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Towards an Algorithmic Construction of Cut-Elimination Procedures

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We investigate cut-elimination in propositional substructural logics. The problem is to decide whether a given calculus admits (reductive) cut-elimination. We show that, for commutative single-conclusion sequent calculi containing generalized knotted structural rules and arbitrary logical rules, the problem can be decided by resolution-based methods. A general cut-elimination proof for these calculi is also provided.

1. Introduction

Gentzen sequent calculi have been the central tool in many proof-theoretical investigations and applications of logic in algebra and computer science. A key property of these calculi is cut-elimination (*Gentzen's Hauptsatz*), first established by Gentzen (1935) for the sequent calculi **LK** and **LJ** for classical and intuitionistic first-order logic. The removal of cuts corresponds to the elimination of intermediate statements (lemmas) from proofs resulting in calculi in which proofs are *analytic* in the sense that all statements in the proofs are subformulae of the result. Analytic proof calculi for logics are not only an important theoretical tool, useful for understanding relationships between logics and proving metalogical properties like consistency, decidability, admissibility of rules and interpolation, but also the key to develop automated reasoning methods. These calculi also provide an alternative representation of varieties of algebras (see e.g. (Galatos and Ono 2006)) which can then be used to give syntactic proofs of algebraic properties, e.g. amalgamation, for which (in particular cases) semantic methods are not known. Cut-elimination is also a powerful tool to prove the completeness of a given analytic sequent calculus with respect to a logic formalized using Hilbert style systems, as the *cut rule* simulates modus ponens.

Cut-elimination proofs have been provided for very many sequent calculi, mainly on a case by case basis (even when the arguments for a given calculus are similar to that of another) and using heavy syntactic arguments usually written without filling in the details. This renders the proof checking difficult and the whole process of eliminating cuts rather opaque.

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In this paper we perform a resolution-based analysis of cut-elimination in *knotted commutative calculi*. These are propositional single-conclusion sequent calculi containing arbitrary logical rules (satisfying suitable conditions), the permutation rule and (possibly) unary structural rules generalizing both the weakening and contraction rules in Gentzen's **LJ**. The considered structural rules are a generalization of the *knotted structural rules* in (Hori, Ono and Schellinx 1994), whose (n, k) type is of the form: from $\Gamma, A, \ldots, A$ (n times) $\rightarrow C$ infer $\Gamma, A, \ldots, A$ (k times)$\rightarrow C$, for all $n \geq 0$ and $k \geq 1$. The (n, k) rule is a restricted form of weakening when $n < k$, and of contraction when $k < n$. In (Hori, Ono and Schellinx 1994) extensions of intuitionistic linear logic without the exponential connectives **ILL** and of its implicational fragment **BCI** with (n, k) have been investigated from the syntactic and semantic point of view. It was shown that **BCI** extended with the (n, k) rule admits cut-elimination if and only if $k = 1$. Moreover **BCI** extended with both weakening and the $(n + 1, n)$ rule admits cut-elimination if and only if $n = 1$ while **BCI** extended with contraction and the $(n, n+1)$, if and only if $n = 0$. A cut-elimination proof working for these cases was also presented. Hori and other's analysis applies only to calculi consisting of *one* knotted structural rule and a *fixed* set of connectives: those of **ILL**.

A larger class of single-conclusion calculi, containing arbitrary structural rules and logical rules satisfying some restrictions, was considered in (Ciabattoni and Terui 2006), where necessary and sufficient conditions for *reductive cut-elimination* were provided. Reductive cut-elimination is a naturally strengthened version of cut-elimination in presence of axioms (see e.g. (Buss 1998)) which encompasses the "standard" cut-elimination methods working by 1. shifting up cuts and 2. replace them with smaller cuts, when the cut formula is introduced by logical rules in both premisses. The syntactic conditions defined there (*reductivity* and *weak substitutivity*) formalize the steps 1. and 2. above. No decision procedure to check whether a calculus admits reductive cut-elimination was defined in (Ciabattoni and Terui 2006).

A decision procedure for cut-elimination is instead contained in (Avron and Lev 2005) for multiple-conclusion calculi with *all* structural rules (weakening, exchange and contraction). Each calculus belonging to this class admits cut-elimination if and only if its logical rules are *coherent*, i.e. for each set of rules introducing a connective, the formulae in their premisse(s) from which the principal formula derives form an inconsistent set of clauses. E.g, the set of clauses $\{\vdash \alpha_1; \vdash \alpha_2; \alpha_1, \alpha_2 \vdash\}$, corresponding to the rules for conjunction in **LK** is inconsistent. The analysis in (Avron and Lev 2005), based on semantic techniques (non-deterministic matrices), strongly relies on the presence of all structural rules. The same holds for Basin and Ganzinger (2001) that use ordered resolution to prove cut-elimination and decide rule dependency in **LK**.

Miller and Pimentel (2002;2005) Extended Avron and Lev's analysis to first-order sequent calculi possibly without the weakening rules and/or the contraction rules. In particular, they introduced a *sufficient* condition for any such a calculus to admit cut-elimination together with an algorithm (based on the encoding of the considered calculi into a linear logic based framework) to check them. Moreover, they provided a decision procedure for derivability of rules in these calculi. Their analysis does not apply however to calculi

with additional structural rules other than standard weakening and contraction, and in particular fails in case of knotted structural rules (even of the form $(n, 1)$ for some $n > 2$). In this paper we provide tools for deciding whether a knotted commutative calculus admits reductive cut-elimination and for automating cut-elimination proofs in these calculi. We define algorithms to check whether rules of knotted commutative calculi satisfy reductivity and weak substitutivity – the necessary and sufficient conditions in Ciabattoni and Terui (2006). To decide reductivity we develop a substructural resolution calculus and make use of normalization of clauses and of subsumption, while for weak substitutivity we use combinatorial arguments; the latter also serve to decide the dependency (derivability) of structural rules, thus obtaining a method to transform knotted commutative calculi which (by their form) do not admit reductive cut-elimination into others which do. Finally we provide a constructive proof of reductive cut-elimination for knotted commutative calculi satisfying reductivity and weak substitutivity.
The long range aim is to develop a uniform method to prove (or disprove) cut-elimination for a wide class of substructural logics. The advantage of such a method would be a twofold one: 1. it becomes easier to prove (or disprove) cut-elimination theorems for new sequent type logic calculi and 2. the construction of the cut-elimination methods can be automatized - provided the general method is computational.

2. Basic Notions

Let us indicate with $\star_1, \star_2, \star_3, \ldots$ logical connectives of suitable arity. A *formula* A is either a propositional variable or a *compound formula* of the form $\star(A_1, \ldots, A_m)$ where $A_1, \ldots, A_m$ are formulae. Let $\Gamma, \Delta, \Pi, \Sigma, \ldots$ stand for (possibly empty) multisets of formulae and S, T for arbitrary sequents. To specify inference rules as rule schemata we will use *meta-variables* (or *formula-variables*) $\alpha, \beta, \ldots$, standing for arbitrary formulae, and (possibly empty) multisets $\Theta, \Xi, \Phi, \Psi, \Upsilon, X, Y, x, y, \ldots$ of meta-variables. ϵ will always denote the empty multiset (of formulae or meta-variables).
When $n \geq 0$, Γ^n and x^n denote $\Gamma, \ldots, \Gamma$ and $x, \ldots, x$ (n times), respectively.
Given a (meta)sequent $\Gamma \Rightarrow \Delta$ ($\Theta \Rightarrow \Xi$)

- Γ (Θ) is called *antecedent*, while Δ (Ξ) *consequent*
- the (meta)sequent is said to be *single-conclusion* if its consequent contains at most one formula (meta-variable).
- the (meta)sequent is called a *clause* if it does not contain logical connectives. A single-conclusion clause is called *Horn clause.* A clause with at most two atoms is called *Krom clause.*

A sequent calculus is *single-conclusion* if so are all its sequents.

Definition 2.1. A *basic calculus* is a single-conclusion sequent calculus that consists of

- axiom schema of *identity* $\alpha \vdash \alpha$
- the (multiplicative version of the) *cut rule* (CUT) and the *permutation* (left) rule

$$\frac{\Theta \vdash \alpha \quad \alpha\, \Theta_1 \vdash \Xi}{\Theta_1\, \Theta \vdash \Xi}\ (CUT) \qquad \frac{\Theta\, \beta\, \alpha\, \Theta' \vdash \Xi}{\Theta\, \alpha\, \beta\, \Theta' \vdash \Xi}\ (e, l)$$

where $\Theta, \Theta_1, \Theta'$ and Ξ are arbitrary (thus (CUT) and (e,l) actually consist of countable sets of inference rules)

— possibly weakening (w,l) and/or *generalized knotted structural rules*

$$\frac{\Theta' \vdash \Xi'}{\Theta'\,\Theta'' \vdash \Xi'}\ (w,l) \qquad \frac{\Theta\, \alpha_1^{n_1} \ldots \alpha_j^{n_j} \vdash \Xi}{\Theta\, \alpha_1^{k_1} \ldots \alpha_j^{k_j} \vdash \Xi}\ ((n_1,k_1), \ldots (n_j,k_j))$$

for any $k_1, \ldots k_j, n_1, \ldots, n_j \geq 1$, $\Theta, \Theta', \Xi, \Xi' \neq \epsilon$, $l \geq 1$ and $\Xi \notin \Theta$

— for each logical connective $\star$, *left logical rules* $\{(\star, l)_j\}_{j \in \Lambda_1}$ and *right logical rules* $\{(\star, r)_k\}_{k \in \Lambda_2}$ (Λ_1, Λ_2 can be empty):

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta\ \star(\vec{\alpha}) \Rightarrow \Xi}\ (\star, l)_j \quad \frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_{n^\circ} \Rightarrow \Psi_{n^\circ}}{\Theta \Rightarrow \star(\vec{\alpha})}\ (\star, r)_k$$

(where $\vec{\alpha} \equiv \alpha_1, \ldots, \alpha_l$ and $n, n^\circ \geq 0$) satisfying the following conditions:

(log0) Any meta-variable in $\Upsilon_1, \ldots, \Upsilon_{n^{(\circ)}}$ (resp. $\Psi_1, \ldots, \Psi_{n^{(\circ)}}$) is either an α_i, with $i = 1, \ldots l$, or it does occur in Θ (resp. Ξ).

(log1) Each meta-variable occurs *at most once* in Θ.

Instances of the identity axiom schema and rules are obtained by substituting arbitrary formulae for meta-variables.
In logical rules the meta-variables (formulae) of the form α_i are called *active meta-variables* (*active formulae*) and the introduced $\star(\vec{\alpha})$ (or the formula of the form $\star(A_1, \ldots, A_l)$) is called *principal formula*, the remaining meta-variables (formulae) are called *contexts*.
In the generalized knotted structural rules the meta-variable (or their instances) in Θ are called *contexts*. Moreover, the two occurrences of the formula instantiating the meta-variable α in (CUT) are called left and right *cut formulae*.

Remark: Henceforth, we will identify (meta)sequents differing only in the order of (meta) formulas in their antecedents and we will therefore not consider explicitly anymore the permutation rule (e,l).

To formalize the class of calculi we will deal with, let us define the closure under cuts of two (meta)sequents. Let S, T be sequents of the form $S = \Gamma \vdash A$ and $T = A^m, \Pi \vdash \Lambda$. Then we define

$$\begin{aligned} Cut_l^0(S,T) &:= \{T\}, \\ Cut_l^{i+1}(S,T) &:= Cut_l^i(S,T) \cup \{A^{m-i-1}, \Gamma^{i+1}, \Pi \vdash \Lambda\}, \\ Cut_l^*(S,T) &:= Cut_l^m(S,T) \end{aligned}$$

We may cut also from the other side. In this case we define

$$\begin{aligned} Cut_r^i(S,T) &:= Cut_l^i(T,S),\ i = 0,1 \\ Cut_r^*(S,T) &:= Cut_l^1(T,S). \end{aligned}$$

The definition above also applies to meta-sequents.

Definition 2.2. Let $\mathcal{R}$ be a set of unary structural rules and $\rho \in \mathcal{R}$. We define $S \rightarrow_\rho S'$

if S' can be obtained from S by one application of ρ. We define $S \rightarrow_{\mathcal{R}} S'$ if there exists a $\rho \in \mathcal{R}$ s.t. $S \rightarrow_{\rho} S'$. $\rightarrow^*_{\rho}$ defines the reflexive transitive closure of $\rightarrow_{\rho}$, $\rightarrow^*_{\mathcal{R}}$ that of $\rightarrow_{\mathcal{R}}$.

Definition 2.3. A *knotted commutative* calculus K is a basic calculus in which each instance of a logical rule ρ with premisses $S_1, \ldots, S_n$, conclusion S and principal formula A, satisfies the additional conditions: for each single conclusion sequent T

(log2) and each $S' \in Cut^*_l(T, S)$ such that the principal formula $A \in S'$ there are $S'_1 \in Cut^*_l(T, S_1), \ldots, S'_n \in Cut^*_l(T, S_n)$ such that $S'_1, \ldots, S'_n \rightarrow_{\rho} S'$

(log3) and each $S' \in Cut^*_r(T, S)$ such that the principal formula $A \in S'$ there are $S'_1 \in Cut^*_r(T, S_1), \ldots, S'_n \in Cut^*_r(T, S_n)$ such that $S'_1, \ldots, S'_n \rightarrow_{\rho} S'$

Remark: Condition **(log0)** ensures that logical rules satisfy the subformula property and do not allow meta-variables (that are not active meta-variables) to move from antecedent to consequent of sequents and vice versa. Conditions **(log2)** and **(log3)** ensure that logical rules allow any (CUT) on a context formula be replaced by (CUT) on its premisse(s) (and one application of the rule).

Definition 2.4. Let $\mathcal{R}$ the set of structural rules of a knotted commutative calculus. Each $\rho \in \mathcal{R}$ is called *regular* and $\mathcal{R}$ is called *regular set.* If $\mathcal{R}$ contains (w, l) then it is called *w-regular*, otherwise *wf-regular* (weakening free regular).

Notice that each generalized knotted structural rule $((n_1, k_1), \ldots (n_j, k_j))$ can be simulated by j knotted structural rules (n_i, k_i), for $i = 1, \ldots, j$

Example 2.1 (Knotted commutative calculi). Many well known sequent calculi fit into our framework. Among them, propositional **LJ** (Gentzen 1935) and the calculi investigated in (Hori, Ono and Schellinx 1994), that are intuitionistic linear logic without the exponentials **ILL** and its implicational fragment extended with the knotted structural rules of the form (n, k) and both (n, k) and (k, n). Notice that $(2, 1)$ is the contraction rule left in **LJ**, $(1, 2)$ is expansion (see (van Benthem 1991)) and $(n + 1, n)$ the so-called n-contraction rule. The latter, investigated in (Prijatelj 1996), is sound for the logic of Łukasiewicz with n truth-values.
Further examples of knotted commutative calculi are, e.g.

— the calculus **LBC-** of Baaz, Ciabattoni and Montagna (2004) whose axioms and rules are exactly those of **ILL** but for the right rule of the $\wedge$ connective that, in the case of **LBC-**, is splitted into the following rules:

$$\frac{\Theta \vdash \alpha_1 \quad \Theta'\, \alpha_1 \vdash \alpha_2}{\Theta\, \Theta' \vdash \alpha_1 \wedge \alpha_2}\ (\wedge, r)_1 \qquad \frac{\Theta \vdash \alpha_2 \quad \Theta'\, \alpha_2 \vdash \alpha_1}{\Theta\, \Theta' \vdash \alpha_1 \wedge \alpha_2}\ (\wedge, r)_2$$

— the calculus K_1 consisting of the following rules

$$\frac{\Theta \vdash \alpha_1 \quad \Theta \vdash \alpha_2}{\Theta \vdash \alpha_1 \bar{\wedge} \alpha_2}\ (\bar{\wedge}, r) \qquad \frac{\Theta\, \alpha_1\, \alpha_2 \vdash \Xi}{\Theta\, \alpha_1 \bar{\wedge} \alpha_2 \vdash \Xi}\ (\bar{\wedge}, l)$$

Definition 2.5. *canonic $\star$ cut-derivation schema φ*:

$$\dfrac{\dfrac{\Theta_1 \vdash \Xi_1 \quad \ldots \quad \Theta_n \vdash \Xi_n}{\gamma \vdash \star(\bar{\alpha})}\,(\star, r)_j \qquad \dfrac{\Phi_1 \vdash \Xi'_1 \quad \ldots \quad \Phi_m \vdash \Xi'_m}{\star(\bar{\alpha})\,\gamma' \vdash \delta}\,(\star, l)_i}{\gamma\,\gamma' \vdash \delta}\,(CUT)$$

The set $\{\Theta_i \vdash \Xi_i, \Phi_j \vdash \Xi'_j\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is called the *reduction set* of φ. A *canonic $\star$ cut-derivation* φ' corresponding to a $\star$ cut-derivation schema φ is an instance of the schema where the (instances of) $\Theta_i \vdash \Xi_i, \Phi_j \vdash \Xi'_j$ are replaced by derivations.

2.1. *Conditions for (reductive) cut-elimination*

For a large class of propositional single-conclusion sequent calculi Ciabattoni and Terui (2006) provide a characterization of the cut-elimination methods that proceed following the "standard" steps: (1) locate canonic cut-derivations and replace them by derivations with "smaller" cuts and (2) shift inferences to achieve canonic cut-derivations. (A different cut-elimination method is e.g. CERES, see (Baaz and Leitsch 2000).)
Necessary and sufficient conditions have been defined for these calculi to admit *reductive cut-elimination* – a naturally strengthened version of cut-elimination in presence of axioms (see e.g. (Buss 1998)) which in addition aims to shift upward non-eliminable cuts *as much as possible*. The defined conditions (*reductivity* and *weak substitutivity*) are recalled below and applied to knotted commutative calculi. Intuitively, logical rules are reductive if they allow the replacement of cuts by "smaller" cuts (this formalizes the step 1. above), and a structural rule is weakly substitutive when any cut can be permuted upward (cf. step 2.). Note that logical rules of knotted commutative calculi are weakly substitutive by definition.

Let K be a knotted commutative calculus and $\mathcal{S}$ a set of sequents (considered as non-logical axioms). A *derivation* in K of a sequent S_0 from $\mathcal{S}$ is a labeled tree whose root is labeled by S_0, the leaves are labeled by an instance of an identity axiom, by an instance of a logical K-rule without premisses or by a sequent in $\mathcal{S}$, and the inner nodes are labeled in accordance with the instances of the K-rules. A derivation in K of a meta-sequent σ from a set of meta-sequents is defined similarly. When there exists such a derivation, we say that S_0 (or σ) is *derivable* from $\mathcal{S}$ in K.

Definition 2.6. An occurrence of (CUT) in a derivation is said to be *reducible* if one of the following holds:

(i) Both cut formulae are the principal formulae of logical rules.
(ii) At least one of the two cut formulae is a context formula of a rule other than (CUT) or an identity axiom.

We say that a knotted commutative sequent calculus K admits *reductive cut-elimination* if whenever a sequent S_0 is derivable in K from a set $\mathcal{S}$ of non-logical axioms, S_0 has a derivation in K from $\mathcal{S}$ without any reducible cuts.

Notice that in a derivation without non-logical axioms, uppermost cuts are always reducible. Hence reductive cut-elimination implies the usual cut-elimination.

Definition 2.7. Let K be a knotted commutative sequent calculus, and $(\star, l)_i$ and $(\star, r)_j$ rules of K introducing a connective $\star$ on the left and right respectively. These rules are *pairwise reductive* in K if for each canonic $\star$ cut-derivation schema Φ, there exists a derivation μ in K of its conclusion $\gamma\,\gamma' \vdash \delta$ (see Def. 2.5) from the reduction set of Φ using no logical rules and all cut-formulae appearing in μ are the active meta-variables of $\star(\bar{\alpha})$. The rules for $\star$ are *reductive* in K, if each left and right rule for $\star$ is pairwise reductive.

Remark: Reductivity corresponds to the principal formula condition in (Restall 1999) and to the coherence criterion of Miller and Pimentel (2002;2005).

Proposition 2.1. If rules for $\star$ are reductive in K, then the end sequent of any canonic $\star$ cut-derivation Φ' can be derived in K from the instances of the reduction set of Φ using no logical rules and all cut-formulae appearing in the derivation are among the active formulas instantiating $\star(\bar{\alpha})$.

Proof. The required derivation is an instance of the derivation μ of Def. 2.7. □

Definition 2.8. Let K be any knotted commutative calculus with structural rules $\mathcal{R}$. $\rho \in \mathcal{R}$ is said to be *weakly substitutive* in K if for all sequents $S, S_1, S_2 \in K$ s.t. $S_1 \rightarrow_\rho S_2$ then $(*)$ for all $S' \in Cut^*_c(S, S_2)$, $c \in \{l, r\}$, there exists an $S'' \in Cut^*_c(S, S_1)$ s.t. $S'' \rightarrow^*_{\mathcal{R}} S'$. $\mathcal{R}$ is said to be *weakly substitutive* if so are all $\rho \in \mathcal{R}$.

Proposition 2.2. Let K be a knotted commutative calculus with structural rules $\mathcal{R}$. $\rho \in \mathcal{R}$ is weakly substitutive if and only if for all $S, S_1, S_2 \in K$ s.t. $S_1 \rightarrow_\rho S_2$,
$(*)'$ for all $S' \in Cut^*_l(S, S_2)$ there exists an $S'' \in Cut^*_l(S, S_1)$ s.t. $S'' \rightarrow^*_{\mathcal{R}} S'$

Proof. Trivially follows by the presence of the "passive" contexts Θ or Θ' in the considered structural rules (Definition 2.1). □

Remark: Weak substitutivity was equivalently defined by Ciabattoni and Terui (2006) using rule *schemas* instead of rule *instances*.

Theorem 2.1. A knotted commutative calculus admits reductive cut-elimination if and only if its logical rules are reductive and its structural rules are weakly substitutive.

Proof. Follows by (Ciabattoni and Terui 2006). □

Given any sequent calculus K'. It is easy to see whether K' belongs to our framework. This can be checked by eye for structural rules and the conditions **(log0)** and **(log1)** of logical rules. Moreover, conditions **(log2)** and **(log3)** for logical rules of K' can be checked in finite time since $Cut^*_l(T, S)$ and $Cut^*_l(T, S_i)$, $i = 1, \ldots, n$, are finite sets.

3. On Regular Sets

Consider the following problem: Let K be a single conclusion calculus and $\mathcal{R} = \{\rho_1, \ldots, \rho_n\}$ a set of unary (i.e. one premiss) structural rules of K. Is a unary structural rule ρ dependent on (or, equivalently, derivable from) $\mathcal{R}$? More formally: the unary rule ρ *depends on* $\mathcal{R}$ if for any S_1 and S_2 s.t. $S_1 \rightarrow_\rho S_2$ we have $S_1 \rightarrow^*_{\mathcal{R}} S_2$.
Since rules in $\mathcal{R}$ can be formulated as Horn/Krom clauses, this problem corresponds to the problem whether a set of Krom clauses S implies another Krom clause. The latter was shown to be undecidable by Schmidt-Schauss (1988) when S contains at least two elements.
Using combinatorial arguments, we prove below that when K is a knotted commutative calculus with regular set $\mathcal{R}$ and ρ any regular rule, the problem above is decidable. Using this result we define a procedure to decide whether a regular set is weakly substitutive. A characterization of weakly substitutive regular sets is also provided.

Definition 3.1. Two regular sets $\mathcal{R}$ and $\mathcal{R}'$ are equivalent (in symbols $\mathcal{R} \sim \mathcal{R}'$) if each $\rho \in \mathcal{R}$ depends on $\mathcal{R}'$ and each $\rho' \in \mathcal{R}'$ depends on $\mathcal{R}$.

Our analysis proceeds by cases according to whether the weakening rule belongs to the regular set $\mathcal{R}$ or it does not.

3.1. *W-regular systems*

Let us assume that $\mathcal{R} = \{(w, l), \rho_i\}$, for $i = 1, \ldots, n$, where ρ_i is

$$\frac{x^{k_i} y \vdash z}{x^{l_i} y \vdash z}\ \rho_i$$

Lemma 3.1. Let w.l.o.g. be $l_1 = \min\{l_1, \ldots, l_n\}$. Then a knotted structural rule (k, l), $k \neq l$, is derivable from $\mathcal{R}$ iff $l \geq l_1$.

Proof. Consider $\mathcal{R}$. Let $p_1 = k_1 - l_1$. We may assume that $k_i > l_i$ for all i, otherwise ρ_i is the identity or an instance of weakening.
$l \geq l_1$:
Then there exists a number r s.t. $l + r * p_1 > k$. We define the derivation

$$\cfrac{\cfrac{x^k y \vdash z}{x^{l+r*p_1} y \vdash z}\ w\colon l}{x^l y \vdash z}\ \rho_1^*$$

As a consequence all rules $\rho_2, \ldots, \rho_n$ are themselves dependent on ρ_1. If $l_1 = 1$ then ordinary contraction can be simulated and then all unary structural rules!
$l < l_1$:
By the presence of weakening we can derive $x^{k_i} y \vdash z$ from $x^{l_i} y \vdash z$ as well. So the rules can be represented by an equational theory

$$\begin{aligned} \mathcal{E} \quad = \quad & \{x^{k_1} = x^{l_1}, \ldots, x^{k_n} = x^{l_n}, \\ & xy = yx,\ x(yz) = (xy)z,\ x\epsilon = x,\ \epsilon x = x\}. \end{aligned}$$

There exists a model of $\mathcal{E}$ with the domain $D = \{\epsilon, d_1, \ldots, d_{l_1}\}$ s.t. $d_i = d^i$ for $i < l_1$ and $d_i = d_{l_1}$ for $i \geq l_1$. In this interpretation the equation $d^l = d^k$ does not hold. In particular $x^l y = x^k y$ is falsified in this model of $\mathcal{E}$. Clearly the derivability of $x^l y$ from $x^k y$ implies the equation $x^l y = x^k y$; therefore the rule is not derivable. □

Proposition 3.1. Let ρ be a unary rule of the form

$$\frac{x_1^{p_1} \ldots x_r^{p_r} y^* \vdash z'}{x_1^{m_1} \ldots x_r^{m_r} y^* z^* \vdash z'}$$

where y^* denotes either y or ϵ (the same for z^*). Then it is decidable whether ρ depends on $\mathcal{R}$.

Proof. Clearly ρ depends on $\mathcal{R}$ iff

$$(+) \quad \min\{m_1, \ldots, m_r\} \geq l_1.$$

Indeed, if (+) holds then we simulate the rule r-times as in Lemma 3.1; otherwise, for $m_i < l_i$ we create a rule instance where all x_j are set to ϵ for $j \neq i$. Then the result follows from Lemma 3.1. □

Remark: In the proposition's claim we do not ask for the existence of an $i = \{1, \ldots, r\}$ such that $p_i = m_i$ and hence ρ might not be a general knotted structural rule.

3.2. *Wf-regular systems*

Let us assume that $\mathcal{R} = \{\rho_i\}$, for $i = 1, \ldots, n$, where each ρ_i is (k_i, l_i). W.l.o.g. we can assume that $k_i \neq l_i$ for all $i \in \{1, \ldots, n\}$, otherwise the corresponding rule is redundant.

We distinguish 3 cases:

(1) $l_i < k_i$ for all $i = 1, \ldots, n$ (that is all the rules are *contractive*),
(2) $l_i > k_i$ for all $i = 1, \ldots, n$,
(3) there exist $i, j < n$ and $i \neq j$ s.t. $l_i < k_i$ and $l_j > k_j$.

The decidability of rule dependency in cases (1) and (2) is easy: in case (1) only finitely many derivations are possible on any sequent; in case (2) we observe the following feature: Let ρ be the rule

$$\frac{x_1^{q_1} \ldots x_m^{q_m} \vdash y}{x_1^{p_1} \ldots x_m^{p_m} \vdash y}$$

First of all note that any rule with different sets of meta-variables on the left side in premiss and conclusion is not derivable: clearly no meta-variable may vanish, and there is no weakening producing additional ones. Thus we may indeed restrict our analysis to rules ρ of the form above. According to the structure of rules in case (2), the sum of powers of the x_i in sequents are strictly increasing with every rule application. So let us assume we have derived

$$s \colon x_1^{r_1} \ldots x_m^{r_m} \vdash y$$

from $x_1^{p_1} \dots x_m^{p_m} \vdash y$ s.t.

$$\sum_{i=1}^{n} r_i > \sum_{i=1}^{n} q_i.$$

Then s is a dead end, as there is no way to reach the rule consequent $x_1^{q_1} \dots x_m^{q_m} \vdash y$ from s, as the sum of the powers increases strictly. On the other hand there are only finitely many derivations ending in sequents $x_1^{r_1} \dots x_m^{r_m} \vdash y$ with

$$\sum_{i=1}^{n} r_i \leq \sum_{i=1}^{n} q_i.$$

So rule dependency is decidable also in case (2).

It remains to investigate case (3):

Definition 3.2. Let $Q\colon \{q_1, \dots, q_n\}$ be a set of integers s.t. $q_i \neq 0$ for all i. We call a number r *representable* by Q if there exist non-negative integers $k_1, \dots, k_n$ s.t.

$$r = k_1 * q_1 + \dots + k_n * q_n.$$

Proposition 3.2. Let $\mathcal{R}$ be any regular system of rules fulfilling restriction (3) above. Then a rule of the form

$$\frac{x_1^{s_1} \dots x_m^{s_m} \vdash y}{x_1^{r_1} \dots x_m^{r_m} \vdash y}$$

is derivable from $\mathcal{R}$ iff $r_i - s_i = 0 \bmod q$ for all $i = 1, \dots, m$ and a number q depending on $\mathcal{R}$.

Proof. Let $\mathcal{R} = \{\rho_1, \dots, \rho_n\}$ where each ρ_i is a knotted structural rule of the form (k_i, l_i). Assume w.l.o.g. that $l_1 < k_1$ and $l_2 > k_2$, and let q be the greatest common divisor of the set $Q\colon \{q_1, \dots, q_n\}$ for $q_i = l_i - k_i$ $(i = 1, \dots, n)$. Then, by elementary number theory, a number r is representable by $\{q_1, \dots, q_n\}$ iff $r = 0 \bmod q$. Note that, if all q_i were positive or all negative, then the representability would hold only above a certain bound. But, due to the presence of different signs, every r with the appropriate modularity is representable.

Now let σ be a generalized knotted structural rule $((s_1, r_1), \dots (s_m, r_m))$. Then σ is a derivable rule iff $r_i - s_i = 0 \bmod q$. Due to the existence of contexts in the rules of $\mathcal{R}$, we can restrict the problem of derivability to simpler rules of the form

$$\frac{x^s \vdash y}{x^r \vdash y}$$

Obviously the conclusion is derivable from the premiss via the rules in $\mathcal{R}$ iff $r = s + k_1 * q_1 + \dots + k_n * q_n$ for non-negative numbers k_i. But this is the case iff $r - s$ is representable by Q. We have seen above that $r - s$ is representable by Q iff $r - s = 0 \bmod q$. □

Example 3.1. Let $\mathcal{R} = \{(3, 1), (1, 5)\}$. Then $Q = \{-2, 4\}$ and $\gcd(\{-2, 4\}) = 2$. So the rule $(1,3)$ is derivable from $\mathcal{R}$ (note that $2 = 0 \bmod 2$) by

$$\dfrac{\dfrac{xy \vdash z}{x^5 y \vdash z}\,(1,5)}{x^3 y \vdash z}\,(3,1)$$

On the other hand the rule $(6,1)$ is not derivable from $\mathcal{R}$, as $5 \neq 0 \bmod 2$.

For $\mathcal{R} = \{(1,3),(6,1)\}$ we obtain $\gcd(\{2,-5\}) = 1$. Therefore all rules (n,k), with $n \geq k$ can be simulated. We show the simulation of ordinary left-contraction:

$$\dfrac{\dfrac{\dfrac{x^2 y \vdash z}{x^4 y \vdash z}\,(1,3)}{x^6 y \vdash z}\,(1,3)}{xy \vdash z}\,(6,1)$$

Theorem 3.1. Let $\mathcal{R}$ be a set of structural rules of a knotted commutative calculus. Then rule dependency from $\mathcal{R}$ is decidable.

Proof. Follows from Sections 3.1 and 3.2. □

As a consequence of this result follows a decision procedure for shifting up (possibly multiple) cuts over regular rules (cf. condition $(*)'$ in Proposition 2.2). Indeed

Theorem 3.2. Let K be a knotted commutative calculus whose set of structural rules is $\mathcal{R}$ and let $\rho \in \mathcal{R}$. Let S, S_1, S_2 be sequents in K and $S_1 \to_\rho S_2$. For each $S' \in Cut_l^*(S, S_2)$ one can decide whether there exists $S'' \in Cut_l^*(S, S_1)$ such that $S'' \to_{\mathcal{R}}^* S'$.

Proof. First notice that $Cut_l^*(S, S_1)$ is a finite set. The claim follows by Theorem 3.1. □

3.3. *Deciding Weak Substitutivity*

Theorem 3.2 ensures that for each regular rule ρ and sequent S condition $(*)'$ in Proposition 2.2 can be checked. However, to conclude that ρ is weakly substitutive, such checking should be done *for all* instances of the rule and *all* sequents $S \in K$. This "brute force" approach results in a semi-decision procedure that eventually finds out if ρ is not weakly substitutive and does not terminate otherwise. To avoid checking possibly infinite instances of regular rules, we introduce below the notion of *most general instance* of a generalized knotted structural rule ρ consisting in a rule schema $\sigma_1 \to_\rho \sigma_2$ such that, $(\star)$ for each instance $S_1 \to_\rho S_2$ and sequent S, for each $S' \in Cut_l^*(S, S_2)$ obtained via cut(s) with (at least a) cut-formula not in the context, there exists $\sigma' \in Cut_l^*(\sigma, \sigma_2)$ where σ is a meta-sequent and S_1, S_2, S and S' are obtained by suitably replacing in $\sigma, \sigma_1, \sigma_2, \sigma'$ meta-variables with formulae, where common meta-variables in $\sigma, \sigma_1, \sigma_2, \sigma'$ are substituted consistently.

Let ρ be the rule (m,l), and σ a meta-sequent $w \vdash x$. E.g. the rule schema

$$\dfrac{x^m y \vdash z}{x^l y \vdash z}\,(m,l)$$

does not satisfy condition ($\star$). For, let $S_1 = A^{m+1}\,\Gamma \vdash \Delta$, $S_2 = A^{l+1}\,\Gamma \vdash \Delta$, $S = \Sigma \vdash A$ and $S' = \Sigma^{l+1}\,\Gamma \vdash \Delta$.

Definition 3.3. The *most general instance* of $\rho = (m, l)$ is obtained by setting $\sigma_1 = x^{m+\mathcal{K}} y \vdash z$ and $\sigma_2 = x^{l+\mathcal{K}} y \vdash z$ where σ_1, σ_2 represent a sequence of meta-sequents for $\mathcal{K}$ ranging over $\mathbb{N}$. Corresponding to $\sigma = w \vdash x, \sigma_1, \sigma_2$ we define

$$\begin{aligned} \tau_1 : \text{cuts}(\sigma, \sigma_1) &= w^I x^{m+\mathcal{K}-I} y \vdash z,\ I \leq m + \mathcal{K}, \\ \tau_2 : \text{cuts}(\sigma, \sigma_2) &= w^J x^{l+\mathcal{K}-J} y \vdash z,\ J \leq l + \mathcal{K}. \end{aligned}$$

cuts(σ, σ_1) and cuts(σ, σ_2) are schemata representing sequences in $CUT_l^*(\sigma, \sigma_1)$ and $CUT_l^*(\sigma, \sigma_2)$, where m, l are fixed and $I, J, \mathcal{K}$ range over $\mathbb{N}$ with the indicated constraints. We call cuts(σ, σ_1) the *first* and cuts(σ, σ_2) the *second cut-schema w.r.t. ρ.* An instance of the schema is a (meta-) sequent obtained by instantiating $I, J, \mathcal{K}$.

Definition 3.4. Let $\mathcal{R}$ be a regular set and τ_1, τ_2 be two meta-sequents. We define $\tau_1 \leq_{\mathcal{R}} \tau_2$ if for every instance τ_2' of τ_2 there exists an instance τ_1' of τ_1 s.t. $\tau_1' \to_{\mathcal{R}}^* \tau_2'$ (where the common meta-variables of τ_1, τ_2 have to be substituted consistently). Otherwise, we call τ_2' a *counterexample schema* and write $\tau_1 \not\leq_{\mathcal{R}} \tau_2$.

Proposition 3.3. Let ρ be a rule in $\mathcal{R}$. Then ρ is weakly substitutive iff cuts$(\sigma, \sigma_1) \leq_{\mathcal{R}}$ cuts(σ, σ_2).

Proof. By Prop. 2.2 and the presence of Θ (see Def. 2.1), ρ is weakly substitutive iff for all $S, S_1, S_2 \in K$ s.t. $S_1 \to_\rho S_2$, for each $S' \in Cut_l^*(S, S_2)$ obtained via cut(s) with (at least a) cut-formula not in the context, there exists an $S'' \in Cut_l^*(S, S_1)$ s.t. $S'' \to_{\mathcal{R}}^* S'$. Each instance of ρ has the form $A^{m+p}\,\Gamma \vdash \Delta \to_\rho A^{l+p}\,\Gamma$, for some $p \in \mathbb{N}$. Hence let S be $\Sigma \vdash A$, each such S' and S'' have the form $\Sigma^i\, A^{l+p-i}\,\Gamma \vdash \Delta$ and $\Sigma^j\, A^{l+p-j}\,\Gamma \vdash \Delta$ for some $i, j \in \mathbb{N}$. The claim then follows by replacing $\mathcal{K}$ with p in cuts(σ, σ_1) and cuts(σ, σ_2). $\square$

To find out which regular rules are "good" for reductive cut-elimination (i.e. weakly substitutive) and which are "bad", we distinguish two cases according to whether the regular system $\mathcal{R}$ contains weakening or it does not. We start considering the latter case that requires an analysis of the subcases (1)-(3) identified in Section 3.2.
Every regular set of rules can be transformed to an equivalent set of rules in "minimal form". These minimal representations will be needed in the proofs of the propositions characterizing weak substitutivity.

Definition 3.5. We define an ordering on rules of type (n, m): let $\rho_1 = (n_1, m_1)$ and $\rho_2 = (n_2, m_2)$ We say that ρ_1 is smaller than ρ_2 (notation $\rho_1 \sqsubset \rho_2$) if the following two conditions hold:

(a) $n_1 \leq n_2$ and $m_1 \leq m_2$, and
(b) either $n_1 < m_1$ or $n_2 < m_2$.

Let $\mathcal{R}'$ be a finite set of rules. We say that $\mathcal{R}' \sqsubset \rho$ if for all $\rho' \in \mathcal{R}'$: $\rho' \sqsubset \rho$.

Definition 3.6. Let $\mathcal{R}$ be a regular system.

— $\mathcal{R}$ is called *minimal* if, for all finite sets of rules $\mathcal{R}'$ which are derivable from $\mathcal{R}$, there exists no $\rho \in \mathcal{R}$ s.t. $\mathcal{R}' \sqsubset \rho$ and ρ is derivable from $\mathcal{R}'$.
— $\mathcal{R}$ is called *normal contractive* if all structural rules (but (e, l)) are of the form $(m_1, 1)$, $\dots (m_n, 1)$.

Note that any regular system $\mathcal{R}$ can be algorithmically transformed into an equivalent minimal system $\mathcal{R}_0$. Indeed, for all $\rho \in \mathcal{R}$, the set of rules $\mathcal{R}'$ with $\mathcal{R}' \sqsubset \rho$ is finite, and the derivability of rules is decidable by Theorem 3.1.

Example 3.2. Let $\mathcal{R} = \{(w, l), (4, 1)\}$. Then $\mathcal{R}$ is not minimal as $(2, 1)$ is derivable from $\mathcal{R}$, $(2, 1) \sqsubset (4, 1)$ and $(4, 1)$ is derivable from $(2, 1)$. The corresponding minimal system $\mathcal{R}_0$ is $\{(w, l), (2, 1)\}$.

Henceforth we will only consider minimal regular sets.

Proposition 3.4 (type-1-bad). Let $\mathcal{R}$ be a system of type (1) with the following properties:

(a) There exists a rule $(m, l) \in \mathcal{R}$, with $l > 1$.
(b) $\mathcal{R}$ is minimal.
(c) No rule (r, s) with $r < m$ and $s > 1$ is derivable in $\mathcal{R}$

Let $\sigma_1 = x^{m+\mathcal{K}} y \vdash z$, $\sigma_2 = x^{l+\mathcal{K}} y \vdash z$, $\sigma = w \vdash x$, and

$$\begin{aligned} \tau_1 &= \text{cuts}(\sigma, \sigma_1) = w^I x^{m+\mathcal{K}-I} y \vdash z,\ I \leq m + \mathcal{K}, \\ \tau_2 &= \text{cuts}(\sigma, \sigma_2) = w^J x^{l+\mathcal{K}-J} y \vdash z,\ J \leq l + \mathcal{K}. \end{aligned}$$

(see Definition 3.3). Then $\tau_1 \not\leq_{\mathcal{R}} \tau_2$.

Proof. Let $\sigma, \sigma_1, \sigma_2$ as above. We instantiate τ_2 to τ_2' by setting $\mathcal{K} = 0$ and $J = l - 1$. Then

$$\tau_2' = w^{l-1} x y \vdash z.$$

We have to consider the instance

$$\tau' = \tau_1\{\mathcal{K} \to 0\} = w^I x^{m-I} y \vdash z$$

We prove that $\tau' \not\leq_{\mathcal{R}} \tau_2'$ and then τ_2' is a counterexample schema.
Let $(m, l) \in \mathcal{R}$ with $l > 1$ (such a rule exists by (a)). We distinguish two cases:

(1) $l = 2$: Then $\tau_2' = wxy \vdash z$. As $\mathcal{R}$ contains no weakening, $I > 0$ and $I < m$ are necessary constraints for the substitution of I.
If $I = 1$ then necessarily $x^{m-1} \to^*_{\mathcal{R}} x$ and, therefore $(m - 1, 1)$ would be a derivable rule. But $(m-1, 1) \sqsubset (m, 2)$ and $(m, 2)$ is derivable from $(m-1, 1)$ contradicting (b).
So let $I = i$ for $i > 1$. Then necessarily $w^i \to^*_{\mathcal{R}} w$ and $x^{m-i} \to^*_{\mathcal{R}} x$. But this is only possible if the rules $(i, 1)$ and $(m-i, 1)$ are derivable; But for $\mathcal{R}' = \{(i, 1), (m-i, 1)\}$ we have $\mathcal{R}' \sqsubset (m, 2)$ and $(m, 2)$ is derivable from $\mathcal{R}'$; again this contradicts (b).
(2) $l > 2$. We check whether there is a i s.t.

$$S_i\colon\ w^i x^{m-i} y \vdash z \to^*_{\mathcal{R}} w^{l-1} x y \vdash z.$$

First of all $i \geq l - 1$ as $\mathcal{R}$ is of type (1).

So let $i = l-1$. Then $S_{l-1} = w^{l-1}x^{m-l+1}y \vdash z$. $S_{l-1} \to^*_{\mathcal{R}} w^{l-1}xy \vdash z$ requires $x^{m-l+1} \to^*_{\mathcal{R}} x$. But this implies that $\rho'\colon (m-l+1, 1)$ is derivable in $\mathcal{R}$. But $\rho' \sqsubset (m,l)$ and (m,l) is derivable from ρ', contradicting (b).
Assume $i > l-1$. Then $S_i \to^*_{\mathcal{R}} w^{l-1}xy \vdash z$ requires $w^i \to^*_{\mathcal{R}} w^{l-1}$. But then the rule $(i, l-1)$ is a derivable rule with $l-1 > 1$; as $i < m$ this contradicts (c).

□

Proposition 3.5 (type-1-good). Let $\mathcal{R}$ be a normal contractive system of type (1). Let $\rho \in \mathcal{R}$ and τ_1, τ_2 be the cut-schemata corresponding to ρ. Then $\tau_1 \leq_{\mathcal{R}} \tau_2$.

Proof. ρ must be of the form $(m,1)$. Let $\sigma, \sigma_1, \sigma_2, \tau_1, \tau_2$ as in Definition 3.3 (with $l=1$). Then for every instance $\{J \to i, \mathcal{K} \to k\}$ we get

$$w^i x^{m+k-i} y \vdash z \to^*_{\mathcal{R}} w^i x^{1+k-i} y \vdash z$$

for $i \leq k$ (setting I to i), and for $i = k+1$ we substitute I by $m+k$ and so obtain $w^{m+k}y \vdash z$, but

$$w^{m+k} y \vdash z \to^*_{\mathcal{R}} w^{k+1} y \vdash z.$$

□

Proposition 3.6 (type-2-bad). Let $\mathcal{R}$ be a minimal system of type (2), $\rho \in \mathcal{R}$ where $\rho = (m,l)$ s.t. $m = \min\{k \mid (k,k') \in \mathcal{R}\}$, $m < l$ and no rule (r,s) with $r < m$ is derivable in $\mathcal{R}$. Let τ_1, τ_2 be the cut-schemata corresponding to ρ. Then $\tau_1 \not\leq_{\mathcal{R}} \tau_2$.

Proof. Let $\sigma, \sigma_1, \sigma_1, \tau_1, \tau_2$ as in Definition 3.3.

(1) $m = 1$. We instantiate τ_2 by $\{\mathcal{K} \to 0, J \to 1\}$. The corresponding instance is $S'\colon\ wx^{l-1}y \vdash z$ (note that $l > 1$). The only possible instances of τ_1 under $\mathcal{K} = 0$ are

$$xy \vdash z,\ wy \vdash z$$

Let S be one of these two sequents. Then, clearly $S \not\to^*_{\mathcal{R}} S'$ $((w,l) \notin \mathcal{R})$.

(2) $m > 1$. Then as $S' = wx^{l-1}y \vdash z$, we must instantiate I to 1 (there is no contractive rule in $\mathcal{R}$). But then

$$S''\colon\ \tau_1\{\mathcal{K} \to 0, J \to 1\} = wx^{m-1}y \vdash z.$$

If $S'' \to^*_{\mathcal{R}} S'$ then $(m-1, l-1)$ must be derivable in $\mathcal{R}$; but $(m-1, l-1) \sqsubset (m,l)$ and (m,l) is derivable from $(m-1, l-1)$, contradicting the minimality of $\mathcal{R}$.

□

Proposition 3.7 (type-3-bad). Let $\mathcal{R}$ be a minimal system of type (3). Let $\rho \in \mathcal{R}$ s.t. $\rho = (1,k)$ for $k > 1$. Let τ_1, τ_2 be the cut-schemata corresponding to ρ. Then $\tau_1 \not\leq_{\mathcal{R}} \tau_2$.

Proof. Like in Proposition 3.6 (type-2-bad) we select the instance

$$S'\colon\ wx^{k-1}y \vdash z$$

from the schema τ_2. Again τ_1 gives only

$$S_1\colon\ xy \vdash z,\ S_2\colon\ wy \vdash z$$

Clearly $S_i \not\to^*_{\mathcal{R}} S'$ for $i = 1, 2$. □

Corollary 3.1. A wf-regular set $\mathcal{R}$ is weakly substitutive if and only if $\mathcal{R}$ is normal contractive.

Proof. Note that every regular system $\mathcal{R}$ of type (1) can be transformed to $\mathcal{R}'$ s.t. $\mathcal{R} \sim \mathcal{R}'$ and either $\mathcal{R}'$ is normal contractive or $\mathcal{R}'$ satisfies the properties (a)-(c) of Proposition 3.4. Moreover if $(m, l) \in \mathcal{R}$ for some $m < l$ then the rule $(1, l - m + 1)$ is derivable in $\mathcal{R}$ by the existence of a characteristic number q (Proposition 3.2). Hence we can always assume that a system of type (3) contains a rule $(1, k)$ for $k > 1$. The claim then follows by Propositions 3.4 - 3.7. □

We consider now the case $(w, l) \in \mathcal{R}$.

Proposition 3.8. A w-regular set $\mathcal{R}$ is weakly substitutive if and only if (a) $\mathcal{R} = \{(w, l)\}$ or (b) $\mathcal{R}$ contains at least a rule $(n, 1)$, with $n > 1$.

Proof. ($\Longrightarrow$) Note that in case (b) by Lemma 3.1 ordinary contraction can be derived.
($\Longleftarrow$) Assume that $\mathcal{R}$ does not contain any $(n, 1)$ rule, with $n > 1$. By Lemma 3.1 the rules in $\mathcal{R}$ are interderivable with those in $\mathcal{R}' = \{(w, l), (l + 1, l)$, for some $l > 1$. In $\mathcal{R}'$ no rule $(k + 1, k)$ for $k < l$ is derivable. It is not hard to see that $(l + 1, l)$ is not weakly substitutive. Indeed, let

$$\begin{aligned} \tau_1 : \mathrm{cuts}(\sigma, \sigma_1) &= w^I x^{l+1+\mathcal{K}-I} y \vdash z,\ I \leq (l+1) + \mathcal{K}, \\ \tau_2 : \mathrm{cuts}(\sigma, \sigma_2) &= w^J x^{l+\mathcal{K}-J} y \vdash z,\ J \leq l + \mathcal{K}. \end{aligned}$$

Select the instance $S' = wx^{l-1}y \vdash z$ by instantiating $\{\mathcal{K} \to 0, J \to 1\}$ in τ_2. Then from $\mathcal{K} \to 0$ in τ_1 we obtain $S = w^I x^{(l+1)-I} y \vdash z$ from τ_1. There is no instance $S'' of S$ such that $S'' \to^*_{\mathcal{R}} S'$. Indeed $I \to 1$ is necessary as $w^{i+1} \not\to^*_{\mathcal{R}} w$ in $\mathcal{R}'$. But with $I = 1$ we obtain $wx^l y \vdash z$ and $x^l \not\to^*_{\mathcal{R}} x^{l-1}$ and hence $S'' \not\to^*_{\mathcal{R}} S'$. □

Remark: Being weakly substitutivity a necessary condition for reductive cut-elimination in knotted commutative calculi (Proposition 2.1), if a regular set cannot be transformed into an equivalent one that is either normal contractive or of the form (a) or (b) (see Theorem 3.8) the corresponding calculus does not admit reductive cut-elimination, no matter which are its logical rules. However, this only says that in such a calculus cuts cannot be removed following the steps 1. and 2. described in Section 2.1 and *not* that applications of (CUT) cannot be removed at all. For instance, consider a calculus whose only structural rule (beside, of course, (e, l)) is $(3, 2)$ and whose logical rules are

$$\frac{\vdash}{\Theta\, \alpha_1 \star \alpha_2 \vdash \Xi}\ (\star, l) \qquad \frac{\vdash}{\Theta \vdash \alpha_1 \star \alpha_2}\ (\star, r)$$

The only sequents provable in this calculus are instances of the identity axiom schema. Hence (CUT) in this calculus is trivially admissible, even though its structural rules are not weakly substitutive.
Note that the counterexample schemas in the propositions above can be turned into counterexamples to cut admissibility along the line of Hori, Ono and Schellinx (1994), if the calculus contains e.g. the implication connective of **ILL**.

4. Deciding Reductivity

A knotted commutative calculus K admits reductive cut-elimination if and only if (a) its structural rules are weakly substitutive and (b) its logical rules are reductive. Given K, a decision procedure for establishing whether (a) holds is contained in the previous section. Here we investigate knotted commutative calculi whose structural rules are weakly substitutive and provide algorithms to decide whether (b) holds, thus deciding the admissibility of reductive cut-elimination for knotted commutative calculi. Our approach is based on substructural (propositional) resolution.

Given a regular set $\mathcal{R}$, we define a structural resolution calculus based on an operator $R_{\mathcal{R}}$.

Definition 4.1. Let S_1: $X \vdash \alpha$ and $S_2 = \alpha, Y \vdash z$ be Horn clauses, where α is a formula variable and X, Y, z multisets of formula variables (z contains at most one element). Then the clause

$$XY \vdash z$$

is called the *resolvent* of S_1, S_2 and is denoted by $\mathcal{R}es(S_1, S_2)$.

Definition 4.2. Let $\mathcal{R}$ be a system of unary structural rules and $\mathcal{S}$ be a set of clauses. Then we define

$$\begin{array}{rcl} X_{\mathcal{R}}(\mathcal{S}) & = & \{S' \mid \text{ there exists an } S \in \mathcal{S} \text{ s.t. } S \rightarrow_{\mathcal{R}} S'\}, \\ \mathcal{R}es(\mathcal{S}) & = & \bigcup\{\mathcal{R}es(S_1, S_2) \mid S_1, S_2 \in \mathcal{S}\}, \\ R_{\mathcal{R}}(\mathcal{S}) & = & \mathcal{S} \cup X_{\mathcal{R}}(\mathcal{S}) \cup \mathcal{R}es(\mathcal{S}). \end{array}$$

The deductive closure under $R_{\mathcal{R}}$ is defined by

$$\begin{array}{rcl} R^0_{\mathcal{R}}(\mathcal{S}) & = & \mathcal{S}, \\ R^{i+1}_{\mathcal{R}}(\mathcal{S}) & = & R_{\mathcal{R}}(R^i_{\mathcal{R}}(\mathcal{S})), \\ R^*_{\mathcal{R}}(\mathcal{S}) & = & \bigcup_{i \in \mathbb{N}} R^i_{\mathcal{R}}(\mathcal{S}). \end{array}$$

Remark: If $\mathcal{S}$ is a set of clauses $R^*_{\mathcal{R}}(\mathcal{S})$ is the set of all clauses derivable by cut (on formula variables) and the structural rules of $\mathcal{R}$.

We distinguish two cases according to whether the weakening rule is in $\mathcal{R}$ or it is not.

4.1. $(w, l) \notin \mathcal{R}$

Let $\mathcal{R}$ be a normal contractive system (otherwise, by Corollary 3.1 $\mathcal{R}$ is not weakly substitutive). As the permutation rules are always available we define two clauses as equal if they are permutation variants of each other.

Definition 4.3. Let C_1, C_2 be Horn clauses where $C_1 = U \vdash \alpha$ and $C_2 = \alpha, V \vdash \gamma$. Then we write the resolvent

$$C\colon\ U, V \vdash \gamma$$

of C_1, C_2 as C_1C_2. We say that C is the product of C_1, C_2. C_1 is called the *active clause* of the product, C_2 the passive one. If C_1, C_2 have no resolvent with C_1 as active clause then we say that C_1C_2 is undefined.

Remark: The multiplication defined above is neither associative nor commutative: If $C_1 = \beta \vdash \alpha$ and $C_2 = \alpha, \alpha \vdash \gamma$. Then $C_1(C_1C_2)$ is defined and is $\beta, \beta \vdash \gamma$, but $(C_1C_1)C_2$ is undefined. Clearly C_1C_2 is defined, but C_2C_1 is not.
On the other hand C_1C_2 is unique if it exists; that justifies the notation as binary function.

Though the product is not associative it is *semi-associative* in the following sense:

Lemma 4.1. Let C_1, C_2, C_3 Horn clauses s.t. $(C_1C_2)C_3$ is defined. Then

$$(C_1C_2)C_3 = C_1(C_2C_3).$$

Proof. The product $(C_1C_2)C_3$ is only defined if the clauses are of the following form

$$\begin{aligned} C_1 &= U \vdash \alpha, \\ C_2 &= \alpha, V \vdash \beta, \\ C_3 &= \beta, W \vdash \gamma. \end{aligned}$$

But then $(C_1C_2)C_3 = C_1(C_2C_3) = UVW \vdash \gamma$. □

Note that the product of Horn clauses represents resolution without structural rules (except permutation which is built in).

Definition 4.4. Let $\mathcal{S}$ be a set of Horn clauses. Then we define

$$R(\mathcal{S}) = \{C_1C_2 \mid C_1, C_2 \in \mathcal{S} \text{ and } C_1C_2 \text{ is defined}\}.$$

Furthermore we define the deductive closure:

$$\begin{aligned} R^0(\mathcal{S}) &= \mathcal{S}, \\ R^{i+1}(\mathcal{S}) &= R(R^i(\mathcal{S})) \cup R^i(\mathcal{S}), \\ R^*(\mathcal{S}) &= \bigcup_{i \in \mathbb{N}} R^i(\mathcal{S}). \end{aligned}$$

The following lemma shows that every derivable Horn clause is a product of another derivable clause and an input clause. This is a standard result in automated deduction implying that there is always an input refutation of a set of Horn clauses, see e.g. (Leitsch 1997).

Definition 4.5. A product of clauses $C_1, \ldots, C_n$ is called in *right-parenthesis form* if

$$C = C_1(C_2 \ldots (C_{n-1}C_n)).$$

Lemma 4.2. Every product of Horn clauses can be transformed into right-parenthesis form.

Proof. By induction on the number n of clauses occurring in the product. The case $n = 1$ is trivial.
(IH) Assume the lemma holds for n.

Let C be a product of $n+1$ Horn clauses which is defined. Then $C = DE$, where D, E are products of $\leq n$ Horn clauses $D_1, \ldots, D_k$ and $E_1, \ldots, E_m$ with $k + m = n + 1$.
By (IH) $D = D_1 D'$ for some D' (which is a product of $k-1$ Horn clauses) and so $C = (D_1 D')E$. By semi-associativity we obtain $C = D_1(D'E)$. But $D'E$ is a product of n Horn clauses and we apply (IH) again. By iteration of the argument we eventually obtain

$$C = D_1(D_2 \ldots (E_1(E_2 \ldots (E_{m-1}E_m) \ldots)$$

□

Corollary 4.1. Let $\mathcal{S}$ be a set of Horn clauses and $C \in R^*(\mathcal{S})$. Then C can be represented in right-parenthesis form over clauses in $\mathcal{S}$.

Proof. By Lemma 4.2 every $C \in R^*(\mathcal{S})$ can be represented in right-parenthesis form. Clearly all the clauses appearing in the product occur in $\mathcal{S}$. □

Definition 4.6. Let $\mathcal{R}$ be a normal contractive system. A clause C is in $\mathcal{R}$ *normal form* if no rule in $\mathcal{R}$ is applicable to C.
Let $\mathcal{S}$ be a set of Horn clauses. Then $\nu_{\mathcal{R}}(\mathcal{S})$ is the set of clauses in normal form which can be obtained by reduction via $\mathcal{R}$.

Note that for normal contractive systems $\nu_{\mathcal{R}}(\mathcal{S})$ is always finite for finite $\mathcal{S}$. But there is even more:

Proposition 4.1. Let $\mathcal{R}$ a normal contractive system and $\mathcal{S}$ be a (possibly infinite) set of Horn clauses over a finite set of variables (formula- and/or multisets of formula-variables). Then $\nu_{\mathcal{R}}(\mathcal{S})$ is finite.

Proof. Let $V\colon \{x_1, \ldots, x_n\}$ be the set of all variables in $\mathcal{S}$. Then every clause over V is of the form

$$C\colon\ x_1^{k_1} \ldots x_n^{k_n} \vdash x_j^p$$

for $k_i \in \mathbb{N}$ and $p \in \{0, 1\}$ (if $k_i = 0$ we omit the element $x_i^{k_i}$ from the sequent). Now let k be the maximal number s.t. $(k, 1) \in \mathcal{R}$. Then $\nu(C)$ consists only of clauses

$$D\colon\ x_1^{r_1} \ldots x_n^{r_n} \vdash x_j^p$$

for $r_i \leq k$ for $i = 1, \ldots, n$. Indeed any larger power of an x_i can be reduced via $\mathcal{R}$. But the number of such clauses is finite and $\leq k^{n+1}$. □

Lemma 4.3. Let $\mathcal{R}$ be a normal contractive system and $\mathcal{S}$ be a set of Horn clauses. Then

$$\nu_{\mathcal{R}}(R^*_{\mathcal{R}}(\mathcal{S})) = \nu_{\mathcal{R}}(R^*(\mathcal{S})).$$

Proof. It is enough to show that for all $C \in R^*_{\mathcal{R}}(\mathcal{S})$ there exists a $D \in R^*(\mathcal{S})$ with $D \to^*_{\mathcal{R}} C$ (then, clearly, $\nu_{\mathcal{R}}(C) \subseteq \nu_{\mathcal{R}}(D)$). We prove this property for $C \in R^i_{\mathcal{R}}(\mathcal{S})$ by induction on i. The case $i = 0$ is trivial, as $\mathcal{S} \subseteq R^*(\mathcal{S})$.
(IH) Assume that for all $C \in R^i_{\mathcal{R}}(\mathcal{S})$ there exists a $D \in R^*(\mathcal{S})$ with $D \to^*_{\mathcal{R}} C$.
Now let $C \in R^{i+1}_{\mathcal{R}}(\mathcal{S}) - R^i_{\mathcal{R}}(\mathcal{S})$.

(a) If $C' \to_{\mathcal{R}} C$ for $C' \in R^i_{\mathcal{R}}(\mathcal{S})$ then, by (IH), there exists a $D' \in R^*(\mathcal{S})$ s.t. $D' \to^*_{\mathcal{R}} C'$. But then clearly $D' \to^*_{\mathcal{R}} C$.

(b) Let $C = C_1 C_2$ for $C_1, C_2 \in R^i_{\mathcal{R}}(\mathcal{S})$ and

$$C_1 = U \vdash \alpha,\ C_2 = \alpha^m Y \vdash z \text{ for some } \alpha \in FV$$

and α not in Y. By (IH) there exist clauses $D_1 \colon U_0 \vdash \alpha$ and $D_2 \colon \alpha^M Y_0 \vdash z$ with $D_1, D_2 \in R^*(\mathcal{S})$ and $D_1 \to^*_{\mathcal{R}} C_1$, $D_2 \to^*_{\mathcal{R}} C_2$. In particular we have

$$U_0 \to^*_{\mathcal{R}} U,\ \alpha^M \to^*_{\mathcal{R}} \alpha^m \text{ and } Y_0 \to^*_{\mathcal{R}} Y.$$

(b1) $m > 1$. Then $C = \alpha^{m-1} U Y \vdash z$. Clearly $\alpha^{M-1} \to^*_{\mathcal{R}} \alpha^{m-1}$ and so the product D of D_1, D_2 fulfills

$$D = \alpha^{M-1} U_0 Y_0 \vdash z \to^*_{\mathcal{R}} C.$$

So $D \in R^*(\mathcal{S})$ and $D \to^*_{\mathcal{R}} C$.

(b2) $m = 1$. Then $C = UY \vdash z$. Note that $\alpha^M \to^*_{\mathcal{R}} \alpha$ and, more generally, $X^M \to^*_{\mathcal{R}} X$ for all sequences X. Resolving D_1 with D_2 M-times, i.e. constructing the product

$$D_1(D_1 \ldots (D_1 D_2) \ldots) \text{ with } M \text{ occurrences of } D_1,$$

results in the clause $D = U_0^M Y_0 \vdash z$ which is in $R^*(\mathcal{S})$. But

$$U_0^M Y_0 \vdash z \to^*_{\mathcal{R}} U_0 Y_0 \vdash z \to^*_{\mathcal{R}} UY \vdash z, \quad \text{so} \quad D \to^*_{\mathcal{R}} C.$$

□

Lemma 4.4. Let $\mathcal{S}$ be a set of Horn clauses and $\mathcal{R}$ be a normal contractive system. Then there exists an algorithm constructing $\nu_{\mathcal{R}}(R^*(\mathcal{S}))$.

Proof. We know that all clauses in $R^*(\mathcal{S})$ can be written in right-parenthesis form (Lemma 4.2). Let $\mathcal{S} = \{C_1, \ldots, C_n\}$. We construct a search tree in the following way:

— Let T_0 be the root.

— T_1 is defined by n edges $E_1, \ldots E_n$ spreading from the root and labeled with the clauses $C_1, \ldots, C_n$. For every end-node N_i in T_1 corresponding to the edge E_i we define $\gamma(N_i) = C_i$, stop(N_i) = false.

— Let T_n be already constructed. We define T_{n+1}: To every end-node M of T_n for which stop(M) = false attach n edges labeled by the clauses $C_1, \ldots, C_n$. For the corresponding end-nodes $N(M, C_i)$ we define

$$\gamma(N(M, C_i)) = C_i \gamma(N(M, C_i))$$

– provided the product is defined. If the product is undefined we delete $N(M, C_i)$. For every end-node N which is not deleted we check whether there exists a predecessor N' on the path from the root to N with $\nu_{\mathcal{R}}(\gamma(N)) = \nu_{\mathcal{R}}(\gamma(N'))$; if the last equation holds we define stop(N) = true.

As $\nu_{\mathcal{R}}(R^*(\mathcal{S}))$ is finite the production of the tree will stop after finitely many steps. Indeed, an infinite path $(N_i)_{i \in \mathbb{N}}$ in the tree can only be constructed if $\nu_{\mathcal{R}}(N_i) \neq \nu_{\mathcal{R}}(N_j)$ for all i, j with $i \neq j$. This is impossible as the set of all subsets of $\nu_{\mathcal{R}}(R^*(\mathcal{S}))$ is finite. If $\gamma(N_i) = \gamma(N_j)$ for $i < j$ we may stop the production of new edges as no new normal forms of clauses will be produced furthermore. The tree T^* produces all clauses in $R^*(\mathcal{S})$ as it produces all products in right-parenthesis form, which is sufficient. □

Corollary 4.2. Let $\mathcal{S}$ be a set of Horn clauses and $\mathcal{R}$ be a normal contractive system. Then there exists an algorithm constructing $\nu_{\mathcal{R}}(R^*_{\mathcal{R}}(\mathcal{S}))$.

Proof. Immediate by Lemma 4.4 and Lemma 4.3. □

4.2. $(w, l) \in \mathcal{R}$

By Proposition 3.8 $\mathcal{R}$ is weakly substitutive if and only if $\mathcal{R} = \{(w, l)\}$ or at least a rule $(n_i, 1) \in \mathcal{R}$, with $n_i > 1$. In both cases, reductivity could be checked using the results in Miller and Pimentel (2002;2005). We give below an alternative proof using resolution.

Theorem 4.1. Let $\mathcal{R} = \{(w, l), (n_i, 1)\}$ for $n_i > 1$. Then $R^*_{\mathcal{R}}(\mathcal{S})$ is decidable.

Proof. We have shown in Section 3 that in this case ordinary contraction can be simulated. Thus the resolution calculus is that of ordinary classical resolution, which is decidable. Indeed, using the contraction normal form of clauses (no repetition of occurring atoms) only finitely many clauses can be derived, or more formally $RN^*(\mathcal{S})$ is finite for the corresponding normal resolution operator RN. Then a clause C is in $R^*_{\mathcal{R}}(\mathcal{S})$ if either the normal form C^* of C occurs in $RN(\mathcal{S})$, or C^* can be obtained from $RN(\mathcal{S})$ via weakening (i.e. subsumption). Obviously this test can be done algorithmically. □

For the proof of the theorem we need the subsumption principle from automated deduction adapted to our purposes:

Definition 4.7. Let $\mathcal{S}_1, \mathcal{S}_2$ be sets of Horn clauses. Then $\mathcal{S}_1 \leq_{ss} \mathcal{S}_2$ if for every $D \in \mathcal{S}_2$ there exists a $C \in \mathcal{S}_1$ s.t. D can be obtained from C by (possibly multiple) applications of (w, l).

Theorem 4.2. Let $\mathcal{R} = \{(w, l)\}$. Then $R^*_{\mathcal{R}}(\mathcal{S})$ is decidable.

Proof.

(a) There are no positive unit clauses in $\mathcal{S}$.
We check whether a clause C is in $R_{\mathcal{R}} * (\mathcal{S})$. To this aim we produce $R^*_{\mathcal{R}}(\mathcal{S})$ but stop the production on clauses D s.t. $l(D) > l(C) + 1$ (where $l(C)$ is the length of C i.e. the number of variables occurring in C). Indeed, if we obtain a clause D with $l(D) > l(C) + 1$ it cannot contribute to a derivation of C. Note that the length can only be decreased by resolution with *negative* unit clauses; but these are only the last elements of resolution products, reducing the length at most by one. So let $\mathcal{S}'$ be the set of clauses produced as indicated above. Then $\mathcal{S}'$ is finite and can be produced in finitely many steps. Finally $C \in \mathcal{S}'$ iff $C \in R^*_{\mathcal{R}}(\mathcal{S})$.

(b) There are positive unit clauses in $\mathcal{S}$.
Produce $R^*_+(\mathcal{S})$, the set of all positive clauses in $R^*(\mathcal{S})$. $R^*_+(\mathcal{S})$ is finite and can be constructed by hyperresolution (see (Leitsch 1997), chapter 3.4).
Let $\mathcal{S}_1 = \mathcal{S}_0 \cup R^*_+(\mathcal{S})$ s.t. $\mathcal{S}_0$ consists of the nonpositive clauses in $\mathcal{S}$. Now perform

resolution only between clauses in $R^*_+(\mathcal{S})$ and $\mathcal{S}_0$ until all formula variables in $R^*_+(\mathcal{S})$ are cut out from (the antecedents of) the clauses in $\mathcal{S}_0$. The result is a finite set of clauses

$$\mathcal{S}_2 = \mathcal{S}' \cup R^*_+(\mathcal{S})$$

s.t. no resolvents are definable between $R^*_+(\mathcal{S})$ and $\mathcal{S}'$, and $\mathcal{S}'$ only consists of non-positive clauses. By definition of $\mathcal{S}_2$ we have

$$\mathcal{S}' \leq_{ss} \mathcal{S}_0.$$

But then $\mathcal{S}_2 \leq_{ss} \mathcal{S}_1$ and, by the subsumption principle in resolution (see (Leitsch 1997), chapter 4.2),

$$R^*(\mathcal{S}_2) \leq_{ss} R^*(\mathcal{S}_1) = R^*(\mathcal{S}) \leq_{ss} R^*_{\mathcal{R}}(\mathcal{S}).$$

By transitivity of subsumption we obtain

$$R^*(\mathcal{S}_2) \leq_{ss} R^*_{\mathcal{R}}(\mathcal{S}).$$

Moreover, by definition of $\mathcal{S}_2$, we have

$$R^*(\mathcal{S}_2) = R^*(\mathcal{S}') \cup R^*_+(\mathcal{S}).$$

Now we check whether a clause C is in $R^*_{\mathcal{R}}(\mathcal{S})$. Clearly $R^*(\mathcal{S}_2)$ subsumes a clause C if either (i) $R^*_+(\mathcal{S}) \leq_{ss} C$ or (ii) $R^*(\mathcal{S}') \leq_{ss} C$. (i) can be checked directly as $R^*_+(\mathcal{S})$ is finite, for (ii) we apply the same method as in (a) (in fact there are no positive unit clauses in $\mathcal{S}'$).

□

Theorem 4.3. Reductivity is decidable for knotted commutative calculi with weakly substitutive regular sets.

Proof. Reductivity for normal contractive systems is decidable. Indeed let $C(\varphi)$ be the reduction set of a $\star$-cut-derivation schema (see Definition 2.5) φ and $S(\varphi)$ be the end-sequent of φ. Then $S(\varphi)$ is in $\mathcal{R}$-normal form. Therefore $S(\varphi) \in \nu_{\mathcal{R}}(R^*_{\mathcal{R}}(C(\varphi)))$ iff $S(\varphi) \in R^*_{\mathcal{R}}(C(\varphi))$.
By Lemma 4.3

$$\nu_{\mathcal{R}}(R^*_{\mathcal{R}}(C(\varphi))) = \nu_{\mathcal{R}}(R^*(C(\varphi))).$$

By Lemma 4.4 the finite set $\nu_{\mathcal{R}}(R^*(C(\varphi)))$ can be constructed algorithmically. This gives a decision procedure for reductivity. The claim follows by Section 3.3, Theorem 4.1 and Theorem 4.2. □

Remark: If rules introducing a connective $\star$ are not reductive, then the corresponding knotted commutative calculus does not admit reductive cut-elimination. As in the case of weak substitutivity, this is not enough to conclude that the calculus does not admit cut-elimination at all. E.g. the rules for $\bar{\wedge}$ in the calculus K_1 of Example 2.1 are not reductive, see e.g. (Ciabattoni and Terui 2006). However K_1 trivially admits cut-elimination since the rule $(\bar{\wedge}, l)$ cannot appear in any derivation and the only sequents provable in K_1 are instances of the identity axiom schema.(Note that applications of (CUT) on these sequents can be easily eliminated).

5. A General Cut-elimination Procedure

Here we provide a constructive proof of cut-elimination for knotted commutative calculi whose structural rules are weakly substitutive and logical rules reductive. Henceforth K will denote any such calculus.

Definition 5.1. The *length* $|d|$ of a derivation d is the maximal number of inference rules + 1 occurring on any branch of d. The *complexity* $|A|$ of a formula A is defined as the number of occurrences of its connectives. The *cut rank* $\rho(d)$ of d is (the maximal complexity of the cut-formulae in d) + 1 ($\rho(d) = 0$ if d has no cuts).

Our cut-elimination procedure for K proceeds by removing cuts which are topmost among all cuts with cut rank equal to the rank of the whole deduction. Let, e.g.

$$\dfrac{\begin{array}{c}\vdots\, d_r \\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\vdots\, d_l \\ \Sigma\, A \vdash \Pi\end{array}}{\Gamma, \Sigma \vdash \Pi}\ \text{(CUT)}$$

a subderivation ending in such a cut. Roughly speaking our strategy is as follows: using the fact that rules are weakly substitutive, (1) we shift up this cut over d_r *as much as possible* until we meet an axiom (or a logical rule with no premiss) or a logical rule introducing the cut formula A (Lemma 5.2). In the former case the cut is easily eliminated. In the latter case we shift this cut upward over d_l and, (2) we replace it by cuts with smaller complexity, when we meet a rule introducing the cut formula A (Lemma 5.1). By Proposition 2.1 this can be done since the logical rules are reductive.

Remark: Using Theorems 3.2 and 4.3 for automating steps (1) and (2), the above strategy (in fact, the formal proof below) can lead to a mechanical construction of a cut-free proof from any proof in a given K.

Henceforth we write $d \vdash_K S$ if d is a derivation in K of S.

Lemma 5.1 (Logical Connectives). Let K be any knotted commutative sequent calculus whose structural rules are weakly substitutive. Let $d_l \vdash_K \Sigma\, A \vdash B$, and $d_r \vdash_K \Gamma \vdash A$, with $\rho(d_l)$, $\rho(d_r) \le |A|$. If $A = \star(A_1, \ldots, A_n)$ is the principal formula of the last rule in the derivation d_r and the rules for the connective $\star$ are reductive in K then for all $T' \in Cut_l^*(\Gamma \vdash A, \Sigma\, A \vdash B)$, we can find a derivation $d \vdash_K T'$ with $\rho(d) \le |A|$.

Of course, one could derive T' by applying (CUT), but the resulting derivation would then have cut rank $|A| + 1$.

Proof. We proceed by induction on $|d_l|$.
Base case: $|d_l| = 1$. Then $\Sigma\, A \vdash B$ is an instance of either an axiom or of a logical rule with no premisses. In the former case $Cut_l^*(\Gamma \vdash A, \Sigma\, A \vdash B) = \{\Sigma\, A \vdash B,\ \Gamma \vdash A\}$, and the required derivation is either d_l or d_r while in the latter case we distinguish two (sub)cases: the rule does not introduces A in the antecedent or it does. In the former (sub)case by condition **(log2)** the required derivation is T' while in the latter (sub)case the claim follows by Proposition 2.1, as rules for $\star$ are reductive.
(IH): $|d_l| > 1$. Let (r) be the last inference rule applied in d_l. Assume w.l.o.g. that (r)

has the form

$$\frac{S_1 \ \dots \ S_m}{S}$$

If (r) is neither (CUT) nor a logical rule introducing A in the antecedent, by (IH) m times for all $T' \in \bigcup_{i=1}^{m} Cut_l^*(\Gamma \vdash A, S_i)$, we can find a derivation $d' \vdash_K T'$ with $\rho(d') \leq |A|$. Now by weak substitutivity (and **(log2)**, if (r) is a logical rule), every $T' \in Cut_l^*(\Gamma \vdash A, \Sigma\, A \vdash B)$ is cut-free derivable from $\bigcup_{i=1}^{m} Cut_l^*(\Gamma \vdash A, S_i)$. Hence T' has a derivation d in K with $\rho(d) \leq |A|$.
If $(r) = (CUT)$ the claim follows by (IH) and an application of (CUT) (note that by hypothesis the cut-formula of this cut is of smaller complexity than A).
Suppose that (r) is a rule introducing A in the antecedent. In this case, by condition **(log2)** the rule

$$\frac{S_1' \ \dots \ S_m'}{S'}$$

where S' is obtained by λ consecutive applications of (CUT) between S and $\Gamma \vdash A$ (where $\lambda{+}1$ is the number of occurrences of A on the antecedent in S) and $S_i' \in Cut_l^*(\Gamma \vdash A, S_i)$, for each $1 \leq i \leq m$, is an instance of (r). Hence the claim follows by Proposition 2.1, being the rules for $\star$ reductive. □

Lemma 5.2 (Shifting Lemma). Let K be any knotted commutative calculus in which (a) logical rules are reductive and (b) structural rules are weakly substitutive. Let $d_r \vdash_K \Gamma \vdash A$ and $d_l \vdash_K \Sigma\, A \vdash B$ with $\rho(d_r), \rho(d_l) \leq |A|$. For all $T' \in Cut_r^*(\Sigma\, A \vdash B, \Gamma \vdash A)$, we can find a derivation $d \vdash_K T'$ with $\rho(d) \leq |A|$.

Proof. Proceed by induction on $|d_r|$, similarly to the previous proof. The main difference is that condition **(log3)** is used and, when the last inference rule (r) applied is a rule (with premisses) introducing A on the consequent the claim follows by Lemma 5.1. □

Theorem 5.1 (Cut Elimination). Any knotted commutative calculus K in which (a) logical rules are reductive and (b) structural rules are weakly substitutive, admits cut-elimination.

Proof. Let d be a derivation in K with $\rho(d) > 0$. The proof proceeds by a double induction on $(\rho(d), n\rho(d))$, where $n\rho(d)$ is the number of cuts in d with cut rank $\rho(d)$. Indeed, let us take in d an uppermost cut with cut rank $\rho(d)$. By applying Lemma 5.2 to its premisses $\Gamma\, A \vdash \Delta$ and $\Sigma \vdash A$ either $\rho(d)$ or $n\rho(d)$ decreases. □

Remarkably enough our cut-elimination procedure can be applied to any single-conclusion sequent calculus whose logical rules, satisfying conditions **(log2)** and **(log3)**, are reductive and structural rules are weakly substitutive. In particular, it does work for the *simple sequent calculi* considered in (Ciabattoni and Terui 2006). The same does not hold for the well known cut-elimination procedures à la Gentzen and à la Schütte-Tait (Schütte 1960; Tait 1968). Indeed Gentzen's method can be applied only when suitable "ad hoc" derivable generalizations of the cut rule (mix-style) are found. These generalizations, needed to cope with rules duplicating formulas are not always easy to define. As an example consider the calculus obtained by extending **ILL** with weak contraction, i.e. the rule

$$\frac{\Theta\,\alpha^2 \vdash \epsilon}{\Theta\,\alpha \vdash \epsilon}$$

On the other hand the applicability of the Schütte-Tait cut-elimination method relies on the inversion of (at least) one of the premises of each canonic cut-derivation. This cannot always be done in calculi that admit reductive cut-elimination. For example, neither of the premises of a canonic $\wedge$ cut-derivation can be inverted in the usual way in the calculus **LBC-** of Baaz, Ciabattoni and Montagna (2004) (see Example 2.1) and hence the Schütte-Tait procedure does not apply to **LBC-** (although its logical rules satisfy conditions **(log2)** and **(log3)** and are reductive while its structural rules are weakly substitutive).

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