Hypersequent and Labelled Calculi for Intermediate Logics*

Agata Ciabattoni[1], Paolo Maffezioli[2], and Lara Spendier[1]

[1] Vienna University of Technology
[2] University of Groningen

Abstract. Hypersequent and labelled calculi are often viewed as antagonist formalisms to define cut-free calculi for non-classical logics. We focus on the class of intermediate logics to investigate the methods of turning Hilbert axioms into hypersequent rules and frame conditions into labelled rules. We show that these methods are closely related and we extend them to capture larger classes of intermediate logics.

1 Introduction

The lack of cut-free sequent calculi for logics having natural semantic characterizations and/or simple axiomatizations has prompted the search for generalizations of the Gentzen sequent framework. Despite the large variety of formalisms introduced in the literature (see e.g., [17]), there are two main approaches. In the *syntactic approach* sequents are generalized by allowing extra structural connectives in addition to sequents' comma; in the *semantic approach* the semantic language is explicit part of the syntax in sequents and rules.

Hypersequent calculus [2] is a prominent example of the syntactic approach, while labelled calculi internalizing Kripke semantics [15, 8, 16, 10] are the most developed systems within the semantic approach. Hypersequent and labelled calculus are general-purpose formalisms powerful enough to capture logics of a different nature ranging from modal to substructural logics [8, 16, 10, 3], and are often viewed as antagonist formalisms to define cut-free calculi.

In this paper we focus on propositional intermediate logics, i.e. logics between intuitionistic and classical logic, in order to analyze and compare the methods in [7, 5] for defining cut-free hypersequent and labelled calculi. Intermediate logics are an adequate case study for two reasons: (i) Although most of them have a simple axiomatization obtained by extending intuitionistic logic IL with suitable axioms, and have a natural Kripke semantics defined by imposing conditions on the standard intuitionistic frame, corresponding cut-free sequent calculi cannot be defined in a modular way by simply extending the Gentzen sequent calculus **LJ** for IL with new axioms or rules, see [5]. (ii) Cut-free hypersequent and labelled systems have been provided for a large class of intermediate logics in a modular and algorithmic way in [5, 7]. The resulting calculi are indeed defined by

* Work supported by FWF START Y544-N23.

adding to the base (hypersequent or labelled) calculus for IL extra *structural* rules corresponding to the additional conditions characterizing the considered logic. The extra rules are constructed in an algorithmic way by turning Hilbert axioms into hypersequent rules [5] and by turning frame conditions –that are formulas of first-order classical logic– into labelled rules [7]. The main differences between these methods are their starting point (syntactic vs. semantic specifications of the considered logics) and their approach: systematic, i.e. based on a syntactic classification of Hilbert axioms in the case of hypersequents, and presenting a specific class of frame conditions (called *geometric formulas*) for which the method works, in the case of labelled sequents.

In this paper we analyze both methods and refine the approach in [5] (and [6]) to introduce new cut-free calculi for intermediate logics. *For hypersequents*: we define a first cut-free hypersequent calculus for the logic Bd_2 [4], one of the seven interpolable intermediate logics and the only one still lacking a cut-free hypersequent calculus. Our calculus is obtained by adapting the method in [6] to extract a *logical* hypersequent rule out of the peculiar axiom of Bd_2, and then modifying the obtained rule to make the cut-elimination go through. *For labelled sequents*: we classify frame conditions according to their quantifier alternation and apply to them the algorithm in [5]; the rules resulting from geometric formulas coincide with those obtained by the method in [7].

2 Preliminaries

The language of propositional intermediate logics consists of infinitely many propositional variables $p, q \ldots$, the connectives & (conjunction), $\vee$ (disjunction), $\supset$ (implication), and the constant $\bot$ for falsity. $\varphi, \psi, \alpha, \beta \ldots$ are formulas built from atoms by using connectives and $\bot$. As usual, $\sim \varphi$ abbreviates $\varphi \supset \bot$.

An intuitionistic frame is a pair $\mathfrak{F} = \langle W, \leqslant \rangle$ where W is a non-empty set, and $\leqslant$ is a reflexive and transitive (accessibility) relation on W. An intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle$ is a frame $\mathfrak{F}$ together with a relation $\Vdash$ (called the forcing) between elements of W and atomic formulas. Intuitively, $x \Vdash p$ means that the atom p is true at x. Forcing is assumed to be monotonic w.r.t. the relation $\leqslant$, namely, if $x \leqslant y$ and $x \Vdash p$ then also $y \Vdash p$. It is defined inductively on arbitrary formulas as follows:

$x \Vdash \bot$	for no x	$x \Vdash \varphi \& \psi$	iff $x \Vdash \varphi$ and $x \Vdash \psi$
$x \Vdash \varphi \vee \psi$	iff $x \Vdash \varphi$ or $x \Vdash \psi$	$x \Vdash \varphi \supset \psi$	iff $x \leqslant y$ and $y \Vdash \varphi$ implies $y \Vdash \psi$

Intermediate logics are obtained from intuitionistic logic IL either by (i) adding suitable axioms to the Hilbert system for IL or (ii) imposing on intuitionistic frames additional conditions on the relation $\leqslant$. The latter conditions are usually expressed as formulas of first-order classical logic CL in which variables are interpreted as elements of W, and the binary predicate $\leqslant$ denotes the accessibility relation of $\mathfrak{F}$. Atomic formulas are *relational atoms* of the form $x \leqslant y$. Compound formulas are built from relational atoms using the propositional connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, and the quantifiers $\forall$ and $\exists$.

Example 1. The intermediate logics below are obtained by extending IL with the given axiom or frame condition for the accessibility relation $\leqslant$.

Logic	Axioms		Frame conditions
Jankov	*(wc)*	$\sim\varphi \vee \sim\sim\varphi$	$\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \rightarrow \exists w(y \leqslant w \wedge z \leqslant w))$
Gödel	*(lin)*	$(\varphi \supset \psi) \vee (\psi \supset \varphi)$	$\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \rightarrow (y \leqslant z \vee z \leqslant y))$
Bd_2	(bd_2)	$\xi \vee (\xi \supset (\varphi \vee (\varphi \supset \psi)))$	$\forall x \forall y \forall z((x \leqslant y \wedge y \leqslant z) \rightarrow (y \leqslant x \vee z \leqslant y))$
CL	*(em)*	$\varphi \vee \sim\varphi$	$\forall x \forall y(x \leqslant y \rightarrow y \leqslant x)$

Hypersequent and labelled calculi. Introduced by Avron in [2], the *hypersequent calculus* is a simple generalization of Gentzen's sequent calculus whose basic objects are finite disjunctions of sequents.

Definition 1. *A hypersequent is a finite multiset* $\Gamma_1 \Rightarrow \Delta_1 | \cdots | \Gamma_n \Rightarrow \Delta_n$ *where each* $\Gamma_i \Rightarrow \Delta_i, i = 1, \ldots, n$ *is a sequent, called a* component *of the hypersequent. If all components of a hypersequent contain at most one formula in the succedent, the hypersequent is called* single-conclusion*, and* multiple-conclusion *otherwise.*

A hypersequent calculus is defined by incorporating Gentzen's original calculus (e.g., **LJ**, **LK** or a substructural version of it) as a sub-calculus and adding an additional layer of information by considering a single sequent to live in the context of hypersequents. This opens the possibility to define new rules that "exchange information" between different sequents. This type of rule increases the expressive power of hypersequent calculi compared to ordinary sequent calculi and allows us to capture the characteristic axioms of several intermediate logics.

Labelled systems are a variant of sequent calculus in which the relational semantics of the formalized logics is made explicit part of the syntax [8, 16, 10]. In a labelled system, each formula φ receives a label x, indicated by $x : \varphi$. The labels are interpreted as possible worlds, and a labelled formula $x : \varphi$ corresponds to $x \Vdash \varphi$. Moreover, labels may occur also in expressions for accessibility relation (relational atoms) like, e.g., $x \leqslant y$ of intuitionistic and intermediate logics.

Definition 2. *A labelled sequent is a sequent consisting of labelled formulas and relational atoms.*

Table 1 depicts the labelled calculus **G3I** for IL. Note that its logical rules are obtained directly from the inductive definition of forcing. The rule $R \supset$ must satisfy the *eigenvariable* condition (y does not occur in the conclusion). The structural rules Ref and $Trans$ for relational atoms correspond to the assumptions of reflexivity and transitivity of $\leqslant$ in $\mathfrak{F}$.

3 Hypersequent Calculi for Intermediate Logics

It was shown in [5] how to transform a large class of Hilbert axioms into structural hypersequent rules in a systematic way. This allowed for the automated definition of cut-free hypersequent calculi for a large class of (substructural) logics. In the case of intermediate logics, the transformation in [5] works for all axioms within the class $\mathcal{P}_3$ of the classification (*substructural hierarchy*) defined

$$x \leqslant y, x : p, \Gamma \Rightarrow \Delta, y : p \qquad \frac{x : \varphi, x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \& \psi, \Gamma \Rightarrow \Delta}\, L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \& \psi}\, R\&$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot \qquad \frac{\Gamma \Rightarrow \Delta, x : \varphi, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \vee \psi}\, R\vee \qquad \frac{x : \varphi, \Gamma \Rightarrow \Delta \quad x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \vee \psi, \Gamma \Rightarrow \Delta}\, L\vee$$

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ref \qquad \frac{x \leqslant y, y : \varphi, \Gamma \Rightarrow \Delta, y : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \supset \psi}\, R\supset \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\, Trans$$

$$\frac{x \leqslant y, x : \varphi \supset \psi, \Gamma \Rightarrow \Delta, y : \varphi \quad x \leqslant y, x : \varphi \supset \psi, y : \psi, \Gamma \Rightarrow \Delta}{x \leqslant y, x : \varphi \supset \psi, \Gamma \Rightarrow \Delta}\, L\supset$$

Table 1. Labelled calculus **G3I** for IL [7]

by the following grammar[3] based on the (propositional) language of **LJ**: $\mathcal{N}_0, \mathcal{P}_0$ contain the set of atomic formulas.

$$\mathcal{P}_{n+1} ::= \bot \mid \top \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \& \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1}$$
$$\mathcal{N}_{n+1} ::= \bot \mid \top \mid P_n \mid \mathcal{N}_{n+1} \& \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$$

The classes $\mathcal{P}_n$ and $\mathcal{N}_n$ contain axioms with leading positive and negative connective, respectively. A connective is positive (negative) if its left (right) logical rule is invertible [1]; note that in the sequent calculus **LJ**, $\vee$ is positive, $\supset$ is negative and & is both positive and negative.

Example 2. The axioms (*lin*), (*wc*) and (*em*) in Example 1 are within the class $\mathcal{P}_3$. The corresponding hypersequent rules can be generated using the PROLOG-system *AxiomCalc*, which implements the algorithm in [5] and is available at `http://www.logic.at/people/lara/axiomcalc.html`.

Theorem 1 ([5]). *Given an axiom $\mathcal{A} \in \mathcal{P}_3$, the rules generated by the algorithm in [5] are sound and complete for the intermediate logic* $\mathrm{IL} + \mathcal{A}$ *and they preserve cut elimination when added to the hypersequent version of* **LJ**.

3.1 Extending the Method - a Case Study

Not all axioms defining intermediate logics are within the class $\mathcal{P}_3$. For instance, (bd_2) (i.e. $\xi \vee (\xi \supset (\varphi \vee (\varphi \supset \psi)))$) is in $\mathcal{P}_4$ and cannot be transformed into an equivalent structural rule using the procedure in [5]. In this section we show how to combine a heuristic method with the procedure in [5] (in fact, its classical and multiple-conclusion version in [6]) to introduce a *logical* rule for (bd_2). We present ad-hoc proofs of soundness, completeness and cut elimination for the resulting calculus.

[3] The substructural hierarchy, as originally defined in [5], is based on the language of Full Lambek calculus with exchange and on the invertibility of its logical rules.

$\dfrac{}{G \mid \varphi \Rightarrow \varphi}$ (*init*)	$\dfrac{}{G \mid \bot \Rightarrow}$ $(\bot, l)$	$\dfrac{G \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}$ (*ee*)	$\dfrac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \varphi \Rightarrow \Delta}$ (w, l)
$\dfrac{G \mid \Gamma \Rightarrow \varphi, \Delta \quad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \supset \psi \Rightarrow \Delta}$ $(\supset, l)$	$\dfrac{G \mid \Gamma, \varphi \Rightarrow \psi}{G \mid \Gamma \Rightarrow \varphi \supset \psi, \Delta}$ $(\supset, r)$	$\dfrac{G \mid \Gamma \Rightarrow \varphi, \varphi, \Delta}{G \mid \Gamma \Rightarrow \varphi, \Delta}$ (c, r)	
$\dfrac{G \mid \Gamma \Rightarrow \varphi, \Delta \quad G \mid \Gamma \Rightarrow \psi, \Delta}{G \mid \Gamma \Rightarrow \varphi \& \psi, \Delta}$ $(\&, r)$	$\dfrac{G \mid \varphi, \psi, \Gamma \Rightarrow \Delta}{G \mid \varphi \& \psi, \Gamma \Rightarrow \Delta}$ $(\&, l)$	$\dfrac{G \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{G \mid \Gamma, \varphi \Rightarrow \Delta}$ (c, l)	
$\dfrac{G \mid \varphi, \Gamma \Rightarrow \Delta \quad G \mid \psi, \Gamma \Rightarrow \Delta}{G \mid \varphi \vee \psi, \Gamma \Rightarrow \Delta}$ $(\vee, l)$	$\dfrac{G \mid \Gamma \Rightarrow \varphi, \psi, \Delta}{G \mid \Gamma \Rightarrow \varphi \vee \psi, \Delta}$ $(\vee, r)$	$\dfrac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \varphi, \Delta}$ (w, r)	
$\dfrac{G \mid \Gamma \Rightarrow \varphi, \Delta \quad H \mid \varphi, \Sigma \Rightarrow \Pi}{G \mid H \mid \Gamma, \Sigma \Rightarrow \Pi, \Delta}$ (*cut*)	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$ (*ec*)	$\dfrac{G}{G \mid \Gamma \Rightarrow \Delta}$ (*ew*)	

Table 2. Hypersequent calculus **HLJ'**

Inspired by [14] we use as base calculus (the hypersequent version of) Maehara's calculus **LJ**' for intuitionistic logic, see [13]. This is a multiple-conclusion version of **LJ** where the intuitionistic restriction, i.e., the consequent of a sequent contains at most one formula, applies only to the right rule of $\supset$ (and $\forall$, in the first order case). The rule schemas for the hypersequent version of **LJ**' (we call this calculus **HLJ'**) are depicted in Table 2. Note that $\Gamma, \Sigma, \Pi, \Delta$ stand for multisets of formulas while G and H denote hypersequents.

The calculus $\mathbf{HBd_2}$ is obtained by extending **HLJ'** with the following rule:

$$\frac{G \mid \Gamma', \Gamma \Rightarrow \Delta' \qquad G \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \varphi \supset \psi, \Delta} \; (bd_2)^*$$

Remark 1. A careful application of the transformation steps of the procedure in [6] to the axiom $\xi \vee (\xi \supset (\varphi \vee (\varphi \supset \psi)))$ yields a similar rule (we call it *(bd_2)'*) with ψ not occurring in the premise. Indeed by using the invertible rules of **HLJ'** ($(\supset, r)$ is when $\Delta = \Gamma = \emptyset$) from $G \mid \Rightarrow \xi \mid \Rightarrow \xi \supset \varphi \vee (\varphi \supset \psi)$ we get

$$G \mid \Rightarrow \xi \mid \xi \Rightarrow \varphi, \varphi \supset \psi$$

which is easily seen to be inter-derivable in **HLJ'** with the following rule:

$$\frac{G \mid \Gamma', \xi \Rightarrow \Delta' \qquad G \mid \Gamma \Rightarrow \xi \qquad G \mid \Gamma, \varphi \Rightarrow \Delta}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \varphi \supset \psi, \Delta}$$

The rule *(bd_2)'* is then obtained by applying cut to the premises $G \mid \Gamma', \xi \Rightarrow \Delta'$ and $G \mid \Gamma \Rightarrow \xi$. However *$(bd_2)$' does not* preserve cut elimination when added to **HLJ'**: e.g., $\Rightarrow \alpha \mid \alpha \Rightarrow \beta, \alpha \supset ((\alpha \supset \beta) \supset \delta)$ can be proved with a cut on $\Rightarrow \alpha \mid \alpha \Rightarrow \beta, \sim \beta$ and $\sim \beta \Rightarrow \alpha \supset ((\alpha \supset \beta) \supset \delta)$ but it has no cut-free proof. The rule $(bd_2)^*$ was obtained by a last heuristic step: by inspecting the counterexample for cut admissibility and changing the rule *(bd_2)'* accordingly.

We show that $\mathbf{HBd}_2$ is sound and complete for the logic Bd_2.

Definition 3. *A hypersequent* $G := \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ *is interpreted as:* $G^I := (\bigwedge \Gamma_1 \supset \bigvee \Delta_1) \vee \cdots \vee (\bigwedge \Gamma_n \supset \bigvee \Delta_n)$ *where* $\bigwedge \Gamma_i$ *is the conjunction & of the formulas in* Γ_i *(*$\top$ *when* Γ_i *is empty), and* $\bigvee \Delta_i$ *is the disjunction of the formulas in* Δ_i *(*$\bot$ *when* Δ_i *is empty).*

The *height* $|d|$ of a derivation d is the maximal number of inference rules + 1 occurring on any branch of d. The *principal formula* of a logical rule is the compound formula introduced in the conclusion. Formulas, which remain unchanged by a rule application, are referred to as *contexts*. Henceforth we use $\vdash_S \varphi$ (or $\vdash_S \Gamma \Rightarrow \Delta$, or $\vdash_S G$) to denote that a formula φ (a sequent $\Gamma \Rightarrow \Delta$, or a hypersequent G) is derivable in the calculus S.

Theorem 2 (Soundness and Completeness). *For any sequent* $\Gamma \Rightarrow \Delta$

$$\vdash_{\mathbf{HBd}_2} \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \vdash_{\mathbf{LJ}+(bd_2)} \Gamma \Rightarrow \Delta$$

Proof. "$\Rightarrow$": We show for any hypersequent G, if $\vdash_{\mathbf{HBd}_2} G$ then $\vdash_{\mathbf{LJ}+(bd_2)} G^I$. By induction on the height of a derivation of G. The base case (G is an initial sequent) is easy. For the inductive case it suffices to see that for each inference rule in $\mathbf{HBd}_2$ with premise(s) G_1 (and G_2), the sequent $G_1^I \Rightarrow G^I$ ($G_1^I, G_2^I \Rightarrow G^I$) is derivable in $\mathbf{LJ}$+(bd_2). The only non-trivial case to show is $(bd_2)^*$:
$\vdash_{\mathbf{LJ}+(bd_2)} (G \mid \Gamma', \Gamma \Rightarrow \Delta')^I, (G \mid \Gamma, \varphi \Rightarrow \psi, \Delta)^I \Rightarrow (G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \varphi \supset \psi, \Delta)^I$
that follows by a (cut) with the axiom (bd_2), i.e., $\Rightarrow \bigwedge \Gamma \vee (\bigwedge \Gamma \supset (\varphi \vee (\varphi \supset \psi)))$.

"$\Leftarrow$": The rules of $\mathbf{LJ}$ are derivable in $\mathbf{HBd}_2$. A proof of the axiom (bd_2) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \varphi, \psi \Rightarrow \psi, \xi}{\Rightarrow \varphi \mid \varphi \Rightarrow \psi, \psi \supset \xi}(bd_2)^*}{\Rightarrow \varphi \mid \varphi \Rightarrow \psi \vee (\psi \supset \xi)}(\vee, r)}{\Rightarrow \varphi \mid \Rightarrow \varphi, \varphi \supset (\psi \vee (\psi \supset \xi))}(\supset, r)}{\Rightarrow \varphi \vee (\varphi \supset (\psi \vee (\psi \supset \xi))) \mid \Rightarrow \varphi \vee (\varphi \supset (\psi \vee (\psi \supset \xi)))}(\vee, r), (w, r)}{\Rightarrow \varphi \vee (\varphi \supset (\psi \vee (\psi \supset \xi)))}(ec)}{}$$

The cut elimination proof for the calculus $\mathbf{HBd}_2$ requires a special strategy. It proceeds by cases according to the cut formula. For non-atomic cut formulas having & and $\vee$ as outermost connective, we use the invertibility of the rules to replace the cut by smaller ones.

Cut formulas having $\supset$ as outermost connective require a different handling. In this case we proceed by shifting the cut upwards in a specific order: First we move the cut upwards in the right derivation d_r which has the cut formula on the right side of the sequent (Lemma 2). If the cut formula is introduced by $(\supset, r)$ or $(bd_2)^*$ we proceed by shifting the cut upwards in the left derivation d_l until the cut formula is introduced and finally cut the premises to replace the cut by smaller ones (Lemma 1). Moving the cut upwards can indeed be problematic in presence of $(\supset, r)$ or $(bd_2)^*$ in d_l. E.g., in the following situation:

$$\dfrac{\begin{array}{c}\vdots d_r \\ G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\vdots d_l \\ H \mid \Sigma, \alpha \supset \beta, \varphi \Rightarrow \psi\end{array}}{H \mid \Sigma, \alpha \supset \beta \Rightarrow \varphi \supset \psi, \Pi}(\supset, r)}{G \mid H \mid \Gamma' \Rightarrow \Delta' \mid \Gamma, \Sigma \Rightarrow \Delta, \varphi \supset \psi, \Pi}(cut)$$

The reason being the presence of the context Δ that does not permit the subsequent application of $(\supset, r)$ to the following derivation

$$\dfrac{\overset{\vdots d_r}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta} \qquad \overset{\vdots d_l}{H \mid \Sigma, \alpha \supset \beta, \varphi \Rightarrow \psi}}{G \mid \Gamma' \Rightarrow \Delta' \mid H \mid \Gamma, \Sigma, \varphi \Rightarrow \Delta, \psi}\ (cut)$$

However, it is always possible to shift the cut upward over d_l when the cut formula in the right premise is introduced by a rule $(\supset, r)$ or $(bd_2)^*$. For instance, in the above case assume that $G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta$ is the conclusion of a $(bd_2)^*$ rule whose premises are

$$d'_r \vdash G \mid \Gamma', \Gamma \Rightarrow \Delta' \quad \text{and} \quad d''_r \vdash G \mid \Gamma, \alpha \Rightarrow \beta, \Delta$$

The original cut above is shifted upwards as follows (we omit the contexts G and H for simplicity):

$$\dfrac{\dfrac{\overset{\vdots d'_r}{\Gamma', \Gamma \Rightarrow \Delta'}}{\Gamma' \Rightarrow \Delta' \mid \Gamma', \Gamma, \Sigma \Rightarrow \Delta'} \qquad \dfrac{\overset{\vdots d_r}{\Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta} \qquad \overset{\vdots d_l}{\Sigma, \alpha \supset \beta, \varphi \Rightarrow \psi}}{\Gamma' \Rightarrow \Delta' \mid \Gamma, \Sigma, \varphi \Rightarrow \Delta, \psi}\ (cut)}{\Gamma' \Rightarrow \Delta' \mid \Gamma' \Rightarrow \Delta' \mid \Gamma, \Sigma \Rightarrow \Delta, \varphi \supset \psi}\ (bd_2)^*$$

Definition 4. *The* complexity $|\varphi|$ *of a formula* φ *is defined as usual:* $|\varphi| = 0$ *if* φ *is atomic,* $|\varphi \& \psi| = |\varphi \vee \psi| = |\varphi \supset \psi| = max(|\varphi|, |\psi|) + 1$. *The* cut-rank $\rho(d)$ *of a derivation* d *is the maximal complexity of cut formulas in* $d + 1$. *(*$\rho(d) = 0$ *if* d *is cut-free).*

We use the following notation where φ is a formula and Σ is a multiset of formulas for $n \geq 0$: $\varphi^n = \overbrace{\{\varphi, \ldots, \varphi\}}^{n}$ *and* $\Sigma^n = \overbrace{\Sigma \cup \ldots \cup \Sigma}^{n}$

Lemma 1 (Shift Left and Reduction of $\supset$). *Let* d_l *and* d_r *be derivations in* **HBd**$_2$ *such that:*

- d_l *is a derivation of* $H \mid \Sigma_1, (\alpha \supset \beta)^{n_1} \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_k, (\alpha \supset \beta)^{n_k} \Rightarrow \Pi_k$,
- d_r *is a derivation of* $G \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta$,
- $\rho(d_l) \leq |\alpha \supset \beta|$ *and* $\rho(d_r) \leq |\alpha \supset \beta|$,
- d_r *ends with an application of* $(\supset, r)$ *or* $(bd_2)^*$ *introducing* $\alpha \supset \beta$.

Then we can find a derivation d *of* $G \mid H \mid \Gamma^{n_1}, \Sigma_1 \Rightarrow \Delta^{n_1}, \Pi_1 \mid \cdots \mid \Gamma^{n_k}, \Sigma_k \Rightarrow \Delta^{n_k}, \Pi_k$ *in* ***HBd***$_2$ *with* $\rho(d) \leq |\alpha \supset \beta|$.

Proof. By induction on $|d_l|$. If $|d_l|$ ends in an axiom, we are done. Otherwise, consider the last inference rule (R) applied in $|d_l|$. Suppose that (R) acts only on H, or (R) is any rule other than $(\supset, l)$ introducing $\alpha \supset \beta$, $(\supset, r)$, or $(bd_2)^*$. Then the claim follows by applications of the inductive hypothesis, (R) and, if needed, weakening and contraction. When $(R) = (\supset, l)$ and $\alpha \supset \beta$ is the principal formula the claim follows by applying the inductive hypothesis and subsequent cuts with cut formulas α and β.

The only interesting cases arise when (R) is $(\supset, r)$ or $(bd_2)^*$.When d_r ends in an application of $(\supset, r)$, the required derivation is simply obtained by applying the inductive hypothesis and (R) (note that in this case Δ is empty and hence no context is added to the premises by the inductive hypothesis).

If d_r ends with $(bd_2)^*$ and $(R) = (\supset, r)$ the case is handled as described on the previous page. Assume that d_r ends with $(bd_2)^*$ and $(R) = (bd_2)^*$ as in the following derivation (we omit the contexts for simplicity):

$$\dfrac{\begin{matrix}\vdots\, d_r \\ \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, d'_l \\ \Sigma', \Sigma_1, (\alpha \supset \beta)^{n_1} \Rightarrow \Pi'\end{matrix} \quad \begin{matrix}\vdots\, d''_l \\ \Sigma_1, (\alpha \supset \beta)^{n_1}, \varphi \Rightarrow \psi, \Pi_1\end{matrix}}{\Sigma' \Rightarrow \Pi' \mid \Sigma_1, (\alpha \supset \beta)^{n_1} \Rightarrow \varphi \supset \psi, \Pi_1}\,(bd_2)^*}{\Gamma' \Rightarrow \Delta' \mid \Sigma' \Rightarrow \Pi' \mid \Gamma^{n_1}, \Sigma_1 \Rightarrow \varphi \supset \psi, \Pi_1, \Delta^{n_1}}\,(cut)$$

where $\Gamma' \Rightarrow \Delta' | \Gamma \Rightarrow \alpha \supset \beta, \Delta$ is the conclusion of a $(bd_2)^*$ rule with premises

$$d'_r \vdash \Gamma', \Gamma \Rightarrow \Delta' \quad \text{and} \quad d''_r \vdash \Gamma, \alpha \Rightarrow \beta, \Delta$$

The cut is moved upwards as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots d'_r \\ \Gamma', \Gamma \Rightarrow \Delta'\end{matrix}}{\Gamma', \Gamma^{n_1}, \Sigma_1 \Rightarrow \Delta'} \qquad \dfrac{\begin{matrix}\vdots d_r \\ \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta\end{matrix} \quad \begin{matrix}\vdots d''_l \\ \Sigma_1, (\alpha \supset \beta)^{n_1}, \varphi \Rightarrow \psi, \Pi_1\end{matrix}}{\Gamma' \Rightarrow \Delta' \mid \Gamma^{n_1}, \Sigma_1, \varphi \Rightarrow \psi, \Delta^{n_1}, \Pi_1}\,(cut)}{\Gamma' \Rightarrow \Delta' \mid \Gamma' \Rightarrow \Delta' \mid \Gamma^{n_1}, \Sigma_1 \Rightarrow \varphi \supset \psi, \Pi_1, \Delta^{n_1}}\,(bd_2)^*$$

Lemma 2 (Shift Right). *Let d_l and d_r be derivations in* $\mathbf{HBd}_2$ *such that:*

- *d_l is a derivation of $H \mid \Sigma, \varphi \Rightarrow \Pi$,*
- *φ is either atomic or of the form $\alpha \supset \beta$,*
- *d_r is a derivation of $G \mid \Gamma_1 \Rightarrow \varphi^{n_1}, \Delta_1 \mid \cdots \mid \Gamma_k \Rightarrow \varphi^{n_k}, \Delta_k$,*
- *$\rho(d_l) \leq |\varphi|$ and $\rho(d_r) \leq |\varphi|$.*

Then we can find a derivation d of $G \mid H \mid \Gamma_1, \Sigma^{n_1} \Rightarrow \Delta_1, \Pi^{n_1} \mid \cdots \mid \Gamma_k, \Sigma^{n_k} \Rightarrow \Delta_k, \Pi^{n_k}$ in $\boldsymbol{HBd}_2$ *with $\rho(d) \leq |\varphi|$.*

Proof. By induction on $|d_r|$. If $|d_r|$ ends in an axiom, we are done. Otherwise, consider the last inference rule (R) in $|d_r|$. If (R) acts only on G or (R) is any rule other than a logical rule introducing φ then the claim follows by applications of the inductive hypothesis, (R) and, if needed, weakening or contraction. If (R) is $(\supset, r)$ or $(bd_2)^*$ and φ is the principal formula. The claim follows by applications of the inductive hypothesis, the corresponding rule (R) and Lemma 1.

Theorem 3 (Cut elimination). *Cut elimination holds for* $\mathbf{HBd}_2$.

Proof. Let d be a derivation in $\mathbf{HBd}_2$ with $\rho(d) > 0$. The proof proceeds by a double induction on $\langle \rho(d), \#\rho(d) \rangle$, where $\#\rho(d)$ is the number of applications of (cut) in d with cut rank $\rho(d)$. Consider an uppermost application of (cut) in d with cut rank $\rho(d)$. Let d_l and d_r be its premises, where d_l is a derivation of $H \mid \Sigma, \varphi \Rightarrow \Pi$, and d_r is a derivation of $G \mid \Gamma \Rightarrow \varphi, \Delta$. We can find a proof of $G \mid H \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi$ in which either $\rho(d)$ or $\#\rho(d)$ decreases. Indeed we distinguish the following cases according to φ:

- φ is an atomic formula or $\varphi = \alpha \supset \beta$. The claim follows by Lemma 2.
- Suppose $\varphi = \alpha \vee \beta$. Being $\vee$ an invertible connective in $\mathbf{HBd}_2$ on the left and on the right (standard proof), we can find the derivations $d'_r \vdash G \mid \Gamma \Rightarrow \alpha, \beta, \Delta$, as well as $d'_l \vdash H \mid \alpha, \Sigma \Rightarrow \Pi$ and $d''_l \vdash H \mid \beta, \Sigma \Rightarrow \Pi$. The claim follows by replacing the cut with cut formula $\alpha \vee \beta$ with cuts on α and β.
- The case $\varphi = \alpha \& \beta$ is similar since & is also invertible on both sides.

4 Labelled Calculi for Intermediate Logics

A methodology to define cut-free labelled calculi for a large class of intermediate logics is contained in [7, 10]. The resulting calculi are obtained by adding to the labelled intuitionistic system **G3I** (see Table 1) new structural rules, corresponding to the peculiar frame conditions of the considered logics.

The (formulas defining) frame conditions, to which the method in [7] applies, are called *geometric formulas*. These consist of conjunctions of formulas of the form $\forall \overline{x}(P_1 \wedge \cdots \wedge P_m \rightarrow \exists \overline{y}(M_1 \vee \cdots \vee M_n))$, where $\overline{x}, \overline{y}$ are sequences of bound variables, each P_i is a relational atom, each M_j is a conjunction of relational atoms $Q_{j_1}, \ldots, Q_{j_k}$ and $\overline{y}$ does not appear in $P_1, \ldots, P_m$. If $\overline{y}$ does not appear in M_i (for all $i = 1, \ldots, n$) the resulting formula is called a *universal axiom*. As shown in [7], the rule scheme corresponding to geometric formulas has the form

$$\frac{\overline{Q}_1[z_1/y_1], P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \cdots \quad \overline{Q}_n[z_n/y_n], P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\ (geom)$$

where each $\overline{Q_j}$ is the multiset of $Q_{j_1}, \ldots, Q_{j_k}$ and $z_1, \ldots, z_n$ are *eigenvariables* (i.e. variables not occurring in the conclusion). The accessibility relation $\leqslant$ in all intermediate logics of Example 1 is characterized by universal or geometric axioms.

Theorem 4 ([7]). *Cut is admissible in any extension of* **G3I** *by rules of the form* (*geom*)*. Weakening and contraction are height-preserving (hp-) admissible, i.e. whenever their premises are derivable, so is their conclusion with at most the same derivation height. All rules are hp-invertible.*

Henceforth we will use $P, Q, \ldots$ (possibly indexed) to indicate relational atoms and $A, B, C, \ldots$ (possibly indexed) for compound formulas.

4.1 Towards a Systematic Approach

Inspired by the algorithms in [5, 6] for hypersequent calculi, we provide a systematic method to transform a large class of frame conditions for intermediate logics into labelled rules. Soundness, completeness and cut-elimination are proved for the generated calculi, that in the case of geometric formulas coincide with those introduced in [7].

We classify the frame conditions characterizing intermediate logics into a hierarchy which intuitively accounts for the difficulty to deal proof theoretically with the corresponding formulas of first-order classical logic. As for the substructural hierarchy in [5] (see Section 3) the classification is based on the invertibility of the logical/quantifier rules of the base calculus, which in our case is **LK'**, i.e., a variant of Gentzen **LK** calculus for first-order classical logic in which all logical rules are invertible, while the universal (existential) quantifier is invertible on the right (respectively on the left). W.l.o.g. we will consider formulas in prenex form. The class to which a formula belongs is determined by the alternation of universal and existential quantifiers in the prefix. The resulting classification is essentially the arithmetical hierarchy.

Definition 5. *The classes Π_k and Σ_k are defined as follows: $A \in \Sigma_0$ and $A \in \Pi_0$, if A is quantifier-free. Otherwise:*

- *if A is classically equivalent to $\exists\overline{x}B$ where $B \in \Pi_n$ then $A \in \Sigma_{n+1}$*
- *if A is classically equivalent to $\forall\overline{x}B$ where $B \in \Sigma_n$ then $A \in \Pi_{n+1}$*

Example 3. Universal axioms are in Π_1, while geometric formulas are in Π_2.

We show below how to transform all formulas within the class Π_2 into structural labelled rules that preserve cut-elimination once added to (a slightly modified version of) **G3I**. The resulting rules are *equivalent* to the corresponding axioms, that is, **LK'** extended with the defined rules or **LK'** extended with the original formula proves the same sequents.

As for the algorithm in [5, 6] (see Remark 1), the key ingredients for our transformation are: (1) the invertibility in **LK'** of the rules $R\forall$ (i.e. introduction of $\forall$ on the right) and $L\exists$ (i.e. introduction of $\exists$ on the left) and of all logical rules; (2) the following lemma that allows formulas to change the side of the (labelled) sequent going from the conclusion to the premises.

Lemma 3 ([5]). *The sequent $A_1, \dots, A_n \Rightarrow B_1, \dots, B_m$ is equivalent to the rule*

$$\frac{B_1, \Gamma \Rightarrow \Delta \quad \cdots \quad B_m, \Gamma \Rightarrow \Delta}{A_1, \dots, A_n, \Gamma \Rightarrow \Delta}$$

where Γ, Δ are fresh metavariables standing for multisets of formulas.

Proof. "$\Rightarrow$": Follows by m applications of CUT (and weakening). "$\Leftarrow$": Follows by instantiating $\Gamma = \emptyset$ and $\Delta = B_1, \dots, B_m$.

Theorem 5. *Every frame condition $\mathcal{F}$ within the class Π_2 can be transformed into a set of equivalent structural rules in labelled calculi.*

Proof. Let $\mathcal{F} = \forall\overline{x}\exists\overline{y}A$, where A is a quantifier-free formula, $\overline{x} = x_1, \dots, x_h$ and $\overline{y} = y_1, \dots, y_l$. W.l.o.g. we assume that A is in disjunctive normal form and has the shape $B_1 \vee \cdots \vee B_k$ where every B_i has the form $Q_{i_1} \wedge \cdots \wedge Q_{i_n} \wedge \neg P_{i_1} \wedge \cdots \wedge \neg P_{i_m}$. By the invertibility of the rule $R\forall$, $\Rightarrow \mathcal{F}$ is equivalent to $\Rightarrow \exists\overline{y}A'$, where A' is obtained by replacing in A all $x_1, \dots, x_h$ with fresh variables $x'_1, \dots, x'_h$ (*eigenvariable* condition). We distinguish two cases according to whether $\mathcal{F}$ contains at least one existential quantifier ($\mathcal{F} \in \Pi_2$) or it does not ($\mathcal{F} \in \Pi_1$).

Assume that $l = 0$ ($\mathcal{F} \in \Pi_1$). By the invertibility of $R\vee$, $R\wedge$ and $R\neg$, $\Rightarrow A'$ is equivalent to a set of *atomic* sequents $\overline{P} \Rightarrow \overline{Q}$ with $\overline{P}, \overline{Q}$ multisets of relational atoms P_{i_r}, Q_{i_s}. By Lemma 3, these sequents are equivalent to rules of the form

$$\frac{\overline{Q}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \; (\Pi_1')$$

Assume that $l > 0$ ($\mathcal{F} \in \Pi_2$). By Lemma 3, $\Rightarrow \exists \overline{y} A'$ is equivalent to $\dfrac{\exists \overline{y} A', \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ which is in turn equivalent to $\dfrac{A'', \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ where A'' is obtained by replacing in A all $y_1, \ldots, y_l$ with fresh variables $y_1', \ldots, y_l'$ (*eigenvariable* condition). By the invertibility of $L\vee, L\wedge$ and $L\neg$ we get

$$\frac{\{Q_{i_1}, \ldots, Q_{i_n}, \Gamma \Rightarrow \Delta, P_{i_1}, \ldots, P_{i_m}\}_{i=1\ldots k}}{\Gamma \Rightarrow \Delta} \; (\Pi_2)$$

The resulting rules are equivalent to $\mathcal{F}$.

Remark 2. The (Π_2) rule (which is, in fact, a rule schema) is invertible. To make (Π_1') invertible we simply repeat $\overline{P}$ in its premises, thus obtaining

$$\frac{\overline{P}, \overline{Q}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \; (\Pi_1)$$

which is interderivable with the rule (Π_1') in **LK'**.

Observe that while (Π_1) coincides with the rule defined in [7] for universal axioms, this is not the case for geometric formulas. Indeed the above procedure applied to a geometric formula generates a rule of the form (Π_2) which might contain relational atoms $(P_{i_1}, \ldots, P_{i_m})$ on the right hand side of premises and is therefore not of the form (*geom*) [7]. Being geometric formulas Π_2 formulas of a *particular* shape, we show below that the (Π_2) rules for them (generated by Th. 5) can be easily transformed into rules with no relational atom on the right hand side; the resulting rules are nothing but the (*geom*) rules in [7].

Corollary 1. *Geometric axioms are equivalent to rules of the form* (*geom*).

Proof. Geometric axioms are formulas in Π_2 of the form $\forall \overline{x} \exists \overline{y} A_G$, where A_G is $B_1 \vee \cdots \vee B_n \vee C_1 \vee \cdots \vee C_m$ where each B_i is $Q_{i_1} \wedge \cdots \wedge Q_{i_k}$ and each C_j is $\neg P_j$. Theorem 5 transforms such an axiom into the equivalent rule

$$\frac{\{Q_{i_1}, \ldots, Q_{i_k}, \Gamma \Rightarrow \Delta\}_{i=1\ldots n} \quad \{\Gamma \Rightarrow \Delta, P_j\}_{j=1\ldots m}}{\Gamma \Rightarrow \Delta} \; (\Pi_2')$$

The claim follows by showing that (Π_2') can be transformed into a rule

$$\frac{\overline{Q}_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \cdots \quad \overline{Q}_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta} \; (\Pi_2^G)$$

where each $\overline{Q}_i$ is a multiset of $Q_{i_1}, \ldots, Q_{i_k}$. Observe that (Π_2^G) is nothing but a (*geom*) rule [7]. To derive (Π_2^G) we use (Π_2') and m initial sequents:

$$\frac{\{\overline{Q}_i, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta\}_{i=1\ldots n} \quad \{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta, P_j\}_{j=1\ldots m}}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta} \; (\Pi_2')$$

To derive (Π_2') we first apply (Π_2^G) followed by m applications of CUT.

Rules for non-geometric Π_2 formulas manipulate relational atoms in both sides of the sequent. We show below that this is not an obstacle for obtaining admissibility results analogous to those in Theorem 4. The base calculus we will work with is a slightly modified version of **G3I** which is obtained by adding initial sequents of the form $x \leqslant y, \Gamma \Rightarrow \Delta, x \leqslant y$ to **G3I**. Note that these sequents, which are needed for our completeness proof (Theorem 6), were first introduced for **G3I** and later removed as they were not needed in the labelled systems for intermediate logics presented in [7]; the reason being that in these systems no rule contains atoms $x \leqslant y$ in the succedent.

Henceforth we denote by **G3SI*** (super-intuitionistic) the system obtained by adding to our base calculus rules of the form (Π_1) and (Π_2) defined by applying Theorem 5 to the set $*$ of formulas within the class Π_2.

Consider the following version of the structural rules for contraction and weakening (Z is either a labelled formula $u : \varphi$ or a relational atom $x \leqslant y$):

$$\frac{\Gamma \Rightarrow \Delta}{Z, \Gamma \Rightarrow \Delta}\,\text{L-W} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, Z}\,\text{R-W} \qquad \frac{Z, Z, \Gamma \Rightarrow \Delta}{Z, \Gamma \Rightarrow \Delta}\,\text{L-C} \qquad \frac{\Gamma \Rightarrow \Delta, Z, Z}{\Gamma \Rightarrow \Delta, Z}\,\text{R-C}$$

Table 3. Structural rules

Lemma 4. *In* **G3SI*** *we have:*

1. *Substitution of variables is hp-admissible;*
2. *Weakening is hp-admissible;*
3. *All the rules are hp-invertible;*
4. *Contraction is hp-admissible.*

Proof. 1. We need to show that if y is free for x in every formula in $\Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta$ is derivable in **G3SI*** then so is $\Gamma[y/x] \Rightarrow \Delta[y/x]$ with the same derivation height. The proof is by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$ and follows mostly the proof of the same theorem in [7].

2. The proof follows the pattern of [7] with the new case of R-W for $x \leqslant y$. Also in this case the proof is by induction on the premise of weakening. When $\Gamma \Rightarrow \Delta$ is concluded by a rule R with variable condition, i.e. by $R \supset$ or (Π_2), we first might need to replace the *eigenvariable* of the rule with a new one. For instance, if R is $R \supset$ and the premise of weakening is a sequent of the form $x \leqslant y, x : \varphi, \Gamma \Rightarrow \Delta', y : \psi$, we first replace y with a new z (Lemma 4.1) and obtain $x \leqslant z, x : \varphi, \Gamma \Rightarrow \Delta', z : \psi$; now by the inductive hypothesis $x \leqslant z, x : \varphi, \Gamma \Rightarrow \Delta', z : \psi, x \leqslant y$; the conclusion follows by $R \supset$.

3. Observe that $L \supset$ and the rules (Π_1) and (Π_2) are hp-invertible since their premises are obtained from the conclusion by applying weakening which is hp-admissible. For the other rules the proof is as in [7].

4. Similar to the case of weakening.

The proof of soundness and completeness of our calculi follows the same pattern of the analogous proof in [11] and it is sketched below. Let $\mathfrak{F}_{SI^*} = \langle W, \leqslant \rangle$ be a frame with the properties of the accessibility relation expressed as (Π_2 and Π_1) formulas in $*$. Let $L = \{x, y, z \dots\}$ be the labels occurring in a **G3SI***-derivation. An *interpretation* I of L in $\mathfrak{F}_{SI^*}$ is a function $I : L \rightarrow W$.

Definition 6. *Let* $\mathfrak{M}_{SI^*} = \langle \mathfrak{F}_{SI^*}, \Vdash \rangle$ *be a model and* I *an interpretation. A labelled sequent* $\Gamma \Rightarrow \Delta$ *is* valid in $\mathfrak{M}_{SI^*}$ *if for every interpretation* I *we have: if for all labelled formulas* $x : \varphi$ *and relational atoms* $y \leqslant z$ *in* Γ*,* $x^I \Vdash \varphi$ *and* $y^I \leqslant z^I$ *hold, then for some* $w : \psi$*,* $u \leqslant v$ *in* Δ *we have* $w^I \Vdash \psi$ *or* $u^I \leqslant v^I$*. A sequent* $\Gamma \Rightarrow \Delta$ *is* valid *in a frame* $\mathfrak{F}_{SI^*}$ *when it is valid in every model* $\mathfrak{M}_{SI^*}$*.*

Theorem 6 (Soundness and Completeness). *For any sequent* $\Gamma \Rightarrow \Delta$

$$\vdash_{\mathbf{G3SI}^*} \Gamma \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \Delta \text{ is valid in every frame } \mathfrak{F}_{SI^*}.$$

Proof. "$\Rightarrow$": By induction on the height of a derivation of $\Gamma \Rightarrow \Delta$ in **G3SI***. The claim is straightforward if $\Gamma \Rightarrow \Delta$ is initial (notice the new case of sequents of the form $x \leqslant y, \Gamma' \Rightarrow \Delta', x \leqslant y$ that are clearly valid). The cases of the rules for **G3I** are as in [11] with $R\supset$ similar to the case $R\Box$, while (Π_2) is handled as the mathematical rules there with *eigenvariable*.

"$\Leftarrow$": We show that each sequent $\Gamma \Rightarrow \Delta$ is either derivable in **G3SI*** or it has a countermodel in a frame with properties expressed by formulas in $*$. We first construct in the usual manner a derivation tree for $\Gamma \Rightarrow \Delta$ by applying the rules of **G3SI*** root first. If the reduction tree is finite, i.e., all leaves are initial or conclusions of $L\bot$, we have a proof in **G3SI***. Assume that the derivation tree is infinite. By König's lemma, it has an infinite branch that is used to build the needed counterexample. Let $\Gamma \Rightarrow \Delta = \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_i \Rightarrow \Delta_i, \ldots$ be one such branch. Consider the sets $\mathbf{\Gamma} \equiv \bigcup \Gamma_i$ and $\mathbf{\Delta} \equiv \bigcup \Delta_i$ for $i \geqslant 0$. We now construct a countermodel, i.e. a model that makes all labelled formulas and relational atoms in $\mathbf{\Gamma}$ true and all labelled formulas and relational atoms in $\mathbf{\Delta}$ false. Let $\mathfrak{F}_{SI^*}$ be a frame, whose elements are all the labels occurring in $\mathbf{\Gamma}, \mathbf{\Delta}$. $\mathfrak{F}_{SI^*}$ is defined as follows: (i) for all $x : p$ in $\mathbf{\Gamma}$ it holds that $x^I \Vdash p$ in $\mathfrak{F}_{SI^*}$; (ii) for all $x \leqslant y$ in $\mathbf{\Gamma}$ we have $x^I \leqslant y^I$ in $\mathfrak{F}_{SI^*}$; (iii) for all $x' : p'$ in $\mathbf{\Delta}$ we have $x'^I \nVdash p'$ in $\mathfrak{F}_{SI^*}$; finally (iv) for all $x' \leqslant y'$ in $\mathbf{\Delta}$ it holds $x'^I \nleqslant y'^I$ in $\mathfrak{F}_{SI^*}$. $\mathfrak{F}_{SI^*}$ is well defined as it is not the case that either $x \leqslant y$ is in Γ_i and $x \leqslant y$ is in Δ_j or $x \leqslant y, x : p$ is in Γ_i and $y : p$ is in Δ_j (for any i and j), as otherwise we would have an initial sequent and therefore the branch would be finite, against the hypothesis. We then show that for any formula φ, φ is forced at x^I if $x : \varphi$ is in $\mathbf{\Gamma}$ and φ is not forced at x^I if $x : \varphi$ is in $\mathbf{\Delta}$. As all relational atoms in $\mathbf{\Gamma}$ are true and those in $\mathbf{\Delta}$ are false by definition of $\mathfrak{F}_{SI^*}$ we have a countermodel to $\Gamma \Rightarrow \Delta$. By induction on the formula φ.

If φ is $\bot$, it cannot be in $\mathbf{\Gamma}$ because no sequent in the branch contains $x : \bot$ in the antecedent, so it is not forced at any node of the model. If φ is an atom p in $\mathbf{\Gamma}$ then $x^I \Vdash p$ by definition; and $x^I \nVdash p$ if it is in Δ.

If $x : \varphi \& \psi$ is in $\mathbf{\Gamma}$, there exists i such that $x : \varphi \& \psi$ appears first in Γ_i, and therefore, for some $j \geqslant 0$, $x : \varphi$ and $x : \psi$ are in Γ_{i+j} . By inductive hypothesis, $x \Vdash \varphi$ and $x \Vdash \psi$ and therefore $x \Vdash \varphi \& \psi$ (analogous for $x : \varphi \vee \psi$ in $\mathbf{\Delta}$).

If $x : \varphi \& \psi$ is in $\mathbf{\Delta}$ then either $x : \varphi$ or $x : \psi$ is in $\mathbf{\Delta}$. By inductive hypothesis, $x \nVdash \varphi$ or $x \nVdash \psi$ and therefore $x \nVdash \varphi \& \psi$ (analogous for $x : \varphi \vee \psi$ in $\mathbf{\Gamma}$).

If $x : \varphi \supset \psi$ is in $\mathbf{\Gamma}$, we consider all the relational atoms $x \leqslant y$ that occur in $\mathbf{\Gamma}$. If there is no such atom then $x \Vdash \varphi \supset \psi$ is in the model. Else, for any occurrence of $x \leqslant y$ in $\mathbf{\Gamma}$, by construction of the tree either $y : \varphi$ is in $\mathbf{\Delta}$ or

$y : \psi$ is in $\mathbf{\Gamma}$. By inductive hypothesis $y \not\Vdash \varphi$ or $y \Vdash \psi$, and since $x \leqslant y$ we have $x \Vdash \varphi \supset \psi$ in the model.

If $x : \varphi \supset \psi$ is in $\mathbf{\Delta}$, at next step of the reduction tree we have that $x \leqslant y$ and $y : \varphi$ are in $\mathbf{\Gamma}$, whereas $y : \psi$ is in $\mathbf{\Delta}$. By inductive hypothesis this gives $x \leqslant y$ and $y \Vdash \varphi$ but $y \not\Vdash \psi$, i.e. $x \not\Vdash \varphi \supset \psi$.

Theorem 7 (Cut elimination). *The cut rule (Z is either $u : \varphi$ or $x \leqslant y$)*

$$\frac{\Gamma \Rightarrow \Delta, Z \quad Z, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \text{CUT}$$

can be eliminated from **G3SI****-derivations.*

Proof. We distinguish two cases according to the cut formula Z. When Z is a labelled formula $u : \varphi$, the proof has the same structure of the cut elimination proof in [7] for **G3I** extended with rules of the form (*geom*). It proceeds by a double induction on the complexity of the cut formula and on the sum of the derivation heights of the premises of cut. We observe that the additional initial sequents, i.e. $x \leqslant y, \Sigma \Rightarrow \Pi, x \leqslant y$, make no trouble as Z belongs to $\Sigma \Rightarrow \Pi$. Moreover, cuts can be permuted upward over any structural rule (Π_1) and (Π_2). To avoid clashes with the variable conditions when permuting a cut with (Π_2) (or with $R\supset$) an appropriate substitution (Lemma 4.1) is used.

When Z is a relational atom $x \leqslant y$ the proof proceeds by induction on the derivation height of the right premises of cut, i.e. $\Gamma \Rightarrow \Delta, x \leqslant y$. The base case is when this is initial; then it is either *(i)* $u \leqslant v, u : p, \Gamma'' \Rightarrow, \Delta'', v : p, x \leqslant y$; or *(ii)* $u \leqslant v, \Gamma'' \Rightarrow \Delta'', u \leqslant v, x \leqslant y$; or else *(iii)* $x \leqslant y, \Gamma'' \Rightarrow \Delta, x \leqslant y$. If *(i)* or *(ii)*, the conclusion of cut is initial. Otherwise, if *(iii)*, the conclusion of cut is obtained by weakening (Lemma 4.2). Assume that $\Gamma \Rightarrow \Delta, x \leqslant y$ is not initial and that R is the last rule applied to derive it. We reason by cases according to R and show that the cut can be shifted upwards over the premise(s) of R. The key observation is that $x \leqslant y$ is left unchanged by the application of R as no rule of **G3SI*** changes the relational atoms appearing on the right hand side of its conclusion. If R is a logical rule other than $R\supset$ or a rule following the (Π_2) scheme then cut is simply permuted upwards with R. For instance let R be (Π_1); then the derivation

$$\frac{\dfrac{Q_1, \dots, Q_m, P_1, \dots, P_n, \Gamma'' \Rightarrow \Delta, x \leqslant y}{P_1, \dots, P_n, \Gamma'' \Rightarrow \Delta, x \leqslant y}\ \Pi_1 \qquad x \leqslant y, \Gamma' \Rightarrow \Delta'}{P_1, \dots, P_n, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\ \text{CUT}$$

is transformed into

$$\frac{\dfrac{Q_1, \dots, Q_m, P_1, \dots, P_n, \Gamma'' \Rightarrow \Delta, x \leqslant y \quad x \leqslant y, \Gamma' \Rightarrow \Delta'}{Q_1, \dots, Q_m, P_1, \dots, P_n, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\ \text{CUT}}{P_1, \dots, P_n, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\ \Pi_1$$

If R is a rule with variable condition as $R\supset$ or a rule following the (Π_2) scheme then we need first to replace the *eigenvariable* in the premise(s) of R and then permute cut and R. Note that the permutation with a $R\supset$ rule is not problematic as the cut formula $x \leqslant y$ on the right hand side always belongs to the context of the rule (i.e., to the Δ in the rule schemas in Table 1).

Open Problems: (1) Characterize the class of axioms that can be transformed into equivalent hypersequent *logical* rules (Section 3.1 shows a particular axiom for which this is the case) and define an algorithm for the transformation. Note that when defining logical rules the cut-admissibility of the resulting calculus needs either an ad-hoc syntactic proof or suitable semantic methods as in [12].

(2) Are there intermediate logics characterized by frame conditions that are Π_2 formulas not equivalent to any geometric formula?

(3) Not all frame conditions are formulas within the class Π_2. As shown in [4], all axiomatizable intermediate logics are definable by *canonical formulas* that are in the class $\mathcal{N}_3$ of the substructural hierarchy (cf. Sec. 3). In light of this result, which is the maximum nesting of quantifiers occurring in formulas defining frame conditions for intermediate logics? How to capture all[4] these formulas?

Acknowledgment: We are grateful to Sara Negri for her suggestions and for pointing out [11] to us.

References

1. J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
2. A. Avron. A Constructive Analysis of RM. *J. of Symb.Logic*, 52(4):939–951. 1987.
3. A. Avron. The method of hypersequents in the proof theory of propositional non-classical logic. In W. Hodges, M. Hyland, C. Steinhorn and J. Truss (eds), *Logic: From Foundations to Applications*. Oxford University Press, pp. 1–32, 1996.
4. A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.
5. A. Ciabattoni, N. Galatos and K. Terui. From axioms to analytic rules in nonclassical logics. In: *Proceedings of LICS'08*, IEEE, 229–240, 2008.
6. A. Ciabattoni, L. Strassburger and K. Terui. Expanding the realm of systematic proof theory. In: *Proceedings of CSL'09*, LNCS, 163–178, 2009.
7. R. Dyckhoff and S. Negri. Proof analysis in intermediate logics. *Archive for Mathematical Logic*, 51(1-2): 71–92, 2012.
8. D. Gabbay. *Labelled Deductive Systems: Foundations*. Oxford Univ, Press, 1996.
9. S. Negri. Proof analysis beyond geometric theories: from rule systems to systems of rules. *Submitted*.
10. S. Negri. Proof analysis in non-classical logics. In *Logic Coll. 2005*, 107–128, 2007.
11. S. Negri. Kripke completeness revisited. In Acts of Knowledge - History, Philosophy and Logic, G. Primiero and S. Rahman (eds.), College Publications, 2009.
12. O. Lahav. From Frame Properties to Hypersequent Rules in Modal Logics. In: *Proceedings of LICS 2013*.
13. G. Takeuti. *Proof Theory*. 2nd edition, North-Holland, 1987.
14. R. Rothenberg. *On the relationship between hypersequent calculi and labelled sequent calculi for intermediate logics with geometric Kripke semantics*. PhD thesis, 2010.
15. A. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, 1994.
16. L. Viganò. *Labelled Non-Classical Logics*. Kluwer, 2000.
17. H. Wansing. Sequent Systems for Modal Logic. In D. Gabbay and F. Guenther (eds), *Handbook of Philosophical Logic*. Kluwer, Dordrecht, pp.61–145, 2002.

[4] See [9] for a recent work in this direction.