytic structural rule (in particular, observe that any structure variable that appears only as an a-part (resp. s-part) structure in every sequent in $\mathcal{S}^i$ has the same property in $((\ldots(\mathcal{S}^i_{p_1})_{p_2}\ldots)_{p_{n-1}})_{p_n}$). By repeating this process to all $\{\rho_i\}_{i \in \Omega}$ we obtain a new set of structural rules $\{\rho'_i\}_{i \in \Omega}$ (Lemma 3.22) such that $\mathcal{C} + \{\rho'_i\}_{i \in \Omega}$ satisfies C1–C8. □

Example 3.32. In Example 3.16 we obtained the semi-structural rule (below left) equivalent to A: $(p \to 0) + ((p \to 0) \to 0)$. In Example 3.25 we saw that the axiom is acyclic by verifying

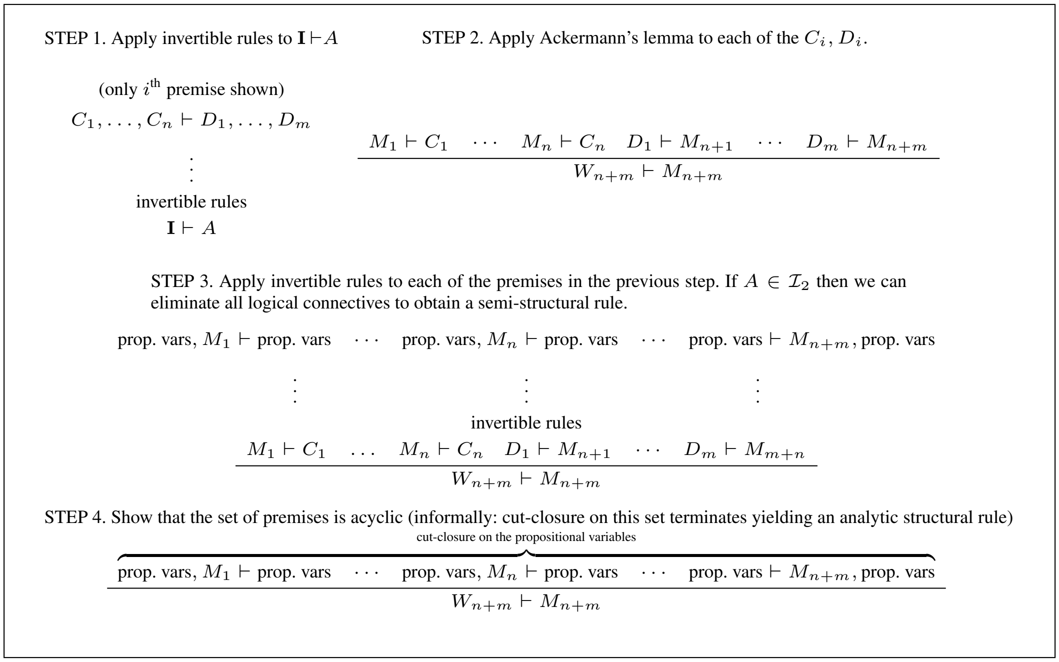


Fig. 1. A summary of the algorithm for converting an initial sequent $\mathbf{I} \vdash A$ into an equivalent analytic structural rule (for simplicity we use comma as the only structural connective).

the acyclicity of the premises $\mathcal{S}$ of that rule. In particular, we computed $\mathcal{S}_p = \{M_2 \vdash \Phi < M_1\}$. Replacing the premises of the semi-structural rule with this set we obtain the equivalent analytic structural rule below right.

$$\frac{M_1 \vdash p > \Phi \quad M_2 \vdash p}{M_2 \vdash \Phi < (\mathbf{I} < (M_1 > \Phi))} \qquad \frac{M_2 \vdash \Phi < M_1}{M_2 \vdash \Phi < (\mathbf{I} < (M_1 > \Phi))}$$

Figure 1 summarises our algorithm for converting initial sequents into analytic structural rules.

3.2. Relating $\mathcal{H} + A$ with $\mathcal{C} + \{\rho_i\}_{i \in \Omega}$

In Theorem 3.31 we established the equivalence between an amenable calculus $\mathcal{C}$ extended by the initial sequent $\mathbf{I} \vdash A$ (for $A \in \mathcal{I}_2(\mathcal{C})$ acyclic) and a suitable analytic rule extension of $\mathcal{C}$. We show below that if the rules of $\mathcal{C}$ and the Hilbert calculus $\mathcal{H}$ for $L_{\mathbf{I}}(\mathcal{C})$ 'simulate' each other (in a sense that is made precise in Definition 3.33) then the logic of $\mathcal{C} + (\mathbf{I} \vdash A)$ coincides with the set of formulae derivable in the axiomatic extension $\mathcal{H} + A$.

Definition 3.33 (*display calculus corresponds to a Hilbert calculus*). Let $\mathcal{C}$ be an amenable calculus and let $\mathcal{L}$ denote the language of $L_{\mathbf{I}}(\mathcal{C})$. Then $\mathcal{C}$ *corresponds* to the Hilbert calculus $\mathcal{H}$ if:

(i) $\mathbb{F}$ is a function mapping sequents of the form $A \vdash B$ ($A, B \in \text{For}\mathcal{L}$) to some $\mathbb{F}(A \vdash B) \in \text{For}\mathcal{L}$ and $\mathbf{I} \vdash B$ to B ($B \in \text{For}\mathcal{L}$).
(ii) For every instance $X_1 \vdash Y_1 \ldots X_N \vdash Y_N / X_{N+1} \vdash Y_{N+1}$ of a rule in $\mathcal{C}$: there is a derivation in $\mathcal{H}$ of $\mathbb{F}(\mathrm{l}(X_{N+1}) \vdash \mathrm{r}(Y_{N+1}))$ assuming $\mathbb{F}(\mathrm{l}(X_1) \vdash \mathrm{r}(Y_1)), \ldots, \mathbb{F}(\mathrm{l}(X_N) \vdash \mathrm{r}(Y_N))$.
(iii) For every instance of a rule $A_1 \ldots A_N / A_{N+1}$ in $\mathcal{H}$: there is a derivation in $\mathcal{C}$ of $\mathbf{I} \vdash A_{N+1}$ assuming $\mathbf{I} \vdash A_1, \ldots, \mathbf{I} \vdash A_N$.

LEMMA 3.34. *If $\mathcal{C}$ corresponds to $\mathcal{H}$ then $L(\mathcal{H})$ is the logic of $\mathcal{C}$ (i.e. $L_{\mathbf{I}}(\mathcal{C}) = L(\mathcal{H})$).*

PROOF. For $B \in L_{\mathbf{I}}(\mathcal{C})$ iff there is a derivation of $\mathbf{I} \vdash B$ in $\mathcal{C}$ iff $\mathbb{F}(\mathbf{I} \vdash B)$ is derivable in $\mathcal{H}$ iff there is a derivation of B in $\mathcal{H}$ iff $B \in L(\mathcal{H})$. $\square$

LEMMA 3.35. *Let $\mathcal{C}$ be an amenable calculus and let $\mathcal{L}$ denote the language of $L_{\mathbf{I}}(\mathcal{C})$. If $\mathcal{C}$ corresponds to the Hilbert calculus $\mathcal{H}$ (for function $\mathbb{F}$), then $\mathcal{C} + (\mathbf{I} \vdash A)$ ($A \in For\mathcal{L}$) corresponds to the axiomatic extension $\mathcal{H} + A$ using the same function $\mathbb{F}$.*

PROOF. Certainly the function $\mathbb{F}$ which witnesses that $\mathcal{C}$ corresponds to $\mathcal{H}$ is a function satisfying condition (i) in the statement '$\mathcal{C} + (\mathbf{I} \vdash A)$ corresponds to $\mathcal{H} + A$'. We now prove (ii), noting that the proof for (iii) is similar. If ρ is a rule in $\mathcal{C} + (\mathbf{I} \vdash A)$ then ρ is a rule in $\mathcal{C}$ or $\rho = (\mathbf{I} \vdash A)$. For every rule $X_1 \vdash Y_1 \ldots X_N \vdash Y_N / X_{N+1} \vdash Y_{N+1}$ in $\mathcal{C}$, there is a derivation in $\mathcal{H}$ of $\mathbb{F}(X_{N+1} \vdash Y_{N+1})$ from $\mathbb{F}(X_1 \vdash Y_1), \ldots, \mathbb{F}(X_N \vdash Y_N)$. By definition this holds in $\mathcal{H} + A$ too. Also $\mathbb{F}(\mathbf{I} \vdash A) = A$ which is certainly derivable in $\mathcal{H} + A$. $\square$

THEOREM 3.36. *Let $\mathcal{C}$ be an amenable calculus and let $\mathcal{L}$ denote the language of $L_{\mathbf{I}}(\mathcal{C})$. Suppose that $\mathcal{C}$ corresponds to the Hilbert calculus $\mathcal{H}$. If Δ is a set of acyclic $\mathcal{I}_2(\mathcal{C})$ formulae in For$\mathcal{L}$ then there is an analytic structural rule extension corresponding to $\mathcal{H} + \Delta$.*

PROOF. Let $A \in \Delta$. By Lemma 3.35 we have that $\mathcal{C} + (\mathbf{I} \vdash A)$ corresponds to $\mathcal{H} + A$. Theorem 3.31 ensures that there are analytic structural rules $\{\rho_i\}$ equivalent to $\mathbf{I} \vdash A$ in $\mathcal{C}$, which implies, by definition of equivalence between rules (Definition 2.5), that $\mathcal{C} + \{\rho_i\}$ corresponds to $\mathcal{H} + A$. $\square$

Example 3.37. It is easy to check that the display calculus δBi-FL from Example 2.10 corresponds to a standard Hilbert calculus $\mathcal{H}$Bi-FL for Bi-FL. Observe here that Bi-FL $= L(\mathcal{H}$Bi-FL$)$. The function $\mathbb{F}$ is defined as follows.

$$\mathbb{F}(\mathbf{I} \vdash B) = B \qquad\qquad \mathbb{F}(A \vdash B) = A \to B$$

Example 3.38 (*Bi-intuitionistic logic*). Bi-intuitionistic logic (Heyting-Brouwer logic) HB is obtained by the addition of the coimplication connective $\leftarrow_d$ to the language of intuitionistic logic.

A Hilbert calculus $\mathcal{H}$HB for HB can be obtained from $\mathcal{H}$Bi-FL by the addition of the axioms (below) for right weakening and left weakening, right exchange and left exchange (first row) and right contraction and left contraction, right associativity and left associativity (second row). Then, using Theorem 3.36, and following some simplification to get the form in [Wansing 2008], we obtain corresponding analytic structural rules (given below each axiom). Their addition to δBi-FL yields a display calculus corresponding to a Hilbert calculus for HB.

$$\begin{array}{cccc}
\mathbf{A} \to (\mathbf{A} + \mathbf{B}) & \mathbf{A} \cdot \mathbf{B} \to \mathbf{A} & \mathbf{A} + \mathbf{B} \to \mathbf{B} + \mathbf{A} & \mathbf{A} \cdot \mathbf{B} \to \mathbf{B} \cdot \mathbf{A} \\
\dfrac{L \vdash M}{L \vdash M, N} & \dfrac{L \vdash M}{L, N \vdash M} & \dfrac{L \vdash M, N}{L \vdash N, M} & \dfrac{L, N \vdash M}{N, L \vdash M} \\[2ex]
\mathbf{A} + \mathbf{A} \to \mathbf{A} & \mathbf{A} \to \mathbf{A} \cdot \mathbf{A} & (\mathbf{A} + \mathbf{B}) + \mathbf{C} \to \mathbf{A} + (\mathbf{B} + \mathbf{C}) & \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \to (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} \\
\dfrac{L \vdash M, M}{L \vdash M} & \dfrac{L, L \vdash M}{L \vdash M} & \dfrac{L \vdash (M_1, M_2), M_3}{L \vdash M_1, (M_2, M_3)} & \dfrac{(M_1, M_2), M_3 \vdash L}{M_1, (M_2, M_3) \vdash L}
\end{array}$$

It is well-known that in the presence of these axioms, the binary connectives $\cdot, +, \leftarrow, \rightarrow_d$ conflate, respectively, with $\wedge, \vee, \to, \leftarrow_d$, and also the constants 1 and 0 conflate with $\top$ and $\bot$, respectively. At the structural level $>$ and $<$ in the antecedent (resp. succedent) conflate, as Φ and $\mathbf{I}$. Hence a display calculus for HB may be obtained from δBi-FL by deleting the logical rules for $\cdot, +, \leftarrow, \rightarrow_d, 1, 0$, and deleting Φ and $>$ (resp. $<$) in the antecedent (succedent). Thus the a-structures and s-structures and functions l and r are defined as follows:

$$\mathfrak{S}_{\text{ant}} ::= A \in \text{For}\mathcal{L}_{\text{HB}} \mid \mathbf{I} \mid \mathfrak{S}_{\text{ant}}, \mathfrak{S}_{\text{ant}} \mid (\mathfrak{S}_{\text{ant}} < \mathfrak{S}_{\text{suc}})$$
$$\mathfrak{S}_{\text{suc}} ::= A \in \text{For}\mathcal{L}_{\text{HB}} \mid \mathbf{I} \mid \mathfrak{S}_{\text{suc}}, \mathfrak{S}_{\text{suc}} \mid (\mathfrak{S}_{\text{ant}} > \mathfrak{S}_{\text{suc}})$$

$$\begin{aligned} l(A) &= A & r(A) &= A \\ l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\ l(X, Y) &= l(X) \cdot l(Y) & r(X, Y) &= r(X) + r(Y) \\ l(X < Y) &= l(X) \leftarrow_d r(Y) & r(X > Y) &= l(X) \rightarrow r(Y) \end{aligned}$$

In the calculus δBi-FL, the ·l and +r rules were invertible but the ∧l and ∨r were not. Since · and ∧ (and + and ∨) conflate we may obtain the display calculus δHB for HB where ∧l and ∨r are invertible from δBi-FL by deleting the logical rules for $\cdot, +, \leftarrow, \rightarrow_d, 1, 0$ and replacing the ∧l and ∨r rules with the following:

$$\frac{A, B \vdash M}{A \wedge B \vdash M}\wedge l \qquad \frac{L \vdash A, B}{L \vdash A \vee B}\vee r$$

The point of having more invertible rules in the calculus is that it enlarges the class $\mathcal{I}_2$. An almost identical calculus for HB appears in [Goré 1998b; Wansing 2008].

The following examples present analytic display calculi for two axiomatic extensions introduced in [Wolter 1998] for the logic HB.

Example 3.39. Let A_1 be the axiom $(p \rightarrow q) \vee (q \rightarrow p)$. From Prop. 3.15 we obtain the equivalent semi-structural rule ρ_1 (below left). The set $\mathcal{S}$ of premises of ρ_1 can be written $\{L \vdash p\} \cup \{p \vdash V\} \cup \{(Z \vdash q), (q \vdash M)\}$. Then $\mathcal{S}_p = \{L \vdash V, Z \vdash q, q \vdash M\}$. Hence $(\mathcal{S}_p)_q = \{L \vdash V, Z \vdash M\}$. So $\mathcal{S}$ is equivalent to the analytic structural rule below right:

$$\frac{L \vdash p \quad q \vdash M \quad Z \vdash q \quad p \vdash V}{\mathbf{I} \vdash (L > M), (Z > V)}\rho_1 \qquad \frac{L \vdash V \qquad Z \vdash M}{\mathbf{I} \vdash (L > M), (Z > V)}\rho_1'$$

Then we have that $\delta\text{HB} + \rho_1'$ is a cut-eliminable display calculus corresponding to $\mathcal{H}\text{HB} + A_1$ with the subformula property.

Example 3.40. Let A_2 be $((p \leftarrow_d q) \wedge (q \leftarrow_d p)) \rightarrow \bot$. $A_2 \in \mathcal{I}_1(\delta\text{HB})$. Then applying our algorithm we get the equivalent rule ρ_2

$$\frac{L \vdash Z \qquad U \vdash M}{(L < M), (U < Z) \vdash \mathbf{I}}\rho_2$$

So $\delta\text{HB} + \rho_2$ is a cut-eliminable calculus corresponding to $\mathcal{H}\text{HB} + A_2$ with the subformula property.

3.3. Related work on (hyper)sequent structural rules

We compare our algorithm for display logic with the algorithm in [Ciabattoni et al. 2008] which computes sequent and hypersequent structural rules to be added to the calculus for intuitionistic Lambek logic (also known as full Lambek calculus) with exchange; the latter are a simple generalization of sequent calculus rules [Avron 1987] acting on basic objects which are disjunctions of sequents (see, e.g. [Ciabattoni et al. 2014]). For the comparison, as a case study we consider calculi for logics between classical and intuitionistic logic (i.e. intermediate logics).

The base calculus that is used [Ciabattoni et al. 2008] is the hypersequent calculus HLJ (see Appendix), essentially obtained by replacing sequents with hypersequents in Gentzen calculus LJ for intuitionistic logic Ip. Structural hypersequent rule extensions of HLJ have been obtained in [Ciabattoni et al. 2008] for intermediate logics extending Ip by formulae in the class $\mathcal{P}_3$. The latter consists of axioms defined by the grammar: $\mathcal{N}_0, \mathcal{P}_0$ contain the set of atomic formulae, and

$$\begin{aligned} \mathcal{P}_{n+1} &::= \bot \mid \top \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \\ \mathcal{N}_{n+1} &::= \bot \mid \top \mid P_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \end{aligned}$$

Clearly $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$, $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$, $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$ and $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$. Also $\mathcal{P}_m, \mathcal{N}_m \subseteq \mathcal{N}_3$ see e.g. [Jeřábek 2015].

The grammar for $\mathcal{P}_{n+1}$ is constructed based on the logical left introduction rules of HLJ that are invertible ($\mathcal{P}$ stands for positive connectives [Andreoli 1992]). Similarly the grammar for $\mathcal{N}_{n+1}$ is based on the logical right introduction rules of HLJ that are invertible ($\mathcal{N}$ stands for negative connectives). Recall that we constructed $\mathcal{I}_{n+1}(\mathcal{C})$ in a similar manner, based on the invertible rules of the calculus $\mathcal{C}$.

3.3.1. Using the display calculus for bi-intuitionistic logic. We show below that by applying our algorithm for display logic to the calculus δHB for Bi-intuitionistic logic (Example 3.38) we can transform into structural rules more Ip formulae than those contained in the class $\mathcal{P}_3$. Note indeed that every left (resp. right) invertible rule in HLJ is left (right) invertible in δHB. In addition, the right introduction rule for disjunction (ie. $\vee$r) is invertible in δHB. Consideration of these facts leads to the conclusion that $\mathcal{P}_3 \subseteq \mathcal{I}_2(\delta\text{HB})$. Furthermore we have

PROPOSITION 3.41. *Every axiom in $\mathcal{P}_3$ is equivalent to an acyclic $\mathcal{I}_2(\delta HB)$ axiom.*

PROOF. Every intuitionistic $\mathcal{P}_3$ formula A is equivalent to a formula A' which is a disjunction of $\mathcal{N}_2$-normal formulae (see [Ciabattoni et al. 2008, Lemma 3.4]). An $\mathcal{N}_2$-normal formula is a conjunction of formulae of the form $\alpha_1 \wedge \ldots \wedge \alpha_n \to \beta$. Here β is $\bot$ or a disjunction $\beta_1 \vee \ldots \vee \beta_k$ with each β_i a conjunction of propositional variables. Also each α_i is of the form $\vee_{1\leq j\leq M_i}\gamma_i^j \to \beta_i^j$ where β_i^j is $\bot$ or a propositional variable and γ_i^j is a conjunction of propositional variables. Noting that the only non-invertible logical rules in δHB are the rules $\to$l and $\leftarrow_d$r, it is easy to check that $A' \in \mathcal{I}_2(\delta\text{HB})$. We show now that A' is acyclic.

In the concrete case of δHB it is easily seen that $\text{inv}^{\text{all}}(\mathbf{I} \vdash A') = \{\mathcal{S}\}$ i.e. a singleton set of N sequents. Let $A_1^j, \ldots, A_n^j$ and $B_1^j, \ldots, B_m^j$ denote, respectively, the a-part formulae and s-part formulae in each of such sequents ($1 \leq j \leq N$). Each $\text{inv}^{\text{all}}(A_k^j \vdash \mathbf{I})$ and $\text{inv}^{\text{all}}(\mathbf{I} \vdash B_l^j)$ has the form $p_1, \ldots, p_n \vdash p_{n+1}, \ldots, p_m$, where p_i are either propositional variables or $\top, \bot$. To show acyclicity of A' we show that $\mathcal{S}_0^j$ is equivalent in δHB to an acyclic set for each j. From $\mathcal{S}_k^j$ ($k \geq 0$) obtain the equivalent set $\mathcal{S}_k'^j$ by:

(i) deleting sequents containing the same propositional variable in the antecedent and succedent (such sequents are derivable in δHB by initial sequents and weakening); and
(ii) repeatedly applying contraction to the sequents to ensure that the propositional variables in the antecedent (resp. succedent) are unique.

Noting that $\mathcal{S}_k'^j$ respects multiplicities wrt any $q_l \in \mathbb{V}(\mathcal{S}_k'^j)$, compute $\mathcal{S}_{k+1}^j = (\mathcal{S}_k'^j)_{q_l}$. Continue in this way to obtain the sequence $\mathcal{S}_0^j, \ldots, \mathcal{S}_m^j$ such that $\mathbb{V}(\mathcal{S}_m'^j) = \emptyset$. It follows that $\mathcal{S}_0^j$ is acyclic. Since j was arbitrary the result is proved. □

So intuitionistic axioms which can be transformed into equivalent structural hypersequent rules can also be transformed into structural display rules. This is no surprise since any hypersequent calculus can be embedded into a display calculus [Ramanayake 2015]. In fact, many more axioms can be transformed into structural rules in the display calculus setting.

Example 3.42 (*Bounded depth axioms*). The axioms Bd_k ($k \geq 1$), defining intermediate logics semantically characterized by Kripke models of depth $\leq k$, belong to the classes $\mathcal{P}_{2k}$ ($\subseteq \mathcal{N}_3$) in the classification in [Ciabattoni et al. 2008]; these axioms are recursively defined as follows:

$$Bd_1: \quad p_1 \vee (p_1 \to \bot) \qquad Bd_{i+1}: \quad p_{i+1} \vee (p_{i+1} \to Bd_i)$$

For $k \geq 2$, no axiom within $\mathcal{P}_3$ is known to be equivalent, yet these all belong to $\mathcal{I}_1(\delta HB)$. As an example: for the case $k = 2$, the analytic structural display rule equivalent to Bd_2 is

$$\frac{M \vdash L \qquad V \vdash U}{\mathbf{I} \vdash L, (M > (U, (V > \mathbf{I})))}\ \rho$$

In contrast *no* equivalent hypersequent structural rule is known.

Conservativity. A logic L is called a *conservative extension* of L' if $L' \subseteq L$ and for every B in the language of L': $B \in L$ implies $B \in L'$. The conservativity of the logic of a display calculus with respect to the logic of a subcalculus (obtained by the deletion of some logical rules) is a delicate point. When conservativity holds, we can obtain a display calculus for the smaller logic. Specifically, let $\mathcal{C}$ be an amenable calculus and let $\mathcal{L}$ denote the language of the logic $L_{\mathbf{I}}(\mathcal{C})$. It is frequently the case that we are interested, not in a calculus for axiomatic extensions of $L_{\mathbf{I}}(\mathcal{C})$, but instead in a calculus for axiomatic extensions of a sublogic $L' \subset L_{\mathbf{I}}(\mathcal{C})$ in a restricted language $\mathcal{L}' \subset \mathcal{L}$. For example, our interest is in intuitionistic, modal or Lambek logic although the logic of the display calculus is usually bi-intuitionistic, tense or Bi-Lambek logic.[5]

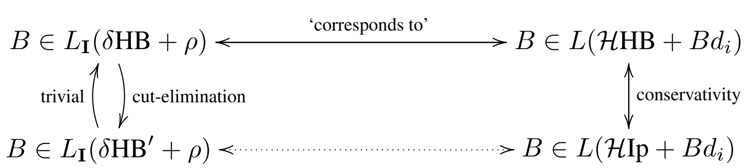
More concretely, we saw in Example 3.42 how to obtain display calculi for $\mathcal{H}\mathrm{HB} + Bd_i$. Let us investigate the conditions under which we can obtain display calculi for $\mathcal{H}\mathrm{Ip} + Bd_i$. Here $\mathcal{H}\mathrm{Ip}$ is a Hilbert calculus for (propositional) intuitionistic logic, so $\mathrm{Ip} = L(\mathcal{H}\mathrm{Ip})$.

Let $\delta\mathrm{HB}'$ be the calculus obtained from $\delta\mathrm{HB}$ by deleting the logical rules for $\leftarrow_d$. Notice that $\delta\mathrm{HB}'$ may not be an amenable calculus. The reason is that it is not clear how to define the functions l and r mapping sequents into formulae in the *intuitionistic* language $\mathcal{L}_{\mathrm{Ip}}$. In particular, it seems not to be possible to interpret the antecedent structural connective $<$ of $\delta\mathrm{HB}$ in $\mathcal{L}_{\mathrm{Ip}}$.

First observe that for every B in the language of $\mathcal{L}_{\mathrm{Ip}}$: $\mathbf{I} \vdash B$ is derivable in $\delta\mathrm{HB}' + \rho$ iff $\mathbf{I} \vdash B$ is derivable in $\delta\mathrm{HB} + \rho$. The forward direction is trivial and the reverse direction makes use of cut-elimination since any cutfree derivation of $\mathbf{I} \vdash B$ cannot use the logical rules for $\leftarrow_d$. It follows that $B \in L_{\mathbf{I}}(\delta\mathrm{HB}' + \rho)$ iff $B \in L_{\mathbf{I}}(\delta\mathrm{HB} + \rho)$. It should be clear that this argument does not work if we had obtained $\delta\mathrm{HB}' + \rho$ by deleting some non-logical rules from $\delta\mathrm{HB} + \rho$ instead.

Next, observe that the languages of $L_{\mathbf{I}}(\delta\mathrm{HB}' + \rho)$ and $L(\mathcal{H}\mathrm{Ip} + Bd_i)$ are identical. Finally, let us recall the fact that $L(\mathcal{H}HB + Bd_i)$ is a conservative extension of $L(\mathcal{H}\mathrm{Ip} + Bd_i)$.

Then we have $B \in L_{\mathbf{I}}(\delta\mathrm{HB}' + \rho)$ iff $B \in L_{\mathbf{I}}(\delta\mathrm{HB} + \rho)$ iff $B \in L(\mathcal{H}\mathrm{HB} + Bd_i)$ iff $B \in L(\mathcal{H}\mathrm{Ip} + Bd_i)$ (the last 'iff' uses conservativity). Therefore $L_{\mathbf{I}}(\delta\mathrm{HB}' + \rho) = L(\mathcal{H}\mathrm{Ip} + Bd_i)$. In other words, $L(\mathcal{H}\mathrm{Ip} + Bd_i)$ is the logic of $\delta\mathrm{HB}' + \rho$. The proof is represented graphically below.



We can generalise the above argument as follows:

THEOREM 3.43. *Let $\mathcal{C}$ be a display calculus and $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert calculi. Suppose that*

(i) $L_{\mathbf{I}}(\mathcal{C} + \{\rho_i\}_{i\in I})$ *corresponds to* $\mathcal{H} + A$,
(ii) The display calculus $\mathcal{C}'$ is obtained from $\mathcal{C}$ by deleting some logical rules,
(iii) the language of $L_{\mathbf{I}}(\mathcal{C}' + \{\rho_i\}_{i\in I})$ and $L(\mathcal{H}' + A)$ are identical, and
(iv) $L(\mathcal{H} + A)$ *is a conservative extension of* $L(\mathcal{H}' + A)$.

Then $L_{\mathbf{I}}(\mathcal{C}' + \{\rho_i\}_{i\in I}) = L(\mathcal{H}' + A)$.

PROOF. By (iii), every formula in the language of $L_{\mathbf{I}}(\mathcal{C}' + \{\rho_i\}_{i\in I})$ is a formula in the language of $L(\mathcal{H}' + A)$ and *vice versa*. Let B be an arbitrary formula from this language. By definition: $B \in L_{\mathbf{I}}(\mathcal{C}' + \{\rho_i\}_{i\in I})$ iff $\mathbf{I} \vdash B$ is derivable in $\mathcal{C}' + \{\rho_i\}_{i\in I}$. Due to cut-elimination and (ii), the latter holds iff $\mathbf{I} \vdash B$ is derivable in $\mathcal{C} + \{\rho_i\}_{i\in I}$. Once again by definition, the latter holds iff $B \in L_{\mathbf{I}}(\mathcal{C} + \{\rho_i\}_{i\in I})$. By the 'corresponds to' relation (i), the latter holds iff $B \in L(\mathcal{H} + A)$. Finally, by conservativity (iv), the latter holds iff $B \in L(\mathcal{H}' + A)$. □

[5]The larger language is needed to interpret the structural connectives that are required to obtain the display property.

Example 3.44. Back to the intermediate logics and analytic structural rule extensions of δHB: conservativity for many logics $L(\mathcal{H}\text{HB} + A)$ over $L(\mathcal{H}Ip + A)$ for acyclic intuitionistic axioms $\mathcal{P}_3 \not\ni A \in \mathcal{I}_2(\delta\text{HB})$ can be established by proving canonicity [Ghilardi and Meloni 1997], and utilising the result that the axiomatic extensions of HB and Ip share the same frame semantics, see [Wolter 1998].

Remark 3.45. Although our algorithm is an abstraction of the algorithm in [Ciabattoni et al. 2008], the key point is that the expressive power of the display calculus permits a base calculus $\delta\text{HB}'$ for Ip (obtained from δHB by deleting the rules for co-implication $\leftarrow_d$) in which the $\vee r$ rule is also invertible, leading to cut-eliminable structural rule extensions for more logics. This justifies the use of the more complex machinery of the display calculus.

4. LIMITS OF STRUCTURAL DISPLAY RULES

Given an amenable calculus $\mathcal{C}$, the previous section presented an algorithm to extract analytic structural rules out of acyclic $\mathcal{I}_2$ axioms. In this section we address the converse problem and show that if the calculus $\mathcal{C}$ satisfies a few natural additional properties then its analytic structural rules are equivalent to acyclic $\mathcal{I}_2$ axioms. The result is a generalisation of Kracht's Display Theorem I for tense logics (see Section 5.1) and applies to calculi for a much larger class of logics including e.g. substructural logics.

Definition 4.1. An amenable calculus $\mathcal{C}$ for L is *well-behaved* if

(i) $\mathcal{C}$ corresponds to some Hilbert calculus $\mathcal{H}$
(ii) $\mathcal{C}$ contains the following rules

$$\frac{A \vdash M \qquad B \vdash M}{A \vee B \vdash M}\ \vee\text{l} \qquad\qquad \frac{L \vdash A \qquad L \vdash B}{L \vdash A \wedge B}\ \wedge\text{r}$$

Here $\wedge$ and $\vee$ are the connectives in the definition of amenable calculus (not necessarily conjunction, disjunction).
(iii) For every sequent $A \vdash B$ ($A, B \in \text{For}\mathcal{L}$) we have $A \vdash B \in \text{inv}(\mathbf{I} \vdash \mathbb{F}(A \vdash B))$ (cf. Def.3.33).

Note that by definition of inv we have: $A \vdash B \in \text{inv}(\mathbf{I} \vdash \mathbb{F}(A \vdash B))$ implies $\mathbf{I} \vdash \mathbb{F}(A \vdash B)$ is derivable from $A \vdash B$.

Example 4.2. It is easy to see that the calculus δBi-FL for non-associative Bi-Lambek logic (Example 2.10) and the calculus δHB for bi-intuitionistic logic (Example 3.38) are well-behaved. In both calculi take indeed $\mathbb{F}(A \vdash B) = A \to B$.

LEMMA 4.3. *Let $\mathcal{C}$ be a well-behaved calculus. Then the rules $\vee$l and $\wedge$r are invertible.*

PROOF. Let us show that $\vee$l is invertible. The case of $\wedge$r is analogous. Suppose we have a concrete sequent of the form $A \vee B \vdash Y$. By Definition 3.1.(1) follows that $A \vdash A$ and $B \vdash B$ are derivable, and then from Definition 3.1.(3) we obtain $A \vdash A \vee B$ and $B \vdash A \vee B$. Then by the cut-rule we obtain derivations of $A \vdash Y$ and $B \vdash Y$. □

Notation. We write $X[U]$ to mean that the structure X contains an occurrence of a substructure U. Then $X[V]$ is the structure obtained by replacing that occurrence with V. Extending this notation, $X[U^1]\dots[U^n]$denotes that X contains occurrences of each U^i. For brevity of notation we write this as $X[U^i]_{i=1}^n$ or simply $X[U^i]_i$. We extend this notation in the obvious way, writing $(X \vdash Y)[U^i]_i$ to mean that the sequent $X \vdash Y$ contains occurrences of the substructures U^i. Finally we write $\text{l}(X) \vdash \text{r}(Y)[U^i]_i$ to mean the sequent obtained by applying the function l (resp. r) to the antecedent (succedent) of $(X \vdash Y)[U^i]_i$

From condition C1 we know that any structure variable in the premise of an analytic structural rule ρ must appear in its conclusion. Suppose that ρ contains a premise s having no structure variables (i.e. s is built using only structural constants). If this premise is derivable in $\mathcal{C}$ then it is clear that an equivalent rule can be obtained by deleting the premise s. On the other hand, if s is not derivable in $\mathcal{C}$, then $\mathcal{C} + \rho = \mathcal{C}$. Thus, without loss of generality we may suppose that each premise of ρ contains (at least) one structure variable which appears in the conclusion. By displaying some structure variable in each premise, we can write any analytic structural rule in the form below.

$$\frac{\left\{X^i_j \vdash L^i\right\}_{ij} \qquad \left\{M^k \vdash Y^k_l\right\}_{kl}}{(X \vdash Y)\left[L^i\right]_i \left[M^k\right]_k \left[P^s\right]_s \left[Q^t\right]_t}\ \rho \tag{3}$$

The i, j, l, k, s, t range over finite index sets. Also X, Y, X^i_j and Y^k_l are structures built from structure variables in $\{L^i\}_i \cup \{M^k\}_k \cup \{P^s\}_s \cup \{Q^t\}_t$ (so *any* structure variable may occur in any X^i_j, Y^k_l structure). By C4, all occurrences of these variables will be a-part or s-part (i.e. a mixture of a-part and s-part is not possible). Here the M^k and P^s variables are a-part and the L^i and Q^t variables are s-part. By C3, $X \vdash Y$ contains only one occurrence of each distinct structure variable.

The key lemma below indicates how to construct the axiom equivalent to ρ.

LEMMA 4.4. *Let $\mathcal{C}$ be a well-behaved calculus for the logic L and ρ an arbitrary analytic structural rule, written in the form (3). Then ρ is equivalent in $\mathcal{C}$ to the sequent below:*

$$(l(X) \vdash r(Y))\left[(L^i\sigma) \vee \bigvee_j l(X^i_j\sigma)\right]_i \left[(M^k\sigma) \wedge \bigwedge_l r(Y^k_l\sigma)\right]_k \left[P^s\sigma\right]_s \left[Q^t\sigma\right]_t \tag{4}$$

Here σ is a function that replaces distinct structure variables with distinct propositional variables.

Note: The rule ρ is constructed from structure variables while (4) is constructed from propositional variables. In order to meaningfully discuss the equivalence of ρ and (4), we must consider concrete instances of the latter, obtained via uniform substitution of formulae for propositional variables.

PROOF. Using the property of the l and r functions (Definition 3.1.1), sequent (4) is equivalent to

$$(X \vdash Y)\left[(L^i\sigma) \vee \bigvee_j \mathrm{l}(X^i_j\sigma)\right]_i \left[(M^k\sigma) \wedge \bigwedge_l \mathrm{r}(Y^k_l\sigma)\right]_k \left[P^s\sigma\right]_s \left[Q^t\sigma\right]_t \tag{5}$$

and, by Ackermann's lemma (Lemma 3.6), to the following rule for fresh structure variables $\{L^i\}_i, \{M^k\}_k, \{P^s\}_s$ and $\{Q^t\}_t$.

$$\frac{\left\{(L^i\sigma) \vee \bigvee_j \mathrm{l}(X^i_j\sigma) \vdash L^i\right\}_i \quad \left\{M^k \vdash (M^k\sigma) \wedge \bigwedge_l \mathrm{r}(Y^k_l\sigma)\right\}_k \quad \left\{P^s \vdash P^s\sigma\right\}_s \quad \left\{Q^t\sigma \vdash Q^t\right\}_t}{(X \vdash Y)\left[L^i\right]_i \left[M^k\right]_k \left[P^s\right]_s \left[Q^t\right]_t} \tag{6}$$

By the invertibility of $\vee$l and $\wedge$r (Lemma 4.3) this is equivalent to the rule:

$$\frac{\left\{L^i\sigma \vdash L^i\right\}_i \quad \left\{\mathrm{l}(X^i_j\sigma) \vdash L^i\right\}_{ij} \quad \left\{M^k \vdash M^k\sigma\right\}_k \quad \left\{M^k \vdash \mathrm{r}(Y^k_l\sigma)\right\}_{kl} \quad \left\{P^s \vdash P^s\sigma\right\}_s \quad \left\{Q^t\sigma \vdash Q^t\right\}_t}{(X \vdash Y)\left[L^i\right]_i \left[M^k\right]_k \left[P^s\right]_s \left[Q^t\right]_t}$$

Using the properties of the l and r functions, this rule is equivalent to

$$\frac{\left\{L^i\sigma \vdash L^i\right\}_i \quad \left\{X^i_j\sigma \vdash L^i\right\}_{ij} \quad \left\{M^k \vdash M^k\sigma\right\}_k \quad \left\{M^k \vdash Y^k_l\sigma\right\}_{kl} \quad \left\{P^s \vdash P^s\sigma\right\}_s \quad \left\{Q^t\sigma \vdash Q^t\right\}_t}{(X \vdash Y)\left[L^i\right]_i \left[M^k\right]_k \left[P^s\right]_s \left[Q^t\right]_t}\ \rho'$$

It remains to show the equivalence between ρ' and (3). Given concrete premises of ρ', apply the cut rule (and display rule where required) to the formulae instantiating the propositional variables $L^i\sigma$, $M^k\sigma$, $P^s\sigma$ and $Q^t\sigma$ to obtain the premises of (3).

Now for the other direction. Suppose that we have concrete premises of (3). Let us notationally distinguish a schematic structure U from its instantiation $\bar{U}$. Convert every instantiation $\bar{N}$ (in $\bar{X}^i_j \vdash \bar{L}^i$) of a structure variable N (in $X^i_j \vdash \bar{L}^i$) to l($\bar{N}$) or r($\bar{N}$) depending on whether N is a-part or s-part, to obtain ultimately a substitution instance of $X^i_j\sigma \vdash \bar{L}^i$. Similarly obtain the substitution instance of $\bar{M}^k \vdash Y^k_l\sigma$ from concrete $\bar{M}^k \vdash \bar{Y}^k_l$. In the substitution instances, every occurrence of $L^i\sigma$, $M^k\sigma$, $P^s\sigma$ and $Q^t\sigma$ is instantiated with the formula r($\bar{L}^i$), l($\bar{M}^k$) r($\bar{P}^s$) and l($\bar{Q}^t$), respectively. We can also derive $\{\text{r}(\bar{L}^i) \vdash \bar{L}^i\}_i$, $\{\bar{M}^k \vdash \text{l}(\bar{M}^k)\}_k$, $\{\text{r}(\bar{P}^s) \vdash P^s\}$ and $\{Q^t \vdash \text{l}(\bar{Q}^t)\}$. We now have concrete instances of the premises of ρ'. Apply ρ' to get the required sequent. □

Example 4.5. The rule ρ_1 below left is equivalent in δHB to the sequent below right (the propositional variable m stands for $M\sigma$)

$$\frac{M \vdash M > \mathbf{I}}{M \vdash \mathbf{I}}\ \rho_1 \qquad\qquad m \wedge (m \to \bot) \vdash \mathbf{I}$$

THEOREM 4.6. *Let $\mathcal{C}$ be a well-behaved calculus (for the Hilbert calculus $\mathcal{H}$) and let Δ be a set of formulae in the language of $L_{\mathbf{I}}(\mathcal{C})$. Then there is an analytic structural extension $\mathcal{C} + \{\rho_i\}_{1\leq i\leq n}$ corresponding to $\mathcal{H} + \Delta$ iff $L(\mathcal{H} + \Delta) = L(\mathcal{H} + \Delta')$ for a set Δ' of acyclic $\mathcal{I}_2(\mathcal{C})$ axioms.*

PROOF. For the 'if' direction, let the set Δ' of acyclic $\mathcal{I}_2(\mathcal{C})$ axioms be $\{A^1, \ldots, A^n\}$. Then by Theorem 3.31 we obtain sets $\mathcal{R}^j$ of analytic structural rules equivalent to $\mathbf{I} \vdash A^j$ for $j \in \{1, \ldots, n\}$. By applying Lemma 3.35 n times we get that $\mathcal{C} + \mathcal{R}^1 + \ldots + \mathcal{R}^n$ corresponds to $\mathcal{H} + A^1 + \ldots + A^n$.

For the 'only if' direction, first note that by Lemma 4.4 we have that each ρ_i ($1 \leq i \leq n$) is equivalent to a sequent $A_i \vdash B_i$ of the form (4). Because $\mathcal{C}$ is well-behaved, it follows that ρ_i is equivalent to $\mathbf{I} \vdash \mathbb{F}(A_i \vdash B_i)$. By Lemma 3.35 $\mathcal{C} + \rho_i$ corresponds to $\mathcal{H} + \mathbb{F}(A_i \vdash B_i)$. Since $\mathcal{C} + \rho_i$ is well-behaved—by inspection, this property is preserved in all extensions of $\mathcal{C}$—we can repeat this argument to ultimately obtain that $\mathcal{C} + \{\rho_i\}_{1\leq i\leq n}$ corresponds to $\mathcal{H} + \Delta'$ where $\Delta' = \{\mathbb{F}(A_i \vdash B_i), \ldots, \mathbb{F}(A_n \vdash B_n)\}$. Since $\mathcal{C} + \{\rho_i\}_{1\leq i\leq n}$ corresponds to $\mathcal{H} + \Delta$ it follows that $L(\mathcal{H} + \Delta) = L(\mathcal{H} + \Delta')$ (cf. Lemma 3.34).

Now it suffices to show that each $\mathbb{F}(A_i \vdash B_i) \in \Delta'$ is an acyclic $\mathcal{I}_2(\mathcal{C})$ formula (we drop the subscript i in the following to simplify the notation). By Definition 4.1(iii) we have $A \vdash B \in$ inv($\mathbf{I} \vdash \mathbb{F}(A \vdash B)$) where $A \vdash B$ has the form (4). To show that $\mathbb{F}(A_i \vdash B_i)$ is an $\mathcal{I}_2(\mathcal{C})$ formula, apply the algorithm transforming an sequent into a semi-structural rule to the sequent (4). To help the reader follow the transformation, we refer now to the labelling of the steps in Figure 1.

Repeated application of the invertible rules to (4)—STEP 1—yields, by inspection, the sequent (5). By repeated application of the display rules and Lemma 3.6 starting with the above sequent—STEP 2—we get the rule (6). Apply invertible rules to each of the premises of this rule (STEP 3), once again by inspection, we get a semi-structural rule. In particular, because this rule contains no logical connectives, we conclude that $\mathbb{F}(A \vdash B)$ is in $\mathcal{I}_2(\mathcal{C})$.

The final step—STEP 4—is to show that the set $\mathcal{S}$ of premises of the semi-structural rule is acyclic. By C4, every propositional variable in a premise $X^i_j\sigma \vdash L^i$ and $M^k \vdash Y^k_l\sigma$ is either a-part or s-part (possibly with multiplicities) but not both. Obviously $P^s \vdash P^s\sigma$ and $Q^t\sigma \vdash Q^t$ also satisfy this condition. Thus $\mathcal{S}$ respects multiplicities wrt every propositional variable occurring in it (see (1) and (2)). By consideration of the simple form of the premises $P^s \vdash P^s\sigma$ and $Q^t\sigma \vdash Q^t$ we see that repeated application of the $\mathcal{S}_p$ operation (see Definition 3.21) yields each time a set that respects multiplicities wrt each variable in it. Then from Definition 3.21 and Definition 3.23, it follows that $\mathcal{S}$ is an acyclic set. By Definition 3.24, $\mathbb{F}(A \vdash B)$ is an acyclic $\mathcal{I}_2$ axiom. □

The following are immediate.

COROLLARY 4.7. *Let $\mathcal{H}$ be an axiomatic extension of intuitionistic non-associative Bi-Lambek logic Bi-FL. There is an analytic structural rule extension of δBi-FL corresponding to $\mathcal{H}$ iff $\mathcal{H}$ is an extension by acyclic $\mathcal{I}_2(\delta$Bi-FL$)$ axioms.*

COROLLARY 4.8. *Let $\mathcal{H}$ be an axiomatic extension of bi-intuitionistic logic HB. There is an analytic structural rule extension of δHB corresponding to $\mathcal{H}$ iff $\mathcal{H}$ is an extension by acyclic $\mathcal{I}_2(\delta HB)$ axioms.*

Example 4.9. In Example 3.42 we saw how to obtain the analytic structural rule ρ (below left) such that $\delta\text{HB} + \rho$ corresponds to $\mathcal{H}\text{HB} + p_2 \vee (p_2 \to (p_1 \vee (p_1 \to \bot)))$.

$$\frac{M \vdash L^1 \qquad V \vdash L^2}{\mathbf{I} \vdash L^1, (M > (L^2, (V > \mathbf{I})))}\,\rho \qquad \frac{M \vdash L^1 \qquad V \vdash L^2}{\mathbf{I} \vdash L^1, (M > (L^2, (V > \mathbf{I})))[L^1][L^2][M][V]}\,\rho$$

This is the 'if' direction of Theorem 4.6.

For the 'only if' direction let us compute the axiom that is equivalent to this rule. Above right we have written ρ in the form (3). From Lemma 4.4, this rule is equivalent to the sequent $\top \vdash (l^1 \vee m) \vee (m \to ((l^2 \vee v) \vee (v \to \bot)))$ where $M\sigma = m$, $V\sigma = v$, $L^1\sigma = l^1$ and $L^2\sigma = l^2$. Using the function $\mathbb{F}$ (see Example 3.37) we have that $\delta\text{HB}+\rho$ corresponds to $\text{HB}+\top \to (l^1 \vee m) \vee (m \to ((l^2 \vee v) \vee (v \to \bot)))$. Although this axiom is not identical to the Bd_2 axiom, it is easy to check that each axiomatisation over HB can derive the other axiom.

5. A CASE STUDY: TENSE LOGICS

The class of tense axioms equivalent to analytic (*proper*, in Kracht's terminology) structural display rules was identified by [Kracht 1996] who called these *primitive tense formulae*. In this section we compare our transformation algorithm with Kracht's method and provide an alternative, fully checkable[6] proof of the converse direction: every analytic structural rule extension of the display calculus δKt corresponds to an axiomatic extension of the Hilbert calculus $\mathcal{H}$Kt for Kt by primitive tense formulae. Here δKt is the display calculus corresponding to $\mathcal{H}$Kt.

Recall that the modal language $\mathcal{L}_K$ is obtained from the propositional classical language by the addition of the modal operators $\Diamond$ and $\Box$. The tense language $\mathcal{L}_{\text{Kt}}$ is obtained from $\mathcal{L}_K$ by the addition of the tense operators $\blacklozenge$ and $\blacksquare$. The Hilbert calculi $\mathcal{H}K$ and $\mathcal{H}$Kt for normal basic modal logic K and tense logic Kt, respectively, are conservative extensions of classical propositional logic obtained by the addition of the usual axioms (see, e.g., [Blackburn et al. 2001]).

The set of a- and s-structures for δKt have the identical grammar $\mathfrak{S}(\mathcal{L}_{\text{Kt}})$:

$$X ::= A \in \text{For}\mathcal{L}_{\text{Kt}} \mid \mathbf{I} \mid (X, X) \mid \bullet X \mid \star X$$

The *display rules* of δKt are:

$$\frac{L, M \vdash Z}{L \vdash Z, \star M} \qquad \frac{L, M \vdash Z}{M \vdash \star L, Z} \qquad \frac{L \vdash M, Z}{L, \star Z \vdash M} \qquad \frac{L \vdash M, Z}{\star M, L \vdash Z} \qquad \frac{\star L \vdash M}{\star M \vdash L}$$

$$\frac{L \vdash \star M}{M \vdash \star L} \qquad \frac{L \vdash \bullet M}{\bullet L \vdash M} \qquad \frac{\star\star L \vdash M}{L \vdash M} \qquad \frac{L \vdash \star\star M}{L \vdash M}$$

Since there is a display rule to remove each structural head connective in the antecedent/succedent to reveal the nested substructure, displaying a substructure of a given sequent is computable (Definition 2.7(ii)). Although the set of sequents display equivalent to a given sequent is not finite—we can repeatedly affix $\star\star$ to any substructure using the display rules—display equivalence is computable as demanded by Definition 2.7(iii). This is because a sequent can be first put in a normal form by removing all occurrences of $(\star\star)^n$ $(n \geq 1)$ that occur in front of a substructure.

The remaining structural rules of δKt are given below.

[6] Unfortunately, a crucial step in the proof of [Kracht 1996] lacks important details, thus making it impossible to check.

Name	Axiom	Rule	Name	Axiom	Rule
D	$\Box A \to \Diamond A$	$(\star \bullet \star)\bullet X \vdash Y / X \vdash Y$	B	$A \to \Box\Diamond A$	$\star \bullet \star X \vdash Y / \bullet X \vdash Y$
G	$\Diamond\Box A \to \Box\Diamond A$	$\bullet X \vdash \star \bullet \star Y / \star \bullet \star X \vdash \bullet Y$	4	$\Box A \to \Box\Box A$	$\bullet X \vdash Y / \bullet \bullet X \vdash Y$
5	$\Diamond A \to \Box\Diamond A$	$\star \bullet \star X \vdash Y / \star \bullet \star X \vdash \bullet Y$	T	$\Box A \to A$	$\bullet X \vdash Y / X \vdash Y$

Fig. 2. Some acyclic $\mathcal{I}_2$ axioms and corresponding analytic structural rules

$$\dfrac{L \vdash Z}{\mathbf{I}, L \vdash Z} \quad \dfrac{L \vdash Z}{L \vdash \mathbf{I}, Z} \quad \dfrac{\mathbf{I} \vdash M}{\star\mathbf{I} \vdash M} \quad \dfrac{L \vdash \mathbf{I}}{L \vdash \star\mathbf{I}} \quad \dfrac{L \vdash Z}{M, L \vdash Z}$$

$$\dfrac{L \vdash Z}{L, M \vdash Z} \quad \dfrac{\mathbf{I} \vdash M}{\bullet\mathbf{I} \vdash M} \quad \dfrac{L \vdash \mathbf{I}}{L \vdash \bullet\mathbf{I}} \quad \dfrac{L, M \vdash Z}{M, L \vdash Z} \quad \dfrac{Z \vdash L, M}{Z \vdash M, L}$$

$$\dfrac{L, L \vdash Z}{L \vdash Z} \quad \dfrac{Z \vdash L, L}{Z \vdash L} \quad \dfrac{L_1, (L_2, L_3) \vdash Z}{(L_1, L_2), L_3 \vdash Z} \quad \dfrac{Z \vdash L_1, (L_2, L_3)}{Z \vdash (L_1, L_2), L_3}$$

The initial sequents of δKt are $p \vdash p$ for any propositional variable p, and $\mathbf{I} \vdash \top$ and $\bot \vdash \mathbf{I}$. Here are the logical rules of δKt (we use the invertible form for $\wedge r$, $\vee l$ and $\to l$).

$$\dfrac{\mathbf{I} \vdash L}{\top \vdash L}\ \top l \qquad \dfrac{L \vdash \mathbf{I}}{L \vdash \bot}\ \bot r \qquad \dfrac{\star A \vdash L}{\neg A \vdash L}\ \neg l$$

$$\dfrac{L \vdash \star A}{L \vdash \neg A}\ \neg r \qquad \dfrac{A, B \vdash L}{A \wedge B \vdash L}\ \wedge l \qquad \dfrac{L \vdash A \qquad L \vdash B}{L \vdash A \wedge B}\ \wedge r$$

$$\dfrac{A \vdash L \qquad B \vdash L}{A \vee B \vdash L}\ \vee l \qquad \dfrac{L \vdash A, B}{L \vdash A \vee B}\ \vee r \qquad \dfrac{\star M \vdash A \qquad B \vdash M}{A \to B \vdash M}\ \to l$$

$$\dfrac{L, A \vdash B}{L \vdash A \to B}\ \to r \qquad \dfrac{A \vdash L}{\Box A \vdash \bullet L}\ \Box l \qquad \dfrac{L \vdash \bullet A}{L \vdash \Box A}\ \Box r$$

$$\dfrac{\star \bullet \star A \vdash L}{\Diamond A \vdash L}\ \Diamond l \qquad \dfrac{L \vdash A}{\star \bullet \star L \vdash \Diamond A}\ \Diamond r \qquad \dfrac{\bullet A \vdash L}{\blacklozenge A \vdash L}\ \blacklozenge l$$

$$\dfrac{L \vdash A}{\bullet L \vdash \blacklozenge A}\ \blacklozenge r \qquad \dfrac{A \vdash L}{\blacksquare A \vdash \star \bullet \star L}\ \blacksquare l \qquad \dfrac{L \vdash \star \bullet \star A}{L \vdash \blacksquare A}\ \blacksquare r$$

Define the functions l and r from $\mathfrak{S}(\mathcal{L}_{\mathrm{Kt}})$ into For$\mathcal{L}_{\mathrm{Kt}}$.

$$\begin{aligned} l(A) &= A & r(A) &= A \\ l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\ l(\star X) &= \neg r(X) & r(\star X) &= \neg l(X) \\ l(X, Y) &= l(X) \wedge l(Y) & r(X, Y) &= r(X) \vee r(Y) \\ l(\bullet X) &= \blacklozenge l(X) & r(\bullet X) &= \Box r(X) \end{aligned}$$

It may be checked that δKt is an amenable well-behaved calculus. Using Theorem 4.6 we have:

COROLLARY 5.1. *Let $\mathcal{H}$ be an axiomatic extension of tense logic Kt. There is an analytic structural rule extension of δKt corresponding to $\mathcal{H}$ iff $\mathcal{H}$ is an extension by acyclic $\mathcal{I}_2(\delta Kt)$ axioms.*

LEMMA 5.2. *Every logical rule with the exception of $\Box l$, $\Diamond r$, $\blacklozenge r$ and $\blacksquare l$ is invertible.*

Example 5.3. Figure 2 displays some examples of acyclic $\mathcal{I}_2(\delta\mathrm{Kt})$ axioms and the corresponding rules generated by our algorithm.

Using the above observation we can give a more explicit description of $\mathcal{I}_2(\delta\mathrm{Kt})$ axioms along the line of the classes in [Ciabattoni et al. 2008] for substructural logics.

Set $\mathcal{P}_0 = \mathcal{N}_0$ as the set of propositional variables.

$$\begin{array}{llll}
\mathcal{P}_1 := \mathcal{P}_0 \mid \top \mid \mathcal{P}_1 \wedge \mathcal{P}_1 \mid \mathcal{P}_1 \vee \mathcal{P}_1 & \mid \neg\mathcal{N}_1 \mid \mathcal{N}_1 \to \mathcal{P}_1 & \mid \Diamond\mathcal{P}_1 \mid \blacklozenge\mathcal{P}_1 \\
\mathcal{N}_1 := \mathcal{N}_0 \mid \bot \mid \mathcal{N}_1 \wedge \mathcal{N}_1 \mid \mathcal{N}_1 \vee \mathcal{N}_1 & \mid \neg\mathcal{P}_1 \mid \mathcal{P}_1 \to \mathcal{N}_1 & \mid \Box\mathcal{N}_1 \mid \blacksquare\mathcal{N}_1 \\
\mathcal{P}_2 := \mathcal{P}_1 \mid \top \mid \mathcal{P}_2 \wedge \mathcal{P}_2 \mid \mathcal{P}_2 \vee \mathcal{P}_2 & \mid \neg\mathcal{N}_2 \mid \mathcal{N}_2 \to \mathcal{P}_2 & \mid \Diamond\mathcal{P}_2 \mid \blacklozenge\mathcal{P}_2 \mid \Box\mathcal{N}_1 \mid \blacksquare\mathcal{N}_1 \\
\mathcal{N}_2 := \mathcal{N}_1 \mid \bot \mid \mathcal{N}_2 \wedge \mathcal{N}_2 \mid \mathcal{N}_2 \vee \mathcal{N}_2 & \mid \neg\mathcal{P}_2 \mid \mathcal{P}_2 \to \mathcal{N}_2 & \mid \Box\mathcal{N}_2 \mid \blacksquare\mathcal{N}_2 \mid \Diamond\mathcal{P}_1 \mid \blacklozenge\mathcal{P}_1
\end{array}$$

It is easy to see that $\mathcal{I}_0(\delta\text{Kt}) = \mathcal{N}_0$; $\mathcal{I}_1(\delta\text{Kt}) = \mathcal{N}_1$; and $\mathcal{I}_2(\delta\text{Kt}) = \mathcal{N}_2$.

Example 5.4. The Scott-Lemmon axioms have the form $\Diamond^i\Box^j A \to \Box^k\Diamond^l A$. It is easy to see that all these axioms are acyclic $\mathcal{I}_2(\delta\text{Kt})$ formulae.

The next section shows that the class of acyclic $\mathcal{I}_2(\delta\text{Kt})$ formulae coincides with Kracht's primitive tense formulae.

Definition 5.5 (*primitive tense formula*). A *primitive tense axiom* is a formula of the form $A \to B$ where both A and B are constructed from propositional variables and $\top$ using $\{\wedge, \vee, \Diamond, \blacklozenge\}$ and A contains each propositional variable at most once.

5.1. Kracht's Display Theorem I revisited

We provide an alternative proof of Kracht' characterisation of analytic structural rule extensions of the display calculus δKt.

THEOREM 5.6 (DISPLAY THEOREM I [KRACHT 1996]). *Let $\mathcal{H}$ be an axiomatic extension of $\mathcal{H}$Kt. There is an analytic structural rule extension of δKt corresponding to $\mathcal{H}$ iff the logic of $\mathcal{H}$ is axiomatisable over $\mathcal{H}$Kt by primitive tense axioms.*

Observe that in the case of δKt, $\text{inv}^{\text{all}}(U \vdash V)$ is a singleton set for any $U \vdash V$ i.e. all possible sequences of applying invertible rules upwards lead to the same set of sequents. With an abuse of notation for the sake of simplicity, in this section we will write $\text{inv}^{\text{all}}(U \vdash V)$ to mean *that* element (rather than the set containing that element).

First note that every primitive tense axiom is an acyclic $\mathcal{I}_2(\delta\text{Kt})$ axiom. To see this, first observe that for any primitive tense formula $A \to B$, both A and B are a-soluble and negation-free. Hence $A \to B \in \mathcal{I}_2(\delta\text{Kt})$. Let $A_1^j, \ldots, A_n^j$ and $B_1^j, \ldots, B_m^j$ denote, respectively, the formulae coming from A and B in a sequent $S^j \in \text{inv}^{\text{all}}(\mathbf{I} \vdash A \to B)$ $(1 \leq j \leq N)$. First note that every B_k^j is an a-part formula and each A_k^i is a (single) propositional variable. The set $\{\text{inv}^{\text{all}}(\mathbf{I} \vdash A_1^j), \ldots, \text{inv}^{\text{all}}(\mathbf{I} \vdash A_n^j), \text{inv}^{\text{all}}(B_1^j \vdash \mathbf{I}), \ldots, \text{inv}^{\text{all}}(B_m^j \vdash \mathbf{I})\}$ is clearly acyclic because every propositional variable in B_k^j is a-part (this is because B is negation-free). It follows that $A \to B$ is acyclic.

To show ($\Leftarrow$), suppose that $\mathcal{H}$ is an axiomatic extension of $\mathcal{H}$Kt and $L(\mathcal{H}) = L(\mathcal{H}\text{Kt} + \Delta)$ where Δ is a set of primitive tense axioms. Due to the above observation and Theorem 3.36 there is an analytic structural rule extension corresponding to $\mathcal{H}\text{Kt} + \Delta$. It may been seen that the structural rule extension also corresponds to $\mathcal{H}$—in particular, note that any derivation from assumptions in $\mathcal{H}\text{Kt} + \Delta$ can be transformed into a derivation from assumptions in $\mathcal{H}$ and *vice versa* since every instance of an axiom in one system must be derivable in the other because $L(\mathcal{H}) = L(\mathcal{H}\text{Kt} + \Delta)$.

Kracht's method for ($\Leftarrow$). Let us simply illustrate Kracht's method for obtaining an analytic structural rule from a primitive tense formula $A \to B$. First write A and B as equivalent disjunctions $\vee_{1\leq i\leq N} C_i$ and $\vee_{1\leq j\leq M} D_j$, respectively, of primitive tense formulae not containing $\vee$. Repeatedly applying the invertible rule $\vee$l starting with $\vee_{1\leq i\leq N} C_i \vdash \vee_{1\leq j\leq M} D_j$, equivalent sequents $C_i \vdash \vee_{1\leq j\leq M} D_j$ are obtained. Although $\vee$r is also invertible, Kracht chooses to apply Ackermann's lemma *directly* to $\vee_{1\leq j\leq M} D_j$ and then applies all possible invertible rules to the premises and conclusion. The resulting rules can be presented in our notation as follows $(1 \leq i \leq N)$:

Name	Axiom	Primitive tense	Name	Axiom	Primitive tense
D	$\Box A \to \Diamond A$	$A \to \Diamond \blacklozenge A$	B	$A \to \Box \Diamond A$	$\blacklozenge A \to \Diamond A$
confluence	$\Diamond \Box A \to \Box \Diamond A$	$\blacklozenge \Diamond A \to \Diamond \blacklozenge A$	4	$\Box A \to \Box \Box A$	$\Diamond \Diamond A \to \Diamond A$
5	$\Diamond A \to \Box \Diamond A$	$\blacklozenge \Diamond A \to \Diamond A$	T	$\Box A \to A$	$A \to \Diamond A$

Fig. 3. Some $\mathcal{I}_2$ axioms and their equivalent primitive tense form.

$$\frac{\text{inv}^{\text{all}}(D_1 \vdash M) \qquad \cdots \qquad \text{inv}^{\text{all}}(D_M \vdash M)}{\text{inv}^{\text{all}}(C_i \vdash M)}$$

Contrast with our procedure where we permit the possibility of applying some invertible right rules to $\vee_{1 \leq j \leq M} D_j$ *before* applying Ackermann's lemma. This is the reason why our procedure can obtain analytic structural rules for a class $\mathcal{I}_2(\delta\text{Kt})$ of axioms that is *syntactically* larger (i.e. in the sense of set containment) than the primitive tense axioms. (Nevertheless, we will see later that every axiomatic extension of $\mathcal{H}$Kt by $\mathcal{I}_2(\delta\text{Kt})$ axioms is equivalent to some extension of $\mathcal{H}$Kt by primitive tense axioms and *vice versa*).

Example 5.7. In general, the Scott-Lemmon axioms in Example 5.4 are not primitive tense axioms since they may contain $\Box$. Nevetheless, an axiomatic extension of $\mathcal{H}$Kt by $\Diamond^i \Box^j A \to \Box^k \Diamond^l A$ is equivalent to an axiomatic extension by the primitive tense formula $\blacklozenge^i \Diamond^k A \to \Diamond^j \blacklozenge^l A$.

Aside from this, it appears that there exists a primitive tense axiom that Kracht's method cannot transform into an equivalent analytic structural rule. Consider the primitive tense formula $\blacklozenge p \to q$. Indeed, if we apply Kracht's method we obtain the rule

$$\frac{Q \vdash M}{\bullet P \vdash M}$$

which is not analytic (C1 is violated since the structure variable Q does not appear in the conclusion). However this shortcoming can be rectified by noting that this rule is equivalent to the rule with empty premise and conclusion $\bullet P \vdash M$. Generalising this argument to handle all primitive tense formulae $A \to B$ where B contains a propositional variable not appearing in A we can complete Kracht's proof of ($\Rightarrow$). Note here that the logic $\mathcal{H}\text{Kt} + \blacklozenge p \to q$ is not the inconsistent logic but instead the tense counterpart of the maximal (in the sense of axiomatic extensions) consistent modal logic Ver which is usually axiomatised as $\mathcal{H}K + \Box\bot$ [Hughes and Cresswell 1996].

Remark 5.8. We have already seen above that a disadvantage of applying Ackermann's lemma directly to $\vee_{1 \leq j \leq M} D_j$ rather than applying invertible right rules first is that the former (Kracht's method) cannot be applied directly to a formula in $\mathcal{I}_2(\delta\text{Kt})$ but instead relies on receiving a primitive tense formula as input. Even the standard presentation of the usual modal axioms are *not* in primitive tense form. This point is illustrated by Figure 3. The need to transform the given axiom into a primitive tense axiom is a disadvantage in Kracht's method—when the axiom is more complicated, it may be rather challenging to do so. In addition, the primitive *modal* equivalent of the given axiom (should it exist) may be considerable more complicated. For example, although the primitive tense equivalent $\blacklozenge A \to \Diamond A$ of the modal axiom $A \to \Box \Diamond A$ is easily derived, it is not immediately clear that the primitive *modal* equivalent is the formula $A \wedge \Diamond B \to \Diamond(\Diamond A \wedge B)$.

Kracht's proof of the (⇒) direction. It cannot be checked because important details are missing. In particular, [Kracht 1996] defines a *special* structural rule as an analytic structural rule containing a structure variable L that (i) is the common antecedent (equivalently, succedent) of every sequent in the rule, and (ii) occurs exactly once in each sequent. A key result is that every analytic structural rule is equivalent to a special structural rule. However Kracht does not give a proof of this equivalence or a method to actually transform each structural rule into a special structural rule.

Proposition 5.17 below provides a new proof of ($\Rightarrow$). The key step is showing that every acyclic $\mathcal{I}_2(\delta\text{Kt})$ axiom is equivalent to a primitive tense formula. This is proved in the crucial Lemma 5.12 in a model-theoretic way, making use of the standard Kripke semantics for tense logics.

We start briefly recalling semantic concepts and terminology; see, e.g., [Blackburn et al. 2001] for more details. A *frame* is a pair $F = (W, R)$ where W is a non-empty set and R is a binary relation on W. A *model* is a pair $M = (F, V)$ where F is a frame (W, R) and V is a *valuation* function assigning to each proposition variable p a subset $V(p)$ of W. Suppose that $M = (F, V)$ is a model. The relation $M, w \models A$ (read as 'formula A holds in M at w') is defined inductively on the structure of A. For the propositional connectives the definition is classical relativised to w. Also

$$
\begin{aligned}
&M, w \models \Diamond A \text{ iff there exists } v \in W \text{ such that } Rwv \text{ and } M, v \models A\\
&M, w \models \blacklozenge A \text{ iff there exists } v \in W \text{ such that } Rvw \text{ and } M, v \models A\\
&M, w \models \Box A \text{ iff for all } v \in W, \text{ if } Rwv \text{ then } M, v \models A\\
&M, w \models \blacksquare A \text{ iff for all } v \in W, \text{ if } Rvw \text{ then } M, v \models A
\end{aligned}
$$

The negation of $M, w \models A$ is written $M, w \not\models A$. A formula A holds on a frame $F = (W, R)$ (denoted $F \models A$) if for all valuations V and $w \in W$: $(F, V), w \models A$. Two formulae A and B are *frame equivalent* if for all frames F: $F \models A$ iff $F \models B$. It is well-known that $A \in \text{Kt}$ iff for all frames F: $F \models A$.

Let α be a formula in the first-order language (of classical logic) with equality and a binary relation R. Viewing a model $M = (F, V)$ as a relational structure, define $M \models \alpha$ in the obvious way to mean that the underlying frame $F = (W, R)$ satisfies α when the symbol R is interpreted as the binary relation on W. For a string σ constructed from $\Diamond$ and $\blacklozenge$ (ϵ is the empty string), define recursively the first-order formulae $\Sigma_\sigma(w, v)$:

$$
\begin{aligned}
\Sigma_\epsilon(w, v) &= (w = v)\\
\Sigma_{\Diamond\sigma}(w, v) &= \exists w'(Rww' \wedge \Sigma_\sigma(w', v))\\
\Sigma_{\blacklozenge\sigma}(w, v) &= \exists w'(Rw'w \wedge \Sigma_\sigma(w', v))
\end{aligned}
$$

Intuitively, $\Sigma_\sigma(w, v)$ converts a path σ between w and v specified in terms of $\Diamond$ and $\blacklozenge$ in terms of existential quantifers. For example,

$$\Sigma_{\Diamond\blacklozenge}(w, s) = \exists w'(Rww' \wedge (\exists w''(Rw''w' \wedge w'' = s)))$$

LEMMA 5.9. *Let σ be a (possibly empty) string constructed from $\Diamond$ and $\blacklozenge$. For any tense formula A, model M and state w:*

$$M, w \models \sigma A \textit{ iff there exists } v \textit{ such that } M \models \Sigma_\sigma(w, v) \textit{ and } M, v \models A$$

PROOF. Induction on the length of σ. $\Box$

We will require the following definitions (see [Blackburn et al. 2001]).

Definition 5.10 (*positive, negative propositional var*). A propositional variable is in *positive* (resp. *negative*) position if it occurs in an implication-free formula under an even (odd) number of negation symbols. A formula is *positive* (resp. *negative*) in p if every occurrence of p is in positive (negative) position.

E.g. the formula $p \wedge q \wedge \neg(q \vee \neg p)$ is positive in p, and neither negative nor positive in q (the first occurrence of q is in positive position and the second is in negative position).

Definition 5.11 (*upward monotone*). A tense formula A is *upward monotone* in p if whenever $V'(p) \subseteq V(p)$ and $V'(q) = V(q)$ for $q \neq p$: $(F, V'), w \models A$ implies $(F, V), w \models A$.

It may be checked easily that if A is positive in p then A is upward monotone in p.

For $A, B \in \mathcal{L}_{\text{Kt}}$, we write $A = B$ to mean $A \to B \in$ Kt and $B \to A \in$ Kt. The proof of the following lemma appears in [Ramanayake 2011] where a second proof using second-order correspondence is also given.

LEMMA 5.12. *Suppose that*

(i) $g(p_1, \ldots, p_N)$ *is a formula constructed from distinct propositional variables* $p_1, \ldots, p_N$ *and* $\top$ *using* $\Diamond, \blacklozenge$ *and* $\wedge$ *such that each propositional variable appears exactly once; and*
(ii) Each of $D_1, \ldots, D_N$ *is either* $\bot$ *or constructed from distinct propositional variables* $p_1, \ldots, p_N, p_{N+1}, \ldots, p_M$ *and* $\top$ *using* $\Diamond, \blacklozenge, \wedge$ *and* $\vee$.

Then $g(p_1 \wedge \neg D_1, \ldots, p_N \wedge \neg D_n) \to \bot$ *is frame-equivalent to*

$$g(p_1, \ldots, p_N) \to g^{\vee}(p_1 \wedge D_1, \ldots, p_N \wedge D_N) \tag{7}$$

where $g^{\vee}(p_1, \ldots, p_N)$ *is obtained by replacing every* $\wedge$ *in* $g(p_1, \ldots, p_N)$ *with* $\vee$*. Also, if some* $D_i \neq \bot$ *then (7) is frame-equivalent to a primitive tense formula.*

PROOF. Let us first show that (7) is equivalent to a primitive tense formula whenever some $D_i \neq \bot$. By inspection, if (7) contains no occurrence of $\bot$ then it is already a primitive tense formula. Next, suppose without loss of generality that $D_N = \bot$. By the hypotheses some $D_k \neq \bot$, hence $N > 1$. Using the equivalence $p_N \wedge \bot = \bot$ followed by repeated applications of $\Diamond\bot = \bot$; $\blacklozenge\bot = \bot$ and finally $\top \vee \bot = \top$ or $(p_j \wedge D_j) \vee \bot = p_j \wedge D_j$ $(j < N)$ we obtain a formula equivalent to (7) of the following form containing one less occurrence of $\bot$.

$$g(p_1, \ldots, p_N) \to g^{\vee\prime}(p_1 \wedge D_1, \ldots, p_{N-1} \wedge D_{N-1})$$

By repeating this procedure we ultimately obtain a primitive tense formula.

We make use of the following notation (left column) in the remainder of this proof.

$g(\mathbf{p_i})$	$g(p_1, \ldots, p_N)$
$g(\mathbf{p_i} \wedge \mathbf{D_i})$	$g(p_1 \wedge D_1, \ldots, p_N \wedge D_N)$

We need to prove that for every frame F: $F \models g(\mathbf{p_i}) \to g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$ iff $F \models g(\mathbf{p_i} \wedge \neg\mathbf{D_i}) \to \bot$. Argue in each direction by contradiction.

Assume that there is some F such that $F \models g(\mathbf{p_i} \wedge \neg\mathbf{D_i}) \to \bot$ and $F \not\models g(\mathbf{p_i}) \to g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$. The latter implies that there exists some model $M = (F, V)$ and state w such that $M, w \not\models g(\mathbf{p_i}) \to g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$. Therefore $M, w \models g(\mathbf{p_i})$ and $M, w \not\models g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$. Starting with $M, w \models g(\mathbf{p_i})$ and making use of Lemma 5.9 it follows that there exist $v_1, \ldots, v_N$ and strings $\sigma_1, \ldots, \sigma_N$ in $\Diamond, \blacklozenge$ such that $M \models \Sigma_{\sigma_i}(w, v_i)$ and $M, v_i \models p_i$ for $1 \leq i \leq N$. Moreover, since $M, w \not\models g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$ it must be the case that $M, v_i \not\models p_i \wedge D_i$ and hence $M, v_i \not\models D_i$ $(1 \leq i \leq n)$. Therefore $M, v_i \models p_i \wedge \neg D_i$ for each i, so $M, w \models g(\mathbf{p_i} \wedge \neg\mathbf{D_i})$. Since $F \models g(\mathbf{p_i} \wedge \neg\mathbf{D_i}) \to \bot$ it follows that $M, w \models \bot$. This is impossible so we have obtained a contradiction.

Now for the other direction. Assume that there is some frame F such that $F \models g(\mathbf{p_i}) \to g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$ and $F \not\models g(\mathbf{p_i} \wedge \neg\mathbf{D_i}) \to \bot$. Then there exists some model $M = (F, V)$ and state w such that $M, w \not\models g(\mathbf{p_i} \wedge \neg\mathbf{D_i}) \to \bot$. Thus $M, w \models g(\mathbf{p_i} \wedge \neg\mathbf{D_i})$. This implies via Lemma 5.9 that there exist $v_1, \ldots, v_N$ and strings $\sigma_1, \ldots, \sigma_N$ constructed from $\Diamond, \blacklozenge$ such that $M \models \Sigma_{\sigma_i}(w, v_i)$ and $M, v_i \models p_i \wedge \neg D_i$. We will assume from here on that

$$(\dagger) \quad \text{no} \quad D_i = \top$$

as this would immediately give us the contradiction. Therefore $M, v_i \models p_i$ for each i, and thus $M, w \models g(\mathbf{p_i})$. Since $F \models g(\mathbf{p_i}) \to g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$ by assumption, we must have $M, w \models g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$. Define the set $\mathbf{u}^i$ $(1 \leq i \leq N)$ ('the set of states that are at the end of a σ_i path from w in which $p_i \wedge D_i$ holds'):

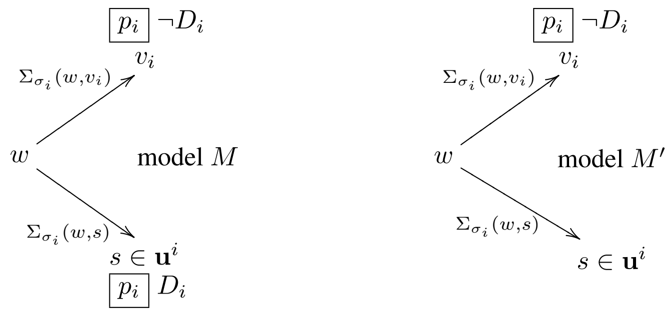
$$s \in \mathbf{u}^i \text{ iff } M \models \Sigma_{\sigma_i}(w, s) \text{ and } M, s \models p_i \wedge D_i$$

Notice that if $\mathbf{u}^i$ is non-empty then $D_i \neq \bot$. Also observe that $v_i \notin \mathbf{u}^i$ for any i since $M, v_i \models p_i \wedge \neg D_i$.

Let $M' = (F, V')$ be the model obtained from $M = (F, V)$ by setting

$$V'(p_i) = V(p_i) \setminus \mathbf{u}^i \text{ for each } p_i;\ V'(q) = V(q) \text{ for other propositional variables}$$

Informally, the model M' is obtained from M by 'switching-off' p_i at states s such that $M \models \Sigma_{\sigma_i}(w, s)$ and $M \models p_i \wedge D_i$. The models M and M' are illustrated below.



Clearly $M', v_i \models p_i$ for each i, so $M', w \models g(\mathbf{p_i})$ and since we have assumed $F \models g(\mathbf{p_i}) \rightarrow g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$ we get $M', w \models g^{\vee}(\mathbf{p_i} \wedge \mathbf{D_i})$. Since no $D_i = \top$ from (†), there must be some i^* and s such that $M' \models \Sigma_{\sigma_{i^*}}(w, s)$ and $M', s \models p_{i^*} \wedge D_{i^*}$. Since the formula $p_{i^*} \wedge D_{i^*}$ is positive in every propositional variable occurring in it, it is upward monotone in all propositional variables. Since $V'(p) \subseteq V(p)$ for every p, by repeated upward monotonicity we have $M, s \models p_{i^*} \wedge D_{i^*}$. Then it must be the case that $s \in \mathbf{u}^{i^*}$ and thus $M', s \not\models p_{i^*}$ by definition of $V'(p_{i^*})$. This is a contradiction since we have already noted that $M', s \models p_{i^*} \wedge D_{i^*}$. □

Example 5.13. Consider the formula $g(p, q, r) = p \wedge \Diamond q \wedge \blacklozenge r$. Suppose that $D_1 = \Diamond q \vee s$; $D_2 = \bot$; and $D_3 = \Diamond \top$. Then the lemma tells us that the formulae below are frame-equivalent. Note that $g^{\vee}(p, q, r) = p \vee \Diamond q \vee \blacklozenge r$.

$$(p \wedge \neg(\Diamond q \vee s)) \wedge \Diamond(q \wedge \neg\bot) \wedge \blacklozenge(r \wedge \neg\Diamond\top) \rightarrow \bot$$
$$p \wedge \Diamond q \wedge \blacklozenge r \rightarrow (p \wedge (\Diamond q \vee s)) \vee \Diamond(q \wedge \bot) \vee \blacklozenge(r \wedge \Diamond\top)$$

The latter is equivalent to the primitive tense $p \wedge \Diamond q \wedge \blacklozenge r \rightarrow (p \wedge (\Diamond q \vee s)) \vee \blacklozenge(r \wedge \Diamond\top)$.

Definition 5.14 (*positive, negative structure var*). A structure variable is in *positive* (*negative*) position if it occurs under an even (odd) number of $\star$ symbols. A schematic structure is *positive* (resp. *negative*) in a structure variable if every occurrence of that variable is positive (negative).

LEMMA 5.15. *Let ρ be an analytic structural rule, or an analytic structural rule minus C1, of δKt containing the structural variable L. Let ρ^* be the rule obtained from ρ by uniformly substituting $\star L$ for L. Then ρ and ρ^* are equivalent.*

PROOF. One direction is trivial since every instance of ρ^* is an instance of ρ. Now suppose that we are given premise instantiations $\{s_i\}_{1 \leq i \leq n}$ of ρ. Let X be the concrete structure instantiating the structure variable L. Apply the display rules to $\{s_i\}_{1 \leq i \leq n}$ to obtain sequents $\{s'_i\}_{1 \leq i \leq n}$ where each X is replaced with $\star\star X$. Apply ρ^* to $\{s'_i\}_{1 \leq i \leq n}$. The conclusion will contain an occurrence of $\star\star X$. It now suffices to apply the display rules to rewrite $\star\star X$ as X. □

A literal has the form p or $\neg p$ where p is a propositional variable.

LEMMA 5.16 (KRACHT). *For any structure X not containing logical connectives:*

(i) $l(X\sigma)$ is equivalent to a formula constructed from literals and $\top$ using $\Diamond, \blacklozenge$ and $\wedge$.
(ii) $r(X\sigma)$ is equivalent to a formula constructed from literals and $\bot$ using $\Box, \blacksquare$ and $\vee$.

Here σ is a function from schematic structures—built from structure variables using structural connectives and constants—to concrete structures, which simply replaces distinct structure variables with distinct propositional variables.

PROOF. The result follows from writing X in a normal form. See [Kracht 1996, Lemma 14]. □

PROPOSITION 5.17. *Let $\mathcal{H}$ be an axiomatic extension of $\mathcal{H}Kt$. If $\delta Kt + \{\rho_i\}_{i\in I}$ is an analytic structural rule extension corresponding to $\mathcal{H}$, then $L(\mathcal{H}) = L(\mathcal{H}Kt + \Delta)$ where Δ is a set of primitive tense formulae.*

PROOF. The idea of the proof is the following: we first use Lemma 4.4 to compute a formula equivalent to each analytic structural rule ρ_i. Then we show that this formula can be transformed into an equivalent formula (this is the formula (8), below) satisfying the hypotheses of Lemma 5.12. It then follows from that lemma that (8) is a primitive tense formula.

To simplify the notation we consider a single rule extension so $\{\rho_i\}_{i\in I} = \{\rho\}$. If ρ contains no premise, then it can be written simply as a sequent $X \vdash \mathbf{I}$. This sequent is equivalent to the following rule for fresh structure variables L and M:

$$\frac{M \vdash L}{M, X \vdash \mathbf{I}}$$

Indeed, applying this rule to the initial sequent $\mathbf{I} \vdash \top$ we get $\mathbf{I}, X \vdash \mathbf{I}$ which is display equivalent to $X \vdash \mathbf{I}$. In the other direction, applying weakening to $X \vdash \mathbf{I}$ we get $M, X \vdash \mathbf{I}$. Note that this rule is an analytic structural rule minus C1.

Hence without loss of generality assume that ρ has at least one premise. Then due to Lemma 5.15 we may assume that ρ has the form below where *every structure variable in the rule is a-part*. Note that we have applied the display rules to the conclusion to move all structures to the antecedent.

$$\frac{\{\quad M^k \vdash Y_l^k \quad\}_{kl}}{X\left[M^k\right]_k \vdash \mathbf{I}}\ \rho$$

Although ρ might be an analytical structural rule minus C1, is easy to see that C1 is not required in the proof of Lemma 4.4. Thus using Lemma 4.4, $\delta\text{Kt} + \rho$ is a calculus corresponding to $\mathcal{H}\text{Kt} + Ax$ where Ax is the formula

$$\text{l}(X)\left[(M^k\sigma) \wedge \bigwedge_l \text{r}(Y_l^k\sigma)\right]_k \rightarrow \bot$$

Here σ is a function from schematic structures—built from structure variables using structural connectives and constants—to concrete structures, which simply replaces distinct structure variables with distinct propositional variables.

Note that $\wedge_l \text{r}(Y_l^k\sigma) = \neg\vee_l \text{l}(\star Y_l^k\sigma)$. From Lemma 5.16 we can write $\vee_l \text{l}(\star Y_l^k\sigma)$ as a formula D^k constructed from *literals* and $\top$ using $\Diamond$, $\blacklozenge$, $\wedge$ and $\vee$. Since every structure variable in ρ is a-part and each Y_l^k is an s-part structure, it follows that Y_l^k is negative in every structure variable and thus $\star Y_l^k$ is positive in every structure variable. Hence D^k is constructed from *propositional variables* and $\top$ using $\Diamond$, $\blacklozenge$, $\wedge$ and $\vee$.

Next we will show that Ax is equivalent to a formula satisfying the hypotheses of Lemma 5.12. Because $X\left[M^k\right]_k \vdash \mathbf{I}$ is the conclusion of an analytic structural rule ρ, due to C3 it contains distinct occurrences of structure variables. Because every structure variable in ρ is a-part, from Lemma 5.16 we can write $\text{l}(X\left[M^k\right]_k\sigma)$ as a formula $g(p_1, \ldots, p_N)$ constructed from distinct propositional variables $\{M^1\sigma, \ldots, M^\mu\sigma, q^1, \ldots, q^\nu\}$ and $\top$ using $\Diamond$, $\blacklozenge$ and $\wedge$ such that each propositional variable occurs exactly once. Here the q^i are propositional variables corresponding to structure variables not in $\{M^k\}_k$. Then the formula Ax is equivalent to the formula α_1:

$$g(M^1\sigma \wedge \neg D^1, \ldots, M^\mu\sigma \wedge \neg D^\mu, q^1 \wedge \neg\bot, \ldots, q^\nu \wedge \neg\bot) \rightarrow \bot \qquad (8)$$

In order to apply Lemma 5.12 it remains to show that each D^k has the proper form. We now show that no $D^k = \bot$. Indeed, suppose that some $D^k = \bot$ so $\wedge_l \mathrm{r}(Y_l^k\sigma) = \top$ and thus $\mathrm{r}(Y_l^k\sigma) = \top$. From Lemma 5.16 and because Y_l^k is negative in every structure variable, we can write $\mathrm{r}(Y_l^k\sigma)$ as a formula constructed from negated propositional variables and $\bot$ using $\Box, \blacksquare$ and $\vee$. Now consider the frame $(\mathbb{Z}, R)$ where $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and R is the binary relation defined by $R(n)(n+1)$ for all $n \in \mathbb{Z}$. Setting the valuation $V(p) = \mathbb{Z}$ for every propositional variable p we obtain the model $M = ((\mathbb{Z}, R), V)$. We claim that for every n: $M, n \not\models \mathrm{r}(Y_l^k\sigma)$. Certainly $M, n \not\models \bot$ and $M, n \not\models \neg p$ (base cases). For the inductive case observe that $M, n \not\models \Box A$ and $M, n \not\models \blacksquare A$ follow, respectively, from $M, n+1 \not\models A$ and $M, n-1 \not\models A$—obtained via the induction hypothesis. Finally $M, n \not\models A \vee B$ since $M, n \not\models A$ and $M, n \not\models B$ by the induction hypothesis. Since there is a model refuting $\mathrm{r}(Y_l^k\sigma)$ it follows that $\mathrm{r}(Y_l^k\sigma) \neq \top$, hence $D^k (= \neg \wedge_l \mathrm{r}(Y_l^k\sigma)) \neq \bot$.

Now from Lemma 5.12 we have that α_1 is frame-equivalent to some primitive tense formula α_2. To complete the proof, define the classes of frames $\mathcal{F}_{\alpha_1} = \{F \mid F \models \alpha_1\}$ and $\mathcal{F}_{\alpha_2} = \{F \mid F \models \alpha_2\}$. We have shown that $\mathcal{F}_{\alpha_1} = \mathcal{F}_{\alpha_2}$. Noting that α_1 and α_2 are Sahlqvist formulae [Blackburn et al. 2001], by the Sahlqvist completeness theorem we have for any formula B: $B \in L(\mathcal{H}\mathrm{Kt} + \alpha_i)$ iff $\mathcal{F}_{\alpha_i} \models B$ ($i \in \{1, 2\}$). Thus $L(\mathcal{H}\mathrm{Kt} + Ax) = L(\mathcal{H}\mathrm{Kt} + \alpha_1) = L(\mathcal{H}\mathrm{Kt} + \alpha_2)$. Since $\delta\mathrm{Kt} + \rho$ is a display calculus corresponding to $\mathcal{H}$ and $\mathcal{H}\mathrm{Kt} + Ax$, from Lemma 3.34 we have $L(\mathcal{H}) = L_{\mathbf{I}}(\delta\mathrm{Kt} + \rho) = L(\mathcal{H}\mathrm{Kt} + Ax) = L(\mathcal{H}\mathrm{Kt} + \alpha_2)$. (Indeed, it even holds that $\delta\mathrm{Kt} + \rho$ corresponds to $\mathcal{H}\mathrm{Kt} + \alpha_2$). $\Box$

Example 5.18. Consider the analytic structural rule below left. The equivalent rule where every structure variable is a-part (see Lemma 5.15) is below right.

$$\frac{P \vdash \bullet\star Q \quad P \vdash \star S \quad \star\bullet\star\mathbf{I} \vdash R}{P, \star\bullet\star Q \vdash \star\bullet\star R} \qquad \frac{P \vdash \bullet\star Q \quad P \vdash \star S \quad R \vdash \bullet\star\mathbf{I}}{P, \star\bullet\star Q, \bullet R \vdash \mathbf{I}}\,\rho$$

From Lemma 4.4, $\mathrm{Kt} + \rho$ is a calculus corresponding to $\mathcal{H}\mathrm{Kt} + Ax$ where Ax is the formula

$$((p \wedge \Box\neg q \wedge \neg s) \wedge \Diamond q \wedge \blacklozenge(r \wedge \Box\neg\top)) \to \bot$$

or equivalently $p \wedge \neg(\Diamond q \vee s) \wedge \Diamond(q \wedge \neg\bot) \wedge \blacklozenge(r \wedge \neg\Diamond\top) \to \bot$. The equivalent primitive tense formula was obtained in Example 5.13 using Lemma 5.12.

Remark 5.19. In addition to the Display Theorem I, Kracht also claimed a '**Display Theorem II**' characterising analytic structural rule extensions of the display calculus δK (obtained from δKt by deleting the rules introducing the connectives $\blacklozenge$ and $\blacksquare$) as axiomatic extensions of the basic modal logic K by *primitive modal* formulae. Here primitive modal formulae refers to the subset of primitive tense formulae in the modal language. A counterexample to Kracht's claim has been known at least as far back as [Wansing 2002] where he credits Rajeev Goré. We note that the calculus δK is not amenable because there do not exist functions l and r satisfying Definition 3.1. Nevertheless we can obtain a display calculus for any acyclic $\mathcal{I}_2(\delta\mathrm{Kt})$ modal axiomatic extension $\mathcal{H}K + A$ if we know that $L(\mathcal{H}\mathrm{Kt} + A)$ is conservative over $L(\mathcal{H}K + A)$ (see Theorem 3.43). In the case that A is a Sahlqvist axiom, conservativity is a direct consequence of the Sahlqvist completeness theorem and the fact that Kt and K share the same frame semantics.

6. SUMMARY AND OPEN PROBLEMS

Given any display calculus satisfying a few (purely syntactic) properties, we introduced an algorithm for transforming large classes of Hilbert axioms into structural rules satisfying Belnap's conditions. The converse direction (from structural rules to axioms) is also shown, thus characterising the class of axioms that can be captured by structural display rules; this class (*acyclic $\mathcal{I}_2$ axioms*) turns out to be a function of the invertible logical rules of the chosen base calculus. Checking if an axiom belongs to this class or not is shown to be decidable.

Our work is a concrete step towards the automated construction of analytic display calculi. This work can be developed in several directions, among them:

Investigating the expressive power of logical rules in cut-eliminable display calculi; the case study of tense logics shows that these rules can formalize Hilbert axioms that cannot be captured by analytic structural rules. As proved by Kracht (and seen in the above section), the latter capture exactly the primitive tense formulae. It is easily verified that every primitive tense formula is a Sahlqvist formula [Blackburn et al. 2001]. On the other hand, extension of the logical calculus by logical rules can capture axioms that are *not* equivalent to Sahlqvist formulae and hence are not primitive tense axioms. An example is provided by the display calculus for provability logic GL [Demri and Goré 2002], obtained by the addition of a logical rule to δK; it is well-known that this logic cannot be axiomatised using Sahlqvist formulae. It would be interesting to develop methods for introducing logical rules preserving cut-elimination and characterising their expressive power. This problem has been already considered [Lellmann and Pattinson 2013; Lellmann 2014] in the context of sequent and hypersequent rules for modal logics.

It would also be interesting to develop (syntactic or semantic) characterisations of acyclic $\mathcal{I}_2$ axioms for specific families of calculi/logics.

The case study of tense logics provides an example of such a syntactic characterization: acyclic $\mathcal{I}_2(\delta\text{Kt})$ axioms coincide with primitive tense formulae (Definition 5.5). For the extensions of the display calculus δHB for Bi-intuitionistic logic (see Example 3.38) we conjecture that all $\mathcal{I}_2(\delta\text{HB})$ axioms are acyclic.

A semantic characterization of acyclicity for the Hilbert axioms in the class $\mathcal{N}_2$ (cf. Section 3.3) that can be captured by structural *sequent calculus* rules is contained in [Ciabattoni et al. 2012]; there, by interweaving proof theoretic and algebraic arguments starting with the observation that axioms over full Lambek calculus FL are precisely algebraic equations over residuated lattices, it is shown that acyclicity is equivalent to the closure under the Dedekind-MacNeille completions for the corresponding varieties of residuated lattices. A similar characterisation of acyclicity for $\mathcal{I}_2(\delta$ Bi-FL) axioms (see Example 2.10) is not yet available.

Incidentally, the steps in our procedure (the usage of the display rules—viewed as residuation properties of the logic—and invertible rules, Ackermann's lemma and the argument from semi-structural rules to structural rules) play a crucial role in the ALBA[7] algorithm [Conradie and Palmigiano 2012] for correspondence theory, although these steps are not explicitly identified there.

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APPENDIX

The hypersequent calculus HLJ is presented below. Note that all hypersequents there are single-conclusioned (Π and Π' are schematic variables to be replaced by the empty set or a single formula).

Initial sequents

$$A \vdash A \qquad \bot \vdash A$$

Cut Rule

$$\frac{G \mid \Gamma' \vdash A \quad G \mid A, \Gamma \vdash \Pi}{G \mid \Gamma, \Gamma' \vdash \Pi}\ (cut)$$

External Structural Rules

$$\frac{G}{G \mid \Gamma \vdash \Pi}\ (ew) \qquad \frac{G \mid \Gamma \vdash \Pi \mid \Gamma \vdash \Pi}{G \mid \Gamma \vdash \Pi}\ (ec) \qquad \frac{G \mid \Gamma' \vdash \Pi' \mid \Gamma \vdash \Pi \mid G'}{G \mid \Gamma \vdash \Pi \mid \Gamma' \vdash \Pi' \mid G'}\ (ee)$$

[7] Ackermann Lemma Based Algorithm.

Internal Structural Rules

$$\frac{G \mid \Gamma \vdash \Pi}{G \mid \Gamma, A \vdash \Pi}\ (w,l) \qquad \frac{G \mid \Gamma \vdash}{G \mid \Gamma \vdash \Pi}\ (w,r)$$

$$\frac{G \mid \Gamma, A, A \vdash \Pi}{G \mid \Gamma, A \vdash \Pi}\ (c,l) \qquad \frac{G \mid \Gamma, B, A, \Delta \vdash \Pi}{G \mid \Gamma, A, B, \Delta \vdash \Pi}\ (e,l)$$

Logical Rules

$$\frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \to B}\ (\to,r) \qquad \frac{G \mid \Gamma \vdash A \quad G \mid B, \Gamma \vdash \Pi}{G \mid \Gamma, A \to B \vdash \Pi}\ (\to,l)$$

$$\frac{G \mid \Gamma \vdash A \quad G \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \wedge B}\ (\wedge,r) \qquad \frac{G \mid \Gamma, A, B \vdash \Pi}{G \mid \Gamma, A \wedge B \vdash \Pi}\ (\wedge,l)$$

$$\frac{G \mid \Gamma \vdash A_i}{G \mid \Gamma \vdash A_1 \vee A_2}\ (\vee_i,r)_{i=1,2} \qquad \frac{G \mid \Gamma, A \vdash C \quad G \mid \Gamma, B \vdash \Pi}{G \mid \Gamma, A \vee B \vdash \Pi}\ (\vee,l)$$

$$\frac{G \mid \Gamma, A \vdash}{G \mid \Gamma \vdash \neg A}\ (\neg,r) \qquad \frac{G \mid \Gamma \vdash A}{G \mid \Gamma, \neg A \vdash}\ (\neg,l)$$

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