Density Elimination [1]

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Abstract

Density elimination, a close relative of cut elimination, consists of removing applications of the Takeuti-Titani density rule from derivations in Gentzen-style (hypersequent) calculi. Its most important use is as a crucial step in establishing standard completeness for syntactic presentations of fuzzy logics; that is, completeness with respect to algebras based on the real unit interval $[0,1]$. This paper introduces the method of density elimination by substitutions. For general classes of (first-order) hypersequent calculi, it is shown that density elimination by substitutions is guaranteed by known sufficient conditions for cut elimination. These results provide the basis for uniform characterizations of calculi complete with respect to densely and linearly ordered algebras. Standard completeness follows for many first-order fuzzy logics using a Dedekind-MacNeille-style completion and embedding.

Key words: Hypersequents, sequents, cut elimination, density elimination, fuzzy logics.

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1 Introduction

Elimination of the cut rule is a fundamental topic in Proof Theory, corresponding to the removal of lemmas from proofs. However, the addition and elimination of other rules also merit investigation. In this paper we consider one such rule, important in the meta-mathematics of Fuzzy Logic: the so-called "density rule" of Takeuti and Titani [18]; formalized Hilbert-style as:

$$\frac{(A \to p) \vee (p \to B) \vee C}{(A \to B) \vee C} \text{ (density)}$$

where p is a propositional variable not occurring in A, B, or C. Ignoring C, this can be read contrapositively as saying (very roughly) "if $A > B$, then $A > p$ and $p > B$ for some p"; hence the name "density".

Adding (density) to Classical Logic leads to inconsistency. Just take A to be $\top$ and B to be $\bot$: the premise is classically equivalent to $\top$ and the conclusion to an arbitrary C. However, for other logics the addition of (density) can be useful, or even "admissible" in the sense that it provides no new theorems. In particular, although the density rule was used by Takeuti and Titani to axiomatize Intuitionistic Fuzzy Logic [18] (better known as first-order Gödel Logic), alternative axiomatizations [13, 17] show that it is redundant. More generally, (density) is a useful tool for axiomatizing fuzzy logics defined via the t-norm based approach of Hájek [11]. Indeed it is shown in [15] that *any* axiomatic extension of the elementary propositional fuzzy logic Uninorm Logic extended with (density) is complete with respect to a corresponding class of linearly and densely ordered algebras. So-called "standard completeness" with respect to algebras with lattice reduct $[0, 1]$ can then be obtained in many cases by means of a Dedekind-MacNeille-style completion.

Density elimination provides a method for showing that (density) is unnecessary in these axiomatizations, and hence for establishing standard completeness for the original systems. This general approach contrasts with more logic-specific algebraic techniques for proving standard completeness, e.g. [11, 14, 9, 16], which encounter problems for logics lacking weakening theorems $A \to (B \to A)$. The first "syntactic elimination" of (density) was provided for first-order Gödel Logic by Baaz and Zach [4] in the framework of *hypersequents*; a generalization of Gentzen sequents to multisets of sequents introduced by Avron in [1]. The elimination method follows the spirit of Gentzen's cut elimination, proceeding by induction on the height of a derivation of the premise and shifting applications of the rule upwards. This approach was extended in [15] to several other propositional logics using calculus-tailored generalizations of the density rule (as in Gentzen's "mix" rule). However, these generalized density rules are of a combinatorial nature and are particularly complicated for logics without weakening.

In this paper we introduce a new method, *density elimination by substitutions*, in which (similarly to normalization for Natural Deduction systems) applications of the density rule are removed from derivations by making suitable substitutions for the new propositional variables. This leads to elegant and uniform density elimination proofs for broad classes of hypersequent calculi and avoids the combinatorial difficulties of the Gentzen-style proofs in [4, 15]. In particular, we show that density elimination by substitutions succeeds for single-conclusion hypersequent calculi with weakening rules that satisfy conditions defined for cut elimination in [5]. We also adapt the method to deal with calculi without weakening rules and show that the same syntactic criteria guarantee density elimination when extended with a further condition. In particular, we obtain uniform density elimination proofs for classes of calculi extending those for first-order Uninorm Logic [15] and first-order Monoidal t-Norm Logic [10, 16, 3].

We also consider the primary application of density elimination. Generalizing the approach of [15] (in particular, to the first-order level), we show that calculi admitting density elimination and some further natural properties are complete with respect to linearly and densely ordered algebras. It follows that Gentzen systems and axiomatizations for many first-order fuzzy logics are complete in this respect. Finally, standard completeness is established for systems for several fuzzy logics including first-order Uninorm Logic and first-order Monoidal t-Norm Logic using a Dedekind-MacNeille-style completion and embedding. [1]

2 Sequent and Hypersequent Calculi

We begin with some preliminary definitions. A *(first-order) (countable) language* $\mathcal{L}$ consists of countable sets of (term) variables $\mathrm{X}_{\mathcal{L}}$; function symbols $\mathrm{F}_{\mathcal{L}}$; predicate symbols $\mathrm{P}_{\mathcal{L}}$; and connectives $\mathrm{C}_{\mathcal{L}}$ with given arities. *$\mathcal{L}$-terms* are constructed as usual from variables and function symbols, while *atomic $\mathcal{L}$-formulas* are constructed from predicate symbols and terms. *$\mathcal{L}$-Formulas* are either atomic or of the form $\star(\vec{A})$ for an m-ary connective $\star \in \mathrm{C}_{\mathcal{L}}$ where $\vec{A} \equiv A_1, \ldots, A_m$, or $\mathcal{Q}xA$ with $\mathcal{Q} \in \{\forall, \exists\}$. We distinguish syntactically between free and bound variables, using a or b to denote the former, and x or y the latter, and recall that an *$\mathcal{L}$-sentence* is an $\mathcal{L}$-formula with no free variables. For convenience we call nullary function symbols, *constants*, and nullary predicate symbols, *propositional variables*. Finally, we define $|A|$ as the number of occurrences of connectives and quantifiers in A.

We indicate with $\Gamma, \Delta, \Pi, \Sigma$ (possibly empty) multisets of formulas $[A_1, \ldots, A_n]$, writing $\Gamma \uplus \Delta$ or sometimes Γ, Δ for the multiset sum of Γ and Δ, letting Γ, A denote $\Gamma \uplus [A]$ for a formula A. We write Γ^n for $\Gamma \uplus \ldots \uplus \Gamma$ (n times) for $n \in \mathbb{N}$ where $\Gamma^0 = []$, and A^n for the multiset $[A]^n$.

[1] An earlier version of this paper introducing density elimination by substitutions for logics with weakening appeared as [7].

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot) \quad \frac{}{\Gamma \Rightarrow \top}\ (\top) \quad \frac{\Gamma, A, A \Rightarrow}{\Gamma, A \Rightarrow}\ (\text{wc}) \quad \frac{\Gamma, \Pi_1^n \Rightarrow \Delta \ \ldots \ \Gamma, \Pi_{n-1}^n \Rightarrow \Delta}{\Gamma, \Pi_1, \ldots, \Pi_{n-1} \Rightarrow \Delta}\ (\text{c}_n)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\text{wl}) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (\text{wr}) \quad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\text{cl}) \quad \frac{\Gamma \Rightarrow \quad \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{mix})$$

Fig. 1. Additional Sequent Rules

2.1 Sequents

A *(single-conclusion) sequent* S in the language $\mathcal{L}$ is an ordered pair consisting of a finite multiset of (antecedent) formulas Γ and a multiset Δ containing at most one (consequent) formula, written $\Gamma \Rightarrow \Delta$.

Defining sequents using multisets rather than sequences or sets ensures that the multiplicity but not the order of formulas matters. Note also that we explicitly define sequents as single-conclusion (multiple-conclusion allows Δ to be a finite multiset), often writing just A for $\Delta = [A]$ and an empty space for $\Delta = []$.

A *sequent rule* is a set of *(sequent rule) instances*: ordered pairs consisting of a finite set of sequents $S_1, \ldots, S_n$ called *premises* and a sequent S called the *conclusion*, written $S_1, \ldots, S_n \ / \ S$ or $\frac{S_1 \ \ldots \ S_n}{S}$. Instances with no premises are called *axioms*. We call a sequent rule *schematic* if it is presented using only multiset variables and propositional formulas built from formula variables. The instances of such a rule are obtained as usual by uniformly replacing the multiset variables by arbitrary multisets of formulas and the formula variables by arbitrary formulas.

In general, a sequent calculus is just any set of sequent rules. Here we are a bit more specific, however. We consider calculi with a basic stock of axioms, a cut rule, and two sets of schematic rules – structural and logical – obeying some natural restrictions. A paradigmatic example of such a calculus is presented in Appendix A: a multiset version (with exchange internalized) of $\forall\mathsf{FL_e}$, the first-order Full Lambek Calculus with Exchange. Further sequent calculi are obtained from this calculus by adding rules such as those in Fig. 1. In particular, adding the axiom schema $(\top)$ and $(\bot)$ gives a calculus $\forall\mathsf{FL_e^\bot}$ (bounded $\forall\mathsf{FL_e}$) for first-order MAILL (Multiplicative Additive Intuitionistic Linear Logic). Adding the weakening rules (wl) and (wr) to $\forall\mathsf{FL_e}$ gives $\forall\mathsf{FL_{ew}}$, a calculus for first-order AMAILL (Affine MAILL), and extending $\forall\mathsf{FL_{ew}}$ with the contraction rule (cl) gives a calculus $\forall\mathsf{FL_{ewc}}$ for first-order Intuitionistic Logic. We will refer to these calculi (collected in Figure 2) and their rules in the definitions below and throughout the paper.

Definition 1 *A* simple sequent calculus L *consists of:*

(1) a stock (id) of basic axioms of the form $A \Rightarrow A$.

Label	Rules	Logic
$\forall\mathsf{FL_e}$	Appendix A	First-Order Full Lambek Calculus with Exchange
$\forall\mathsf{FL_e^\bot}$	$\forall\mathsf{FL_e} + (\bot) + (\top)$ of Fig. 1	First-Order MAILL
$\forall\mathsf{FL_{ew}}$	$\forall\mathsf{FL_e}$ + (wl) + (wr) of Fig. 1	First-Order AMAILL
$\forall\mathsf{FL_{ewc}}$	$\forall\mathsf{FL_{ew}}$ + (cl) of Fig. 1	First-Order Intuitionistic Logic

Fig. 2. Sequent Calculi Reference Chart

(2) a set of schematic structural rules *each satisfying the* local subformula property*: any formula appearing on the left (right) in the premise of a rule instance should occur as a subformula on the left (right) in its conclusion.*

(3) a set of schematic logical rules *consisting for each connective* $\star$ *of* left logical rules *labelled* $\{(\star\Rightarrow)_j\}_{j\in I_l^\star}$ *and* right logical rules *labelled* $\{(\Rightarrow\star)_k\}_{k\in I_r^\star}$ *for (possibly empty) finite index sets* $I_l^\star$*,* $I_r^\star$*, with instances of the form (*$n \geq 0$*):*

$$\frac{\Pi_1 \Rightarrow \Sigma_1 \quad \cdots \quad \Pi_n \Rightarrow \Sigma_n}{\Gamma, \star(\vec{A}) \Rightarrow \Delta}\ (\star\Rightarrow)_j \qquad \frac{\Pi_1 \Rightarrow \Sigma_1 \quad \cdots \quad \Pi_n \Rightarrow \Sigma_n}{\Gamma \Rightarrow \star(\vec{A})}\ (\Rightarrow\star)_k$$

where:

(i) $\star(\vec{A})$ *is called the* principal formula *of the rule instance.*

(ii) Π_i *and* Σ_i *for* $i = 1 \ldots n$ *consist of* active formulas *taken from* $\vec{A}$ *together with other* context formulas.

(iii) the rule instance obtained by removing the principal formula from the conclusion and the active formulas from the premises satisfies the local subformula property (see (2)).

(4) the (multiplicative version of the) cut rule*, with instances:*

$$\frac{\Gamma \Rightarrow A \quad \Pi, A \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ \textit{(cut)}$$

where A *is called the* cut formula.

L *is called* w-simple *if it contains the weakening rules (wl) and (wr) of Fig. 1.*

Definition 2 *A* first-order (w-)simple sequent calculus *is a (w-)simple sequent calculus plus the quantifier rules* $(\forall\Rightarrow)$, $(\Rightarrow\forall)$, $(\exists\Rightarrow)$*, and* $(\Rightarrow\exists)$ *of Appendix A.*

A *derivation* d of a sequent S from sequents $S_1, \ldots, S_n$ in a sequent calculus L is a labelled tree with the root labelled by S, and for each node labelled S' with parent nodes labelled $S'_1, \ldots, S'_m$ (where possibly $m = 0$), $S'_1, \ldots, S'_m \;/\; S'$ is an instance of a rule of L. In this case, we write:

$$d, S_1, \ldots, S_n \vdash_{\mathsf{L}} S$$

$$\frac{\mathcal{G}}{\mathcal{G} \mid S}\ (\text{ew}) \qquad \frac{\mathcal{G} \mid S \mid S}{\mathcal{G} \mid S}\ (\text{ec}) \qquad \frac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow \Delta_2}\ (\text{com})$$

Fig. 3. Additional Hypersequent Rules

or $S_1, \ldots, S_n \vdash_{\mathsf{L}} S$ to denote just that there exists such a derivation. The *height* $|d|$ of the derivation is the height of the labelled tree.

2.2 Hypersequents

A *hypersequent* $\mathcal{G}$ is a finite multiset of sequents (called the *components* of $\mathcal{G}$) [1]:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

Since sequents are assumed to be single-conclusion, hypersequents are likewise single-conclusion.

Notions of rules, rule instances, derivations, and so on, defined for sequents and sequent calculi transfer unscathed to hypersequents and hypersequent calculi: just replace all mention of sequents with hypersequents. Moreover, the hypersequent calculi that we are interested in arise uniformly from simple sequent calculi via a simple transfer principle. First we take *hypersequent versions* of sequent rules, obtained intuitively by adding a "side hypersequent" $\mathcal{G}$ to both the premises and the conclusion. For example, the hypersequent versions of the implication rules in Appendix A are:

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta \quad \mathcal{G} \mid \Pi \Rightarrow A}{\mathcal{G} \mid \Gamma, \Pi, A \to B \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \to B}\ (\Rightarrow\to)$$

More precisely, the *hypersequent version* of a sequent rule consists of all instances $(\mathcal{G} \mid S_1), \ldots, (\mathcal{G} \mid S_n)\ /\ (\mathcal{G} \mid S)$ for any hypersequent $\mathcal{G}$ and instance $S_1, \ldots, S_n\ /\ S$ of the sequent rule, calling $S_1, \ldots, S_n, S$ the *active components* of the instance. We qualify the hypersequent versions of the quantifier rules $(\exists \Rightarrow)$ and $(\Rightarrow \forall)$ (for convenience, without introducing new terminology) by extending the eigenvariable condition that the variable a in the premise does not occur in the conclusion to the side-hypersequent $\mathcal{G}$. The hypersequent version of a (first-order) simple sequent calculus L consists of the hypersequent versions of the rules of L.

Taking hypersequent versions alone is not enough to obtain calculi for new logics, however. We require further "external" structural rules that operate on components of the hypersequent. The external weakening and contraction rules (ew) and (ec) of Fig. 3 add and contract components respectively while the key rule to deal with the prelinearity axiom schema $(A \to B) \vee (B \to A)$ is Avron's "communication" rule (com) which permits interaction between components [2].

L	HL^{C} (the hypersequent version of L plus (ec), (ew), and (com) of Fig. 3)
$\forall\mathsf{FL}_{\mathsf{e}}^{\perp}$	First-Order Uninorm Logic
$\forall\mathsf{FL}_{\mathsf{ew}}$	First-Order Monoidal t-Norm Logic
$\forall\mathsf{FL}_{\mathsf{ew}}$ + (wc)	First-Order Strict Monoidal t-Norm Logic
$\forall\mathsf{FL}_{\mathsf{ew}}$ + (c_n)	First-Order n-Contractive Monoidal t-Norm Logic
$\forall\mathsf{FL}_{\mathsf{ewc}}$	First-Order Gödel Logic

Fig. 4. Hypersequent Calculi Reference Chart

Following [5], we formulate our transfer principle as follows.

Definition 3 *Let* L *be any (first-order) simple sequent calculus. Then* HL^{C} *is the hypersequent version of* L *plus (ec), (ew), and (com) of Fig. 3.*

For example, $\mathsf{H}\forall\mathsf{FL}_{\mathsf{ewc}}^{\mathsf{C}}$ is a hypersequent calculus for first-order Gödel Logic [2, 4]. Removing the contraction rule (cl) gives a calculus $\mathsf{H}\forall\mathsf{FL}_{\mathsf{ew}}^{\mathsf{C}}$ for first-order Monoidal t-Norm Logic [10, 3], and removing also the weakening rules (wl) and (wr) gives a calculus $\mathsf{H}\forall\mathsf{FL}_{\mathsf{e}}^{\perp\mathsf{C}}$ for first-order Uninorm Logic [15]. Hypersequent calculi for first-order Strict Monoidal t-Norm Logic [10] and first-order n-Contractive Monoidal t-Norm Logic [6] are obtained by extending $\mathsf{H}\forall\mathsf{FL}_{\mathsf{ew}}^{\mathsf{C}}$ with hypersequent versions of (wc) and (c_n), respectively [5, 6]. In all of these calculi, (com) can be used to prove instances of the prelinearity axiom schema as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\text{id}) \quad \dfrac{}{B \Rightarrow B}\,(\text{id})}{A \Rightarrow B \mid B \Rightarrow A}\,(\text{com})}{A \Rightarrow B \mid \Rightarrow B \to A}\,(\Rightarrow\to)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\,(\Rightarrow\to)}{\Rightarrow A \to B \mid \Rightarrow (A \to B) \vee (B \to A)}\,(\Rightarrow\vee)_2}{\Rightarrow (A \to B) \vee (B \to A) \mid \Rightarrow (A \to B) \vee (B \to A)}\,(\Rightarrow\vee)_1}{\Rightarrow (A \to B) \vee (B \to A)}\,(\text{ec})$$

For convenience, we collect the definitions of the various hypersequent calculi mentioned above in Fig. 4.

2.3 Criteria for Cut-Elimination

Syntactic criteria for preserving cut elimination when a sequent calculus L is “lifted” to HL^{C} were introduced in [5]. Intuitively, it should be possible (a) to shift applications of (cut) upwards over the premises of rule instances where the cut formula is not principal (rules are *substitutive*), and (b) to replace applications of (cut) in which the cut formula is principal in both premises by applications of (cut) with smaller cut formulas (logical rules are *reductive*). These notions are formalized for the underlying simple sequent calculus as follows.

Definition 4 *Logical rules* $\{(\Rightarrow\star)_j\}_{j\in I_l^\star}$ *and* $\{(\star\Rightarrow)_k\}_{k\in I_r^\star}$ *for* $\star$ *are* reductive *in a (first-order) simple sequent calculus* L *if for any* $j \in I_l^\star$, $k \in I_r^\star$, *and instances:*

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \cdots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma, \star(\vec{A}) \Rightarrow \Delta}\ (\star\Rightarrow)_j \qquad \frac{\Gamma'_1 \Rightarrow \Delta'_1 \quad \cdots \quad \Gamma'_m \Rightarrow \Delta'_m}{\Gamma' \Rightarrow \star(\vec{A})}\ (\Rightarrow\star)_k$$

$\Gamma, \Gamma' \Rightarrow \Delta$ *is derivable from* $\{\Gamma_i \Rightarrow \Delta_i\}_{1\le i\le n}$ *and* $\{\Gamma'_i \Rightarrow \Delta'_i\}_{1\le i\le m}$ *using only (cut) with cut formulas from* $\vec{A}$ *and the structural rules of* L.

Example 5 *The logical rules of* $\forall\mathsf{FL_e}$ *are reductive in this calculus. Consider, e.g. instances of* $(\to\Rightarrow)$ *and* $(\Rightarrow\to)$*:*

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Pi \Rightarrow A}{\Gamma, \Pi, A \to B \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \textit{and} \qquad \frac{\Sigma, A \Rightarrow B}{\Sigma \Rightarrow A \to B}\ (\Rightarrow\to)$$

Then we have the required derivation:

$$\frac{\Gamma, B \Rightarrow \Delta \qquad \dfrac{\Sigma, A \Rightarrow B \quad \Pi \Rightarrow A}{\Pi, \Sigma \Rightarrow B}\ \textit{(cut)}}{\Gamma, \Pi, \Sigma \Rightarrow \Delta}\ \textit{(cut)}$$

To deal with shifting applications of (cut) upwards, we require some way of indicating a particular formula in hypersequents; either the cut formula or the principal formula of some rule. A *marked hypersequent* $\mathcal{G}$ is a hypersequent with one occurrence of a formula A distinguished, written $\mathcal{G}' \mid \Gamma, \underline{A} \Rightarrow \Delta$ or $\mathcal{G}' \mid \Pi \Rightarrow \underline{A}$. A *marked instance* of a rule is an instance with the occurrence of the principal formula marked, if there is one. We will assume that all notions pertaining to usual hypersequents also apply in the same way to marked hypersequents.

It is now straightforward to define the result of multiple applications of (cut) with one fixed premise. We mark the cut formula in the fixed premise, while in the other premise, a marked hypersequent indicates a formula not to be used in the applications of (cut).

For a (marked or unmarked) hypersequent $\mathcal{G}$ and a marked hypersequent $\mathcal{H}$, we define $\mathsf{CUT}(\mathcal{G}, \mathcal{H})$ as the smallest set satisfying:

1. $(\mathcal{H}' \mid \mathcal{G}) \in \mathsf{CUT}(\mathcal{G}, \mathcal{H})$ where $\mathcal{H} = (\mathcal{H}' \mid \Pi \Rightarrow \underline{A})$ or $\mathcal{H} = (\mathcal{H}' \mid \Pi, \underline{A} \Rightarrow \Delta)$.

2. $(\mathcal{G}' \mid \Gamma, \Pi \Rightarrow \Delta) \in \mathsf{CUT}(\mathcal{G}, \mathcal{H})$ if:

$$\text{either } (\mathcal{G}' \mid \Gamma, A \Rightarrow \Delta) \in \mathsf{CUT}(\mathcal{G}, \mathcal{H}) \text{ and } \mathcal{H} = (\mathcal{H}' \mid \Pi \Rightarrow \underline{A})$$

$$\text{or } (\mathcal{G}' \mid \Gamma \Rightarrow A) \in \mathsf{CUT}(\mathcal{G}, \mathcal{H}) \text{ and } \mathcal{H} = (\mathcal{H}' \mid \Pi, \underline{A} \Rightarrow \Delta)$$

noting that the occurrence of A in $\mathcal{G}' \mid \Gamma, A \Rightarrow \Delta$ or $\mathcal{G}' \mid \Gamma \Rightarrow A$ is unmarked.

Notice that for a sequent $S = (\Gamma, A^n \Rightarrow \Delta)$ and marked sequent $S' = (\Pi \Rightarrow \underline{A})$ where A does not occur unmarked in Γ, the set $\mathsf{CUT}(S, S')$ consists of all sequents of the form $\Gamma, A^{n-k}, \Pi^k \Rightarrow \Delta$ for $k = 1 \dots n$. For $S = (\Gamma \Rightarrow A)$ and $S' = (\Pi, \underline{A} \Rightarrow \Sigma)$, the only member of $\mathsf{CUT}(S, S')$ is $\Gamma, \Pi \Rightarrow \Sigma$.

Definition 6 *A rule (r) is* substitutive *if for any marked instance $S_1, \dots, S_n \ / \ S$ of (r), marked sequent S' and $S'' \in \mathsf{CUT}(S, S')$:*

$S'_1, \dots, S'_n \ / \ S''$ is an instance of (r) for some $S'_i \in \mathsf{CUT}(S_i, S')$ for $i = 1 \dots n$

The logical rules of $\forall\mathsf{FL_e}$ are substitutive. However, for the standard weakening and contraction rules of this calculus, we require a slightly weaker condition.

Definition 7 *A rule (r) is* weakly substitutive *in a sequent calculus L if for any marked instance $S_1, \dots, S_n \ / \ S$ of (r), marked sequent S', and $S'' \in \mathsf{CUT}(S, S')$:*

S'' is derivable from $S'_1, \dots, S'_n$ for some $S'_i \in \mathsf{CUT}(S_i, S')$ for $i = 1 \dots n$ using only the structural rules of L and (r).

Example 8 *The rules (wl), (wr), (cl), (wc), (mix), and (c_n) ($n \geq 2$) of Fig. 1 are all weakly substitutive in $\forall\mathsf{FL_e}$. E.g., for (mix), suppose that we have an instance:*

$$\frac{\Gamma, A^k \Rightarrow \quad \Pi, A^{n-k} \Rightarrow \Delta}{\Gamma, \Pi, A^n \Rightarrow \Delta}$$

Consider $S' = \Sigma \Rightarrow \underline{A}$ and $S'' = \Gamma, \Pi, \Sigma^m, A^{n-m} \Rightarrow \Delta$. We obtain the following derivations for $k \leq m$ (left) and $k > m$ (right):

$$\frac{\Gamma, \Sigma^k \Rightarrow \quad \Pi, \Sigma^{m-k}, A^{n-m} \Rightarrow \Delta}{\Gamma, \Pi, \Sigma^m, A^{n-m} \Rightarrow \Delta} \qquad \frac{\Gamma, \Sigma^m, A^{k-m} \Rightarrow \quad \Pi, A^{n-k} \Rightarrow \Delta}{\Gamma, \Pi, \Sigma^m, A^{n-m} \Rightarrow \Delta}$$

However, the following variant of (mix) is not weakly substitutive in $\forall\mathsf{FL_e}$:

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (mix)'$$

To see why, consider an instance where $\Delta = [A]$ and cut the conclusion with $\Sigma, A \Rightarrow$: it is not possible to derive $\Gamma, \Pi, \Sigma \Rightarrow$ from $\Gamma, \Sigma \Rightarrow$ and $\Pi, \Sigma \Rightarrow$ in $\forall\mathsf{FL_e}$ +

(mix)′. On the other hand, this rule is *weakly substitutive in* $\forall\mathsf{FL}_{\mathsf{ewc}}$*: in this case we can apply (cl) repeatedly to obtain the required derivations.*

Definition 9 *A (first-order) simple calculus* L *is called* reductive *if it has reductive logical rules, and* substitutive *if it has substitutive logical rules and weakly substitutive structural rules.*

Let us now assume that L is any (first-order) simple reductive and substitutive sequent calculus. We will show (following the proof of [5]) that the transferred hypersequent calculus HL^{C} admits cut elimination. First, we state a technical lemma asserting the "substitutivity" of calculi with substitutive rules, easily proved by induction on the height of a derivation. Let $d(t)$ and $\mathcal{G}(t)$ denote the results of substituting the term t for all free occurrences of a in the derivation $d(a)$ and hypersequent $\mathcal{G}(a)$, respectively.

Lemma 10 *If* $d_1(a), \mathcal{G}_1(a), \ldots, \mathcal{G}_n(a) \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}(a)$ *and* t *is a term whose variables are all free and do not occur in* $d_1(a)$*, then* $d_1(t), \mathcal{G}_1(t), \ldots, \mathcal{G}_n(t) \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}(t)$*.*

Theorem 11 HL^{C} *admits cut elimination.*

Proof It is sufficient to show that an "uppermost" application of (cut) in any HL^{C}-derivation can be eliminated without introducing new applications of (cut). Hence, letting $\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}$ be HL^{C} without (cut), we prove the following:

Claim: For any hypersequent $\mathcal{G}$ and hypersequent $\mathcal{H}$ with marked formula A:

$$\text{if } d_{\mathcal{G}} \vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{G} \text{ and } d_{\mathcal{H}} \vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{H}, \text{ then } \vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{G}' \text{ for all } \mathcal{G}' \in \mathsf{CUT}(\mathcal{G}, \mathcal{H})$$

Note that using Lemma 10 we can assume without loss of generality that any free variables other than those in A that occur in $d_{\mathcal{G}}$ and $d_{\mathcal{H}}$ are distinct. We prove the claim by induction on the lexicographically ordered triple:

$$\langle |A|, e(d_{\mathcal{H}}), |d_{\mathcal{G}}| \rangle$$

$$\text{where} \quad e(d) = \begin{cases} 0 \text{ if } d \text{ ends with a logical rule applied to a marked formula} \\ 1 \text{ otherwise} \end{cases}$$

We begin by considering the last application of a rule (r) in $d_{\mathcal{G}}$. If (r) is (id), then a member of $\mathsf{CUT}(\mathcal{G}, \mathcal{H})$ is either $\mathcal{H}$ or of the form $\mathcal{H}' \mid \mathcal{G}$: the claim follows immediately for the former and by (ew) for the latter. If (r) is (ec), (ew), or (com), then the claim follows by applying the induction hypothesis followed by (r).

Otherwise, (r) contains only one active component in its premises and conclusion. We distinguish two cases:

(a) The application of (r) is of the form:

$$\frac{\mathcal{G} \mid S_1 \quad \dots \quad \mathcal{G} \mid S_n}{\mathcal{G} \mid S_0}$$

and the principal formula (if there is one) is *not* A on the opposite side to the marked occurrence in $\mathcal{H}$. Let $\mathcal{H} = \mathcal{H} \mid S$, where S is of the appropriate form: $\Pi \Rightarrow \underline{A}$ or $\Pi, \underline{A} \Rightarrow \Sigma$. Pick $\mathcal{G}' \mid S_0' \in \mathsf{CUT}(\mathcal{G} \mid S_0, \mathcal{H})$. If (r) is a quantifier rule, then the claim follows by the induction hypothesis and an application of (r), using Lemma 10 to take care of renaming variables when needed. Otherwise, (r) is (weakly) substitutive and there exists a derivation $d', S_1', \dots, S_n' \vdash_{\mathsf{L}} S_0'$, with $S_i' \in \mathsf{CUT}(S_i, S)$ for $i = 0, \dots, n$, that uses only the structural rules of L and (r). By the induction hypothesis $\vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{G}''$ for all $\mathcal{G}'' \in \mathsf{CUT}(\mathcal{G} \mid S_i, \mathcal{H})$ for $i = 1 \dots n$. The claim follows by lifting the derivation d' to hypersequents (i.e. $\mathcal{G}' \mid S_1', \dots, \mathcal{G}' \mid S_n' \vdash_{\mathsf{L}} \mathcal{G}' \mid S_0'$), since each $\mathcal{G}' \mid S_i' \in \mathsf{CUT}(\mathcal{G} \mid S_i, \mathcal{H})$.

(b) (r) is a logical or a quantifier rule whose application is of the form:

$$\frac{\mathcal{G}_1 \quad \dots \quad \mathcal{G}_m}{\mathcal{G}' \mid \Gamma, A^n \Rightarrow \Delta} \qquad \text{or} \qquad \frac{\mathcal{G}_1 \quad \dots \quad \mathcal{G}_m}{\mathcal{G}' \mid \Gamma \Rightarrow A}$$

where for the case on the left $A \notin \Gamma$, and in both cases A is the principal formula of the application on the opposite side to the marked occurrence in $\mathcal{H}$.

Let (r) be a logical rule and let $\mathcal{G}^{\mathcal{H}} \in \mathsf{CUT}(\mathcal{G}, \mathcal{H})$ where $\mathcal{H}$ is of the form:

$$\mathcal{H}' \mid \Pi \Rightarrow \underline{A} \qquad \text{or} \qquad \mathcal{H}' \mid \Pi, \underline{A} \Rightarrow \Sigma$$

The only tricky case (others follow as above using substitutivity) is when $\mathcal{G}^{\mathcal{H}}$ is:

$$\mathcal{G}'' \mid \Gamma, \Pi^n \Rightarrow \Delta \qquad \text{or} \qquad \mathcal{G}'' \mid \Gamma, \Pi \Rightarrow \Sigma$$

where $\mathcal{G}'' \in \mathsf{CUT}(\mathcal{G}', \mathcal{H})$. Notice that also:

$$\mathcal{G}'' \mid \Gamma, \Pi^{n-1}, A \Rightarrow \Delta \qquad \text{or} \qquad \mathcal{G}'' \mid \Gamma \Rightarrow A$$

is a member of $\mathsf{CUT}(\mathcal{G}, \mathcal{H})$. So by the substitutivity of (the logical rule) (r), there exist $\mathcal{G}_i' \in \mathsf{CUT}(\mathcal{G}_i, \mathcal{H})$ for $i = 1 \dots m$ such that:

$$\frac{\mathcal{G}_1' \quad \dots \quad \mathcal{G}_m'}{\mathcal{G}'' \mid \Gamma, \Pi^{n-1}, A \Rightarrow \Delta} \qquad \text{or} \qquad \frac{\mathcal{G}_1' \quad \dots \quad \mathcal{G}_m'}{\mathcal{G}'' \mid \Gamma \Rightarrow A}$$

is an instance of (r). Moreover, by the induction hypothesis $\vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{G}_i'$ for $i = 1 \dots m$. Hence there is a $\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}$ derivation d ending with such a rule application.

Now we consider two subcases:

(1) $e(d_{\mathcal{H}}) = 1$: i.e. $d_{\mathcal{H}}$ does not end with the application of a logical rule to the marked occurrence of A. Mark the remaining occurrence of A on the left or right as appropriate in d to give a derivation of $\mathcal{G}'' \mid \Gamma, \Pi^{n-1}, \underline{A} \Rightarrow \Delta$ or $\mathcal{G}'' \mid \Gamma \Rightarrow \underline{A}$, and remove the underlining in $d_{\mathcal{H}}$. Observe that:

$$\langle |A|, e(d), |d_{\mathcal{H}}| \rangle < \langle |A|, e(d_{\mathcal{H}}), |d_{\mathcal{G}}| \rangle$$

Hence by the induction hypothesis and further applications of (ec), $\vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{G}^{\mathcal{H}}$.

(2) $e(d_{\mathcal{H}}) = 0$: i.e. $d_{\mathcal{H}}$ ends with the application of a logical rule to the marked occurrence of $A = \star(\vec{A})$ of the form:

$$\frac{\mathcal{H}_1 \quad \ldots \quad \mathcal{H}_l}{\mathcal{H}' \mid \Pi \Rightarrow A} \quad \text{or} \quad \frac{\mathcal{H}_1 \quad \ldots \quad \mathcal{H}_l}{\mathcal{H}' \mid \Pi, A \Rightarrow \Sigma}$$

Then by reductivity (of the sequent version of (r)) and lifting to hypersequents, $\mathcal{G}^{\mathcal{H}}$ is derivable from $\mathcal{G}'_1, \ldots, \mathcal{G}'_m, \mathcal{H}_1, \ldots, \mathcal{H}_l$ with cut formulas from $\vec{A} = A_1, \ldots, A_k$. But:

$$\langle |A_i|, e(d_{\mathcal{H}}), |d| \rangle < \langle |A|, e(d_{\mathcal{H}}), |d_{\mathcal{G}}| \rangle \quad \text{for } i = 1 \ldots k$$

So by several applications of the induction hypothesis and (ec), $\vdash_{\mathsf{HL}^{\mathsf{C}}_{\mathsf{cf}}} \mathcal{G}^{\mathcal{H}}$.

Cases where (r) is a quantifier rule are very similar, except that in case (2), Lemma 10 is used to replace the new variable a in $(\Rightarrow \forall)$ or $(\exists \Rightarrow)$ with the new term t in $(\forall \Rightarrow)$ or $(\Rightarrow \exists)$. $\square$

Corollary 12 HL^{C} *has the* subformula property*; i.e. if* $\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}$*, then there exists a cut-free derivation* d *of* $\mathcal{G}$ *in* HL^{C} *such that any formula occurring in* d *is a subformula of a formula in* $\mathcal{G}$*.*

3 Density Elimination by Substitutions

We now turn our attention to the main topic of this paper: the density rule, utilized (in a different form) by Takeuti and Titani to axiomatize first-order Gödel Logic [18]. Following Baaz and Zach [4], the hypersequent version is written as follows:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta} \text{(density)}$$

where p is a propositional variable not occurring in Γ, Σ, Δ, or $\mathcal{G}$.

To gain an intuitive understanding of the rule, consider a simple instance:

$$\frac{A \Rightarrow p \mid p \Rightarrow B}{A \Rightarrow B}$$

Since p does not occur in A or B we can read the premise as universally quantified: "for all p" $A \Rightarrow p \mid p \Rightarrow B$. Now interpret $\Rightarrow$ as "$\leq$" and $\mid$ as "or". Contrapositively, the rule says "if $A > B$, then $A > p$ and $p > B$ for some p".

Adding the density rule to a hypersequent calculus can have a dramatic effect. Consider e.g. a calculus with the split rule:

$$\frac{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow} \text{ (split)}$$

Extending the calculus with (density), we are able to prove the empty sequent:

$$\dfrac{\dfrac{\overline{p \Rightarrow p} \text{ (id)}}{p \Rightarrow \mid \Rightarrow p} \text{ (split)}}{\Rightarrow} \text{ (density)}$$

If the calculus also has weakening rules (see Fig. 1), then any hypersequent is derivable: just apply (ew), (wl), and (wr) to the empty sequent and proceed as above.

However, for many calculi, adding (density) has no effect on which hypersequents are derivable: applications of (density) can be *eliminated* from derivations. In [4, 15], "Gentzen-style" (by analogy with cut elimination) elimination procedures are defined. These proceed by induction on the height of a derivation of the premise and shift applications of (density) upwards. The main difficulty, as for the corresponding cut elimination method, is the duplication of components or formulas in the derivation. For example, if the p in $\Sigma, p \Rightarrow \Delta$ in the premise of (density) is derived by internal contraction (cl), or one of the components $\Sigma, p \Rightarrow \Delta$ or $\Gamma \Rightarrow p$ is derived by external contraction (ec), the permutation of (density) with (cl) or (ec) does not necessarily move (density) higher up in the derivation. To solve this problem, ad hoc (Gentzen mix-style) rules are used that allow applications of (density) to be handled "in parallel". For example, density elimination can be established for the hypersequent calculus $\mathsf{H\forall FL^{C}_{ewc}}$ for first-order Gödel Logic using the following generalization of (density) [4]:

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow p \mid \ldots \mid \Gamma_n \Rightarrow p \mid \Sigma_1, p^* \Rightarrow \Delta_1 \mid \ldots \mid \Sigma_m, p^* \Rightarrow \Delta_m}{\mathcal{G} \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_1, \ldots, \Gamma_n, \Sigma_m \Rightarrow \Delta_m}$$

where p does not occur in the conclusion and p^* stands for any multiset of ps.

The above rule is not suitable, however, in the absence of either contraction or weakening rules. In this situation, more complicated "combinatorial" induction hypotheses are required, involving many hard-to-check cases [15].

Here we present a new method for removing applications of (density) from hypersequent derivations: *density elimination by substitutions*. Similarly to normalization for Natural Deduction systems (and the cut elimination method in [5]), topmost applications of (density) are removed by making suitable substitutions in the derivation for the new propositional variables. Proceeding "by substitutions" instead of shifting applications of (density) upwards avoids the need for complicated rules as induction hypotheses and leads to uniform density elimination proofs for large classes of calculi. In particular, we are able to show density elimination for *all* (first-order) w-simple hypersequent calculi with substitutive and reductive rules, and obtain some general (but more limited) results for calculi without weakening.

3.1 Calculi with Weakening

In this section we show that the sufficient conditions for cut elimination of Section 2.3 guarantee that HL^{C} extended with (density) admits density elimination by substitutions for any (first-order) w-simple sequent calculus L. For an intuitive view of the elimination procedure, consider a cut-free derivation d in such a calculus HL^{C} that ends in a topmost application of (density):

$$\dfrac{\begin{array}{c}\vdots\, d' \\ \mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta\end{array}}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta}\ \text{(density)}$$

Let us call sequents of the form $\Gamma, p \Rightarrow p$ *quasi-p-axioms*, reflecting the fact that such sequents are derivable in w-simple calculi using (wl) and (id). For all sequents occurring in hypersequents in d that are not quasi-p-axioms, occurrences of p are replaced in an "asymmetric" way according to whether the occurrence is on the left or the right in the sequent: with Γ if p occurs on the left, and, if p occurs on the right, with Σ on the left and Δ on the right (i.e. $\Pi \Rightarrow p$ becomes $\Pi, \Sigma \Rightarrow \Delta$). Or, to put matters another way, we perform repeated *cuts* on hypersequents occurring in d with the sequents $\Gamma \Rightarrow p$ and $\Sigma, p \Rightarrow \Delta$.

Following this replacement, the last step in the derivation becomes an application of (ec) rather than (density). However, the resulting labelled tree is no longer a derivation; it requires some further "correction" steps:

(a) applications of logical and structural rules are replaced by suitable inferences guaranteed by (weak) substitutivity.

(b) each subtree ending in an application of (com) involving a quasi-p-axiom:

$$\dfrac{\begin{array}{c}\vdots \\ \mathcal{G}' \mid \Gamma_1, \Pi_1, p^l \Rightarrow p\end{array} \qquad \begin{array}{c}\vdots \\ \mathcal{G}' \mid \Gamma_2, \Pi_2, p^{(k-l)} \Rightarrow \Delta_1\end{array}}{\mathcal{G}' \mid \Gamma_1, \Gamma_2 \Rightarrow p \mid \Pi_1, \Pi_2, p^k \Rightarrow \Delta_1}\ \text{(com)}$$

after applying the substitutions becomes:

$$\dfrac{\dfrac{\vdots}{\mathcal{G}'' \mid \Gamma_1, \Pi_1, p^l \Rightarrow p} \qquad \dfrac{\vdots}{\mathcal{G}'' \mid \Gamma_2, \Pi_2, \Gamma^{(k-l)} \Rightarrow \Delta_1}}{\mathcal{G}'' \mid \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Delta \mid \Pi_1, \Pi_2, \Gamma^k \Rightarrow \Delta_1}\,(\text{com})$$

and is then replaced using (cut) and (wl) by a derivation of the form:

$$\begin{array}{c} \vdots \\ \mathcal{G}'' \mid \Gamma_2, \Pi_2, \Gamma^{(k-l)} \Rightarrow \Delta_1 \\ \hline \\ \vdots \\ \mathcal{G}'' \mid \Gamma_1, \Gamma_2, \Sigma \Rightarrow \Delta \mid \Pi_1, \Pi_2, \Gamma^k \Rightarrow \Delta_1 \end{array}$$

We are ready to make this more formal. For any hypersequent $\mathcal{H}$, let:

$$\mathcal{H}[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$$

be the hypersequent obtained by first replacing all occurrences of p on the left in components of $\mathcal{H}$ with Γ, and then all components $\Pi \Rightarrow p$ with $\Pi, \Sigma \Rightarrow \Delta$. Also let $\mathcal{H}[A/p]$ be $\mathcal{H}$ with all occurrences of p replaced by A. The following lemma is then proved by a straightforward induction on the height of a derivation, making use of the fact that the logical and structural rules of the underlying sequent calculus are schematic and (weakly) substitutive.

Lemma 13 *Let* L *be a (first-order) simple reductive and substitutive sequent calculus and let* $\mathcal{G}$ *be a hypersequent where* p *occurs only as a propositional variable. If* $d \vdash_{\mathsf{HL^C}} \mathcal{G}$*, then* $\vdash_{\mathsf{HL^C}} \mathcal{G}[A/p]$ *for any formula* A*.*

We can now prove our main theorem, noting that for convenience, we write $(r)^*$ for an application of a rule (r) with extra applications of (ew) and (ec).

Theorem 14 *Let* L *be a (first-order) w-simple reductive and substitutive sequent calculus. Then* $\mathsf{HL^C}$ *plus (density) admits density elimination.*

Proof For technical reasons, it will be useful in the proof to mimic the "," occurring in hypersequents and its unit by suitable connectives $\odot$ and t (or different symbols, if these are already taken). To this end, notice that we can assume that L contains the (reductive and weakly substitutive) rules $(\odot \Rightarrow)$, $(\Rightarrow \odot)$, $(t \Rightarrow)$, and $(\Rightarrow t)$ of Appendix A. If not, then suppose that the theorem holds for the calculus extended with these rules. Since this extended calculus has the subformula property by Corollary 12, the theorem holds for the original calculus.

To perform density elimination it is sufficient to remove topmost applications of (density). Hence by Theorem 11 we can consider a cut-free derivation d ending:

$$\dfrac{\begin{array}{c}\vdots\, d' \\ \mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta\end{array}}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta}\ \text{(density)}$$

Claim: if $d_{\mathcal{H}}$ is a cut-free derivation in HL^{C} of a hypersequent $\mathcal{H}$ where p occurs only as a propositional variable and no component of $\mathcal{H}$ is a quasi-p-axiom, then:

$$\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid \mathcal{H}[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$$

The desired result follows easily from this claim. Just let $\mathcal{H}$ be $\mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta$. We get $\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid \mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma, \Sigma \Rightarrow \Delta$ (noting that $\mathcal{G}[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$ is just $\mathcal{G}$). So, by multiple applications of (ec), we have $\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta$.

The proof of the claim proceeds by induction on $|d_{\mathcal{H}}|$. For the base case, $\mathcal{H}$ is either of the form $\mathcal{H}' \mid B \Rightarrow B$ or is (the hypersequent version of) a substitutive logical rule with no premise. In both cases, the claim follows by applying (ew).

For the inductive step, we distinguish cases according to the last rule (r) applied in $d_{\mathcal{H}}$. If (r) is (ec), (ew) or a quantifier rule then the claim follows by applying the induction hypothesis and (r), using Lemma 10 to take care of renaming variables when needed. The remaining cases are as follows:

- Let (r) be a rule other than (ec), (ew), or (com) with an instance (since there is only one active component in the premises and conclusion):

  $$\dfrac{\mathcal{G}' \mid S_1 \ \ldots\ \mathcal{G}' \mid S_m}{\mathcal{G}' \mid S}$$

  By assumption $\mathcal{G}' \mid S$ does not contain any quasi-p-axiom. Hence by the local subformula property and the absence of cuts in d, also no $\mathcal{G}' \mid S_i$ for $i \in \{1, \ldots, m\}$ contains a quasi-p-axiom. So by the induction hypothesis:

  $$\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid (\mathcal{G}' \mid S_1)[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}] \quad \ldots \quad \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid (\mathcal{G}' \mid S_m)[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$$

  But now, since the rules of L are (weakly) substitutive and obey the local subformula property there exists a derivation for:

  $$S_1[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}], \ldots, S_m[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}] \vdash_{\mathsf{L}} S[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$$

  that uses only the structural rules of L and (r). The claim then follows by (ew), lifting the above derivation from L to HL^{C} (i.e. adding $\mathcal{G} \mid \mathcal{G}'[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$ to both the premises and conclusion).

- If (r) is (com), two cases can occur: (a) none of the premises contains a quasi-p-axiom or (b) one of the (active) premises does. For (a), the claim follows by applying the induction hypothesis and then (com). As an example, consider:

$$\dfrac{\dfrac{\vdots\, d_1}{\mathcal{G}' \mid \Gamma_1, \Pi_1 \Rightarrow p} \qquad \dfrac{\vdots\, d_2}{\mathcal{G}' \mid \Gamma_2, \Pi_2, p^k \Rightarrow \Delta_1}}{\mathcal{G}' \mid \Gamma_1, \Gamma_2, p^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow p}\ \text{(com)}$$

Let $\mathcal{G}^* = \mathcal{G}'[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$. Then by the induction hypothesis.:

$$\vdash_{\mathsf{HL^C}} \mathcal{G} \mid \mathcal{G}^* \mid \Gamma_1, \Pi_1, \Sigma \Rightarrow \Delta \quad \text{and } \vdash_{\mathsf{HL^C}} \mathcal{G} \mid \mathcal{G}^* \mid \Gamma_2, \Pi_2, \Gamma^k \Rightarrow \Delta_1$$

Hence by (com), $\vdash_{\mathsf{HL^C}} \mathcal{G} \mid \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2, \Sigma \Rightarrow \Delta$.

For (b), we have an application of (com) of the form:

$$\dfrac{\mathcal{G}' \mid \Gamma_1, \Pi_1, p^l \Rightarrow p \qquad \mathcal{G}' \mid \Gamma_2, p^{(k-l)}, \Pi_2 \Rightarrow \Delta_1}{\mathcal{G}' \mid \Gamma_1, \Gamma_2, p^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow p}\ \text{(com)}$$

Let $\mathcal{G}^* = \mathcal{G}'[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}]$. Then by the induction hypothesis:

$$d_1 \vdash_{\mathsf{HL^C}} \mathcal{G} \mid \mathcal{G}^* \mid \Gamma_2, \Gamma^{(k-l)}, \Pi_2 \Rightarrow \Delta_1$$

Our aim is to find a derivation for $\mathcal{G} \mid \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2, \Sigma \Rightarrow \Delta$.

Let $\odot[A_1, \ldots, A_m]$ stand for $A_1 \odot \ldots \odot A_m$ if $m \geq 1$ and t if $m = 0$. Consider the original derivation d' ending with the premise $\mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta$ of (density). By Lemma 13, we can substitute $\odot\Pi_2$ for p in this derivation to get:

$$d_2 \vdash_{\mathsf{HL^C}} \mathcal{G} \mid \Gamma \Rightarrow \odot\Pi_2 \mid \Sigma, \odot\Pi_2 \Rightarrow \Delta$$

Let d_3 be the (easy) derivation of $\Pi_2 \Rightarrow \odot\Pi_2$ using $(\Rightarrow \odot)$, $(\Rightarrow t)$, and (id), and let d_2' be the derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\, d_2}{\mathcal{G} \mid \Gamma \Rightarrow \odot\Pi_2 \mid \Sigma, \odot\Pi_2 \Rightarrow \Delta}}{\mathcal{G} \mid \Gamma^l \Rightarrow \odot\Pi_2 \mid \Pi_1, \odot\Pi_2, \Sigma \Rightarrow \Delta}\ \text{(wl)} \qquad \dfrac{\vdots\, d_3}{\Pi_2 \Rightarrow \odot\Pi_2}}{\mathcal{G} \mid \Gamma^l \Rightarrow \odot\Pi_2 \mid \Pi_1, \Pi_2, \Sigma \Rightarrow \Delta}\ \text{(cut)}^*$$

Now let d_1' be the derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\, d_1}{\mathcal{G} \mid \mathcal{G}^* \mid \Gamma_2, \Gamma^{(k-l)}, \Pi_2 \Rightarrow \Delta_1}}{\mathcal{G} \mid \mathcal{G}^* \mid \Gamma_2, \Gamma^{(k-l)}, \odot\Pi_2 \Rightarrow \Delta_1}\ (\odot\Rightarrow) + (t\Rightarrow)}{\mathcal{G} \mid \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{(k-l)}, \odot\Pi_2 \Rightarrow \Delta_1}\ \text{(wl)}$$

We obtain the required derivation:

$$\dfrac{\mathcal{G} \mid \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{(k-l)}, \odot\Pi_2 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma^l \Rightarrow \odot\Pi_2 \mid \Pi_1, \Pi_2, \Sigma \Rightarrow \Delta}{\mathcal{G} \mid \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2, \Sigma \Rightarrow \Delta}\ \text{(cut)}^* \qquad \square$$

For calculi without weakening rules, matters are a bit more complicated. Left as it is, the density elimination method of Section 3.1 does not work in such cases. For these calculi, quasi-p-axioms are not always derivable, and cannot therefore be removed quite so easily. A further substitution step is required. To formalize this step, we introduce the following notation. Let t be a constant and $\mathcal{G}$ any hypersequent:

$\mathcal{H}^t$ is $\mathcal{H}$ in which each component of the form $\Gamma, p \Rightarrow p$ is replaced by $\Gamma \Rightarrow t$.

The idea is to perform the asymmetric substitutions of the previous section to $\mathcal{H}^t$ rather than $\mathcal{H}$. However, to obtain an analogue of Theorem 14 for calculi without weakening, we also require a further condition:

Definition 15 *Let* L *be a (first-order) simple sequent calculus. A rule instance:*

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$$

is premise-balanced *if one of the following hold:*

(i) $\Delta = []$.
(ii) $\Gamma_i = \Gamma$ *and* $\Delta_i = \Delta$ *for* $i = 1 \dots n$ *(in particular, if* $n = 0$*).*
(iii) $\Gamma_1 \uplus \dots \uplus \Gamma_n = \Gamma$ *and* $\Delta_1 \uplus \dots \uplus \Delta_n = \Delta$.

A (first-order) simple calculus is premise-balanced if all instances of its structural rules and logical rules with the principal formula and active formulas removed are premise-balanced.

Example 16 $\forall\mathsf{FL_e}$ *(see Appendix A) is premise-balanced (the trivial-seeming condition (ii) is needed, with (iii), to ensure that instances of the logical rules with the principal formula and active formulas removed are premise-balanced). Also the structural rules (wc) and (mix) in Fig. 1 are premise-balanced: (wc) satisfies (i), while (mix) satisfies* (iii)*). However, none of the conditions (i)-(iii) hold for the contraction rules (c*$_n$*) and (cl).*

Theorem 17 *Let* L *be a (first-order) simple reductive, substitutive, and premise-balanced sequent calculus. Then* $\mathsf{HL^C}$ *plus (density) admits density elimination.*

Proof As in the proof of Theorem 14, we can assume that the calculus includes rules for $\odot$ and t. Then, as before we proceed by removing applications of (density) which are topmost. Let d be a cut-free derivation:

$$\dfrac{\begin{array}{c}\vdots\, d' \\ \mathcal{G} \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta\end{array}}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta}\ \text{(density)}$$

Claim: if $\mathrm{d}_{\mathcal{H}}$ is a cut-free derivation of a hypersequent $\mathcal{H}$ where p occurs only as a propositional variable, then:

$$\vdash_{\mathsf{HL^c}} \mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta \mid \mathcal{H}^t[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}]$$

The result follows from this claim exactly as in the proof of Theorem 14.

The proof of the claim proceeds by induction on $|d_{\mathcal{H}}|$. If $\mathcal{H}$ is an instance of (id), then for the case where $\mathcal{H}$ is $p \Rightarrow p$, we have that $\mathcal{G} \mid \Rightarrow t$ is derivable by (ew) and $(\Rightarrow t)$, and otherwise $\mathcal{H}^t$ is $\mathcal{H}$ and the result follows using (ew) and (id). We distinguish other cases according to the last rule (r) applied in $d_{\mathcal{H}}$. The cases where (r) is (ec), (ew) or a quantifier rule proceed as in Theorem 14.

Suppose that (r) is any rule instance other than (ec), (ew), or (com), without loss of generality of the form (since there is only one active component in the premises and conclusion):

$$\frac{\mathcal{G}_1 \mid S_1 \ \ldots \ \mathcal{G}_1 \mid S_m}{\mathcal{G}_1 \mid S}$$

If S is *not* a quasi-p-axiom, then we proceed as in Theorem 14. Otherwise, if at least one of $S_1 \ldots S_m$ is a quasi-p-axiom (hence (r) satisfies Conditions (ii) or (iii) of Definition 15), the claim follows by the induction hypothesis and a subsequent application of (r). Hence assume that none of $S_1 \ldots S_m$ is a quasi-p-axiom and $S = (\Pi, p^{k+1} \Rightarrow p)$ where p does not occur in Π. Let $\mathcal{H}' = (\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta)$ and observe that $S_i^t = S_i$ for $i = 1 \ldots m$. By the induction hypothesis:

$$\vdash_{\mathsf{HL^c}} \mathcal{H}' \mid (\mathcal{G}_1^t \mid S_1)[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}] \quad \ldots \quad \vdash_{\mathsf{HL^c}} \mathcal{H}' \mid (\mathcal{G}_1^t \mid S_m)[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}]$$

Using the (weak) substitutivity and the local subformula property of the rules of L we have $\vdash_{\mathsf{HL^c}} \mathcal{H}' \mid (\mathcal{G}_1^t \mid S)[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}]$; i.e.

$$\vdash_{\mathsf{HL^c}} \mathcal{H}' \mid (\mathcal{G}_1^t)[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}] \mid \Pi, \Gamma^{k+1}, \Sigma \Rightarrow \Delta$$

and we can complete the required derivation as follows:

$$\frac{\mathcal{H}' \mid (\mathcal{G}_1^t)[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}] \mid \Pi, \Gamma^{k+1}, \Sigma \Rightarrow \Delta \qquad \overline{\Rightarrow t}\ (\Rightarrow t)}{\mathcal{H}' \mid (\mathcal{G}_1^t)[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}] \mid \Pi, \Gamma^{k} \Rightarrow t \mid \Gamma, \Sigma \Rightarrow \Delta}\ (\text{com})^*$$

Now assume that (r) is (com). Two cases can occur: (a) neither of the active components in the rule conclusion contains a quasi-p-axiom; (b) at least one does.

For (a), if no active component in the premises contain a quasi-p-axiom, then the claim easily holds by applying the induction hypothesis followed by an application of (com). Otherwise, we have:

$$\frac{\begin{array}{c}\vdots\ d_1 \\ \mathcal{G}_1 \mid \Gamma_1, \Pi_1, p^{(k-l)} \Rightarrow \Delta_1\end{array} \qquad \begin{array}{c}\vdots\ d_2 \\ \mathcal{G}_1 \mid \Gamma_2, \Pi_2, p^l \Rightarrow p\end{array}}{\mathcal{G}_1 \mid \Gamma_1, \Gamma_2, p^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow p}\ (\text{com})$$

where $p \notin \Gamma_1 \uplus \Gamma_2 \uplus \Delta_1 \uplus \Pi_1 \uplus \Pi_2$. Let $\mathcal{G}^* = \mathcal{G}_1^t[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}] \mid \mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta$. Our aim is to prove:

$$\vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2, \Sigma \Rightarrow \Delta$$

starting from the derivations (obtained by the induction hypothesis):

$$d_1' \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_1, \Pi_1, \Gamma^{(k-l)} \Rightarrow \Delta_1 \quad \text{and} \quad d_2' \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_2, \Pi_2, \Gamma^{(l-1)} \Rightarrow t$$

We first apply the rule $(t \Rightarrow)$ to the end sequent of d_1', obtaining a derivation of $\mathcal{G}^* \mid \Gamma_1, \Gamma^{(k-l)}, \Pi_1, t \Rightarrow \Delta_1$. By (cut) with the end sequent of d_2':

$$d_1^* \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{k-1}, \Pi_1, \Pi_2 \Rightarrow \Delta_1$$

Now let $P = \odot(\Pi_1 \uplus \Pi_2)$, letting $P = t$ when $\Pi_1 = \Pi_2 = []$, and consider:

$$\dfrac{\mathcal{G}^* \mid \Gamma \Rightarrow P \mid \Sigma, P \Rightarrow \Delta \quad \mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{(k-1)}, P \Rightarrow \Delta_1}{\mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta_1 \mid \Sigma, P \Rightarrow \Delta}\ (\text{cut})^*$$

The left premise is derivable by (ew) and Lemma 13, and the right premise is derivable by extending d_1^* with $(t \Rightarrow)$ and $(\odot \Rightarrow)$ as necessary. The required derivation is then obtained by applying (cut)* to the conclusion and $\Pi_1, \Pi_2 \Rightarrow P$.

Now consider case (b) for (com). Assume first that just one active component in the conclusion of (com) contains a quasi-p-axiom. If one of the active components in the premise contains a quasi-p-axiom, then the claim easily follows by applying the induction hypothesis and (com). Otherwise, the application is of the form:

$$\dfrac{\begin{matrix}\vdots\, d_1 \\ \mathcal{G}_1 \mid \Gamma_1, \Pi_1 \Rightarrow p\end{matrix} \qquad \begin{matrix}\vdots\, d_2 \\ \mathcal{G}_1 \mid \Gamma_2, \Pi_2, p^{(k+j)} \Rightarrow \Delta_1\end{matrix}}{\mathcal{G}_1 \mid \Gamma_1, \Gamma_2, p^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2, p^j \Rightarrow p}\ (\text{com})$$

Let $\mathcal{G}^* = \mathcal{G}_1^t[{}^{\Gamma}/_{p^l}, {}^{\Sigma \Rightarrow \Delta}/_{p^r}] \mid \mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta$. By the induction hypothesis:

$$d_1' \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_1, \Pi_1, \Sigma \Rightarrow \Delta \quad \text{and} \quad d_2' \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_2, \Pi_2, \Gamma^{(k+j)} \Rightarrow \Delta_1$$

The required derivation can be given as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots\, d_1' \\ \mathcal{G}^* \mid \Gamma_1, \Pi_1, \Sigma \Rightarrow \Delta\end{matrix} \qquad \begin{matrix}\vdots\, d_2' \\ \mathcal{G}^* \mid \Gamma_2, \Pi_2, \Gamma^{(k+j)} \Rightarrow \Delta_1\end{matrix}}{\mathcal{G}^* \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma_1, \Gamma_2, \Gamma^{(k+j-1)}, \Pi_1, \Pi_2 \Rightarrow \Delta_1}\ (\text{com}) \qquad \dfrac{}{\Rightarrow t}\ (\Rightarrow t)}{\mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2, \Gamma^{(j-1)} \Rightarrow t}\ (\text{com})^*$$

Now, again for case (b), assume that both active components in the conclusion of (com) contain a quasi-p-axiom. If the active components in both premises also contain a quasi-p-axiom, then the claim easily follows by applying the induction hypothesis followed by an application of (com). Assume instead that only one active component in the premise of (com) contains a quasi-p-axiom, as in:

$$\dfrac{\begin{matrix}\vdots\, d_1 \\ \mathcal{G}_1 \mid \Gamma_1, \Pi_1, p^{(k+j)} \Rightarrow p\end{matrix} \qquad \begin{matrix}\vdots\, d_2 \\ \mathcal{G}_1 \mid \Gamma_1, \Pi_2 \Rightarrow p\end{matrix}}{\mathcal{G}_1 \mid \Gamma_1, \Sigma_1, p^k \Rightarrow p \mid \Pi_1, \Pi_2, p^j \Rightarrow p}\ (\text{com})$$

Let $\mathcal{G}^* = \mathcal{G}_1^t[{}^{\Gamma}/_{p^l}, {}^{\Sigma\Rightarrow\Delta}/_{p^r}] \mid \mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta$. By the induction hypothesis:

$$d_1' \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_1, \Pi_1, \Gamma^{(k+j-1)} \Rightarrow t \quad \text{and} \quad d_2' \vdash_{\mathsf{HL^C}} \mathcal{G}^* \mid \Gamma_2, \Pi_2, \Sigma \Rightarrow \Delta$$

The required derivation of $\mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{(k-1)} \Rightarrow t \mid \Pi_1, \Pi_2, \Gamma^{(j-1)} \Rightarrow t$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, d_1' \\ \mathcal{G}^* \mid \Gamma_1, \Pi_1, \Gamma^{(k+j-1)} \Rightarrow t\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, d_2' \\ \mathcal{G}^* \mid \Gamma_2, \Pi_2, \Sigma \Rightarrow \Delta\end{matrix}}{\mathcal{G}^* \mid \Gamma_2, \Pi_2, \Sigma, t \Rightarrow \Delta}\ (\text{t}\Rightarrow)}{\mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Pi_1, \Pi_2, \Gamma^{(k+j-1)}, \Sigma \Rightarrow \Delta}\ (\text{cut}) \qquad \dfrac{}{\Rightarrow t}\ (\Rightarrow\text{t})}{\mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{(k-1)} \Rightarrow t \mid \Pi_1, \Pi_2, \Gamma^{j}, \Sigma \Rightarrow \Delta}\ (\text{com})^* \qquad \dfrac{}{\Rightarrow t}\ (\Rightarrow\text{t})}{\mathcal{G}^* \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma_1, \Gamma_2, \Gamma^{(k-1)} \Rightarrow t \mid \Pi_1, \Pi_2, \Gamma^{(j-1)} \Rightarrow t}\ (\text{com})^*}{\mathcal{G}^* \mid \Gamma_1, \Gamma_2, \Gamma^{(k-1)} \Rightarrow t \mid \Pi_1, \Pi_2, \Gamma^{(j-1)} \Rightarrow t}\ (\text{ec}) \qquad \square$$

4 Standard Completeness

We turn our attention now to the main application of density elimination: establishing standard completeness for syntactic presentations of (first-order) fuzzy logics. To better explain what we mean by this, consider the following axiom system MTL for Monoidal t-Norm Logic in a language with connectives $\odot$, $\to$, $\wedge$, and f:

(A1) $(A \to B) \to ((B \to C) \to (A \to C))$

(A2) $(A \odot B) \to A$

(A3) $(A \odot B) \to (B \odot A)$

(A4) $(A \wedge B) \to A$

(A5) $(A \wedge B) \to (B \wedge A)$

(A6) $(A \odot (A \to B)) \to (A \wedge B)$

(A7) $((A \odot B) \to C) \to (A \to (B \to C))$

(A8) $(A \to (B \to C)) \to ((A \odot B) \to C)$

(A9) $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

(A10) $f \to A$

$$\frac{A \quad A \to B}{B} \text{ (mp)}$$

An alternative axiomatization is obtained by extending any axiom system for $\mathsf{FL_{ew}}$ with the axiom schema $(A9)$.

It was conjectured by Godo and Esteva in [10] that a formula A is derivable in MTL iff it is valid (always evaluates to 1) in all algebras $\langle [0,1], \odot, \to, \wedge, \vee, f, t\rangle$ where $\wedge$ and $\vee$ are interpreted by $\min$ and $\max$, f and t by 0 and 1, $\odot$ by a left-continuous *t-norm* (an increasing commutative associative binary function on $[0,1]$ with unit 1), and $\to$ by the *residuum* of $\odot$; a binary function satisfying $x \odot y \leq z$ iff $x \leq y \to z$.

To put this another way, consider the class of MTL-*algebras* $\langle L, \wedge, \vee, \odot, \to, f, t\rangle$ where $\langle L, \odot, t\rangle$ is a commutative monoid, $\langle L, \wedge, \vee, f, t\rangle$ is a bounded lattice, and $\to$ is the residuum of $\odot$, satisfying the prelinearity condition $t \leq (x \to y) \vee (y \to x)$ for all $x, y \in L$. Since MTL is sound and complete with respect to MTL-algebras (A is derivable in MTL iff A is valid in all MTL-algebras), the conjecture becomes that A is valid in all MTL-algebras iff it is valid in all "standard" MTL-algebras; those MTL-algebras with $L = [0,1]$.

A proof of Godo and Esteva's conjecture was provided by Jenei and Montagna in [14]. Their method consists of three parts. First it is shown that if a formula is not valid in an MTL-algebra, then it is not valid in a countable MTL-chain (linearly ordered MTL-algebra). Next it is shown that any countable MTL-chain can be embedded into a countable dense MTL-chain by adding countably many new elements to the algebra and extending the operations appropriately. This establishes "rational completeness" for MTL: a formula is derivable iff it is valid in all dense MTL-chains. Finally, a countable dense MTL-chain is embedded into a standard MTL-algebra using a Dedekind-MacNeille-style completion. This method has been extended to first-order MTL in [16] (making use of a different completion), and adapted to prove standard completeness for other axiomatizations of fuzzy logics with weakening in [9, 6]. It relies however on finding the "right extension" of operations from chains to dense chains for each logic. Indeed, no such extension has been found for logics without weakening such as Uninorm Logic, axiomatized by extending an axiom system for $\mathsf{FL_e^\perp}$ with the prelinearity schema $(A \to B) \vee (B \to A)$ and distributivity schema $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$ [15].

Density elimination provides an alternative and more general method for establishing rational completeness. Instead of treating MTL directly, we can consider the corresponding hypersequent calculus $\mathsf{HFL^C_{ew}}$. As we show below, a sequent $\Rightarrow A$ is derivable in $\mathsf{HFL^C_{ew}}$ extended with (density) iff A is valid in all dense MTL-chains. But then by density elimination, this holds iff $\Rightarrow A$ is derivable in $\mathsf{HFL^C_{ew}}$ and hence iff A is derivable in MTL. More generally, we show that any suitable hypersequent calculus $\mathsf{HL^C}$ extended with (density) is sound and complete with respect to a class of dense chains obtained via a Lindenbaum algebra construction. Density elimination then tells us that this completeness result holds also for $\mathsf{HL^C}$. Finally, we can use a Dedekind-MacNeille-style completion and embedding to obtain uniform standard completeness proofs for a wide range of logics, including first-order Uninorm Logic and first-order Monoidal t-Norm Logic.

4.1 Hypersequent Theories

We adapt the usual notion of a theory here to consist of hypersequents of sentences (rather than just sentences). Hence an *$\mathcal{L}$-theory* T for a language $\mathcal{L}$ is a set of hypersequents containing only $\mathcal{L}$-sentences, recalling that since we deal only with countable languages, theories will also be countable. As usual we write T_1, T_2 and $T, \mathcal{G}$ to denote $T_1 \cup T_2$ and $T \cup \{\mathcal{G}\}$, respectively.

A hypersequent calculus H for a language $\mathcal{L}$ has the:

- *Proof by cases property* $\mathcal{PCP}$ if whenever $T, \mathcal{G}_1 \vdash_{\mathsf{H}} \mathcal{H}$ and $T, \mathcal{G}_2 \vdash_{\mathsf{H}} \mathcal{H}$, then $T, (\mathcal{G}_1 \mid \mathcal{G}_2) \vdash_{\mathsf{H}} \mathcal{H}$.
- *Prelinearity property* $\mathcal{PP}$ if whenever $T, (A \Rightarrow B) \vdash_{\mathsf{H}} \mathcal{H}$ and $T, (B \Rightarrow A) \vdash_{\mathsf{H}} \mathcal{H}$, then $T \vdash_{\mathsf{H}} \mathcal{H}$.
- *Density property* $\mathcal{DP}$ if whenever $T \vdash_{\mathsf{H}} \mathcal{G} \mid A \Rightarrow p \mid p \Rightarrow B$ for some p not occurring in T, $\mathcal{G}$, A, or B, then $T \vdash_{\mathsf{H}} \mathcal{G} \mid A \Rightarrow B$.
- *Local deduction property* $\mathcal{LDP}$ if $T \vdash_{\mathsf{H}} \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta$ iff there exists a multiset of formulas Π with predicate symbols restricted to those in T such that $\vdash_{\mathsf{H}} \Gamma_1, \Pi \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Pi \Rightarrow \Delta_n$ and $T \vdash_{\mathsf{H}} \Rightarrow A$ for all $A \in \Pi$.

Let us assume now that L is a (first-order) simple calculus for some language $\mathcal{L}$, recalling that $\mathsf{HL^C}$ is the hypersequent version of L plus (ew), (ec), and (com).

We write $\mathcal{L}' \geq \mathcal{L}$ to mean that the language $\mathcal{L}'$ is an extension of the language $\mathcal{L}$ with at most countably many new propositional variables and constants. Since all the rules of $\mathsf{HL^C}$ except the quantifier rules are schematic, we can extend $\mathsf{HL^C}$ for $\mathcal{L}$ to any $\mathcal{L}' \geq \mathcal{L}$ in the usual way with the extra substitution instances of the rules.

Lemma 18

(a) $\mathsf{HL^C}$ *satisfies the* $\mathcal{PCP}$ *and* $\mathcal{PP}$.
(b) If $\mathsf{HL^C}$ *plus (density) admits density elimination and the* $\mathcal{LDP}$*, then* $\mathsf{HL^C}$ *satisfies the* $\mathcal{DP}$.

Proof For (a), observe first that the $\mathcal{PP}$ follows from the $\mathcal{PCP}$. Suppose that:

$$T, (A \Rightarrow B) \vdash_{\mathsf{HL^C}} \mathcal{H} \quad \text{and} \quad T, (B \Rightarrow A) \vdash_{\mathsf{HL^C}} \mathcal{H}$$

Then $T, (A \Rightarrow B \mid B \Rightarrow A) \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{H}$ by the $\mathcal{PCP}$. But also $\vdash_{\mathsf{HL}^{\mathsf{C}}} A \Rightarrow B \mid B \Rightarrow A$, so $T \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{H}$. For the $\mathcal{PCP}$, consider a derivation of $T, \mathcal{G}_1 \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{H}$: a tree where the leaves are labelled either with axioms, members of T, or $\mathcal{G}_1$ (we can assume by Lemma 10 that none of the new variables in the derivation occur in $\mathcal{H}$). We alter this tree as follows to obtain a derivation for $T, (\mathcal{G}_1 \mid \mathcal{G}_2) \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{H}$:

(1) Re-label nodes labelled $\mathcal{G}$ with $\mathcal{G} \mid \mathcal{H}$, and extend the tree at the root from $\mathcal{H} \mid \mathcal{H}$ to a new root $\mathcal{H}$ by removing sequents from $\mathcal{H}$; i.e. by applying (ec).
(2) Extend every leaf labelled $\mathcal{G} \mid \mathcal{H}$ where $\mathcal{G}$ is an axiom or member of T to a new leaf $\mathcal{G}$ by removing sequents from $\mathcal{H}$; i.e. with applications of (ew).
(3) Extend the remaining leaves of the form $\mathcal{G}_1 \mid \mathcal{H}$ by placing them as roots of derivations of $T, \mathcal{G}_2 \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{H}$ with every node labelled $\mathcal{G}$ re-labelled $\mathcal{G}_1 \mid \mathcal{G}$.
(4) Extend every leaf labelled $\mathcal{G}_1 \mid \mathcal{G}$ where $\mathcal{G}$ is an axiom or member of T to a new leaf $\mathcal{G}$ by removing the elements of $\mathcal{G}_1$; i.e. with applications of (ew).

The only leaves not labelled with an axiom or member of T are labelled $\mathcal{G}_1 \mid \mathcal{G}_2$, so we have a derivation for $T, (\mathcal{G}_1 \mid \mathcal{G}_2) \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{H}$.

(b) Suppose that $T \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid A \Rightarrow p \mid p \Rightarrow B$ for some propositional variable p not occurring in T, $\mathcal{G}$, A, or B. By the $\mathcal{LDP}$, there exists a set of formulas Π with predicate symbols restricted to those occurring in T such that:

$$T \vdash_{\mathsf{HL}^{\mathsf{C}}} \Rightarrow C \text{ for all } C \in \Pi \quad \text{and} \quad \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}^{\Pi} \mid \Pi, A \Rightarrow p \mid \Pi, p \Rightarrow B$$

where $\mathcal{G}^{\Pi}$ is obtained by adding Π to the left of all the sequents in $\mathcal{G}$. But then by density elimination $\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}^{\Pi} \mid \Pi, \Pi, A \Rightarrow B$. Hence, since $T \vdash_{\mathsf{HL}^{\mathsf{C}}} \Rightarrow C$ for all $C \in \Pi$, by multiple applications of (cut), $T \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G} \mid A \Rightarrow B$ as required. □

We define an $\mathcal{L}$-theory T to be:

- *$\mathcal{L}$-linear* if for all $\mathcal{L}$-sentences A, B, either $T \vdash_{\mathsf{HL}^{\mathsf{C}}} A \Rightarrow B$ or $T \vdash_{\mathsf{HL}^{\mathsf{C}}} B \Rightarrow A$.
- *$\mathcal{L}$-dense* if for all $\mathcal{L}$-sentences A, B, whenever $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} A \Rightarrow B$, then $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} A \Rightarrow C$ and $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} C \Rightarrow B$ for some $\mathcal{L}$-sentence C.
- *$\mathcal{L}$-Henkin* if for all $\mathcal{L}$-sentences C, $\forall x A(x)$, and $\exists x A(x)$, whenever $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} C \Rightarrow \forall x A(x)$, then $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} C \Rightarrow A(c)$ for some constant c of $\mathcal{L}$, and whenever $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} \exists x A(x) \Rightarrow C$, then $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} A(d) \Rightarrow C$ for some constant d of $\mathcal{L}$.

We now come to our crucial lemma, relating density elimination to the extension of theories to dense linear Henkin theories.

Lemma 19 *If* HL^{C} *satisfies the* $\mathcal{DP}$ *and* $T \not\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}$ *for some* $\mathcal{L}$*-theory* T*, then* $\hat{T} \not\vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}$ *for some* $\hat{\mathcal{L}}$*-linear* $\hat{\mathcal{L}}$*-dense* $\hat{\mathcal{L}}$*-Henkin* $\hat{\mathcal{L}}$*-theory* $\hat{T} \supseteq T$ *where* $\hat{\mathcal{L}} \geq \mathcal{L}$*.*

Proof We construct $\hat{T}$ in countably many steps. First let $\hat{\mathcal{L}}$ be the extension of

$\mathcal{L}$ with countably infinitely many new propositional variables and constants not occurring in T or $\mathcal{G}$. In the construction of $\hat{T}$ we have to:

(a) deal with $\hat{\mathcal{L}}$-linearity and $\hat{\mathcal{L}}$-density for each pair of $\hat{\mathcal{L}}$-sentences A and B.
(b) deal with the $\hat{\mathcal{L}}$-Henkin property for each pair of $\hat{\mathcal{L}}$-sentences C and $\forall x A$.
(c) deal with the $\hat{\mathcal{L}}$-Henkin property for each pair of $\hat{\mathcal{L}}$-sentences C and $\exists x A$.

Since these are countably many tasks we can interleave them.

We let $T_0 = T$ and $\mathcal{G}_0 = \mathcal{G}$. For the induction step, assume that T_n and $\mathcal{G}_n$ have been constructed such that $T_n \not\vdash_{\mathsf{HL^C}} \mathcal{G}_n$. We construct $T_{n+1} \supseteq T_n$ and $\mathcal{G}_{n+1} \supseteq \mathcal{G}_n$ such that $T_{n+1} \not\vdash_{\mathsf{HL^C}} \mathcal{G}_{n+1}$ and T_{n+1} fulfills the n-th task.

(a) Suppose that the n-th task is dealing with $\hat{\mathcal{L}}$-linearity and $\hat{\mathcal{L}}$-density for A and B. If $T_n, (A \Rightarrow B), (B \Rightarrow A) \not\vdash_{\mathsf{HL^C}} \mathcal{G}_n$, then it is sufficient to define:

$$T_{n+1} = T_n \cup \{A \Rightarrow B, B \Rightarrow A\} \quad \text{and} \quad \mathcal{G}_{n+1} = \mathcal{G}_n$$

Otherwise, we claim that one of the following holds:

(1) $T_n, (A \Rightarrow B) \not\vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid B \Rightarrow p \mid p \Rightarrow A$.
(2) $T_n, (B \Rightarrow A) \not\vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid A \Rightarrow p \mid p \Rightarrow B$.

for some p not occurring in T_n, A, B, or $\mathcal{G}_n$. If not, then by the $\mathcal{DP}$:

$$T_n, (A \Rightarrow B) \vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid B \Rightarrow A \quad \text{and} \quad T_n, (B \Rightarrow A) \vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid A \Rightarrow B$$

But now since $T_n, (A \Rightarrow B), (B \Rightarrow A) \vdash_{\mathsf{HL^C}} \mathcal{G}_n$ and $T_n, \mathcal{G}_n \vdash_{\mathsf{HL^C}} \mathcal{G}_n$, by the $\mathcal{PCP}$:

$$T_n, (A \Rightarrow B), (\mathcal{G}_n \mid B \Rightarrow A) \vdash_{\mathsf{HL^C}} \mathcal{G}_n \quad \text{and} \quad T_n, (\mathcal{G}_n \mid A \Rightarrow B), (B \Rightarrow A) \vdash_{\mathsf{HL^C}} \mathcal{G}_n$$

and so, since $T_n, (A \Rightarrow B) \vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid B \Rightarrow A$ and $T_n, (B \Rightarrow A) \vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid A \Rightarrow B$:

$$T_n, A \Rightarrow B \vdash_{\mathsf{HL^C}} \mathcal{G}_n \quad \text{and} \quad T_n, B \Rightarrow A \vdash_{\mathsf{HL^C}} \mathcal{G}_n$$

But then $T_n \vdash_{\mathsf{HL^C}} \mathcal{G}_n$ by the $\mathcal{PP}$, contradicting the induction hypothesis.

If (1) holds, let $T_{n+1} = T_n \cup \{A \Rightarrow B\}$ and $\mathcal{G}_{n+1} = \mathcal{G}_n \mid B \Rightarrow p \mid p \Rightarrow A$.
If (2) holds, let $T_{n+1} = T_n \cup \{B \Rightarrow A\}$ and $\mathcal{G}_{n+1} = \mathcal{G}_n \mid A \Rightarrow p \mid p \Rightarrow B$.

Clearly T_{n+1} fulfills the $\hat{\mathcal{L}}$-linearity condition for A and B, and $T_{n+1} \not\vdash_{\mathsf{HL^C}} \mathcal{G}_{n+1}$. Moreover, if $T_{n+1} \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$, then $T_{n+1} \not\vdash_{\mathsf{HL^C}} \mathcal{G}_n \mid A \Rightarrow p \mid p \Rightarrow B$ and so by (ew), $T_{n+1} \not\vdash_{\mathsf{HL^C}} A \Rightarrow p$ and $T_{n+1} \not\vdash_{\mathsf{HL^C}} p \Rightarrow B$. The case where $T_{n+1} \not\vdash_{\mathsf{HL^C}} B \Rightarrow A$ is symmetrical, so T_{n+1} fulfills the $\hat{\mathcal{L}}$-density condition for A and B.

(b) If the n-th task is dealing with the $\hat{\mathcal{L}}$-Henkin property for C and $\forall x A(x)$, then let c be a constant not occurring in T_n, $\mathcal{G}_n$, C, or A. There are two cases:

(1) If $T_n \nvdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_n \mid C \Rightarrow A(c)$, then $T_n \nvdash_{\mathsf{HL}^{\mathsf{C}}} C \Rightarrow \forall x A(x)$, and let:

$$T_{n+1} = T_n \quad \text{and} \quad \mathcal{G}_{n+1} = \mathcal{G}_n \mid C \Rightarrow A(c)$$

(2) If $T_n \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_n \mid C \Rightarrow A(c)$, then $T_n \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_n \mid C \Rightarrow \forall x A(x)$, so let:

$$T_{n+1} = T_n \cup \{C \Rightarrow \forall x A(x)\} \quad \text{and} \quad \mathcal{G}_{n+1} = \mathcal{G}_n$$

Suppose that $T_{n+1} \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_{n+1}$. Since also $T_n, \mathcal{G}_n \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_n$, by the $\mathcal{PCP}$, we get $T_n, (\mathcal{G}_n \mid C \Rightarrow \forall x A(x)) \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_n$. Hence $T_n \vdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}_n$, a contradiction.

(c) Dealing with the $\hat{\mathcal{L}}$-Henkin property for C and $\exists x A(x)$ is very similar to (b).

Finally, $\hat{T} = \bigcup_{n \in \mathbb{N}} T_n$ is $\hat{\mathcal{L}}$-linear, $\hat{\mathcal{L}}$-dense, and $\hat{\mathcal{L}}$-Henkin, and $\hat{T} \nvdash_{\mathsf{HL}^{\mathsf{C}}} \mathcal{G}$. $\square$

4.2 Algebraic Semantics

Let us begin by reviewing some algebraic notions for first-order logics (see e.g. [8]). A *(partially-ordered) algebra* $\mathbf{A}$ for a language $\mathcal{L}$ is a poset $\langle L_{\mathbf{A}}, \leq \rangle$ equipped with operations corresponding to the connectives $\mathrm{C}_{\mathcal{L}}$ of $\mathcal{L}$. $\mathbf{A}$ is called a *chain* if it is linearly ordered, i.e. $x \leq y$ or $y \leq x$ for all $x, y \in L_{\mathbf{A}}$, and *dense* if whenever $x \not\leq y$ for $x, y \in L_{\mathbf{A}}$, there exists $z \in L_{\mathbf{A}}$ such that $x \not\leq z$ and $z \not\leq y$.

An $\mathbf{A}$-*structure* is a triple $\mathbf{M} = (M, (p_{\mathbf{M}})_{p \in \mathrm{P}_{\mathcal{L}}}, (f_{\mathbf{M}})_{f \in \mathrm{F}_{\mathcal{L}}})$ where M is a non-empty set called the *domain*, $p_{\mathbf{M}}$ is a function $M^n \to L_{\mathbf{A}}$ for each $p \in \mathrm{P}_{\mathcal{L}}$ with arity n, and $f_{\mathbf{M}}$ is a function $M^n \to M$ for each $f \in \mathrm{F}_{\mathcal{L}}$ with arity n.

An $\mathbf{M}$-*valuation* v is a mapping from object variables $\mathrm{X}_{\mathcal{L}}$ to M. For any variable x and $u \in M$, $v[x \to u]$ is the $\mathbf{M}$-valuation defined by $v[x \to u](x) = u$ and $v[x \to u](y) = v(y)$ for any y not equal to x.

$\|A\|^{\mathbf{A}}_{\mathbf{M},v}$ is then defined inductively as follows, stipulating that the value is undefined if either one of the required arguments is undefined or the needed infimum or supremum does not exist in $L_{\mathbf{A}}$:

$$\begin{aligned}
\|x\|^{\mathbf{A}}_{\mathbf{M},v} &= v(x) && \text{for } x \in \mathrm{X}_{\mathcal{L}}\\
\|f(t_1,\ldots,t_n)\|^{\mathbf{A}}_{\mathbf{M},v} &= f_{\mathbf{M}}(\|t_1\|^{\mathbf{A}}_{\mathbf{M},v},\ldots,\|t_n\|^{\mathbf{A}}_{\mathbf{M},v}) && \text{for } f \in \mathrm{F}_{\mathcal{L}} \text{ with arity } n\\
\|p(t_1,\ldots,t_n)\|^{\mathbf{A}}_{\mathbf{M},v} &= p_{\mathbf{M}}(\|t_1\|^{\mathbf{A}}_{\mathbf{M},v},\ldots,\|t_n\|^{\mathbf{A}}_{\mathbf{M},v}) && \text{for } p \in \mathrm{P}_{\mathcal{L}} \text{ with arity } n\\
\|\star(A_1,\ldots,A_n)\|^{\mathbf{A}}_{\mathbf{M},v} &= \star(\|A_1\|^{\mathbf{A}}_{\mathbf{M},v},\ldots,\|A_n\|^{\mathbf{A}}_{\mathbf{M},v}) && \text{for } \star \in \mathrm{C}_{\mathcal{L}} \text{ with arity } n\\
\|(\forall x)A\|^{\mathbf{A}}_{\mathbf{M},v} &= \inf\{\|A\|^{\mathbf{A}}_{\mathbf{M},v[x\to u]} : u \in M\}\\
\|(\exists x)A\|^{\mathbf{A}}_{\mathbf{M},v} &= \sup\{\|A\|^{\mathbf{A}}_{\mathbf{M},v[x\to u]} : u \in M\}
\end{aligned}$$

$\mathbf{M}$ is called *safe* if $\|A\|^{\mathbf{A}}_{\mathbf{M},v}$ is defined for each formula A and $\mathbf{M}$-valuation v, and in this case we define $\|A\|^{\mathbf{A}}_{\mathbf{M}} = \inf\{\|A\|^{\mathbf{A}}_{\mathbf{M},v} : v$ is an $\mathbf{M}$-valuation$\}$.

We call an $\mathcal{L}$-theory T *equational* if it contains only sequents of the form $A \Rightarrow B$ where A and B are $\mathcal{L}$-sentences.

An $\mathbf{A}$-*model* of such an equational theory T is a safe $\mathbf{A}$-structure $\mathbf{M}$ such that $\|A\|^{\mathbf{A}}_{\mathbf{M}} \leq \|B\|^{\mathbf{A}}_{\mathbf{M}}$ for all $A \Rightarrow B \in T$.

For a class of algebras $\mathcal{K}$ for a language $\mathcal{L}$, we write $T \models_{\mathcal{K}} A \Rightarrow B$ to mean that for all $\mathbf{A} \in \mathcal{K}$, any $\mathbf{A}$-model of T is an $\mathbf{A}$-model of $\{A \Rightarrow B\}$.

We obtain corresponding algebras for suitable hypersequent calculi via a Lindenbaum algebra construction. Call a (first-order) sequent calculus L for a language $\mathcal{L}$ *regular* if it is simple and for each $\star \in \mathrm{C}_{\mathcal{L}}$ with arity n, for $i = 1 \ldots n$:

$$(A \Rightarrow B), (B \Rightarrow A) \vdash_{\mathsf{L}} \star(C_1, \ldots, C_n)[A/C_i] \Rightarrow \star(C_1, \ldots, C_n)[B/C_i]$$

Example 20 *Any simple sequent calculus with rules for connectives in* $\{\wedge, \vee, \rightarrow, \odot, t, f, \bot, \top\}$ *from Appendix A and Figure 1 is regular. E.g., for* $\rightarrow$ *we have:*

$$\dfrac{\dfrac{\dfrac{}{C \Rightarrow C}\,(id) \quad B \Rightarrow A}{A \rightarrow C, B \Rightarrow C}\,(\rightarrow\Rightarrow)}{A \rightarrow C \Rightarrow B \rightarrow C}\,(\Rightarrow\rightarrow) \quad \textit{and} \quad \dfrac{\dfrac{A \Rightarrow B \quad \dfrac{}{C \Rightarrow C}\,(id)}{C \rightarrow A, C \Rightarrow B}\,(\rightarrow\Rightarrow)}{C \rightarrow A \Rightarrow C \rightarrow B}\,(\Rightarrow\rightarrow)$$

That is, $(A \Rightarrow B), (B \Rightarrow A) \vdash_{\mathsf{L}} A \rightarrow C \Rightarrow B \rightarrow C$ *and* $(A \Rightarrow B), (B \Rightarrow A) \vdash_{\mathsf{L}} C \rightarrow A \Rightarrow C \rightarrow B$ *as required.*

For a (first-order) regular sequent calculus L for a language $\mathcal{L}$ and $\mathcal{L}$-theory T, let:

$$\textsc{Lind}^{\mathcal{L}}_T =_{\text{def}} \langle L^{\mathcal{L}}_T, \{\star^{\mathcal{L}}_T : \star \in \mathrm{C}_{\mathcal{L}}\}\rangle \quad \text{where:}$$

1. $[A]^{\mathcal{L}}_T =_{\text{def}} \{B$ is an $\mathcal{L}$-sentence : $T \vdash_{\mathsf{HL^C}} A \Rightarrow B$ and $T \vdash_{\mathsf{HL^C}} B \Rightarrow A\}$.
2. $L^{\mathcal{L}}_T =_{\text{def}} \{[A]^{\mathcal{L}}_T : A$ is an $\mathcal{L}$-sentence$\}$ and $[A]^{\mathcal{L}}_T \leq [B]^{\mathcal{L}}_T$ iff $T \vdash_{\mathsf{HL^C}} A \Rightarrow B$.
3. $\star^{\mathcal{L}}_T([A_1]^{\mathcal{L}}_T, \ldots, [A_n]^{\mathcal{L}}_T) =_{\text{def}} [\star(A_1, \ldots, A_n)]^{\mathcal{L}}_T$ for each n-ary connective $\star$ of $\mathcal{L}$.

The definition is justified by the fact that L is regular.

An $\mathsf{HL^C}$-*algebra* $\mathbf{A}$ is any algebra for $\mathcal{L}$ such that for all equational $\mathcal{L}$-theories T' and $\mathcal{L}$-sentences A and B, if $T' \vdash_{\mathsf{HL^C}} A \Rightarrow B$, then $T' \models_{\{\mathbf{A}\}} A \Rightarrow B$.

We call a linearly ordered $\mathsf{HL^C}$-algebra, an $\mathsf{HL^C}$-*chain*, and a *standard* $\mathsf{HL^C}$-algebra if its universe is the real unit interval $[0, 1]$ equipped with the usual order. We let $\mathsf{DEN}(\mathsf{HL^C})$ and $\mathsf{STAN}(\mathsf{HL^C})$ denote the classes of all dense $\mathsf{HL^C}$-chains and standard $\mathsf{HL^C}$-algebras, respectively.

Lemma 21 *(a)* $\textsc{Lind}^{\mathcal{L}}_T$ *is a countable* $\mathsf{HL^C}$*-algebra. (b)* $\textsc{Lind}^{\mathcal{L}}_T$ *is a chain iff* T *is* $\mathcal{L}$*-linear, and dense iff* T *is* $\mathcal{L}$*-dense.*

Proof (a) Let T' be an equational $\mathcal{L}$-theory and let A and B be $\mathcal{L}$-sentences. Suppose that $T' \vdash_{\mathsf{HL^C}} A \Rightarrow B$. Let $\mathbf{M}$ be a $\textsc{Lind}^{\mathcal{L}}_T$-model of T'. I.e. $\|C\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}} \leq \|D\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}}$ for all $C \Rightarrow D \in T'$. But then $T \vdash_{\mathsf{HL^C}} \|C\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}} \Rightarrow \|D\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}}$ for all $C \Rightarrow D \in T'$. So $T \vdash_{\mathsf{HL^C}} \|A\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}} \Rightarrow \|B\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}}$ which means that $\|A\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}} \leq \|B\|^{\textsc{Lind}^{\mathcal{L}}_T}_{\mathbf{M}}$. That is, $\mathbf{M}$ is a $\textsc{Lind}^{\mathcal{L}}_T$-model of $\{A \Rightarrow B\}$.

(b) $\textsc{Lind}^{\mathcal{L}}_T$ is a chain iff for all $\mathcal{L}$-sentences A and B either $[A]^{\mathcal{L}}_T \leq [B]^{\mathcal{L}}_T$ or $[B]^{\mathcal{L}}_T \leq [A]^{\mathcal{L}}_T$. But this holds iff $T \vdash_{\mathsf{HL^C}} A \Rightarrow B$ or $T \vdash_{\mathsf{HL^C}} B \Rightarrow A$; i.e. iff T is $\mathcal{L}$-linear. $\textsc{Lind}^{\mathcal{L}}_T$ is dense iff whenever $[A]^{\mathcal{L}}_T \not\leq [B]^{\mathcal{L}}_T$, there exists $[C]^{\mathcal{L}}_T$ such that $[A]^{\mathcal{L}}_T \not\leq [C]^{\mathcal{L}}_T$ and $[C]^{\mathcal{L}}_T \not\leq [B]^{\mathcal{L}}_T$. But this holds iff whenever $T \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$, then $T \not\vdash_{\mathsf{HL^C}} A \Rightarrow C$ and $T \not\vdash_{\mathsf{HL^C}} C \Rightarrow B$ for some $\mathcal{L}$-sentence C; i.e. iff T is $\mathcal{L}$-dense. $\square$

Lemma 22 *For an $\mathcal{L}$-Henkin theory T and formula $A(a)$ with one free variable:*

(a) $[\forall x A(x)]^{\mathcal{L}}_T = \inf\{[A(c)]^{\mathcal{L}}_T : c$ *is a constant of* $\mathcal{L}\}$.
(b) $[\exists x A(x)]^{\mathcal{L}}_T = \sup\{[A(c)]^{\mathcal{L}}_T : c$ *is a constant of* $\mathcal{L}\}$.

Proof We will consider just (a) since (b) is very similar. Easily $[\forall x A(x)]^{\mathcal{L}}_T \leq [A(c)]^{\mathcal{L}}_T$ for all constants c of $\mathcal{L}$ since $T \vdash_{\mathsf{HL^C}} \forall x A(x) \Rightarrow A(c)$. Now suppose that $[C]^{\mathcal{L}}_T \leq [A(c)]^{\mathcal{L}}_T$ for all c but $[C]^{\mathcal{L}}_T \not\leq [\forall x A(x)]^{\mathcal{L}}_T$. We get $T \not\vdash_{\mathsf{HL^C}} C \Rightarrow \forall x A(x)$, so by the $\mathcal{L}$-Henkin property $T \not\vdash_{\mathsf{HL^C}} C \Rightarrow A(d)$ for some d, a contradiction. $\square$

Now we can use Lemma 19 to obtain completeness results with respect to dense chains. Let L be a (first-order) regular sequent calculus such that $\mathsf{HL^C}$ plus (density) admits density elimination and the $\mathcal{LDP}$.

Lemma 23 *If $T \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$ for some equational $\mathcal{L}$-theory T and $\mathcal{L}$-sentences A and B, then $T \not\models_{\{\mathbf{A}\}} A \Rightarrow B$ for some countable dense $\mathsf{HL^C}$-chain $\mathbf{A}$.*

Proof Let T be an equational $\mathcal{L}$-theory, let A and B be $\mathcal{L}$-sentences, and suppose that $T \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$. By Lemma 19, $\hat{T} \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$ for some $\hat{\mathcal{L}}$-linear $\hat{\mathcal{L}}$-dense $\hat{\mathcal{L}}$-Henkin $\hat{\mathcal{L}}$-theory $\hat{T} \supseteq T$ where $\hat{\mathcal{L}} \geq \mathcal{L}$. Let $\mathbf{A} = \textsc{Lind}^{\hat{\mathcal{L}}}_{\hat{T}}$. By Lemma 21, $\mathbf{A}$ is a countable dense $\mathsf{HL^C}$-chain. Let D be the set of closed terms of $\hat{\mathcal{L}}$. Define an $\mathbf{A}$-structure $\mathbf{M}$ with domain D such that $f_{\mathbf{M}}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ for each m-ary function symbol f and $p_{\mathbf{M}}(t_1, \ldots, t_m)) = [p(t_1, \ldots, t_m)]^{\hat{\mathcal{L}}}_{\hat{T}}$ for each m-ary predicate symbol p. Then, proceeding by induction on formula complexity, using regularity to take care of the propositional connectives and Lemma 22 to take care of the quantifiers, $\|B\|^{\mathbf{A}}_{\mathbf{M}} = [B]^{\hat{\mathcal{L}}}_{\hat{T}}$ for all $\hat{\mathcal{L}}$-sentences B. Hence for each $C \Rightarrow D \in T$, since $\hat{T} \vdash_{\mathsf{HL^C}} C \Rightarrow D$, $[C]^{\hat{\mathcal{L}}}_{\hat{T}} \leq [D]^{\hat{\mathcal{L}}}_{\hat{T}}$ and so $\|C\|^{\mathbf{A}}_{\mathbf{M}} \leq \|D\|^{\mathbf{A}}_{\mathbf{M}}$. Similarly, since $\hat{T} \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$, it follows that $[A]^{\hat{\mathcal{L}}}_{\hat{T}} \not\leq [B]^{\hat{\mathcal{L}}}_{\hat{T}}$ and $\|A\|^{\mathbf{A}}_{\mathbf{M}} \not\leq \|B\|^{\mathbf{A}}_{\mathbf{M}}$. $\square$

Corollary 24 *For any equational $\mathcal{L}$-theory T and $\mathcal{L}$-sentences A, B:*

$$T \vdash_{\mathsf{HL^C}} A \Rightarrow B \quad \textit{iff} \quad T \models_{\mathsf{DEN(HL^C)}} A \Rightarrow B$$

We can use the very general results established above to obtain standard completeness results for first-order fuzzy logics, including first-order Uninorm Logic and first-order Monoidal t-Norm Logic. Rather than deal directly with axiom systems, we treat their algebraic counterparts; (bounded) pointed commutative residuated lattices, investigated in detail by Tsinakis and co-workers (see e.g. [12]).

A *pointed commutative residuated lattice (p.c.r.l.)* is an algebra $\langle L, \wedge, \vee, \odot, \to, t, f\rangle$ with binary operations $\wedge$, $\vee$, $\odot$, $\to$, and constants t, f such that:

(1) $\langle L, \wedge, \vee\rangle$ is a lattice with order defined by $x \leq y$ iff $x \wedge y = x$.
(2) $\langle L, \odot, t\rangle$ is a commutative monoid.
(3) $x \odot y \leq z$ iff $x \leq y \to z$ for all $x, y, z \in L$.

A *bounded p.c.r.l.* is an algebra $\langle L, \wedge, \vee, \odot, \to, t, f, \bot, \top\rangle$ where $\langle L, \wedge, \vee, \odot, \to, t, f\rangle$ is a p.c.r.l. with top and bottom elements $\top$ and $\bot$, respectively.

The classes of (bounded) p.c.r.l.s that we are interested in are classes of algebras based on hypersequent calculi. Let us fix the sequent calculus L to be (referring to Appendix A and Fig. 1):

$$\forall\mathsf{FL}_{\mathsf{ew}} + K \text{ for } K \subseteq \{(\text{cl}), (\text{wc}), (\text{c}_n)\} \quad \text{or} \quad \forall\mathsf{FL}_{\mathsf{e}}^{\bot} + K \text{ for } K \subseteq \{(\text{wc}), (\text{mix})\}$$

Lemma 25 *The class of* HL^{C}*-algebras consists of all (bounded, if the language of* L *contains* $\top$ *and* $\bot$*) p.c.r.l.s satisfying:*

(i) $t \leq (x \to y) \vee (y \to x)$ *and* $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$.
(ii) $x \leq t$ *and* $f \leq x$ *if* L *extends* $\forall\mathsf{FL}_{\mathsf{ew}}$.
(iii) $x \leq x \odot x$ *if (cl)* $\in K$.
(iv) $f \leq t$ *if (mix)* $\in K$.
(v) $x \wedge (x \to f) \leq f$ *if (wc)* $\in K$.
(vi) $x^{n-1} \leq x^n$ *if (c$_n$)* $\in K$ *for* $n \geq 2$ *(where* $x^1 = x$ *and* $x^{k+1} = x \odot x^k$*).*

Proof Let $\mathbf{A}$ be a (bounded) p.c.r.l. satisfying the appropriate conditions for L stated above. Using soundness results in the literature (consult e.g. [15]), if $T \vdash_{\mathsf{HL}^{\mathsf{C}}} A \Rightarrow B$, then every $\mathbf{A}$-model of T is an $\mathbf{A}$-model of $A \Rightarrow B$. Hence $\mathbf{A}$ is an HL^{C}-algebra. For the other direction, note that for each inequation in (i)-(vi), a corresponding sequent (replacing $\leq$ by $\Rightarrow$) is derivable in HL^{C} (again consult [15]). For example, both $t \Rightarrow (A \to B) \vee (B \to A)$ and $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ are $\mathsf{HFL}_{\mathsf{e}}^{\mathsf{C}}$-derivable. Hence each HL^{C}-algebra satisfies the appropriate conditions. □

In particular, bounded p.c.r.l.s satisfying the prelinearity and distributivity conditions of (i) (algebras for Uninorm Logic) are precisely $\mathsf{HFL}_{\mathsf{e}}^{\bot\mathsf{C}}$-algebras and vice versa, while p.c.r.l.s satisfying conditions (i)-(ii) (algebras for Monoidal t-Norm Logic) constitute the class of $\mathsf{HFL}_{\mathsf{ew}}^{\mathsf{C}}$-algebras.

Since $\mathsf{HL^C}$ admits density elimination, by Corollary 24, we can establish completeness of $\mathsf{HL^C}$ with respect to dense $\mathsf{HL^C}$-chains by showing that $\mathsf{HL^C}$ has the local deduction property.

Lemma 26 $\mathsf{HL^C}$ *has the* $\mathcal{LDP}$.

Proof Let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. Suppose that for some finite multiset of formulas Π with predicate symbols restricted to those in T:

$$\vdash_{\mathsf{HL^C}} \Gamma_1, \Pi \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Pi \Rightarrow \Delta_n \quad \text{and} \quad T \vdash_{\mathsf{HL^C}} \Rightarrow A \ \text{ for all } A \in \Pi$$

We obtain a derivation for $T \vdash_{\mathsf{HL^C}} \mathcal{G}$ using these derivations and multiple applications of (cut) and (ew).

Now suppose conversely that $T \vdash_{\mathsf{HL^C}} \mathcal{G}$. Compactness follows from the definition of a derivation, so we can assume that T is finite. Define a function I from hypersequents to formulas as follows: (a) $I(\Gamma \Rightarrow \Delta) = \odot\Gamma \to \oplus\Delta$ where $\star[A_1, \ldots, A_k] = (A_1 \star \ldots \star A_k)$ for $\star \in \{\odot, \oplus\}$, $\odot[] = t$, and $\oplus[] = f$; (b) $I(S_1 \mid \ldots \mid S_m) = I(S_1) \vee \ldots \vee I(S_m)$. Let $\Pi = [I(\mathcal{H}) \wedge t : \mathcal{H} \in T]$. Note that the following rule is $\mathsf{HL^C}$-derivable using (ec), $(\Rightarrow\vee)_1$, and $(\Rightarrow\vee)_2$:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B} \ (\Rightarrow\vee)$$

We obtain a derivation for $T \vdash_{\mathsf{HL^C}} \Rightarrow A$ for each $A \in \Pi$ by applying (backwards) the rules $(\Rightarrow\wedge)$, $(\Rightarrow t)$, $(\Rightarrow\vee)$, $(\Rightarrow\to)$, and $(\odot\Rightarrow)$. Moreover, we can show that if $d, T \vdash_{\mathsf{HL^C}} \mathcal{G}$, then $\vdash_{\mathsf{HL^C}} \Gamma_1, \Pi^m \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Pi^m \Rightarrow \Delta_n$ for some $m \in \mathbb{N}$, proceeding by induction on $|d|$. For the base case, if d ends with (id) or a logical rule, then the result follows immediately, taking $m = 0$. If $\mathcal{G}$ is a member of T, then we take $m = 1$. By the invertibility of the rules $(\Rightarrow\vee)$, $(\Rightarrow\to)$, $(\odot\Rightarrow)$, $(t\Rightarrow)$, and $(\Rightarrow f)$ (see [15] for details), $\vdash_{\mathsf{HL^C}} \Gamma_1, I(\mathcal{G}) \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, I(\mathcal{G}) \Rightarrow \Delta_n$, and the result follows by multiple applications of $(\wedge\Rightarrow)$ and $(t\Rightarrow)$.

For the inductive step, we consider as an example the case where d ends with an application of $(\to\Rightarrow)$:

$$\frac{\Sigma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \quad \Sigma_2, B \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}{\Sigma_1, \Sigma_2, A \to B \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

where $\Gamma_1 = \Sigma_1 \uplus \Sigma_2$. Then by the induction hypothesis twice, $\vdash_{\mathsf{HL^C}} \Sigma_1, \Pi^k \Rightarrow A \mid \Gamma_2, \Pi^k \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n, \Pi^k \Rightarrow \Delta_n$ and $\vdash_{\mathsf{HL^C}} \Sigma_2, B, \Pi^l \Rightarrow \Delta_1 \mid \Gamma_2, \Pi^l \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n, \Pi^l \Rightarrow \Delta_n$ for some $k, l \in \mathbb{N}$. Let $m = k + l$. Then by multiple applications of $(\wedge\Rightarrow)$ and $(t\Rightarrow)$, we have $\vdash_{\mathsf{HL^C}} \Sigma_1, \Pi^k \Rightarrow A \mid \Gamma_2, \Pi^m \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n, \Pi^m \Rightarrow \Delta_n$ and $\vdash_{\mathsf{HL^C}} \Sigma_2, B, \Pi^l \Rightarrow \Delta_1 \mid \Gamma_2, \Pi^m \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n, \Pi^m \Rightarrow \Delta_n$. The result follows by a single application of $(\to\Rightarrow)$. $\Box$

So by Corollary 24, for any equational theory T and sentences A, B: $T \vdash_{\mathsf{HL^C}} A \Rightarrow B$ iff $T \models_{\mathsf{DEN(HL^C)}} A \Rightarrow B$. As a further step, we now use a Dedekind-MacNeille-

L	$\mathsf{HL^C}$	Original Proofs
$\forall\mathsf{BFL_e}$	First-Order Uninorm Logic	[15] (propositional)
$\forall\mathsf{FL_{ew}}$	First-Order Monoidal t-Norm Logic	[14, 16]
$\forall\mathsf{FL_{ew}}$ + (wc)	First-Order Strict Monoidal t-Norm Logic	[9] (propositional)
$\forall\mathsf{FL_{ew}}$ + (c_n)	First-Order n-Contractive Monoidal t-Norm Logic	[6] (propositional)
$\forall\mathsf{FL_{ewc}}$	First-Order Gödel Logic	[13]

Fig. 5. Standard Completeness Results for First-Order Fuzzy Logics

style completion (following [15]) to show that $\mathsf{HL^C}$ is complete with respect to *standard* $\mathsf{HL^C}$-algebras.

Let $\mathbf{A}$ be a (bounded) p.c.r.l. For $X \subseteq L_\mathbf{A}$, let X^u denote the set of upper bounds of X, and X^l, the set of lower bounds of X. Let $\mathsf{DM}(\mathbf{A})$ be the algebra with universe $\mathsf{DM}(L_\mathbf{A}) =_{\text{def}} \{X \subseteq L_\mathbf{A} : (X^u)^l = X\}$ ordered by $\subseteq$ with constants $t_{\mathsf{DM}} = \{t\}^l$ and $f_{\mathsf{DM}} = \{f\}^l$ (and $\bot_{\mathsf{DM}} = \{\bot\}$ and $\top_{\mathsf{DM}} = L_\mathbf{A}$ if $\mathbf{A}$ is bounded) and binary operations:

$$X \wedge_{\mathsf{DM}} Y = X \cap Y \qquad X \odot_{\mathsf{DM}} Y = (\{x \odot y : x \in X, y \in Y\}^u)^l$$
$$X \vee_{\mathsf{DM}} Y = ((X \cup Y)^u)^l \qquad X \to_{\mathsf{DM}} Y = \{x \in L_\mathbf{A} : x \odot y \in Y \text{ for all } y \in Y\}$$

Lemma 27 *Every countable dense $\mathsf{HL^C}$-chain can be embedded into a standard $\mathsf{HL^C}$-algebra by a complete embedding.*

Proof Let $\mathbf{A}$ be a countable dense $\mathsf{HL^C}$-chain. Since $L_\mathbf{A}$ is order-isomorphic to $\mathbb{Q} \cap [0,1]$ with the usual order, the Dedekind-MacNeille completion $\mathsf{DM}(L_\mathbf{A})$ is order-isomorphic to $[0,1]$ with the usual order. Moreover, as shown in e.g. [15], $\mathsf{DM}(\mathbf{A})$ is a (bounded) p.c.r.l. satisfying the appropriate conditions for $\mathsf{HL^C}$-algebras of Lemma 25. Finally, $\Phi(x) = \{x\}^l$ is a complete embedding (preserving $\inf$s and $\sup$s of elements in $L_\mathbf{A}$) of $\mathbf{A}$ into $\mathsf{DM}(\mathbf{A})$. □

Theorem 28 *For any equational $\mathcal{L}$-theory T and $\mathcal{L}$-sentences A, B:*

$$T \vdash_{\mathsf{HL^C}} A \Rightarrow B \text{ iff } T \models_{\mathsf{STAN(HL^C)}} A \Rightarrow B$$

Proof The left-to-right direction follows from the definition of an $\mathsf{HL^C}$-algebra. For the other direction, suppose that $T \not\vdash_{\mathsf{HL^C}} A \Rightarrow B$. By Theorems 14 and 17, $\mathsf{HL^C}$ plus (density) admits density elimination. Also by Lemma 26, these systems admit the $\mathcal{LDP}$. Hence by Lemma 23, there is a countable dense $\mathsf{HL^C}$-chain $\mathbf{A}$ and an $\mathbf{A}$-model of T that is not an $\mathbf{A}$-model of $\{A \Rightarrow B\}$. But by the previous lemma, there is a complete embedding of $\mathbf{A}$ into a standard $\mathsf{HL^C}$-algebra. Hence $T \not\models_{\mathsf{STAN(HL^C)}} A \Rightarrow B$. □

In particular, we obtain standard completeness results for the first-order fuzzy logics displayed with references to the original proofs (some just at the propositional level) in Figure 5.

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A The Sequent Calculus $\forall\mathsf{FL_e}$

Initial Sequents

$$\frac{}{A \Rightarrow A}\ \text{(id)}$$

Logical Rules

$$\frac{}{f \Rightarrow}\ (f\Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow f}\ (\Rightarrow f)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, t \Rightarrow \Delta}\ (t\Rightarrow) \qquad \frac{}{\Rightarrow t}\ (\Rightarrow t)$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \odot B \Rightarrow \Delta}\ (\odot\Rightarrow) \qquad \frac{\Gamma \Rightarrow A \quad \Pi \Rightarrow B}{\Gamma, \Pi \Rightarrow A \odot B}\ (\Rightarrow\odot)$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow\vee)_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Pi \Rightarrow A}{\Gamma, \Pi, A \rightarrow B \Rightarrow \Delta}\ (\rightarrow\Rightarrow) \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ (\Rightarrow\rightarrow)$$

Quantifier Rules

$$\frac{\Gamma, A(t) \Rightarrow \Delta}{\Gamma, \forall x A(x) \Rightarrow \Delta}\ (\forall\Rightarrow) \qquad \frac{\Gamma \Rightarrow A(a)}{\Gamma \Rightarrow \forall x A(x)}\ (\Rightarrow\forall)$$

$$\frac{\Gamma, A(a) \Rightarrow \Delta}{\Gamma, \exists x A(x) \Rightarrow \Delta}\ (\exists\Rightarrow) \qquad \frac{\Gamma \Rightarrow A(t)}{\Gamma \Rightarrow \exists x A(x)}\ (\Rightarrow\exists)$$

where a does not occur in the conclusions of $(\exists\Rightarrow)$ or $(\Rightarrow\forall)$.

Cut Rule

$$\frac{\Gamma \Rightarrow A \quad \Pi, A \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ \text{(cut)}$$