Bunched Hypersequent Calculi for Distributive Substructural Logics

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Abstract

We introduce a new proof-theoretic framework which enhances the expressive power of bunched sequents by extending them with a hypersequent structure. A general cut-elimination theorem that applies to bunched hypersequent calculi satisfying general rule conditions is then proved. We adapt the methods of transforming axioms into rules to provide cutfree bunched hypersequent calculi for a large class of logics extending the distributive commutative Full Lambek calculus $\mathsf{DFL_e}$ and Bunched Implication logic BI. The methodology is then used to formulate new logics equipped with a cutfree calculus in the vicinity of Boolean BI.

1 Introduction

The wide applicability of logical methods and their use in new subject areas has resulted in an explosion of new logics. The usefulness of these logics often depends on the availability of an *analytic proof calculus* (formal proof system), as this provides a natural starting point for investigating metalogical properties such as decidability, complexity, interpolation and conservativity, for developing automated deduction procedures, and for establishing semantic properties like standard completeness [26]. A calculus is *analytic* when every derivation (formal proof) in the calculus has the property that every formula occurring in the derivation is a subformula of the formula that is ultimately proved (i.e. the *subformula property*). The use of an analytic proof calculus tremendously restricts the set of possible derivations of a given statement to derivations with a discernible structure (in certain cases this set may even be finite). Gentzen [17] presented the first analytic calculi, for classical and intuitionistic logic—in his *sequent calculus* formalism—by proving the celebrated *Hauptsatz*. Subsequently the formalism was extended to obtain analytic proof calculi for other logics. The key idea for obtaining an analytic calculus is the use of generalised proof rules that preserve the subformula property and make the subformula property-violating *cut rule* redundant. The cut rule is itself a generalisation of the rule of *modus ponens*, which is, of course, the crucial rule in a standard (Hilbert) axiomatisation of a logic. By deleting the now redundant cut rule, an analytic calculus is obtained.

Unfortunately there are many logics which do not support such cut-elimination in the sequent calculus formalism due to inherent technical restrictions in the formalism. In the last three decades this has led to the introduction of many other formalisms; prominent examples include the hypersequent and display calculus [3, 2, 12, 19, 6], labelled calculus [36, 28] and bunched sequent calculus [14, 27]. These formalisms feature different expressive power and can be useful for proving different computational and metalogical properties of the formalised logics. In general, expressive formalisms support analytic calculi for more logics. Since this expressivity is typically obtained by the addition of new non-logical symbols (structural connectives), it becomes harder to control the form of derivations in the calculus, making it more difficult to

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prove metalogical properties and perform automated proof search. For this reason, the most suitable formalism for investigating the properties of a logic is often the simplest one (in terms of structure) which permits analyticity.

Bunched sequent calculi, also known as Dunn-Mints systems [14, 27, 31], were developed to provide a cutfree formulation of logics lacking weakening/integrality but satisfying the distributivity axiom $(p \wedge (q \vee r)) \multimap ((p \wedge q) \vee (p \wedge r))$. The idea behind these systems is to consider sequents of the form $X \Rightarrow A$ where X is permitted to contain structural symbols for both the additive/lattice conjunction (";") and multiplicative/monoidal conjunction (","). These structural symbols correspond, respectively, to the logical symbols $\wedge$ and $\otimes$.

Bunched calculi have been used to define analytic calculi for relevance logics and for the logic of Bunched Implication BI [33] introduced to reason about dynamic data structures [30]. Although these logics can be formalised using the more powerful formalism of display calculi,[1] the advantage of using a simpler formalism is evident, e.g., when searching for proofs of decidability of the logic (see [18, 5, 22]). Unlike display calculi, bunched calculi do not require that the structural connectives appear in residuated pairs, and the structural connectives occur in a derivation only when the corresponding logical connectives appear in the formula to be proved.

In this paper we introduce the new proof theoretic framework of *bunched hypersequents*, which generalises the bunched sequents. Bunched hypersequents are defined by extending bunched sequents with a hypersequent structure. I.e. use a finite non-empty multiset of bunched sequents rather than just a single bunched sequent. This allows the definition of new rules which apply to several bunched sequents simultaneously, thus increasing the expressive power of the framework. Although a bunched hypersequent is a more complex data structure than a bunched sequent, it is simpler in structural terms than a display sequent. Indeed, by adapting the method in [34], the bunched hypersequent formalism can be embedded into the display calculus formalism. Our aim is to capture a large class of those logics that defy an analytic bunched calculus while retaining much of the structural simplicity of the bunched calculus.

The expressive power of the new formalism is demonstrated by introducing cutfree bunched hypersequent calculi for a large class of extensions of distributive commutative Full Lambek calculus $\mathsf{DFL_e}$. The calculi are obtained by suitably extending the procedure in [8] for transforming Hilbert axioms into structural rules. We then consider the case of extensions of the logic of bunched implication BI. Aside from its theoretical interest, the recent applications [13] of BI-related logics illustrates the importance of having available general methods for the construction of analytic calculi. Extensions of BI by a certain class of axioms including restricted weakening or contraction are presented. However, our attempt to extend the BI calculus to obtain a simple analytic calculus for Boolean Bunched Implication BBI encountered a surprising obstacle: while a hypersequent structure extending the bunched calculus for BI *can* be defined (and hence also logics extending BI via the exploitation of the hypersequent structure), there are technical difficulties associated with its interpretation. This demonstrates the the importance of verifying the sequent-to-hypersequent, axioms-to-structural rules paradigm for each framework of interest. In response, motivated by the perspective gained from the hypersequent calculi for extensions of $\mathsf{DFL_e}$, we turn the investigation on its head and formulate an analytic hypersequent calculus for a consistent extension of BI which derives a limited boolean principle—$\mathbf{1} \Rightarrow p \vee (p \rightarrow \bot)$ but not $\top \Rightarrow p \vee (p \rightarrow \bot)$—and hence is not BBI, and whose properties, including its decidability problem, invite further investigation.

[1] A display calculus typically contains many structural connectives because the structural connectives come in pairs [3, 19, 12]: whenever there is a structural connective corresponding to a logical connective, the calculus will also contain a structural connective corresponding to the algebraic residual of that logical connective.

2 Preliminaries

The base logic we consider in this paper is the substructural logic called distributive commutative Full Lambek calculus $\mathsf{DFL_e}$[2]. A $\mathsf{DFL_e}$ *formula* is built from propositional variables in a countably infinite set $\mathcal{V}$, using the binary connectives $\vee, \wedge, \otimes, \multimap$ and the constants $\mathbf{1}, \mathbf{0}$. The algebraic semantics of $\mathsf{DFL_e}$ (see [15]) are given by the class of algebras $\mathbf{A} = (A, \wedge, \vee, \otimes, \multimap, \mathbf{1}, \mathbf{0})$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \otimes, \mathbf{1})$ is a commutative monoid, $\mathbf{0}$ an arbitrary element of A and $x \otimes y \leq z$ iff $x \leq y \multimap z$ for all $x, y, z \in A$. Let $\mathbb{K}$ be a class of algebras. Then $\models_{\mathbb{K}} C \leq D$ denotes that for every $\mathbf{A} \in \mathbb{K}$ and valuation V (i.e. the map from $\mathcal{V}$ to A lifted to formulae) it is the case that $C^V \leq D^V$ holds on $\mathbf{A}$. Distributive Full Lambek logic with exchange $\mathsf{DFL_e}$ is the logic of $\mathsf{FL_e}$-algebras satisfying *distributivity* $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ (the reverse inequality holds on all $\mathsf{FL_e}$-algebras).

The analytic sequent calculus for $\mathsf{FL_e}$ is well-known. It can be obtained from Gentzen's sequent calculus **LJ** [17] for propositional intuitionistic logic by deleting the structural rules of weakening and contraction. Then the structural connective "," in the antecedent of sequents stands for $\otimes$. As shown in [8], analytic calculi for many axiomatic extensions of $\mathsf{FL_e}$ can be defined in a modular way by adding structural rules to the sequent calculus for $\mathsf{FL_e}$. However, an analytic calculus for $\mathsf{DFL_e}$ cannot be obtained by structural rule extension in this way (see Corollary 7.4 in [9]). Instead it is necessary to augment the structural symbol (";") in the sequent calculus with a second symbol (",") thus leading to the bunched calculi. A *bunch* is a finite term from the following grammar:

$$X := A \text{ is a formula} \quad | \quad \varnothing_m \quad | \quad (X, X) \quad | \quad (X; X)$$

A *(bunched) sequent* (denoted $X \Rightarrow \psi$) is an ordered pair where X (the *antecedent*) is a bunch and ψ (the *succedent*) is a formula or $\mathcal{O}_m$. The structural constant $\varnothing_m$ ($\mathcal{O}_m$) will be interpreted as $\mathbf{1}$ (resp. $\mathbf{0}$). Notice that $\mathcal{O}_m$ appears only in the succedent. A *context* $\Gamma[\,]$ is an extension of the bunch grammar to permit the occurrence of a single 'hole' $[\,]$. When $\Gamma[\,]$ is a context, then $\Gamma[\Delta]$ is the bunch obtained by filling the hole with the bunch Δ.

Example 1. *Consider the bunch $\alpha; (\beta, \gamma)$. We may write this as $\Gamma[\Delta]$ where Δ is any one of the following: $\alpha \quad \beta \quad \gamma \quad \beta, \gamma \quad \alpha; (\beta, \gamma)$. These are the only possibilities. Then $\Gamma[\,]$ is, respectively: $[\,]; (\beta, \gamma) \quad \alpha; ([\,], \gamma) \quad \alpha; (\beta, [\,]) \quad \alpha; [\,] \quad [\,]$.*

Definition 2 (Sequent calculus $s\mathsf{DFL_e}$)**.** *The rules of $s\mathsf{DFL_e}$ are given in Fig. 1 by deleting the hypersequent context "$g\,|$" and the rules (EW) and (EC).*

A *rule instance* is obtained from a rule (schema) by suitably instantiating the various metavariables with formulae/bunches/hypersequents. Following standard practice, we do not explicitly distinguish between a rule and its instance.

Notation. $\mathcal{C} + r$ denotes the extension of the calculus $\mathcal{C}$ with rule r. Also $\models_{\mathsf{DFL_e} + \{A_i \leq B_i\}_{i \in I}}$ denotes the semantic consequence relation restricted to $\mathsf{DFL_e}$ algebras satisfying $\{A_i^V \leq B_i^V\}_{i \in I}$ under every valuation V.

Theorem 3. *Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be finite sets of $\mathsf{DFL_e}$ formulae. Then $C \Rightarrow D$ is derivable in $s\mathsf{DFL_e} + (cut) + \{A_i \Rightarrow B_i\}_{i \in I}$ iff $\models_{\mathsf{DFL_e} + \{A_i \leq B_i\}_{i \in I}} C \leq D$.*

Proof. Straightforward generalisation of the proof for $\mathsf{FL_e}$ [15]. □

[2] FL_e is also known as Intuitionistic Linear Logic without the exponential connectives.

If $\models_{\mathsf{DFL_e}+\{\mathbf{1}\leq B_i\}_{i\in I}} \mathbf{1} \leq D$ then D is said to be a *theorem* of the Hilbert calculus for $\mathsf{DFL_e}$ [15] extended with axioms $\{B_i\}_{i\in I}$ (henceforth denoted $D \in \mathsf{DFL_e} + \{B_i\}_{i\in I}$).

The sequent calculus $\mathbf{s}\mathsf{DFL_e} + (cut)$ has *cut-elimination*: all instances of the cut rule in a derivation can be eliminated to obtain a *cutfree* derivation of the same sequent. So $\mathbf{s}\mathsf{DFL_e}+(cut)$ and $\mathbf{s}\mathsf{DFL_e}$ derive the same sequents. Nevertheless, in general, the calculus $\mathbf{s}\mathsf{DFL_e} + (cut) + \{\varnothing_m \Rightarrow B_i\}_{i\in I}$ does not have cut-elimination. It is precisely this failure of cut-elimination and the desirability of the subformula property (typically a consequence of cut-elimination) that motivates the work here on the bunched hypersequent formalism.

3 Bunched hypersequent calculi

Hypersequents [1, 32] extend the sequent formalism by considering multiple sequents rather than just a single one. Although a hypersequent is a more complex data structure than a sequent, it is not *much* more complicated, and it goes in fact just one step further. Nevertheless, the use of hypersequents has yielded analytic calculi for many more logics of interest with respect to sequents (see e.g. [2, 8, 26]). In this paper we extend the bunched sequent formalism to obtain *bunched hypersequents*.

Definition 4. *A* bunched hypersequent *is a multiset* $\Gamma_1 \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Pi_n$ *of bunched sequents. Each* $\Gamma_i \Rightarrow \Pi_i$ *is a* component *of the bunched hypersequent.*

The hypersequent version of the bunched calculus $\mathbf{s}\mathsf{DFL_e}$ is obtained simply by adding a context g to all sequents in $\mathbf{s}\mathsf{DFL_e}$, and the rules (EW) and (EC) which behave like weakening and contraction over whole components of bunched hypersequents. More precisely:

Definition 5 (Bunched hypersequent calculus $\mathbf{h}\mathsf{DFL_e}$)**.** *The rules of the hypersequent calculus* $\mathbf{h}\mathsf{DFL_e}$ *are given in Fig. 1.*

As with bunched sequent calculi, the rules of bunched hypersequent calculi consist of initial bunched hypersequents (i.e. axioms), the cut rule as well as logical and structural rules. These inference rules are usually presented as rule schemata. A rule instance is obtained by instantiating the schematic variables. The structural rules are divided into *internal* and *external rules*. The former deal with formulae within one component of the conclusion, e.g. in Fig. 1 (m-ex), (ex), (c) and (w) are internal structural rules, while (EW) and (EC) are external ones.

A *derivation* is defined in the usual way as a finite tree of bunched hypersequents constructed from the rules starting with the initial sequents.

Definition 6 (Interpretation)**.** *The interpretation* h^I *of a bunched hypersequent* $s_1|\ldots|s_{n+1}$ *of* $\mathbf{h}\mathsf{DFL_e}$ *is the formula* $s_1^I \vee \ldots \vee s_{n+1}^I$*, where* $(X \Rightarrow A)^I$ *is* $X^I \multimap A$*,* $(X \Rightarrow \mathcal{O}_m)^I$ *is* $X^I \multimap \mathbf{0}$ *and the interpretation of bunches is*

$$A^I := A \qquad \varnothing_m^I := \mathbf{1} \qquad (\Gamma;\Delta)^I := \Gamma^I \wedge \Delta^I \qquad (\Gamma,\Delta)^I := \Gamma^I \otimes \Delta^I$$

Remark 7. *The interpretation of* $s_1|\ldots|s_{n+1}$ *in the hypersequent version* $\mathbf{h}\mathsf{FL_e}$ *of the calculus for* FL_e *is instead the formula* $(s_1^I \wedge \mathbf{1}) \vee \ldots \vee (s_{n+1}^I \wedge \mathbf{1})$ *[10].*

We may apply (EC), (EW),($\varnothing_m$l),($\varnothing_m$l$'$) rules without explicit mention.

Remark 8. *A bunched sequent calculus can be viewed trivially as a bunched hypersequent calculus. The added expressive power of the latter is due to the possibility of defining new external structural rules that can act simultaneously on several components of a bunched hypersequent (see Section 5).*

$$\frac{g \mid X \Rightarrow A \qquad g \mid \Gamma[A] \Rightarrow \psi}{g \mid \Gamma[X] \Rightarrow \psi}\ (\text{cut}) \qquad \frac{}{A \Rightarrow A}\ (\text{init})$$

Units:

$$\frac{g \mid \Gamma[\varnothing_m] \Rightarrow \psi}{g \mid \Gamma[\mathbf{1}] \Rightarrow \psi}\ (\mathbf{1}\text{l}) \qquad \frac{}{\varnothing_m \Rightarrow \mathbf{1}}\ (\mathbf{1}\text{r}) \qquad \frac{}{\mathbf{0} \Rightarrow \mathcal{O}_m}\ (\mathbf{0}\text{l}) \qquad \frac{g \mid \Gamma \Rightarrow \mathcal{O}_m}{g \mid \Gamma \Rightarrow \mathbf{0}}\ (\mathbf{0}\text{r})$$

Multiplicatives:

$$\frac{g \mid X \Rightarrow A \qquad g \mid \Gamma[B] \Rightarrow \psi}{g \mid \Gamma[X, A \multimap B] \Rightarrow \psi}\ (\multimap l) \qquad \frac{g \mid A, X \Rightarrow B}{g \mid X \Rightarrow A \multimap B}\ (\multimap r)$$

$$\frac{g \mid \Gamma[A, B] \Rightarrow \psi}{g \mid \Gamma[A \otimes B] \Rightarrow \psi}\ (\otimes\text{l}) \qquad \frac{g \mid X \Rightarrow A \qquad g \mid Y \Rightarrow B}{g \mid X, Y \Rightarrow A \otimes B}\ (\otimes\text{r})$$

Additives:

$$\frac{g \mid \Gamma[A; B] \Rightarrow \psi}{g \mid \Gamma[A \wedge B] \Rightarrow \psi}\ (\wedge\text{l}) \qquad \frac{g \mid X \Rightarrow A \qquad g \mid X \Rightarrow B}{g \mid X \Rightarrow A \wedge B}\ (\wedge\text{r})$$

$$\frac{g \mid \Gamma[A] \Rightarrow \psi \qquad g \mid \Gamma[B] \Rightarrow \psi}{g \mid \Gamma[A \vee B] \Rightarrow \psi}\ (\vee\text{l}) \qquad \frac{X \Rightarrow A_i}{X \Rightarrow A_1 \vee A_2}\ (\vee\text{r})$$

Internal and external structural rules:

$$\frac{g \mid \Gamma[X] \Rightarrow \psi}{g \mid \Gamma[\varnothing_m, X] \Rightarrow \psi}\ (\varnothing_m\text{l}) \qquad \frac{g \mid \Gamma[X] \Rightarrow \psi}{g \mid \Gamma[X; Y] \Rightarrow \psi}\ (\text{w}) \qquad \frac{g \mid \Gamma[X; X] \Rightarrow \psi}{g \mid \Gamma[X] \Rightarrow \psi}\ (\text{c}) \qquad \frac{g}{g \mid X \Rightarrow \psi}\ (\text{EW})$$

$$\frac{g \mid \Gamma[\varnothing_m, X] \Rightarrow \psi}{g \mid \Gamma[X] \Rightarrow \psi}\ (\varnothing_m\text{l}') \qquad \frac{g \mid \Gamma[X, Y] \Rightarrow \psi}{g \mid \Gamma[Y, X] \Rightarrow \psi}\ (\text{m-ex}) \qquad \frac{g \mid \Gamma[X; Y] \Rightarrow \psi}{g \mid \Gamma[Y; X] \Rightarrow \psi}\ (\text{ex}) \qquad \frac{g \mid X \Rightarrow \psi \mid X \Rightarrow \psi}{g \mid X \Rightarrow \psi}\ (\text{EC})$$

Figure 1: The bunched hypersequent calculus $\mathbf{h}\mathsf{DFL_e} + (cut)$ for $\mathsf{DFL_e}$.

3.1 Soundness and Completeness of $\mathbf{h}\mathsf{DFL_e} + (cut)$

In the absence of further external structural rules, a sequent is derivable in $\mathbf{h}\mathsf{DFL_e}$ if and only if is it derivable in $\mathbf{s}\mathsf{DFL_e}$; Theorem 3 then guarantees the soundness and completeness of $\mathbf{h}\mathsf{DFL_e}$ w.r.t. $\mathsf{DFL_e}$. In this section we prove the stronger result that for any set of hypersequents g_j ($j \in J$) $\mathbf{h}\mathsf{DFL_e} + (cut) + \{g_j\}_{j \in J}$ derives precisely the theorems of $\mathsf{DFL_e} + \{g_j^I\}$ (Cor. 11); this result will be used in Section 5 to extract bunched hypersequent rules from axioms.

Notation. For a formula A, let $A_{\wedge\mathbf{1}}$ to denote the formula $A \wedge \mathbf{1}$.

Lemma 9. *Let g be a bunched hypersequent, X a bunch, $\Gamma[\,]$ a context and A, B and C formulae. Then*

(i) **Add context to antecedent and succedent of a disjunct**
If $\varnothing_m \Rightarrow g^I \vee (X \multimap A)^I_{\wedge\mathbf{1}}$ is derivable in $\mathbf{s}\mathsf{DFL_e} + (cut)$ then so is the bunched sequent $\varnothing_m \Rightarrow g^I \vee (\Gamma[X]^I \multimap \Gamma[A]^I)_{\wedge\mathbf{1}}$.

(ii) **Transitivity of $\multimap$ under disjunction**
The following sequent is derivable in $\mathbf{s}\mathsf{DFL_e}$.

$$(g^I_{\wedge\mathbf{1}} \vee (A \multimap B)_{\wedge\mathbf{1}}), (g^I_{\wedge\mathbf{1}} \vee (B \multimap C)_{\wedge\mathbf{1}}) \Rightarrow g^I \vee (A \multimap C) \tag{1}$$

(iii) **Modus ponens for $\multimap$ inside context**
If below left is derivable in $\mathbf{s}\mathsf{DFL_e} + (cut)$ then so is below right.

$$\varnothing_m \Rightarrow g^I_{\wedge\mathbf{1}} \vee (X^I \multimap \Gamma[A \otimes (A \multimap B)]^I)_{\wedge\mathbf{1}} \qquad \varnothing_m \Rightarrow g^I \vee (X^I \multimap \Gamma[B]^I)$$

(iv) **Add $\wedge\mathbf{1}$ to disjuncts**
Let $A^1, \ldots, A^{n+1}$ be arbitrary formulae. If $\varnothing_m \Rightarrow A^1 \vee \ldots \vee A^{n+1}$ is derivable in $\mathsf{sDFL_e} + (cut)$ *then so is $\varnothing_m \Rightarrow A^1_{\wedge\mathbf{1}} \vee \ldots \vee A^{n+1}_{\wedge\mathbf{1}}$.*

Proof. In this proof we will write the interpretation of a bunch *dropping the superscript "I"* e.g. we write $X^I \multimap A$ as $X \multimap A$. This slight abuse of notation is employed to avoid clutter and aid readability. The identity of the object as bunch or formula will be inferable from the context. Similar conventions will be applied to the hypersequent g and context $\Gamma[]$.

Proof of (i). Induction on the structure of $\Gamma[]$. The base case, when $\Gamma[]$ is $[]$, is trivial.

Now suppose that $\Gamma[]$ is $U, V[]$. By the IH we have that $\varnothing_m \Rightarrow g \vee (V[X] \multimap V[A])_{\wedge\mathbf{1}}$ is derivable in $\mathsf{sDFL_e}$. The result then follows by cut on the following derivation (we have omitted some steps).

$$\dfrac{\dfrac{\dfrac{\dfrac{U \Rightarrow U \qquad V[X], V[X] \multimap V[A] \Rightarrow V[A]}{V[X] \multimap V[A], U, V[X] \Rightarrow U \otimes V[A]}}{(V[X] \multimap V[A])_{\wedge\mathbf{1}} \Rightarrow U \otimes V[X] \multimap U \otimes V[A]}}{(V[X] \multimap V[A])_{\wedge\mathbf{1}} \Rightarrow (U \otimes V[X] \multimap U \otimes V[A])_{\wedge\mathbf{1}}}}{g \vee (V[X] \multimap V[A])_{\wedge\mathbf{1}} \Rightarrow g \vee (U \otimes V[X] \multimap U \otimes V[A])_{\wedge\mathbf{1}}}$$

Finally suppose that $\Gamma[]$ is $U; V[]$. By the IH we have that $\varnothing_m \Rightarrow g \vee (V[X] \multimap V[A])_{\wedge\mathbf{1}}$ is derivable in $\mathsf{sDFL_e}$. The result then follows by cut on the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1, U \Rightarrow U}{(V[X] \multimap V[A])_{\wedge\mathbf{1}}, U \wedge V[X] \Rightarrow U} \qquad \dfrac{V[X] \multimap V[A], V[X] \Rightarrow V[A]}{(V[X] \multimap V[A])_{\wedge\mathbf{1}}, U \wedge V[X] \Rightarrow V[A]}}{(V[X] \multimap V[A])_{\wedge\mathbf{1}}, U \wedge V[X] \Rightarrow U \wedge V[A]}}{(V[X] \multimap V[A])_{\wedge\mathbf{1}} \Rightarrow U \wedge V[X] \multimap U \wedge V[A]}}{(V[X] \multimap V[A])_{\wedge\mathbf{1}} \Rightarrow (U \wedge V[X] \multimap U \wedge V[A])_{\wedge\mathbf{1}}}}{g \vee (V[X] \multimap V[A])_{\wedge\mathbf{1}} \Rightarrow g \vee (U \wedge V[X] \multimap U \wedge V[A])_{\wedge\mathbf{1}}}$$

Proof of (ii). Omitting some steps for brevity:

$$\dfrac{\cdots \qquad \dfrac{\dfrac{A, A \multimap B, B \multimap C \Rightarrow C}{A \multimap B, B \multimap C \Rightarrow A \multimap C}}{A \multimap B, B \multimap C \Rightarrow g \vee (A \multimap C)}}{(g_{\wedge\mathbf{1}} \vee (A \multimap B)_{\wedge\mathbf{1}}), (g_{\wedge\mathbf{1}} \vee (B \multimap C)_{\wedge\mathbf{1}}) \Rightarrow g \vee (A \multimap C)}$$

Proof of (iii). It is easy to see that $A \otimes (A \multimap B)$ is derivable in $\mathsf{sDFL_e} + (cut)$. From (i) we have that $\Gamma[A \otimes (A \multimap B)] \Rightarrow \Gamma[B]$ and hence $\varnothing_m \Rightarrow g_{\wedge\mathbf{1}} \vee (\Gamma[A \otimes (A \multimap B)] \multimap \Gamma[B])_{\wedge\mathbf{1}}$ is derivable in $\mathsf{sDFL_e} + (cut)$. Then by (ii) we have that $\varnothing_m \Rightarrow g \vee (X \Rightarrow \Gamma[B])$ is derivable in $\mathsf{sDFL_e} + (cut)$.

Proof of (iv). From $\varnothing_m \Rightarrow A^1 \vee \ldots \vee A^{n+1}$ we have $\varnothing_m \Rightarrow (A^1 \vee \ldots \vee A^{n+1}) \wedge \mathbf{1}$. Now apply cut with

$$\dfrac{\dfrac{\cdots \qquad \dfrac{\dfrac{A^j \Rightarrow A^j \quad \mathbf{1} \Rightarrow \mathbf{1}}{A^j; 1 \Rightarrow A^j_{\wedge\mathbf{1}}}}{A^j; 1 \Rightarrow A^1_{\wedge\mathbf{1}} \vee \ldots \vee A^{n+1}_{\wedge\mathbf{1}}} \qquad \cdots}{(A^1 \vee \ldots \vee A^{n+1}); 1 \Rightarrow A^1_{\wedge\mathbf{1}} \vee \ldots \vee A^{n+1}_{\wedge\mathbf{1}}}}{(A^1 \vee \ldots \vee A^{n+1}) \wedge 1 \Rightarrow A^1_{\wedge\mathbf{1}} \vee \ldots \vee A^{n+1}_{\wedge\mathbf{1}}}$$

□

As usual, the *height* of a derivation is the number of rules on its longest branch.

Lemma 10. *Let A be a formula and $\{g_j\}_{j\in J}$ a finite set of hypersequents. Then $\varnothing_m \Rightarrow A$ is derivable in $\mathcal{H} = \boldsymbol{h}\mathsf{DFL_e}+(cut)+\{g_j\}_{j\in J}$ iff $\varnothing_m \Rightarrow A$ is derivable in $\mathcal{S} = \boldsymbol{s}\mathsf{DFL_e}+(cut)+\{\varnothing_m \Rightarrow g_j^I\}_{j\in J}$.*

Proof. The result follows from the following two statements.

(a) If the sequent $X \Rightarrow \psi$ is derivable in $\mathcal{S}$ then $X \Rightarrow \psi$ is derivable in $\mathcal{H}$.

(b) If the hypersequent h is derivable in $\mathcal{H}$ then $\varnothing_m \Rightarrow h^I$ is derivable in $\mathcal{S}$.

In particular, if $\varnothing_m \Rightarrow A$ is derivable in $\mathcal{H}$, then (b) implies that $\varnothing_m \Rightarrow \mathbf{1} \multimap A$ is derivable in $\mathcal{S}$. It follows then that $\varnothing_m \Rightarrow A$ is derivable in $\mathcal{S}$.

Proof of (a). It is straightforward to simulate each of the rules of $\mathcal{S}$ in $\mathcal{H}$. Every initial sequent $\varnothing_m \Rightarrow g_j^I$ in $\mathcal{S}$ can be derived in $\mathcal{H}$ starting from the hypersequent g_j. The idea is to rewrite every component $Y \Rightarrow \phi$ in g_j as $\varnothing_m \Rightarrow Y^I \multimap \phi$ using ($\wedge$l) and ($\otimes$l) and ($\multimap$r). The result then follows from repeated use of ($\vee$r) and (EC).

Proof of (b). Induction on the height of the $\mathcal{H}$-derivation of h. If the derivation is an initial sequent then the argument is trivial. Otherwise consider the last rule in the derivation. By the induction hypothesis, we may assume that the result holds for the premises of the rule. Let us illustrate some of the cases.

(cut). Suppose that $\varnothing_m \Rightarrow g^I \vee (X \multimap A)$ and $\varnothing_m \Rightarrow g^I \vee (\Gamma[A] \Rightarrow C)$ are derivable in $\mathcal{S}$. By Lemma 9(iv) we have that $\varnothing_m \Rightarrow g^I_{\wedge\mathbf{1}} \vee (X \multimap A)_{\wedge\mathbf{1}}$ and $\varnothing_m \Rightarrow g^I_{\wedge\mathbf{1}} \vee (\Gamma[A] \Rightarrow C)_{\wedge\mathbf{1}}$ are derivable in $\mathcal{S}$. Applying Lemma 9(i) to the former we obtain $\varnothing_m \Rightarrow g^I_{\wedge\mathbf{1}} \vee (\Gamma[X] \multimap \Gamma[A])_{\wedge\mathbf{1}}$. Now obtain the required $\varnothing_m \Rightarrow g^I \vee (\Gamma[X] \Rightarrow C)$ by (cut) on (1) (Lemma 9(ii)).

($\otimes$r). Suppose that $\varnothing_m \Rightarrow g^I \vee (X \multimap A)$ and $\varnothing_m \Rightarrow g^I \vee (Y \multimap B)$ are derivable in $\mathcal{S}$. It follows from Lemma 9(iv) that we have a derivation of

$$\varnothing_m \Rightarrow (g^I_{\wedge\mathbf{1}} \vee (X \multimap A)_{\wedge\mathbf{1}}) \otimes (g^I_{\wedge\mathbf{1}} \vee (Y \multimap B)_{\wedge\mathbf{1}})$$

The result follows from an application of cut with the following derivation.

$$\dfrac{\dfrac{\dfrac{X \multimap A, Y \multimap B, X, Y \Rightarrow A \otimes B}{(X \multimap A)_{\wedge\mathbf{1}}, (Y \multimap B)_{\wedge\mathbf{1}} \Rightarrow X \otimes Y \multimap A \otimes B}}{(g^I_{\wedge\mathbf{1}} \vee (X \multimap A)_{\wedge\mathbf{1}}) \otimes (g^I_{\wedge\mathbf{1}} \vee (Y \multimap B)_{\wedge\mathbf{1}}) \Rightarrow g^I \vee (X \otimes Y \multimap A \otimes B)}}{(g^I_{\wedge\mathbf{1}} \vee (X \multimap A)_{\wedge\mathbf{1}}) \otimes (g^I_{\wedge\mathbf{1}} \vee (Y \multimap B)_{\wedge\mathbf{1}}) \Rightarrow g^I \vee (X \otimes Y \multimap A \otimes B)}$$

($\multimap$l). We have derivations of $g^I \vee (Y \multimap A)$ and $g^I \vee (\Gamma[B] \Rightarrow C)$ in $\mathbf{s}\mathsf{DFL_e} + (cut)$. From the former, by Lemma 9(iv) and (i) we obtain a derivation of $g^I \vee (\Gamma[Y \otimes A \multimap B] \multimap \Gamma[A \otimes A \multimap B])$. Then by Lemma 9(iv) and (iii) we obtain a derivation of $g^I \vee (\Gamma[Y \otimes A \multimap B] \multimap \Gamma[B])$. From Lemma 9(iv) and (ii) using $g^I_{\wedge\mathbf{1}} \vee (\Gamma[B] \Rightarrow C)_{\wedge\mathbf{1}}$ we derive $g^I \vee (\Gamma[Y \otimes A \multimap B] \Rightarrow C)$. □

Notation. Let $A =_{\mathsf{DFL_e}} B$ denote $(A \multimap B) \wedge (B \multimap A) \in \mathsf{DFL_e}$.

Corollary 11. *Let $\{A_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be finite sets of formulae and hypersequents such that $A_j =_{\mathsf{DFL_e}} g_j^I$ for each $j \in J$. Then for every formula B:*

$$B \in \mathsf{DFL_e} + \{A_i\}_{i\in I} \quad \textit{iff} \quad \varnothing_m \Rightarrow B \textit{ derivable in } \boldsymbol{h}\mathsf{DFL_e} + (cut) + \{g_j\}_{j\in J}$$

Proof. From Theorem 3 $B \in \mathsf{DFL_e} + \{A_i\}_{i\in I}$ iff $\varnothing_m \Rightarrow B$ is derivable in $\mathbf{s}\mathsf{DFL_e} + (cut) + \{\varnothing_m \Rightarrow A_j\}_{j\in J}$ iff $\mathbf{s}\mathsf{DFL_e} + (cut) + \{\varnothing_m \Rightarrow g_j^I\}_{j\in J}$. From Lemma 10 we have the latter iff $\varnothing_m \Rightarrow B$ derivable in $\mathbf{h}\mathsf{DFL_e} + (cut) + \{g_j\}_{j\in J}$. □

4 Cut-elimination for structural rule extensions of $\mathbf{h}\mathsf{DFL_e}$

We present a uniform cut-elimination proof applying to structural rule extensions of $\mathbf{h}\mathsf{DFL_e}$ ($\mathbf{s}\mathsf{DFL_e}$ is then a special case of the theorem). Our proof applies to structural rules satisfying conditions inspired by Belnaps [3] conditions for cut-elimination in the display calculus.[3] The advantage of presenting a general cut-elimination theorem is that it enables the readers to check that it holds for a structural rule extension of their choice by simply verifying on sight the (sufficient) conditions given below.

The *parametric ancestors* [3] of a formula occurrence A in the conclusion of a rule instance are those occurrences of A in the premises occurring in the *same position* i.e. instantiating the same schematic variable in the same position.

Example 12. *A formula occurrence and its parametric ancestors (highlighted with $\hat{A}$):*

$$\frac{C, (A; (\hat{A}, B)) \Rightarrow D}{A; (\hat{A}, B) \Rightarrow C \multimap D} \qquad \frac{B; (C, D) \Rightarrow \hat{A}}{B; C \otimes D \Rightarrow \hat{A}} \qquad \frac{\hat{A}, A \Rightarrow \mathcal{O}_m \mid \hat{A}, A \Rightarrow \mathcal{O}_m}{\hat{A}, A \Rightarrow \mathcal{O}_m}$$

Definition 13 (Permutative)**.** *Let r be a bunched hypersequent rule instance and $\hat{A}$ an occurrence of a non-principal formula in the conclusion. The rule r is* permutative *if the premises and conclusion have the same context g and:*

(C4) Polarity preserving: *If $\hat{A}$ is in the antecedent (succedent) then all its parametric ancestors (if any) are also in the antecedent (resp. succedent).*

(C6) Substitution closed: *If $\hat{A}$ is in the antecedent, substituting $\hat{A}$ and also all its parametric ancestors (if any) in the premise(s) with an arbitrary bunch X is a legal rule instance of r.*

(C7) Closed under arbitrary contexts: *If $\hat{A}$ is in the succedent, replacing every component of the form $U \Rightarrow \hat{A}$ in the premise and conclusion of r with $\Gamma[U] \Rightarrow \psi$ yields a legal rule instance of r for any context $\Gamma[\,]$ and ψ.*

The labels C4, C6, C7 correspond to Belnap's [3] conditions in the display calculus which are used to permute a cut upwards in a derivation.

Lemma 14. *Every rule in $\boldsymbol{h}\mathsf{DFL_e} + (cut)$ and $\boldsymbol{s}\mathsf{DFL_e} + (cut)$ is permutative.*

Proofs of cut-elimination in (bunched) hypersequent calculi are similar to those for (bunched) sequent calculi; the additional difficulty arises due to the (EC) rule which duplicates components that should be handled in parallel, see e.g. [26, 12]. A solution is to consider generalisations of Gentzen's multicut rule in which one of the premises necessarily contains a single occurrence of the cut formula (every indicated occurrence of A in the rules below is called a *cut-formula*):

$$\frac{h \mid X \Rightarrow A \qquad h \mid \Gamma_1[\![A]\!] \Rightarrow \psi_1 \mid \cdots \mid \Gamma_{M+1}[\![A]\!] \Rightarrow \psi_{M+1}}{h \mid \Gamma_1[\![X]\!] \Rightarrow \psi_1 \mid \cdots \mid \Gamma_{M+1}[\![X]\!] \Rightarrow \psi_{M+1}}\ (\mathrm{cut}_{\Rightarrow\bullet})$$

$$\frac{h \mid X_1 \Rightarrow A \mid \cdots \mid X_{N+1} \Rightarrow A \qquad h \mid \Gamma[A] \Rightarrow \psi}{h \mid \Gamma[X_1] \Rightarrow \psi \mid \cdots \mid \Gamma[X_{N+1}] \Rightarrow \psi}\ (\mathrm{cut}_{\bullet\Rightarrow})$$

[3] Although the rule conditions are inspired by the display calculus, the same cut-elimination argument cannot be used here. Display calculi are equipped with 'enough' structural connectives to allow the isolation of any formula as the whole of the antecedent (succedent) in a sequent. This is called the *display property* and it simplifies the cut-elimination argument.

The subscript $\bullet\!\Rightarrow$ ($\Rightarrow\!\bullet$) identifies that the single occurrence is in the antecedent (resp. succedent). Since (cut) can be viewed as a special instance of either of these rules, cut-elimination follows from elimination of ($\text{cut}_{\Rightarrow\bullet}$) and ($\text{cut}_{\bullet\Rightarrow}$).

Algorithm description: (1) Choose a topmost cut. (2) Permute $\text{cut}_{\Rightarrow\bullet}$ ($\text{cut}_{\bullet\Rightarrow}$) *upwards* in its right (resp. left) premise until some cut-formula in the right (resp. left) premise of $\text{cut}_{\Rightarrow\bullet}$ becomes principal. (3) Cut all the non-principal occurrences by shifting the cut upwards. (4) Cut the remaining (principal) occurrence using $\text{cut}_{\bullet\Rightarrow}$ (resp. $\text{cut}_{\Rightarrow\bullet}$). *Notice we have switched the cut rule*! (5) Permute the cut upwards—this time in the left (resp. right) premise derivation—and repeat argument. (6) When the cut is principal in both left and right premise, transform into cuts on smaller subformulae as in Gentzen's original proof.

Notation and terminology. The *cut-height* is the sum of the heights of the derivations ending in the premise of the cut. We write $\Gamma\llbracket X\rrbracket$ to mean the bunch $\Gamma[X]\cdots[X]$ containing some number of occurrences of X.

Theorem 15. *The rule (cut) is eliminable in* $\boldsymbol{h}\mathsf{DFL_e}$ + *(cut)* + $\{r_i\}_{i\in I}$ *for any set* $\{r_i\}_{i\in I}$ *of permutative rules.*

Proof. We prove the stronger statement that the rules ($\text{cut}_{\bullet\Rightarrow}$) and ($\text{cut}_{\Rightarrow\bullet}$) are eliminable in $\mathbf{h}\mathsf{DFL_e}+(\text{cut}_{\bullet\Rightarrow})+(\text{cut}_{\Rightarrow\bullet})+\{r_i\}_{i\in I}$. As usual, it suffices to eliminate topmost cuts. For $\text{cut}_{\Rightarrow\bullet}$, set $e=0$ if some cut-formula in the right premise is principal else set $e=1$. For $\text{cut}_{\bullet\Rightarrow}$, set $e=0$ if some cut-formula in the left premise is principal else set $e=1$. The elimination of ($\text{cut}_{\bullet\Rightarrow}$) and ($\text{cut}_{\Rightarrow\bullet}$) is by primary induction on the size of the cut formula, secondary induction on the e-value of the cut and tertiary induction on its cut-height.

Consider an arbitrary topmost instance of ($\text{cut}_{\Rightarrow\bullet}$) (the case of ($\text{cut}_{\bullet\Rightarrow}$) is symmetric). First suppose that none of the occurrences of the cut-formula in the premise $h\,|\,\Gamma_1\llbracket A\rrbracket \Rightarrow \psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket A\rrbracket\Rightarrow\psi_{M+1}$ is principal. Let r_R be the rule above this sequent (we illustrate with a unary rule, below left). Identify the parametric ancestors of the cut-formula in the premises of r_R. Proceed as below right (applications of (EW) are not indicated).

$$\dfrac{h\,|\,X\Rightarrow A \qquad \dfrac{g\llbracket\hat{A}\rrbracket}{h\,|\,\Gamma_1\llbracket\hat{A}\rrbracket\Rightarrow\psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket\hat{A}\rrbracket\Rightarrow\psi_{M+1}}\,r_R}{h\,|\,\Gamma_1\llbracket X\rrbracket\Rightarrow\psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket X\rrbracket\Rightarrow\psi_{M+1}}\,(\text{cut}_{\Rightarrow\bullet}) \qquad \dfrac{\dfrac{h\,|\,X\Rightarrow A \qquad h\,|\,g\llbracket A\rrbracket}{h\,|\,g\llbracket X\rrbracket}\,(\text{cut}_{\Rightarrow\bullet})^{\dagger}}{h\,|\,\Gamma_1\llbracket X\rrbracket\Rightarrow\psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket X\rrbracket\Rightarrow\psi_{M+1}}\,r_R$$

The $(\text{cut}_{\Rightarrow\bullet})^{\dagger}$ has lesser cut-height than the original and hence is eliminable by the induction hypothesis. Note that the new instance of r_R is legal due to permutativity (condition C6).

Next suppose that a cut-formula A^* in $h\,|\,\Gamma_1\llbracket A\rrbracket\Rightarrow\psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket A\rrbracket\Rightarrow\psi_{M+1}$ is principal by r_R (we illustrate with a unary rule). There are two subcases.

(i) The cut formula is not principal in $h\,|\,X\Rightarrow A$ (below left). Proceed below right:

$$\dfrac{h\,|\,X\Rightarrow A \qquad \dfrac{g'\llbracket A\rrbracket}{h\,|\,g\llbracket A\rrbracket[A^*]}\,r_R}{h\,|\,\Gamma_1\llbracket X\rrbracket\Rightarrow\psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket X\rrbracket\Rightarrow\psi_{M+1}}\,(\text{cut}_{\Rightarrow\bullet}) \qquad \dfrac{h\,|\,X\Rightarrow A \qquad \dfrac{\dfrac{h\,|\,X\Rightarrow A \qquad h\,|\,g'\llbracket A\rrbracket}{h\,|\,g'\llbracket X\rrbracket}\,(\text{cut}_{\Rightarrow\bullet})^{\dagger}}{h\,|\,g\llbracket X\rrbracket[A^*]}\,r_R}{h\,|\,\Gamma_1\llbracket X\rrbracket\Rightarrow\psi_1\,|\cdots|\,\Gamma_{M+1}\llbracket X\rrbracket\Rightarrow\psi_{M+1}}\,(\text{cut}_{\bullet\Rightarrow})^{\ddagger}$$

The $(\text{cut}_{\Rightarrow\bullet})^{\dagger}$ has lesser cut-height than the original cut hence eliminable by induction hypothesis. Also r_R is legal as it is permutative. Observe that we switched from the original ($\text{cut}_{\Rightarrow\bullet}$) where the left premise was *not* principal to $(\text{cut}_{\bullet\Rightarrow})^{\ddagger}$ where right premise *is* principal. Thus the new cut has lesser e-value and is hence eliminable via the induction hypothesis.

(ii) The cut formula is principal in $h\,|\,X\Rightarrow A$. First cut the non-principal occurrences of $\hat{A}$ as in $(\text{cut}_{\Rightarrow\bullet})^{\dagger}$ in subcase (i). What remains then is the familiar case of a principal-principal cut encountered in cut-elimination for the sequent calculus. The idea is to apply cuts to the *subformulae* of A. The cut will be eliminable via the induction hypothesis because the cut-formula has smaller size. □

5 From axioms to rules for extensions of $\mathsf{DFL_e}$

We now turn to the task of obtaining cutfree calculi for axiomatic extensions of $\mathsf{DFL_e}$. We adapt the method introduced in [8] and applied to various other formalisms e.g. [11], to compute permutative structural rules from axioms. We then identify the class of axioms amenable to this algorithm. We advise the reader that from an applicative perspective, it is straightforward to apply the algorithm 'by hand' to an axiom of interest and so determine if a permutative structural rule can be obtained from it or not.

Lemma 16 (Ackermann Lemma)**.** *Let $\mathcal{C}$ be an extension of $\boldsymbol{h}\mathsf{DFL_e} + (cut)$. Then $\mathcal{C} + r_1$ and $\mathcal{C} + r_1'$ (resp. $\mathcal{C} + r_2$ and $\mathcal{C} + r_2'$) derive the same set of bunched hypersequents.*

$$\frac{h_1 \quad \cdots \quad h_n}{g_1 \,|\, g_2 \,|\, X \Rightarrow A}\, r_1 \qquad \frac{h_1 \quad \cdots \quad h_n \qquad g_1 \,|\, \Delta[A] \Rightarrow \psi}{g_1 \,|\, g_2 \,|\, \Delta[X] \Rightarrow \psi}\, r_1'$$

$$\frac{h_1 \quad \cdots \quad h_n}{g_1 \,|\, g_2 \,|\, \Gamma[A] \Rightarrow \psi}\, r_2 \qquad \frac{h_1 \quad \cdots \quad h_n \qquad g_1 \,|\, \Sigma \Rightarrow A}{g_1 \,|\, g_2 \,|\, \Gamma[\Sigma] \Rightarrow \psi}\, r_2'$$

In the right column above $\Delta[\,], \Sigma$ and ψ are (meta)variables not appearing in r_i.

Proof. We prove the first claim. The proof of the second is similar.

First suppose that we are given concrete instances of the premises $\{h_i\}_{1\leq i\leq n}$ of r_1. Apply r_1' with premises $\{h_i\}_{1\leq i\leq n}$ and $g_1 \,|\, A \Rightarrow A$ (so $\Delta[\,] := [\,]$) to obtain $g \,|\, X \Rightarrow A$. This is the conclusion of r_1, as required.

Other direction: suppose that are given the premises $\{h_i\}_{1\leq i\leq n}$ and $g \,|\, \Delta[A] \Rightarrow \psi$ of r_1'. Apply r_1 with premises $\{h_i\}_{1\leq i\leq n}$ to obtain $g_1 \,|\, g_2 \,|\, X \Rightarrow A$. Apply the cut rule to this and $g_1 \,|\, \Delta[A] \Rightarrow \psi$ to obtain $g_1 \,|\, g_2 \,|\, \Delta[X] \Rightarrow \psi$ as required. □

Note: the proof of the above lemma establishes an even stronger proof-theoretic property: the inter-derivability of r_i and r_i' ($i \in \{1, 2\}$) in $\mathbf{h}\mathsf{DFL_e} + (cut)$. For our purposes here, it suffices that the set of derivable bunched sequents is identical.

Obtaining a structural rule from an axiom

The *invertible logical rules* of $\mathbf{h}\mathsf{DFL_e}$ are those logical rules whose premises are derivable whenever the conclusion is derivable. Applying such a rule backwards to a hypersequent thus preserves derivability. The rules are: ($\mathbf{1}$l) ($\mathbf{0}$r) ($\multimap$r) ($\otimes$l) ($\wedge$l) ($\wedge$r) ($\vee$l)

PREPROCESSING Let A be an axiom of the form $B_1 \vee \ldots \vee B_{n+1}$. Then define h_0 as $\varnothing_m \Rightarrow B_1 \,|\, \ldots \,|\, \varnothing_m \Rightarrow B_{n+1}$ so $h_0^I = (\mathbf{1} \multimap B_1) \vee \ldots \vee (\mathbf{1} \multimap B_{n+1})$. Now $A =_{\mathsf{DFL_e}} h_0^I$. By Cor. 11 we have that $\mathbf{h}\mathsf{DFL_e} + (cut) + h_0$ is a calculus for $\mathsf{DFL_e} + A$. Then the calculus $\mathbf{h}\mathsf{DFL_e} + (cut) + g|h_0$ derives exactly the same hypersequents. Context g is required for cut-elimination.

Example 17. *Set $A = (p \multimap \mathbf{0}) \vee (p \multimap \mathbf{0}) \multimap \mathbf{0}$. Then we have that $g \,|\, h_0$ is the hypersequent $g \,|\, \varnothing_m \Rightarrow p \multimap \mathbf{0} \,|\, \varnothing_m \Rightarrow (p \multimap \mathbf{0}) \multimap \mathbf{0}$.*

STEP 1 Repeatedly apply all possible invertible logical rules backwards i.e. from conclusion to premise(s), starting with $g \mid h_0$ and collect the set of hypersequent premises (it will be a singleton set only if a binary invertible rule is not applied). Then proceed with Step 2 on each hypersequent in the set.

For illustration, suppose that we complete Step 1 obtaining the singleton set $\{g \mid h_1\}$.

Example 17 (cont.). *Applying ($\multimap r$) to $g \mid h_0$ we get below left. Applying ($\multimap r$) again we get below center. Indeed, we want to apply all possible invertible rules and that ultimately yields below right ($g \mid h_1$).*

$$g \mid p \Rightarrow \mathbf{0} \mid \varnothing_m \Rightarrow (p \multimap \mathbf{0}) \multimap \mathbf{0} \qquad g \mid p \Rightarrow \mathbf{0} \mid p \multimap \mathbf{0} \Rightarrow \mathbf{0} \qquad g \mid p \Rightarrow \mathcal{O}_m \mid p \multimap \mathbf{0} \Rightarrow \mathcal{O}_m$$

STEP 2 Compute a rule from each hypersequent in the set obtained in Step 1 by replacing every formula therein with a fresh structure variable via Ackermann's lemma (thereby adding a new premise).

For illustration, suppose that we obtained the set $\{g \mid h_1\}$ in Step 1. Then let ρ_2 denote the rule computed at the end of Step 2. Then $\mathsf{DFL_e} + (cut) + \varnothing_m \Rightarrow A$ and $\mathsf{DFL_e} + (cut) + \rho_2$ derive the same hypersequents.

Example 17 (cont.). *A single application of Ackermann's lemma to $g \mid h_1$ yields below left. Ackermann's lemma once more yields ρ_2 below right.*

$$\frac{g \mid \Sigma \Rightarrow p}{g \mid \Sigma \Rightarrow \mathcal{O}_m \mid p \multimap \mathbf{0} \Rightarrow \mathcal{O}_m} \qquad \frac{g \mid \Sigma \Rightarrow p \qquad g \mid \Pi \Rightarrow p \multimap \mathbf{0}}{g \mid \Sigma \Rightarrow \mathcal{O}_m \mid \Pi \Rightarrow \mathcal{O}_m}\,\rho_2$$

STEP 3 Apply all possible invertible logical rules backwards to the premises of ρ_2 to decompose the formulae occurring in the premises, to ultimately obtain ρ_3 (binary invertible rules result in more premises in the rule). Then $\mathsf{DFL_e} + (cut) + \rho_2$ and $\mathsf{DFL_e} + (cut) + \rho_3$ derive the same hypersequents.

Example 17 (cont.). *Applying the invertible rules ($\multimap r$) and ($\mathbf{0}r$) to the premise of ρ_2 we get the rule below (ρ_3).*

$$\frac{g \mid \Sigma \Rightarrow p \qquad g \mid p, \Pi \Rightarrow \mathcal{O}_m}{g \mid \Sigma \Rightarrow \mathcal{O}_m \mid \Pi \Rightarrow \mathcal{O}_m}\,\rho_3$$

The ρ_3 rule is not structural as its premises contain formulae (specifically, propositional variables). A prerequisite for obtaining a structural rule is that every formula in the premise of ρ_3 is a *propositional variable* (else the procedure fails).

STEP 4 Remove all the propositional variables that appear in the premises and not in the conclusion. When those variables appear only in the antecedent (or only in the consequent) of different premises we simply remove such premises. Otherwise apply cut in 'all possible ways' among the premises of ρ_3 (analogous to the procedure and proof detailed in [8, 11]). This means selecting a propositional variable p and applying (cut) to all premises of ρ_3 containing p. The step succeeds if none of the bunched hypersequents thus obtained contains p. Consider all the p-free bunched hypersequents as the new premises of the rule, select a new propositional variable and repeat.

Example 17 (cont.). *There is just a single cut that can be made to the premises of ρ_3, yielding the structural rule ρ_4 below.*

$$\frac{g \mid \Sigma, \Pi \Rightarrow \mathcal{O}_m}{g \mid \Sigma \Rightarrow \mathcal{O}_m \mid \Pi \Rightarrow \mathcal{O}_m}\,\rho_4$$

Rules ρ_3 and ρ_4 are equivalent in $\boldsymbol{h}\mathsf{DFL_e} + (cut)$: the set of derivable hypersequents is the same under the addition of either rule. Indeed, the rule ρ_4 is at least as powerful as ρ_3: apply cut to concrete instances of the premises of ρ_3 followed by ρ_4 to obtain the conclusion of ρ_3. For the other direction, suppose we are given a concrete instance of the premise of ρ_4. Apply ρ_3 to this sequent using the derivable hypersequent $\Sigma \Rightarrow \Sigma^I$ as the other premise to obtain $\Sigma \Rightarrow \mathcal{O}_m \mid \Pi \Rightarrow \mathcal{O}_m$. Then $B \in \mathsf{DFL_e} + (p \multimap \mathbf{0}) \vee (p \multimap \mathbf{0}) \multimap \mathbf{0}$ iff $\varnothing_m \Rightarrow B$ is derivable in $\boldsymbol{h}\mathsf{DFL_e} + r$.

If applying cuts in all possible ways *does* terminate in a structural rule ρ_4, then we say that r *is the structural rule computed from* $\varnothing_m \Rightarrow A$.

Note. The procedure also applies to $\mathbf{s}\mathsf{DFL_e}$ by omitting the preprocessing.

The success of the procedure depends on

(I) Every formula in the rule obtained after Step 3 being a propositional variable, and

(II) Termination of 'cut in all possible ways'.

Set $\mathcal{N}_0^d$ and $\mathcal{P}_0^d$ as the set of propositional variables. Then define:

$$\mathcal{P}_{n+1}^d ::= \mathbf{1} \mid \mathcal{N}_n^d \mid \mathcal{P}_{n+1}^d \otimes \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \wedge \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \vee \mathcal{P}_{n+1}^d$$
$$\mathcal{N}_{n+1}^d ::= \mathbf{0} \mid \mathcal{P}_n^d \mid \mathcal{N}_{n+1}^d \wedge \mathcal{N}_{n+1}^d \mid \mathcal{P}_{n+1}^d \multimap \mathcal{N}_{n+1}^d$$

The *positive* classes $\mathcal{P}_i$ (the *negative* classes $\mathcal{N}_i$) contain formulae whose most external connective is invertible on the left (resp. right).

Following [8], we can identify those axioms that satisfy (I):

Lemma 18. *Every axiom in $\mathcal{N}_2^d$ satisfies (I) for $\boldsymbol{s}\mathsf{DFL_e}$. Furthermore, every disjunction of $\mathcal{N}_2^d$ axioms satisfies (I) for $\boldsymbol{h}\mathsf{DFL_e}$.*

Proof. The first claim states that every formula in the rule ρ_3 obtained after Step 3 is a propositional variable when the procedure in $\mathbf{s}\mathsf{DFL_e}$ is applied to a $\mathcal{N}_2^d$ formula. Applying all possible invertible rules (Step 1) to $\mathbf{1} \Rightarrow A$ for $A \in \mathcal{N}_2^d$, we obtain sequents where every antecedent (succedent) formula is in $\mathcal{P}_2^d$ (resp. $\mathcal{N}_2^d$). If α is a (non-invertible, $\mathcal{P}_2^d$) formula in the antecedent that is not a propositional variable, then α must be either $\mathbf{0}$ or $\beta \multimap \gamma$. It must be the case then that $\alpha \in \mathcal{N}_1^d$ since the $\mathcal{P}$-classes do not have a $\mathbf{0}, \multimap$-constructor. Similarly, if α is a (non-invertible, $\mathcal{N}_2^d$) formula in the succedent that is not a propositional variable, then α must be either $\mathbf{1}$ or $\beta \otimes \gamma$ or $\beta \vee \gamma$, and $\alpha \in \mathcal{P}_1^d$ since the $\mathcal{N}$-classes do not have a $\mathbf{1}$, $\otimes$, $\vee$-constructor.

Applying Ackermann's Lemma (Step 2) to these formulae, we ultimately obtain a rule ρ_2 whose conclusion contains no formulae, and every antecedent (succedent) formula in the premises is in $\mathcal{P}_1^d$ (resp. $\mathcal{N}_1^d$). Apply all possible invertible rules to each of the premises (Step 3) we ultimately obtain sequents whose only formulae are propositional variables. Indeed, if α is a (non-invertible) formula in the antecedent of this sequent that is not a propositional variable, then $\alpha = \mathbf{0}$ or $\alpha = \beta \multimap \gamma$ and $\alpha \in \mathcal{N}_0^d$ but this is a contradiction because $\mathcal{N}_0^d$ is the set of propositional variables. Similarly, if α is a (non-invertible) formula in the succedent of this sequent that is not a propositional variable, then α is either $\mathbf{1}$ or $\beta \otimes \gamma$ or $\beta \vee \gamma$, and $\alpha \in \mathcal{P}_0^d$. This is a contradiction because $\mathcal{P}_0^d$ is the set of propositional variables.

The second claim states that property (I) holds in $\mathbf{h}\mathsf{DFL_e}$ when the axiom is a disjunction of $\mathcal{N}_2^d$ formulae. The preprocessing step takes a formula $B_1 \vee \ldots \vee B_{n+1}$ and returns the hypersequent $g \mid \varnothing_m \Rightarrow B_1 \mid \ldots \mid \varnothing_m \Rightarrow B_{n+1}$. If each $B_i \in \mathcal{N}_2^d$, using the argument above we obtain a rule ρ_3 such that every formula in the rule is a propositional variable, as claimed. □

The above lemma tells us that axiomatic extensions of $\mathsf{DFL_e}$ by finite disjunctions of $\mathcal{N}_2^d$ formulae can be presented over $\mathbf{h}\mathsf{DFL_e}$ whenever property (II) holds. Of course, it suffices that the logic is *expressible* as an axiomatic extension of finite disjunctions of $\mathcal{N}_2^d$ formulae. The following lemma shows that this is the case for extensions of $\mathsf{DFL_e}$ axiomatised by formulae from the grammar $\mathcal{P}_3^{d'}$ below.

$$\mathcal{P}_3^{d'} ::= \mathbf{1} \mid \mathcal{N}_2^d \wedge \mathbf{1} \mid \mathcal{P}_3^{d'} \otimes \mathcal{P}_3^{d'} \mid \mathcal{P}_3^{d'} \vee \mathcal{P}_3^{d'}$$

Lemma 19. *Let L be an extension of $\mathsf{DFL_e}$. Then every extension of L by $\mathcal{P}_3^{d'}$ axioms is equivalent to an extension by a finite set of disjunctions of $\mathcal{N}_2^d$ axioms.*

Proof. First we claim that the formula β obtained by augmenting every subformula A in $\alpha \in \mathcal{P}_3^{d'}$ as $A \wedge \mathbf{1}$ satisfies $\alpha =_{\mathsf{DFL_e}} \beta$. Induction on the size of the axiom. Since $\mathbf{1} =_{\mathsf{DFL_e}} \mathbf{1} \wedge \mathbf{1}$, the claim holds for the base cases $\mathbf{1} \wedge \mathbf{1}$ and $\mathcal{N}_2^d \wedge \mathbf{1}$. If the axiom is $A \otimes B$, then by the induction hypothesis we obtain A' and B' such that $A =_{\mathsf{DFL_e}} A' \wedge \mathbf{1}$ and $B =_{\mathsf{DFL_e}} B' \wedge \mathbf{1}$. It may then be verified that $A'_{\wedge\mathbf{1}} \otimes B'_{\wedge\mathbf{1}} =_{\mathsf{DFL_e}} (A'_{\wedge\mathbf{1}} \otimes B'_{\wedge\mathbf{1}})_{\wedge\mathbf{1}}$. If the axiom is $A \vee B$, then by the induction hypothesis we obtain A' and B' such that $A =_{\mathsf{DFL_e}} A' \wedge \mathbf{1}$ and $B =_{\mathsf{DFL_e}} B' \wedge \mathbf{1}$. It may then be verified that $A'_{\wedge\mathbf{1}} \vee B'_{\wedge\mathbf{1}} =_{\mathsf{DFL_e}} (A'_{\wedge\mathbf{1}} \vee B'_{\wedge\mathbf{1}})_{\wedge\mathbf{1}}$. Let us suppose that we have obtained β as above. It suffices to show that the addition of β to L is equivalent to the extension of L by a finite set of disjunctions of $\mathcal{N}_2^d$ axioms. Proof by induction on the size of the axiom. If β is a disjunction of $\mathcal{N}_2^d$ formulae we are already done. Otherwise β is $(A_{\wedge\mathbf{1}} \otimes B_{\wedge\mathbf{1}})_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}}$ for some γ. By inspection, $L + \beta$ is equivalent to $L + A_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}} + B_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}}$ using the observation that $((A_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}}) \otimes (B_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}})) \multimap \alpha \in \mathsf{DFL_e}$. The result follows by applying the induction hypothesis to $A_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}}$ and $B_{\wedge\mathbf{1}} \vee \gamma_{\wedge\mathbf{1}}$. □

The proofs above simplify the argument in [8, Lem. 3.5, Thm. 5.6] as they alleviate the need to introduce a $\mathcal{N}_2$-normal form.

Property (II) holds when cuts on the premises of the rule ρ_3 obtained in Step 3 do not lead to bunched hypersequents containing components of the form $\Gamma[X, p] \Rightarrow p$. However, in the presence of the internal weakening rule for sequents (below left) and hypersequents (below right) such bunched (hyper)sequents are derivable and hence can be deleted from the premises of the computed rules.

$$\frac{\Gamma[X] \Rightarrow \psi}{\Gamma[X, Y] \Rightarrow \psi} \qquad\qquad \frac{g \mid \Gamma[X] \Rightarrow \psi}{g \mid \Gamma[X, Y] \Rightarrow \psi}$$

The presence of the weakening rule thus guarantees the success of Step 4.

Notation. $p^k \equiv p \otimes \ldots \otimes p$ and $\Sigma^k \equiv \Sigma, \ldots, \Sigma$ (k times).

Example 20. *The rules for the $\mathcal{N}_2^d$ axioms for restricted weakening $(1 \wedge (p \otimes q)) \multimap p$ and n-contraction ($n > 2$ fixed) $p^{n-1} \multimap p^n$, obtained by applying Steps 1–4 are given below.*

$$\frac{\Gamma[X] \Rightarrow \psi}{\Gamma[\emptyset_m; (X, Y)] \Rightarrow \psi}\, eW \qquad \frac{\left\{\Gamma[\Sigma_1^{k_1}, \ldots, \Sigma_{n-1}^{k_{n-1}}] \Rightarrow \psi\right\}_{k_1 + \cdots + k_{n-1} = n}}{\Gamma[\Sigma_1, \ldots, \Sigma_{n-1}] \Rightarrow \psi}\, (nctr)$$

Lemma 21. *Any structural rule r computed from $\varnothing_m \Rightarrow A$ is permutative.*

Proof. Follows from: all metavariables in r are introduced via Ackermann's lemma and are polarity preserving, r is a structural rule, and the hypersequent premise and conclusion contexts can be made identical by using (EW) and (EC). □

We are ready to present the main theorem.

Theorem 22. *Let* $\{r_i\}_{i\in J}$ *be the structural rules computed from the set* $\{\varnothing_m \Rightarrow A_i\}_{i\in I}$*. Then* $B \in \mathsf{DFL_e} + \{A_i\}_{i\in I}$ *iff* $\varnothing_m \Rightarrow B$ *is derivable in* $\boldsymbol{h}\mathsf{DFL_e} + \{r_i\}_{i\in J}$*.*

Proof. By definition of "structural rule computed from", Cor. 11 and the fact that Step 4 preserves the rule equivalence (the argument is similar to that in [11] Prop. 3.30): $B \in \mathsf{DFL_e} + \{A_i\}_{i\in I}$ iff $\varnothing_m \Rightarrow B$ is derivable in $\mathbf{h}\mathsf{DFL_e} + (cut) + \{r_j\}_{i\in J}$. By Lemma 21 and Theorem 15 the latter occurs iff $\varnothing_m \Rightarrow B$ is derivable in $\mathbf{h}\mathsf{DFL_e} + \{r_i\}_{i\in J}$. □

Related work:

A method was introduced in [29] to transform axioms of a simple form into structural rules such that cut-elimination is preserved when added to sequent calculi for classical and intuitionistic first-order logic. The work [8] generalizes the idea and presents a systematic procedure for extracting structural rules preserving cut-elimination for sequent and hypersequent calculi for commutative substructural logics (i.e. axiomatic extensions of $\mathsf{FL_e}$). The adaptation described in this section shows that the presence of distributivity, and hence the use of bunched hypersequents, allows us to capture more logics. Specifically, the class of axioms that can be presented via analytic structural rules over $\mathbf{h}\mathsf{DFL_e}$ is broader than the class—denoted $\mathcal{P}'_3$ in [8]—over $\mathbf{h}\mathsf{FL_e}$. The reasons are: (i) in $\mathbf{h}\mathsf{FL_e}$ the connective $\wedge$ is only invertible in the succedent (i.e. the $(\wedge r)$ rule) while the use of ";" in bunched sequents makes $\wedge$ invertible on both sides, and (ii) the failure of Lemma 9(iv) for $\mathbf{h}\mathsf{FL_e}$ means that, in absence of weakening, this calculus requires that disjuncts of $\mathcal{N}_2^d$ formulas have $\wedge\mathbf{1}$ appended (see Remark 7). E.g. $(p \to q) \vee (q \to p)$ can be transformed into an analytic structural rule over $\mathbf{h}\mathsf{DFL_e}$ but not over $\mathbf{h}\mathsf{FL_e}$ (which instead can transform $(p \to q)\wedge\mathbf{1} \vee (q \to p)\wedge\mathbf{1}$).

An abstraction (and reformulation) of the algorithm in [8] was applied to display calculi in [11]. There, the use of an additional structural connective for the residual of disjunction (i.e. the bi-implication) makes the $\vee$ rules of the display calculus for $\mathsf{FL_e}$ invertible both in the antecedent and succedent of display sequents. This enables the transformation of more axioms into analytic structural (display) rules; the price to pay is a more complicated formalism, and having to use structural connectives that are not interpretable as $\mathsf{FL_e}$ formulas. The latter can lead to conservativity issues for the introduced calculi (e.g. at the first-order level, intuitionistic logic extended with the bi-implication connective is not conservative over intuitionistic logic [25]).

Non-commutative case:

The algorithm can be generalised to non-commutative bunched calculi similar to the generalisation of [8] in [9] for structural *sequent* rules. In the non-commutative case, the implication connective is replaced by two connectives: left and right implication. We can then handle all $\mathcal{N}_2^d$ axioms (the classes are modified in the natural way to accommodate the new connectives in FL). A generalisation to non-commutative bunched *hypersequent* calculi is also possible; as shown in [10], the interpretation of the hypersequent "|" then requires a special form ∇ [4] of disjunction (also considered in the setting of abstract algebraic logic) which consists of a combination of $\vee$ and iterated conjugates.

6 The case of Bunched Implication logics

The logic BI [30, 33] of bunched implication extends the language of $\mathsf{DFL_e}$ with $\bot$, $\top$ and a connective $\to$ for intuitionistic implication. The interest in BI and its extensions is due to the fact that these logics allow us to reason about resource composition and systems modelling and provide a basis for an assertion language of separation logic [21], and even more recently, for reasoning about systems architecture layers [13]. An analytic calculus for BI is obtained by the addition of the following rules to $\mathsf{sDFL_e}$ [33]. There are two new structural connectives: $\varnothing_a$ is antecedent-interpreted as $\top$ and $\mathcal{O}_a$ is succedent-interpreted as $\bot$.

$$\frac{X \Rightarrow \mathcal{O}_a}{X \Rightarrow \psi}\,(\mathrm{wr}) \qquad \frac{\Gamma[X] \Rightarrow \psi}{\Gamma[\varnothing_a; X] \Rightarrow \psi}\,(\varnothing_a \mathrm{l}) \qquad \frac{\Gamma[\varnothing_a; X] \Rightarrow \psi}{\Gamma[X] \Rightarrow \psi}\,(\varnothing_a \mathrm{l}') \qquad \frac{G \,|\, \Gamma[\emptyset_a] \Rightarrow \psi}{G \,|\, \Gamma[\top] \Rightarrow \psi}\,(\top \mathrm{l})$$

$$\frac{}{\emptyset_a \Rightarrow \top}\,(\top \mathrm{r}) \qquad \Gamma[\bot] \Rightarrow \psi \qquad \frac{X \Rightarrow A \qquad \Gamma[B] \Rightarrow \psi}{\Gamma[X; A \to B] \Rightarrow \psi}\,(\to \mathrm{l}) \qquad \frac{A; X \Rightarrow B}{X \Rightarrow A \to B}\,(\to \mathrm{r})$$

The algebraic semantics of BI are Heyting algebras (intuitionistic implication is denoted by $\to$) equipped with a commutative monoidal operation $\otimes$ with identity $\mathbf{1}$ and associated multiplication $\multimap$ satisfying $x \otimes y \leq z$ iff $x \leq y \multimap z$.

The soundness and completeness theorem for sBI can be obtained from Theorem 3 by uniformly replacing "$\mathsf{DFL_e}$" with "BI". The hypersequent calculus $\mathbf{h}\mathsf{BI}$ is obtained from sBI by adding a hypersequent context "$g \,|$" to each rule in analogy with $\mathsf{sDFL_e}$ and $\mathbf{h}\mathsf{DFL_e}$. The cut-elimination method of Section 4 also applies:

Theorem 23. *The rule (cut) is eliminable in* $\boldsymbol{h}\mathsf{BI} + (cut) + \{r_i\}_{i\in I}$ ($\boldsymbol{s}\mathsf{BI} + (cut) + \{r_i\}_{i\in I}$) *for any set* $\{r_i\}_{i\in I}$ *of permutative rules.*

It is easy to see the procedure in the previous section can be used to compute structural rule extensions of sBI from initial sequents in the "$\mathcal{N}_2^d$" class in the extended grammar for BI formulae. More precisely, consider ($\mathcal{N}_0^{BI}$ and $\mathcal{P}_0^{BI}$ are the set of propositional variables):

$$\mathcal{P}_{n+1}^{BI} ::= \mathbf{1} \mid \top \mid \mathcal{N}_n^{BI} \mid \mathcal{P}_{n+1}^{BI} \otimes \mathcal{P}_{n+1}^{BI} \mid \mathcal{P}_{n+1}^{BI} \wedge \mathcal{P}_{n+1}^{BI} \mid \mathcal{P}_{n+1}^{BI} \vee \mathcal{P}_{n+1}^{BI}$$
$$\mathcal{N}_{n+1}^{BI} ::= \mathbf{0} \mid \bot \mid \mathcal{P}_n^{BI} \mid \mathcal{N}_{n+1}^{BI} \wedge \mathcal{N}_{n+1}^{BI} \mid \mathcal{P}_{n+1}^{BI} \multimap \mathcal{N}_{n+1}^{BI} \mid \mathcal{P}_{n+1}^{BI} \to \mathcal{N}_{n+1}^{BI}$$

Theorem 24. *Every* $\alpha \Rightarrow \beta$ *where* $\alpha \in \mathcal{P}_2^{BI}$ *and* $\beta \in \mathcal{N}_2^{BI}$ *can be transformed into equivalent structural bunched sequent rules that are permutative.*

Example 25 (Extensions of BI). *Analytic bunched calculi for BI extended with* $p^{n-1} \Rightarrow p^n$ *or with restricted weakening eW (see, e.g., [7]) can be obtained by adding to* $\boldsymbol{s}\mathsf{BI}$ *the rules (nctr) or (eW) in Ex. 20.*

Structural rules for bunched hypersequents

Analytic calculi for extensions of $\mathsf{DFL_e}$ by disjunctions of $\mathcal{N}_2^d$ axioms were obtained by exploiting the hypersequent structure. Some obstacles are encountered, however, when we attempt to extend Theorem 24 in the same manner to present extensions of BI beyond $\alpha \Rightarrow \beta$ ($\alpha \in \mathcal{P}_2^{BI}$ and $\beta \in \mathcal{N}_2^{BI}$).

As an example, consider the following sequent, whose addition to sBI+(cut) yields a bunched calculus for Boolean Bunched Implication logic BBI where the cut rule is not eliminable.

$$\top \Rightarrow p \vee (p \to \bot)$$

It would be tempting to preprocess this initial sequent (analogous to how we handled top-level disjunctions in axioms in the $\mathsf{DFL_e}$ case) as the initial hypersequent $\top \Rightarrow p \,|\, \top \Rightarrow p \to \bot$ (two components). However, it is not clear how to interpret such a hypersequent in sBI, as required in order to prove a soundness and completeness statement corresponding to Lemma 10.

Recall that in $\mathbf{h}\mathsf{DFL_e}$, we interpreted the initial hypersequent g as the initial sequent $\varnothing_m \Rightarrow g^I$ thus allowing us to prove Lemma 10 asserting equi-derivability of the bunched hypersequent and bunched sequent calculi. We are unable to define a corresponding $(\cdot)^I$ function in $\mathbf{h}\mathsf{BI}$ because the presence of the two different implication connectives that both utilise the turnstile $\Rightarrow$ in their right introduction rule—i.e. ($\multimap$r) and ($\rightarrow$r)—prohibits the interpretation of $\Rightarrow$ as either connective. The lack of a soundness and completeness theorem impedes in turn a general theory relating initial sequent extensions of $\mathbf{s}\mathsf{BI} + cut$ with analytic structural rule extensions of $\mathbf{h}\mathsf{BI}$.[4]

Despite this, there is a way to obtain new logics extending BI sidestepping the hypersequent interpretation-dependent preprocessing step. Motivated by the analytic structural rules for $\mathbf{h}\mathsf{DFL_e}$, we may construct analytic hypersequent structural rules for $\mathbf{h}\mathsf{BI}$ and investigate the set of sequents (i.e. hypersequents with exactly one component) that are derivable. For example, we already can compute that the addition of the structural rule below left to $\mathbf{h}\mathsf{DFL_e}$ yields an analytic hypersequent calculus for $\mathsf{DFL_e} + p \vee (p \multimap \mathbf{0})$. Motivated by this rule, consider the rule below right.

$$\frac{g \,|\, \Gamma[\Sigma] \Rightarrow \psi}{g \,|\, \Gamma[\varnothing_m] \Rightarrow \psi \,|\, \varnothing_m, \Sigma \Rightarrow \mathcal{O}_m} \qquad \frac{g \,|\, \Gamma[\Sigma] \Rightarrow \psi}{g \,|\, \Gamma[\varnothing_m] \Rightarrow \psi \,|\, \varnothing_m; \Sigma \Rightarrow \mathcal{O}_a}\ (\mathrm{cl})$$

It may be verified that $\mathbf{h}\mathsf{BI} + (cl)$ derives $\mathbf{1} \Rightarrow p \vee (p \rightarrow \bot)$. Moreover this logic is consistent in the sense that $\top \Rightarrow \bot$ is not derivable: to see this, argue backwards from $\top \Rightarrow \bot$ in $\mathbf{h}\mathsf{BI} + (cl)$ and observe that there is no way of obtaining the semicolon-separated $\varnothing_m$ that is required for an application of (cl). By a similar argument the sequent $\top \Rightarrow p \vee (p \rightarrow \bot)$ is not derivable and hence the logic of $\mathbf{h}\mathsf{BI} + (cl)$ cannot be the logic BBI!

The connective $\rightarrow$ is intuitionistic implication in BI and classical implication in BBI. Since spatial and separation logics as well as epistemic resource reasoning [16] typically use the classical implication [23], BBI has greater applicative importance than BI. For this reason, several analytic proof calculi for BBI have been presented, such as an attractive display calculus [6] which makes use of involutive negation as a structural connective, and also a somewhat more complicated labelled calculus [20] and nested sequent calculus [35]. Despite its great applicative interest, a drawback of BBI is that it is undecidable [24, 7]. This undecidability motivates the interest in finding potentially decidable logics in the vicinity of BBI which are equipped with analytic proof calculi to assist further investigation. Based on the discussion here, the logic of $\mathbf{h}\mathsf{BI} + (cl)$ emerges as such a candidate. It is worth noting that the hypersequent structure is *crucial* for formulating this logic; in particular, the display calculus in [6] cannot be used because it relies crucially on the involutive negation structural connective (which builds-in BBI).

Incidentally observe that if we want $\top \Rightarrow p \vee (p \rightarrow \bot)$ to be derivable, then we might be tempted to amend (cl) to (cl′) as follows:

$$\frac{g \,|\, \Gamma[\Sigma] \Rightarrow \psi}{g \,|\, \Gamma[\varnothing_a] \Rightarrow \psi \,|\, \Sigma \Rightarrow \mathcal{O}_a}\ (\mathrm{cl'})$$

In turns out then that $p \Rightarrow p \otimes p$ is derivable in $\mathbf{h}\mathsf{BI} + (cl')$ which means that the formalised logic must be an extension of $\otimes$-contractive BBI. We leave the systematic study of the logics obtained by analytic structural rule extension over $\mathbf{h}\mathsf{BI}$ and their potential resource-sensitive applications as future work.

[4] The inability to give a fixed definition to the interpretation of a hypersequent has already been recognised, and the interpretation is known to be strongly related to the underlying semantics of the logic. See [10] where it is shown that in the context of FL-algebras, the semantic interpretation of | is not a disjunction unless the algebraic models are commutative and integral.

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