Herbrand's Theorem for Prenex Gödel Logic and its Consequences for Theorem Proving ⋆

M. Baaz A. Ciabattoni⋆⋆ C.G. Fermüller

Technische Universität Wien, Austria

Abstract. Herbrand's Theorem for $G^{\triangle}_{\infty}$, i.e., Gödel logic enriched by the projection operator $\triangle$ is proved. As a consequence we obtain a "chain normal form" and a translation of prenex $G^{\triangle}_{\infty}$ into (order) clause logic, referring to the classical theory of dense total orders with endpoints. A chaining calculus provides a basis for efficient theorem proving.

1 Introduction

Fuzzy logic formalizes reasoning in the context of vague (imprecise) information. (See the introduction of [21].) Automated reasoning in first order fuzzy logic(s) is a big and important challenge. Among the three fundamental fuzzy logics — Łukasiewicz logic Ł, Product logic P, and Gödel logic G_{∞} — only G_{∞} (also called "intuitionistic fuzzy logic"[26]) is recursively axiomatizable (see [21]). In fact, even Gödel logic is incomplete if either certain "0-1-relativizations" are added to the language (see [4]) or the topological structure of the truth value is changed (see [8]). In any case, in contrast to propositional logics, efficient proof search at the (general) first order level seems to be beyond the current state of the art, if possible at all. Thus it is reasonable to consider natural, non-trivial fragments.

Here we focus on the *prenex* fragment of $G^{\triangle}_{\infty}$; i.e., G_{∞} enriched by the relativisation operator $\triangle$. $\triangle$ allows to make "fuzzy" statements "crisp" by mapping $\triangle P$ to the distinguished truth value 1 if the value of P equals 1, and to 0 otherwise. (See [4, 11] and Section 2, below, for more information about $\triangle$.)

We demonstrate (in Section 3) that Herbrand's Theorem holds for $G^{\triangle}_{\infty}$. This has important consequences not only from a theoretical point of view, but also for automated proof search. Indeed, we will use Herbrand's Theorem to show (in Section 5) that all prenex formulas P from $G^{\triangle}_{\infty}$ can be translated faithfully and efficiently (in linear time) into corresponding sets of "order clauses". The latter are classical clauses with predicate symbols $<$ and $\leq$ interpreted as total dense orders (strict and reflexive, respectively). "Chaining calculi" for efficient deduction in such a context have been introduced (among others) in [13, 14]. We will focus on one of these calculi (in Sections 6) and argue (in Section 7) that it is a suitable basis for handling translated formulas from prenex $G^{\triangle}_{\infty}$; in particular for the *monadic* fragment of prenex $G^{\triangle}_{\infty}$, which we will also show to be undecidable. See [20] for another approach applying chaining techniques to deduction in many-valued logics.

Another consequence of Herbrand's Theorem for $G^{\triangle}_{\infty}$ is the existence of a "chain normal form" for prenex formulas. This is investigated in Section 4.

⋆ Partly supported by the Austrian Science Fund under grant P–12652 MAT

⋆⋆ Research supported by EC Marie Curie fellowship HPMF–CT–1999–00301

2 Preliminaries

First-order Gödel logics G_∞, sometimes also called intuitionistic fuzzy logic [26] or Dummett's *LC* (eg. in [1, 19], referring to [16]), arises from intuitionistic logic by adding the axiom of linearity $(P \supset Q) \vee (Q \supset P)$ and the axioms $\forall x(P(x) \vee Q^{(x)}) \supset (\forall x P(x)) \vee Q^{(x)}$ and $\exists x(P(x) \vee Q^{(x)}) \supset (\exists x P(x)) \vee Q^{(x)}$ ($\vee$-shift), where the notation $A^{(x)}$ indicates that x does not occur free in A. Semantically Gödel logic is viewed as infinite-valued logic with the real interval $[0, 1]$ as set of truth values[1] .

An *interpretation* $\mathcal{I}$ consists of a non-empty *domain* D and a *valuation function* $\mathrm{val}_\mathcal{I}$ that maps constants and object variables to elements of D and n-ary function symbols to functions from D^n into D. $\mathrm{val}_\mathcal{I}$ extends in the usual way to a function mapping all terms of the language to an element of the domain. Moreover, $\mathrm{val}_\mathcal{I}$ maps every n-ary predicate symbol p to a fuzzy relation, i.e., a function from D^n into $[0, 1]$. The truth-value of an atomic formula (*atom*) $A = p(t_1, \dots , t_n)$ is thus defined as

$$\mathrm{val}_\mathcal{I}(A) = \mathrm{val}_\mathcal{I}(p)(\mathrm{val}_\mathcal{I}(t_1), \dots , \mathrm{val}_\mathcal{I}(t_n)).$$

For the truth constants $\bot$ and $\top$ we have $\mathrm{val}_\mathcal{I}(\bot) = 0$ and $\mathrm{val}_\mathcal{I}(\top) = 1$.

The semantics of propositional connectives is given by

$$\mathrm{val}_\mathcal{I}(P \supset Q) = \begin{cases} 1 & \text{if } \mathrm{val}_\mathcal{I}(P) \le \mathrm{val}_\mathcal{I}(Q) \\ \mathrm{val}_\mathcal{I}(Q) & \text{otherwise ,} \end{cases}$$
$$\mathrm{val}_\mathcal{I}(P \wedge Q) = \min(\mathrm{val}_\mathcal{I}(P), \mathrm{val}_\mathcal{I}(Q))$$
$$\mathrm{val}_\mathcal{I}(P \vee Q) = \max(\mathrm{val}_\mathcal{I}(P), \mathrm{val}_\mathcal{I}(Q)).$$

$\neg A$ and $A \leftrightarrow B$ are abbreviations for $A \supset \bot$ and $(A \supset B) \wedge (B \supset A)$, respectively.

To assist a concise formulation of the semantics of quantifiers we define the *distribution* of a formula P and a free variable x with respect to an interpretation $\mathcal{I}$ as $\mathrm{Distr}_\mathcal{I}(P(x)) \stackrel{\text{def}}{=} \{\mathrm{val}_{\mathcal{I}'}(P(x)) \mid \mathcal{I}' \sim_x \mathcal{I}\}$, where $\mathcal{I}' \sim_x \mathcal{I}$ means that $\mathcal{I}'$ is exactly as $\mathcal{I}$ with the possible exception of the domain element assigned to x. The semantics of quantifiers is given by the infimum and supremum of the corresponding distribution:

$$\mathrm{val}_\mathcal{I}((\forall x)P(x)) = \inf \mathrm{Distr}_\mathcal{I}(P(x)) \quad \mathrm{val}_\mathcal{I}((\exists x)P(x)) = \sup \mathrm{Distr}_\mathcal{I}(P(x)).$$

Following [4] we extend G_∞ with the "projection modalities" $\bigtriangledown$ and $\triangle$:

$$\mathrm{val}_\mathcal{I}(\bigtriangledown P) = \begin{cases} 1 & \text{if } \mathrm{val}_\mathcal{I}(P) = 0 \\ 0 & \text{if } \mathrm{val}_\mathcal{I}(P) \neq 0 \end{cases} \qquad \mathrm{val}_\mathcal{I}(\triangle P) = \begin{cases} 1 & \text{if } \mathrm{val}_\mathcal{I}(P) = 1 \\ 0 & \text{if } \mathrm{val}_\mathcal{I}(P) \neq 1 \end{cases}$$

A formula P is called *valid* in $G_\infty^\triangle$ — we write: $\models_{G_\infty^\triangle} P$ — if $\mathrm{val}_\mathcal{I}(P) = 1$ for all interpretations $\mathcal{I}$.

Whereas $\bigtriangledown P$ can already be defined in G_∞ as $\neg P$, the extension including $\triangle$, called $G_\infty^\triangle$ here, is strictly more expressive. $\triangle$ allows to recover classical reasoning inside "fuzzy reasoning" in a very simple and natural manner: If all atoms are prefixed by $\triangle$ then $G_\infty^\triangle$ coincides with classical logic. However, the expressive power of $\triangle$ goes much beyond this. In particular, observe that $\triangle \exists x P(x) \supset \exists x \triangle P(x)$ is not valid in $G_\infty^\triangle$. In fact, as shown in [4], $G_\infty^\triangle$ is not even recursively axiomatizable if a certain

[1] For more information about Gödel logic—its winding history, importance, variants, alternative semantics and proof systems—see, e.g., [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 16, 17, 18, 21, 22, 26].

"relativization operator" is present. (The recursive axiomatizability of $G_\infty^\triangle$ itself still seems to be an open problem; compare [4].) This motivates the interest in fragments of $G_\infty^\triangle$ in the context of effective theorem proving. A natural (syntactically simple) and non-trivial (see below) fragment of $G_\infty^\triangle$ is *prenex* $G_\infty^\triangle$, i.e., all quantifiers in a formula are assumed to occur at the left hand side of the formula.

Remark 1. Whereas the prenex fragment of intuitionistic logic is PSPACE-complete [15], prenex $G_\infty^\triangle$ is undecidable. In fact, we will show in Section 7 that prenex $G_\infty^\triangle$ is already undecidable for signatures with only *monadic* predicate symbols and no function symbols. On the other hand — like in intuitionistic logic — quantifiers cannot be shifted arbitrarily in G_∞ and $G_\infty^\triangle$. In other words, arbitrary formulas cannot be reduced to provably equivalent prenex formulas (in contrast to classical logic).

3 Herbrand's Theorem

In this section we show how to effectively associate with each prenex formula P of $G_\infty^\triangle$ a propositional (variable free) formula P^* which is valid if and only if P is valid.

Definition 2. *Let $\mathsf{Q}_1 y_1 \dots \mathsf{Q}_n y_n P$, with $\mathsf{Q}_i \in \{\forall, \exists\}$ be a (prenex) formula, where P is quantifier free. Its* Skolem form, *denoted by $\exists \overline{x} P^F(\overline{x})$[2], is obtained by rewriting $\exists \overline{z} \forall u Q(\overline{z}, u)$ to $\exists \overline{z} Q(\overline{z}, f(\overline{z}))$ as often as possible.*

Lemma 3. *Let P be a quantifier free formula:*

$$\models_{G_\infty^\triangle} \mathsf{Q}_1 y_1 \dots \mathsf{Q}_n y_n P(y_1, \dots, y_n) \ \Rightarrow \ \models_{G_\infty^\triangle} \exists \overline{x} P^F(\overline{x}).$$

Proof. Follows from the usual laws of quantification. □

Let P be a formula. The *Herbrand universe* $U(P)$ of P is the set of all *ground* terms (those with no variables) which can be constructed from the set of function symbols occurring in P. To prevent $U(P)$ from being finite or empty we add a constant and a function symbol of positive arity if no such symbols appear in P. The *Herbrand base* $\mathcal{B}(P)$ is the set of atoms constructed from the predicate symbols in P and the terms of the Herbrand universe. A *Herbrand expansion* of P is a disjunction of instances of P where free variables are replaced with terms in $U(P)$.

Remark 4. We make use of the fact that the truth-value of any formula P of $G_\infty^\triangle$ under a given interpretation only depends on the ordering of the respective values of atoms occurring in P.

Lemma 5. *Let P be a quantifier-free formula. If $\models_{G_\infty^\triangle} \exists \overline{x} P(\overline{x})$ then there exist tuples $\overline{t_1}, \dots \overline{t_n}$ of terms in $U(P)$, such that $\models_{G_\infty^\triangle} \bigvee_{i=1}^n P(\overline{t_i})$.*

[2] The notation hides the fact that the Skolem form also depends on the quantifier prefix. However, below, the context will always provide the relevant information.

Proof. Let $A_1, A_2, \ldots$ be a non-repetitive enumeration of (the infinite set) $\mathcal{B}(P)$. We construct a "semantic tree" T; i.e., a systematic representation of all possible order types of interpretations. T is a rooted tree whose nodes appear at levels. Each node at level ℓ is labelled with an expression, called *constraint*, of form

$$c_\ell^\pi \stackrel{\text{def}}{=} 0 \bowtie_0 A_{\pi(1)} \bowtie_1 \ldots \bowtie_{\ell-1} A_{\pi(\ell)} \bowtie_\ell 1,$$

where $\bowtie$ is either $=$ or $<$ and π is a permutation of $\{1,\ldots,\ell\}$. We say that an interpretation $\mathcal{I}$ of $P(\overline{x})$ *fulfills* the constraint c_ℓ^π if

$$0 \bowtie_0 \text{val}_\mathcal{I}(A_{\pi(1)}) \bowtie_1 \ldots \bowtie_{\ell-1} \text{val}_\mathcal{I}(A_{\pi(\ell)}) \bowtie_\ell 1$$

holds. We say that the constraint $c_{\ell+1}^{\pi'} \stackrel{\text{def}}{=} 0 \bowtie_0 A_{\pi'(1)} \bowtie_1 \ldots \bowtie_\ell A_{\pi'(\ell+1)} \bowtie_{\ell+1} 1$ *extends* c_ℓ^π if every interpretation fulfilling $c_{\ell+1}^{\pi'}$ also fulfills c_ℓ^π.

T is constructed inductively as follows:

– The root of T is at level 0 and is labelled with the constraint $0 < 1$.
– Let ν be a node at level ℓ with label c_ℓ^π. If for all interpretations $\mathcal{I}$ that fulfill c_ℓ^π we have $\text{val}_\mathcal{I}(P(\overline{t})) = 1$ for some instance $P(\overline{t})$ of $P(\overline{x})$, where the atoms of $P(\overline{t})$ are among $A_1, \ldots, A_\ell$, then ν is a leaf node of T. Otherwise, for each constraint $c_{\ell+1}^{\pi'}$ that extends c_ℓ^π a successor node ν' labelled with this constraint is appended to ν (at level $\ell+1$).

Observe that for all interpretations $\mathcal{I}$ of $\mathcal{B}(P)$ there is branch of T such that $\mathcal{I}$ fulfills all constraints at all nodes of this branch. Two cases arise:

1. T is finite. Let $\nu_1, \ldots, \nu_m$ be the leaf nodes of T. Then $\models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m P(\overline{t_i})$, where $P(\overline{t_i})$ is an instance of $P(\overline{x})$ such that $\text{val}_\mathcal{I}(P(\overline{t_i})) = 1$ for all interpretations $\mathcal{I}$ that fulfill the constraint at ν_i.
2. T is infinite. By König's lemma, T has an infinite branch. This implies that there is an interpretation $\mathcal{I}$ such that $\text{val}_\mathcal{I}(P(\overline{t_i})) < 1$ for every tuple $\overline{t_i}$ of terms of $U(P)$. Now we use the following

   Claim. For every propositional formula P of $\mathrm{G}_\infty^\triangle$ and interpretation $\mathcal{I}$ such that $\text{val}_\mathcal{I}(P) < 1$, one can find an interpretation $\mathcal{I}^c$ such that $\text{val}_{\mathcal{I}^c}(P) < c$, for an arbitrary constant $0 < c < 1$.

   The claim is easily proved by structural induction on P. It follows that there is an interpretation $\mathcal{I}'$ with $\text{val}_{\mathcal{I}'}(\exists\overline{x}P(\overline{x})) < 1$. This contradicts the assumption that $\models_{\mathrm{G}_\infty^\triangle} \exists\overline{x}P(\overline{x})$. □

The following lemma establishes sufficient conditions for a logic to allow *reverse Skolemization*. By this we mean the re-introduction of quantifiers in Herbrand expansions. Here, by a *logic* $\mathcal{L}$ we mean a set of formulas that is closed under modus ponens, generalization and substitutions (of both formulas and terms). We call a formula P *valid* in $\mathcal{L}$ — and write: $\models_\mathcal{L} P$ — if $P \in \mathcal{L}$.

Lemma 6. *Let $\mathcal{L}$ be a logic satisfying the following properties:*

1. $\models_\mathcal{L} Q \vee P \Rightarrow \models_\mathcal{L} P \vee Q$ *(commutativity of $\vee$)*
2. $\models_\mathcal{L} (Q \vee P) \vee R \Rightarrow \models_\mathcal{L} Q \vee (P \vee R)$ *(associativity of $\vee$)*
3. $\models_\mathcal{L} Q \vee P \vee P \Rightarrow \models_\mathcal{L} Q \vee P$ *(idempotency of $\vee$)*
4. $\models_\mathcal{L} P(y) \Rightarrow \models_\mathcal{L} \forall x[P(x)]^{(y)}$
5. $\models_\mathcal{L} P(t) \Rightarrow \models_\mathcal{L} \exists x P(x)$

6. $\models_{\mathcal{L}} \forall x(P(x) \vee Q^{(x)}) \ \Rightarrow \models_{\mathcal{L}} (\forall x P(x)) \vee Q^{(x)}$
7. $\models_{\mathcal{L}} \exists x(P(x) \vee Q^{(x)}) \ \Rightarrow \models_{\mathcal{L}} (\exists x P(x)) \vee Q^{(x)}$.

Let $\exists \overline{x} P^F(\overline{x})$ be the Skolem form of $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$. For all tuples of terms $\overline{t_1}, \ldots, \overline{t_m}$ of the Herbrand universe of $P^F(\overline{x})$

$$\models_{\mathcal{L}} \bigvee_{i=1}^{m} P^F(\overline{t_i}) \quad \Rightarrow \quad \models_{\mathcal{L}} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n).$$

Proof. To re-introduce quantifiers we proceed as follows. Every instance of a Skolem term $s = f(t'_1, \ldots, t'_k)$ in $\bigvee_{i=1}^{m} P^F(\overline{t_i})$ is replaced by a new variable x_s. We denote the resulting formula by $\bigvee_{i=1}^{m} P^F(\overline{t_i})[\overline{s}/\overline{x_s}]$. Let $\mathcal{V}_{SK}$ be the set of such new variables. We define $x_s \leq x_t$ iff either s is a subterm of t or $s = f(t_1, \ldots, t_a)$ and $t = g(t'_1, \ldots, t'_b)$ and $a \leq b$.

Starting with the innermost quantifier occurrence Q_n we re-introduce all quantifiers in k steps from $k = n$ down to $k = 1$. We use $\overline{\mathsf{Q}}P_i^{(k)}$ to denote the result of applying the substitutions from step n down to $k+1$ to the disjunct $P^F(\overline{t_i})[\overline{s}/\overline{x_s}]$ and prefixing it with $\mathsf{Q}_{k+1} y_{k+1} \ldots \mathsf{Q}_n y_n$. m_k is the number of disjuncts remaining before step k is applied.

If $\mathsf{Q}_k y_k = \exists y_k$: Re-substitute y_k for the variable $z_s \in \mathcal{V}_{SK}$ that occurs in $\overline{\mathsf{Q}}P_j^{(k)}$ at the positions where s has replaced y_k in $P^F(\overline{t_j})$. By hypothesis 5 we obtain

$$\models_{\mathcal{L}} \exists y_k (\bigvee_{i=1}^{i=j-1} \overline{\mathsf{Q}}P_i^{(k)} \vee \overline{\mathsf{Q}}P_j^{(k)}[y_k/z] \vee \bigvee_{i=j}^{i=m_k} \overline{\mathsf{Q}}P_i^{(k)})$$

By hypotheses 1, 2, and 7 one has

$$\models_{\mathcal{L}} (\bigvee_{i=1}^{i=j-1} \overline{\mathsf{Q}}P_i^{(k)} \vee \exists y_k \overline{\mathsf{Q}}P_j^{(k)}[y_k/z] \vee \bigvee_{i=j}^{i=m_k} \overline{\mathsf{Q}}P_i^{(k)})$$

This is repeated for all m_k disjuncts until $\exists y_k$ is re-introduced everywhere.

If $\mathsf{Q}_k y_k = \forall y_k$: First eliminate redundant copies of identical disjuncts. This can be done by hypotheses 1, 2 and 3. Observe that, by the special form of Skolem terms, any *maximal* variable $z_s \in \mathcal{V}_{SK}$ can now occur only in a single disjunct $\overline{\mathsf{Q}}P_j^{(k)}$. Analogously to the case above, we can apply hypotheses 1, 2, 4 and 6 to re-introduce Q_k and shift it to the appropriate disjunct to obtain:

$$\models_{\mathcal{L}} (\bigvee_{i=1}^{i=j-1} \overline{\mathsf{Q}}P_i^{(k)} \vee \forall y_k \overline{\mathsf{Q}}P_j^{(k)}[y_k/z] \vee \bigvee_{i=j}^{i=m_k} \overline{\mathsf{Q}}P_i^{(k)})$$

This is repeated for all m_k disjuncts until $\forall y_k$ is re-introduced everywhere.

Finally, $\models_{\mathcal{L}} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$ follows from contracting identical disjuncts (i.e., applying hypotheses 1, 2, and 3). □

Corollary 7. *Let $\exists \overline{x} P^F(\overline{x})$ be the Skolem form of $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$. For all tuples $\overline{t_1}, \ldots \overline{t_m}$ of terms of the Herbrand universe of $P^F(\overline{x})$:*

$$\models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^{m} P^F(\overline{t_i}) \ \Rightarrow \ \models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n).$$

Corollary 8. *Let P be a quantifier free formula of $\mathrm{G}_\infty^\triangle$:*

$$\models_{\mathrm{G}_\infty^\triangle} \exists \overline{x} P^F(\overline{x}) \;\Rightarrow\; \models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n).$$

Proof.

$\models_{\mathrm{G}_\infty^\triangle} \exists \overline{x} P^F(\overline{x})$
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^n P^F(\overline{t_i})$ for appropriate $\overline{t_1}, \ldots, \overline{t_m}$ by Lemma 5
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$ by Corollary 7 $\square$

Proposition 9. *For all formulas P and Q of $\mathrm{G}_\infty^\triangle$*

1. $\models_{\mathrm{G}_\infty^\triangle} P \Leftrightarrow \models_{\mathrm{G}_\infty^\triangle} \triangle P$
2. $\models_{\mathrm{G}_\infty^\triangle} \triangle(P \vee Q) \Leftrightarrow \models_{\mathrm{G}_\infty^\triangle} (\triangle P \vee \triangle Q)$.

Theorem 10. *Let P be a quantifier-free formula of $\mathrm{G}_\infty^\triangle$ and $\mathsf{Q}_i \in \{\forall, \exists\}$*

$$\models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$$

if and only if there exist tuples $\overline{t_1}, \ldots \overline{t_m}$ of terms of the Herbrand universe of $\exists \overline{x} P(\overline{x})$, such that

$$\models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m \triangle P^F(\overline{t_i}).$$

Proof.

($\Rightarrow$) $\models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \exists \overline{y} P^F(\overline{y})$ by Lemma 3
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m P^F(\overline{t_i})$ by Lemma 5
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \triangle(\bigvee_{i=1}^m P^F(\overline{t_i}))$ by Proposition 9.1
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m \triangle P^F(\overline{t_i})$ by Proposition 9.2

($\Leftarrow$) $\models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m \triangle P^F(\overline{t_i})$
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \triangle(\bigvee_{i=1}^m P^F(\overline{t_i}))$ by Proposition 9.2
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m P^F(\overline{t_i})$ by Proposition 9.1
$\Rightarrow \models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$ by Corollary 7 $\square$

Remark 11. For G_∞ (without $\triangle$), an alternative proof of Herbrand's theorem can be obtained using the analytic calculus *HIF* ("Hypersequent calculus for Intuitionistic Fuzzy logic") introduced in [12].

Corollary 12. *Let P be a quantifier-free formula of $\mathrm{G}_\infty^\triangle$:*

$$\models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n) \quad \Leftrightarrow \quad \models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n \triangle P(y_1, \ldots, y_n).$$

Proof.

$\models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$
$\Leftrightarrow \models_{\mathrm{G}_\infty^\triangle} \bigvee_{i=1}^m \triangle P^F(\overline{t_i})$ for appropriate $\overline{t_1} \ldots \overline{t_n}$ by Theorem 10
$\Leftrightarrow \models_{\mathrm{G}_\infty^\triangle} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n \triangle P(y_1, \ldots, y_n)$ by Corollary 7 and Lemma 3 $\square$

4 A Chain Normal Form for Prenex $\mathrm{G}_\infty^\triangle$

We define a normal form for formulas P of prenex $\mathrm{G}_\infty^\triangle$, that is based on the fact that the truth-value of P under a given interpretation only depends on the ordering of the respective values of atoms occurring in P. We exploit the fact that the corresponding order relation is expressible in $\mathrm{G}_\infty^\triangle$. (This is not true for G_∞.) More formally, we use

$$\begin{array}{lll} P \lessdot Q & \text{as an abbreviation for} & \neg \triangle (Q \supset P), \text{ and} \\ P \equiv_\triangle Q & \text{as an abbreviation for} & \triangle(P \supset Q) \wedge \triangle(Q \supset P). \end{array}$$

These formulas express strict linear order and equality, respectively, in the following sense. For every interpretation $\mathcal{I}$ of $\mathrm{G}_\infty^\triangle$ one has

$$\begin{array}{lll} \mathrm{val}_\mathcal{I}(P \lessdot Q) & \text{iff} & \mathrm{val}_\mathcal{I}(P) < \mathrm{val}_\mathcal{I}(Q), \text{ and} \\ \mathrm{val}_\mathcal{I}(P \equiv_\triangle Q) & \text{iff} & \mathrm{val}_\mathcal{I}(P) = \mathrm{val}_\mathcal{I}(Q). \end{array}$$

Definition 13. *Let P be a quantifier-free formula of $\mathrm{G}_\infty^\triangle$ and $A_1, \dots, A_n$ the atoms occurring in P except $\bot$ and $\top$. A* $\triangle$-chain *over P is any formula of the form*

$$(\bot \bowtie_0 A_{\pi(1)}) \wedge (A_{\pi(1)} \bowtie_1 A_{\pi(2)}) \wedge \dots \wedge (A_{\pi(n-1)} \bowtie_{n-1} A_{\pi(n)}) \wedge (A_{\pi(n)} \bowtie_n \top)$$

where π is a permutation of $\{1, \dots, n\}$, $\bowtie_i$ is either $\lessdot$ or $\equiv_\triangle$, and at least one of the $\bowtie_i$'s stands for $\lessdot$.

Every $\triangle$-chain describes a possible ordering of the values of atoms of P. By $\xi(P)$ we denote the set of all $\triangle$-chains over P. For any $C \in \xi(P)$, we define

$$\{P\}^C \stackrel{\text{def}}{=} \begin{cases} \top \text{ if } \mathrm{val}_\mathcal{I}(P) = 1 \\ \bot \text{ if } \mathrm{val}_\mathcal{I}(P) < 1 \end{cases}$$

for all interpretations $\mathcal{I}$ that satisfy the ordering conditions expressed by C. Observe that $\{P\}^C$ is always defined.

Proposition 14. *For all quantifier free formulas P, Q and F of $\mathrm{G}_\infty^\triangle$*

$$\models_{\mathrm{G}_\infty^\triangle} P \leftrightarrow Q \;\Rightarrow \models_{\mathrm{G}_\infty^\triangle} F[P] \leftrightarrow F[Q],$$

where $F[Q]$ denotes the formula arising from $F[P]$ by replacing some occurrences of the subformula P by Q.

Lemma 15. *For every quantifier free formula P and $\triangle$-chain $C \in \xi(P)$*

$$\models_{\mathrm{G}_\infty^\triangle} C \supset (\triangle P \leftrightarrow \{P\}^C).$$

Proof. By induction on the structure of P using the following tautologies of $\mathrm{G}_\infty^\triangle$:

$$\begin{array}{rr} (P \lessdot Q) \supset \triangle((P \supset Q) \leftrightarrow \top) & (Q \lessdot P) \supset \triangle((P \supset Q) \leftrightarrow Q) \\ (P \equiv_\triangle Q) \supset \triangle((P \supset Q) \leftrightarrow \top) & (P \lessdot Q) \supset \triangle((P \vee Q) \leftrightarrow Q) \\ (Q \lessdot P) \supset \triangle((P \vee Q) \leftrightarrow P) & (Q \equiv_\triangle P) \supset \triangle((P \vee Q) \leftrightarrow P) \\ (P \lessdot Q) \supset \triangle((P \wedge Q) \leftrightarrow P) & (Q \lessdot P) \supset \triangle((P \wedge Q) \leftrightarrow Q) \\ (Q \equiv_\triangle P) \supset \triangle((P \wedge Q) \leftrightarrow P) & (P \lessdot \top) \supset \triangle(\triangle P \leftrightarrow \bot) \\ (P \equiv_\triangle \top) \supset \triangle(\triangle P \leftrightarrow \top) & \end{array}$$

as well as $\models_{\mathrm{G}_\infty^\triangle} \triangle(P \leftrightarrow Q) \supset (\triangle P \leftrightarrow \triangle Q)$ together with Proposition 14. $\square$

Lemma 16. *For every quantifier free formula P and $C \in \xi(P)$*

$$\models_{G_\infty^\triangle} (C \wedge \triangle P) \leftrightarrow (C \wedge \{A\}^C).$$

Proof. It is easy to check that $\models_{G_\infty^\triangle} (P_1 \wedge P_2) \wedge (P_1 \supset (P_2 \leftrightarrow P_3)) \supset (P_1 \wedge P_3)$. We instantiate the above formula by setting $P_1 = C$, $P_2 = \triangle P$ and $P_3 = \{P\}^C$. By using Lemma 15 we obtain $\models_{G_\infty^\triangle} (C \wedge \triangle P) \supset (C \wedge \{P\}^C)$. The converse implication follows analogously. □

Theorem 17. *For every quantifier free formula P there exists $\Gamma(P) \subseteq \xi(P)$ such that*

$$\models_{G_\infty^\triangle} \triangle P \leftrightarrow \bigvee_{C \in \Gamma(P)} C$$

Proof. First note that $\models_{G_\infty^\triangle} \bigvee_{C \in \xi(P)} C$. Therefore we have

$$\models_{G_\infty^\triangle} \triangle P \leftrightarrow [(\bigvee_{C \in \xi(P)} C) \wedge \triangle P].$$

By moving $\triangle P$ into the disjunction and using Lemma 16, one obtains

$$\models_{G_\infty^\triangle} \triangle P \leftrightarrow [\bigvee_{C \in \xi(P)} (C \wedge \{P\}^C]$$

The claim follows by Proposition 14 since for every $C \in \xi(P))$ we have either $\models_{G_\infty^\triangle} (C \wedge \{P\}^C) \leftrightarrow (C \wedge \top)$ or $\models_{G_\infty^\triangle} (C \wedge \{P\}^C) \leftrightarrow (C \wedge \bot)$. □

Remark 18. A related normal form has been introduced for propositional Gödel logic without $\triangle$ in [11]. There, the total order of the truth values is expressed using the formulas $A \leftrightarrow B$ and $A \prec B$, where the latter abbreviates $(A \supset B) \wedge ((B \supset A) \supset A)$.

As a corollary to this normal form theorem and Herbrand's theorem (Theorem 10) we obtain:

Corollary 19. *Let P be a quantifier-free formula of $G_\infty^\triangle$. There exist tuples of terms $\overline{t_1}, \dots, \overline{t_n}$ of the Herbrand universe of P,*

$$\models_{G_\infty^\triangle} Q_1 y_1 \dots Q_n y_n P(y_1, \dots, y_n) \Leftrightarrow \models \bigvee_{i=1}^{n} \bigvee_{C \in \Gamma(P^F)} C[{}^{\overline{t_i}}/_{\overline{y_i}}]$$

where $C[{}^{\overline{t_i}}/_{\overline{y_i}}]$ is the chain obtained by substituting $\overline{t_i}$ for $\overline{y_i}$.

5 Translation into Order Clauses

The chain normal form for prenex formulas P of $G_\infty^\triangle$, introduced in Section 4 above, can be used to reduce the validity problem for P into the problem of detecting unsatisfiability of a corresponding set of "order clauses" with respect to the (classical) theory of dense total orders with endpoints 0 and 1. However, the computation of the chain normal form is quite inefficient in general. Therefore we use properties of $\triangle$ to introduce also a "definitional normal form", similar to the one for classical or intuitionistic logic (see, e.g., [6]).

Definition 20. *For any formula F of form $F_1 \circ F_2$, where $\circ \in \{\wedge, \vee, \supset\}$, let*

$$\mathrm{df}(F) \stackrel{\mathrm{def}}{=} [p_F(\overline{x}) \equiv_\triangle (p_{F_1}(\overline{x_1}) \circ p_{F_2}(\overline{x_2}))]$$

where p_F, p_{F_1}, p_{F_2} are new predicate symbols and $\overline{x}, \overline{x_1}, \overline{x_2}$ are the tuples of variables occurring in F, F_1, F_2, respectively. If F is of form $\triangle F_1$ then

$$\mathrm{df}(F) \stackrel{\mathrm{def}}{=} [p_F(\overline{x}) \equiv_\triangle \triangle p_{F_1}(\overline{x_1})].$$

If F is atomic then $p_F(\overline{x})$ is used as an alternative denotation for $F(\overline{x})$.

For any quantifier free formula P the definitional normal form *is defined as*

$$\mathrm{DEF}(P) \stackrel{\mathrm{def}}{=} [(\bigwedge_{F \in \mathrm{nasf}(P)} \mathrm{df}(F)) \supset \triangle p_P(\overline{x})]$$

where $\mathrm{nasf}(P)$ denotes the set of all non-atomic subformulas of P, $\overline{x}$ is the tuple of variables occurring in P, and p_P is a new predicate symbol.

Remark 21. Certain optimizations, using tautologies of $\mathrm{G}_\infty^\triangle$, will lead to shorter definitional normal forms in general. However, in any case the logical complexity (i.e. the number of connectives) of $\mathrm{DEF}(P)$ is *linear* in the logical complexity of P.

Lemma 22. *For all quantifier free formulas P of $\mathrm{G}_\infty^\triangle$:*

$$\models_{\mathrm{G}_\infty^\triangle} \exists \overline{x} P(\overline{x}) \;\; \Leftrightarrow \models_{\mathrm{G}_\infty^\triangle} \exists \overline{x} \mathrm{DEF}(P(\overline{x})).$$

Proof. By Corollary 12, $\models_{\mathrm{G}_\infty^\triangle} \exists \overline{x} P(\overline{x})$ iff $\models_{\mathrm{G}_\infty^\triangle} \exists \overline{x} \triangle P(\overline{x})$. For every interpretation $\mathcal{I}$: $\mathrm{val}_\mathcal{I}(A \equiv_\triangle B) = 1$ if $\mathrm{val}_\mathcal{I}(A) = \mathrm{val}_\mathcal{I}(B)$ and $\mathrm{val}_\mathcal{I}(A \equiv_\triangle B) = 0$, otherwise. Consequently, the proof proceeds exactly as in the case for classical logic (see [24, 6]). I.e., for all non-atomic quantifier free formulas $F(\overline{x})$, one can show by induction on the complexity of F that $\mathrm{val}_\mathcal{I}(\mathrm{df}(F(\overline{x}))) = 1$ iff $\mathrm{val}_\mathcal{I}(F(\overline{x})) = \mathrm{val}_\mathcal{I}(p_F(\overline{x}))$. □

We translate prenex $\mathrm{G}_\infty^\triangle$-formulas into sets of clauses of the following form.

Definition 23. *Let the sign $\lhd$ stands for either $<$ or $\leq$. An* inequality *is an expression of form $s \lhd t$, where $s, t \in T(\mathbf{F}, \mathcal{X})$, i.e., the set of all terms over function symbols* **F** *(including constants) and variables $\mathcal{X}$. An* (order) clause *is a finite set of inequalities.*

Definition 24. *By a* dense total order *$\mathcal{O}$ we mean a (classical) interpretation of the signature $<$, $\leq$, and* **F**, *where $<$ is interpreted as strict and dense total (linear) order over the elements assigned to $T(\mathbf{F}, \mathcal{X})$ and $\leq$ is interpreted as the reflexive closure of $<$. If also the* endpoint *axioms $\forall x(0 \leq x)$, $\forall x(x \leq 1)$, and $0 < 1$ are satisfied we call $\mathcal{O}$ a* DTOE*-model. A set of order clauses $\mathcal{S}$ is* DTOE-satisfiable *if $\mathcal{S}$ has a dense total order with endpoints 0 and 1, respectively, as model.*

In the following we also allow equalities $s = t$ to occur in clauses. However, a clause of form $\{s = t\} \cup C$ is considered here as an abbreviation for the two clauses $\{s \leq t\} \cup C$ and $\{t \leq s\} \cup C$.

Remark 25. In implementing the proof procedure, equalities can and should be handled more efficiently than indicated above. In particular, combinations of chaining and superposition along the line of [13, 14] should be applied.

Definition 26. *We define sets of clauses that correspond to the various forms of formulas of type* $\mathrm{df}(F)$*:*

$$\begin{array}{ll} \mathrm{cl}(A \equiv_{\triangle} (B \wedge C)) & \stackrel{\text{def}}{=} \{\{A \leq B\},\ \{A \leq C\},\ \{A = B, A = C\}\} \\ \mathrm{cl}(A \equiv_{\triangle} (B \vee C)) & \stackrel{\text{def}}{=} \{\{B \leq A\},\ \{C \leq A\},\ \{A = B, A = C\}\} \\ \mathrm{cl}(A \equiv_{\triangle} (B \supset C)) & \stackrel{\text{def}}{=} \{\{1 \leq A, A = C\},\ \{B \leq C, A = C\},\ \{1 \leq A, C < B\}\} \\ \mathrm{cl}(A \equiv_{\triangle} \triangle B) & \stackrel{\text{def}}{=} \{\{A < 1, 1 \leq B\},\ \{B < 1, 1 \leq A\}\} \end{array}$$

where A*,* B *and* C *are atoms, considered as terms.*

The clause form *for formulas* $\exists\overline{x}P(\overline{x})$ *is given by*

$$\mathrm{CF}^{\mathrm{d}}(\exists\overline{x}P(\overline{x})) \stackrel{\text{def}}{=} \{\{p_P(\overline{x}) < 1\}\} \cup \bigcup_{F \in \mathrm{nasf}(P)} \mathrm{cl}(\mathrm{df}(F))$$

To define the alternative clause normal form $\mathrm{CF}^{\mathrm{c}}(\exists\overline{x}P(\overline{x}))$ *based on chains, let* $[A < B]^{\#} \stackrel{\text{def}}{=} \{B \leq A\}$ *and* $[A \equiv_{\triangle} B]^{\#} \stackrel{\text{def}}{=} \{A < B, B < A\}$*.*

$$\mathrm{CF}^{\mathrm{c}}(\exists\overline{x}P(\overline{x})) \stackrel{\text{def}}{=} \{\{\bigcup_{A \bowtie_i B\ in\ C} [A \bowtie_i B]^{\#}\} \mid C \in \Gamma(P)\}$$

where $\Gamma(P)$ *is the subset of* $\xi(P)$ *given by Theorem 17.*

Lemma 27. *For every interpretation* $\mathcal{I}$ *there is a* DTOE*-model* $\mathcal{O}_{\mathcal{I}}$*, such that for all non-atomic* F*:* $\mathrm{val}_{\mathcal{I}}(\mathrm{df}(F)) = 1$ *iff* $\mathcal{O}_{\mathcal{I}}$ *satisfies* $\mathrm{cl}(\mathrm{df}(F))$*; and vice versa.*

Proof. We only present the case for $F = [A \equiv_{\triangle} (B \wedge C)]$. The other cases are similar. We have:

$$\begin{array}{rl} \mathrm{val}_{\mathcal{I}}(A \equiv_{\triangle} (B \wedge C)) = 1 & \Leftrightarrow \mathrm{val}_{\mathcal{I}}(A) = \min\{\mathrm{val}_{\mathcal{I}}(B), \mathrm{val}_{\mathcal{I}}(C)\} \\ & \Leftrightarrow \mathrm{val}_{\mathcal{I}}(A) \leq \mathrm{val}_{\mathcal{I}}(B) \text{ and } \mathrm{val}_{\mathcal{I}}(A) \leq \mathrm{val}_{\mathcal{I}}(C) \text{ and} \\ & \quad (\mathrm{val}_{\mathcal{I}}(A) = \mathrm{val}_{\mathcal{I}}(B) \text{ or } \mathrm{val}_{\mathcal{I}}(A) = \mathrm{val}_{\mathcal{I}}(C)) \end{array}$$

Therefore $\mathcal{I}$ induces an DTOE-model $\mathcal{O}_{\mathcal{I}}$ satisfying the order clauses

$$\{A \leq B\},\ \{A \leq C\},\ \text{and}\ \{A = B, A = C\}.$$

Conversely, every DTOE-model for this clause set induces an interpretation that evaluates $A \equiv_{\triangle} (B \wedge C)$ to 1. □

Theorem 28. *Any prenex formula* $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$ *of* $\mathrm{G}_{\infty}^{\triangle}$ *is valid if and only if* $\mathrm{CF}^{\mathrm{d}}(\exists\overline{x}P^F(\overline{x}))$ *is* DTOE*-unsatisfiable.*

Proof. By Lemma 3 and Corollary 8 we have: $\models_{\mathrm{G}_{\infty}^{\triangle}} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$ iff $\models_{\mathrm{G}_{\infty}^{\triangle}} \exists\overline{x}P^F(\overline{x})$. By Lemma 22 we have: $\models_{\mathrm{G}_{\infty}^{\triangle}} \exists\overline{x}P^F(\overline{x})$ iff $\models_{\mathrm{G}_{\infty}^{\triangle}} \exists\overline{x}\mathrm{DEF}(P^F(\overline{x}))$.

Since the conclusion as well as the conjuncts in the premise of $\mathrm{DEF}(P^F(\overline{x}))$ are prefixed by $\triangle$, those subformulas behave like in classical logic. Hence the validity problem can be dualized; i.e., $\exists\overline{x}\mathrm{DEF}(P^F(\overline{x}))$ is valid iff

$$\forall\overline{x}\neg \triangle p_P(\overline{x}) \wedge \bigwedge_{F \in \mathrm{nasf}(P)} \mathrm{df}(F)$$

is unsatisfiable. By Lemma 27 the latter is equivalent to the DTOE-unsatisfiability of $\mathrm{CF}^{\mathrm{d}}(\exists\overline{x}P^F(\overline{x}))$. □

Remark 29. By similar arguments Theorem 28 also holds for $\mathrm{CF}^{\mathrm{c}}(\exists\overline{x}P^F(\overline{x}))$.

6 Using an Ordered Chaining Calculus

In the previous sections, we have reduced the validity problem for prenex $\mathrm{G}_\infty^\triangle$ to checking DTOE-unsatisfiability of certain sets of *order clauses*. Fortunately, efficient theorem proving for (various types of) order clauses has already received considerable attention in the literature; see [14, 13] (and the references given there).

Some familiarity with basic notions from automated deduction, in particular the concept of a *most general unifier (mgu)* of two or more terms, is assumed in the following (see, e.g., [23].) We will identify a substitution σ with a set $\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ and define $codom(\sigma) = \{t_1, \ldots, t_n\}$.

We consider the following rules (cf. [13]) for order clauses:

Irreflexivity Resolution:

$$\frac{C \cup \{s < t\}}{C\sigma}$$

where σ is the mgu of s and t

(Factorized) Chaining:

$$\frac{C \cup \{u_1 \lhd_1 s_1, \ldots, u_m \lhd_m s_m\} \quad D \cup \{t_1 \lhd'_1 v_1, \ldots, t_n \lhd'_n v_n\}}{C\sigma \cup D\sigma \cup \{u_i\sigma \lhd_{i,j} v_j\sigma \mid 1 \leq i \leq m, 1 \leq n\}}$$

where σ is the mgu of $s_1, \ldots, s_m, t_1, \ldots, t_n$ and $\lhd_{i,j}$ is $<$ if and only if either $\lhd_i$ is $<$ or $\lhd'_j$ is $<$. Moreover, $t_1\sigma$ occurs in $D\sigma$ only in inequalities $v \lhd t_1\sigma$.

These two rules constitute a refutationally complete inference system for the theory of all total orders in presence of set $\mathcal{E}q^{\mathbf{F}}$ of clauses

$$\{x_i < y_i, y_i < x_i \mid 1 \leq i \leq n\} \cup \{f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)\}$$

where f ranges the set $\mathbf{F}$ of function symbols of the signature. Observe that, in translating a formula P from prenex $\mathrm{G}_\infty^\triangle$ into a set of order clauses $\mathrm{CF}^{\mathrm{d}}(P)$, we treat the predicate symbols of P as function symbols. Additional function symbols occur from Skolemization.

The inference system is not yet sufficiently restrictive for efficient proof search. We follow [13] and add conditions to the rules that refer to some complete reduction order $\succ$ (on the set of all terms). We write $s \not\preceq t$ if $\neg(s \succ t)$ and $s \neq t$; and "*t is basic in (clause) C*" if $t \lhd s \in C$ or $s \lhd t \in C$.

Maximality Condition for Irreflexivity Resolution: $s\sigma$ is a maximal term in $C\sigma$.
Maximality Condition for Chaining: (1) $u_i\sigma \not\succeq s_1\sigma$ for all $1 \leq i \leq n$, (2) $v_i\sigma \not\succeq t_1\sigma$ for all $1 \leq i \leq m$, (3) $u\sigma \not\succeq s_1\sigma$ for all terms u that are basic in C, and (4) $v\sigma \not\succeq t_1\sigma$ for all terms v that are basic in D.

For our purposes it is convenient to view the resulting inference system $\mathrm{MC}_\succ$ as a set operator.

Definition 30. $\mathrm{MC}_\succ(\mathcal{S})$ *is the set of all conclusions of Irreflexivity Resolution or Maximal Chaining where the premises are (variable renamed copies of) members of the set of clauses* $\mathcal{S}$*. Moreover,* $\mathrm{MC}^0_\succ(\mathcal{S}) = \mathcal{S}$, $\mathrm{MC}^{i+1}_\succ(\mathcal{S}) = \mathrm{MC}_\succ(\mathrm{MC}^i_\succ(\mathcal{S})) \cup \mathrm{MC}^i_\succ(\mathcal{S})$, *and* $\mathrm{MC}^*_\succ(\mathcal{S}) = \bigcup_{i\geq 0} \mathrm{MC}^i_\succ(\mathcal{S})$.

The set consisting of the three clauses $\{0 \leq x\}$, $\{x \leq 1\}$, and $\{0 < 1\}$, corresponding to the endpoint axioms, is called $\mathcal{E}p$. The set consisting of $\{y \leq x, d(x,y) < y\}$ and $\{y \leq x, x < d(x,y)\}$, corresponding to the usual density axiom, is called $\mathcal{D}o$.

The following completeness theorem follows directly from Theorem 2 of [13].

Theorem 31. $\mathcal{S}$ *has a dense total order with endpoints* 0 *and* 1 *as a model if and only if* $\mathrm{MC}^*_\succ(\mathcal{S} \cup \mathcal{E}q^{\mathbf{F}} \cup \mathcal{E}p \cup \mathcal{D}o)$ *does not contain the empty clause.*

Remark 32. Even more refined "chaining calculi" for handling orders have been defined by Bachmair and Ganzinger in [13, 14]. However, $\mathrm{MC}_\succ$ turns out to be quite appropriate for our context. (In particular, since the problem of "variable chaining" does not occur for the sets of clauses considered here).

7 The Monadic Prenex Fragment

A formula is called *monadic* if all predicate symbols are monadic (unary) and no function symbols occur in it.

To support the claim that $\mathrm{MC}_\succ$ provides an efficient proof system for prenex $\mathrm{G}^\triangle_\infty$, we conclude by investigating the special case of *monadic* formulas.

To appreciate the importance of this fragment, remember that monadic predicates are interpreted as fuzzy sets. We will show that $\mathrm{MC}_\succ$ allows to prevent the nesting of function symbols (beyond the level of the input set) in clauses derivable from chain-based clause normal forms of the Skolem form of a prenex and monadic formula.

To characterize the syntactic restrictions obeyed by clauses arising from translating prenex monadic formulas we need some additional notation.

From now on we assume that the set of function symbols $\mathbf{F}$ consists in the disjoint union $\mathbf{S} \cup \mathbf{P} \cup \{0,1\} \cup \{d\}$, where $\mathbf{S}$ are the function symbols and constants arising from Skolemizing the original formula P, and $\mathbf{P}$ is the set of monadic predicate symbols occurring in P. We will distinguish the different types of function symbols syntactically by using lower case letters for symbols in $\mathbf{S}$ and upper case letters for symbols in $\mathbf{P}$. Moreover, we assume the set of variables $\mathcal{X}$ to be *stratified* in the following sense: $\mathcal{X}$ is the disjoint union $\biguplus_{1\leq i\leq p} \mathcal{X}_i$, where each $\mathcal{X}_i$ is infinite and p is the maximal arity of function symbols in $\mathbf{F}$.

Definition 33. *We call a term* simple *if it is either a variable or a constant or of form* $f(x_1, \dots, x_n)$ *where* $x_i \in \mathcal{X}_i$ *for* $1 \leq i \leq n$. *(We call terms of the latter type* stratified.*) A term is called* atom-like *if it is of form* $P(s)$, *where* P *is a monadic function symbol and* s *is a simple term.*

An inequality $t_1 \lhd t_2$ *is called* monadic *if* t_1, t_2 *are either simple or atom-like. A* clause *is called* monadic *if all its inequalities are monadic. Finally, a* set of clauses *is called* monadic *if all its clauses are monadic.*

Proposition 34. *Let* $\mathsf{Q}_1 y_1 \dots \mathsf{Q}_n y_n P(y_1, \dots, y_n)$ *be a monadic and prenex formula of* $\mathrm{G}_\infty^\triangle$. *Then* $\mathrm{CF}^c(\exists \overline{x} P^F)$ *is monadic. Moreover,* $\mathcal{E}q^{\mathbf{F}}$, $\mathcal{E}p$, *and* $\mathcal{D}o$ *are monadic too, up to renaming of variables.*

To obtain the closure of the class of monadic sets of clauses with respect to $\mathrm{MC}_\succ$, we have to choose the reduction order $\succ$ appropriately. From now on we assume that $\succ$ fulfills all of the following, where $x \in \mathcal{X}$ and $P, Q \in \mathbf{P}$:

(a) $t \succ x$ if x is a proper subterm of t, and
(b) $t \succ P(x)$ if t is a simple term containing x as a proper subterm.
(c) $Q(t) \succ P(x)$ if t is simple term containing x as a proper subterm.

It is easy to check that these conditions are fulfilled if $\succ$ is a lexicographic path order based on a strict order $>_{\mathbf{F}}$ of the signature where $f >_{\mathbf{F}} p$ whenever $f \in \mathbf{S}$ and $p \in \mathbf{P}$. (See, e.g., [3].)

Lemma 35. *If* $\mathcal{S}$ *is monadic then* $\mathrm{MC}_\succ(\mathcal{S})$ *is monadic too.*

Proof. Consider *irreflexivity resolution*: i.e., $C\sigma$ where σ is the mgu of s and t in the monadic clause $C \cup \{s < t\}$.

(1) If $codom(\sigma)$ contains only variables (or σ is the empty substitution) then the only condition on monadicity that is not already obviously fulfilled by $C\sigma$ is that that all terms of form $f(x_1, \dots, x_n)$ occurring in $C\sigma$ are stratified. We have to check the following cases
  (1.1) s and t are variables: By the maximality condition, no term of form $f(x_1, \dots, x_n)$ in C can contain s or t as a subterm. Therefore such terms remain unchanged and, in particular, stratified.
  (1.2) $s = f(x_1, \dots, x_n)$ and $t = f(y_1, \dots, y_n)$. Since $x_i \in \mathcal{X}_i$ and $y_i \in V_i$, stratification is preserved in $C\sigma$.
  (1.3) $s = P(f(x_1, \dots, x_n))$ and $t = P(f(y_1, \dots, y_n))$. Like case (1.2).
  (1.4) $s = P(x)$ and $t = P(y)$; $\sigma = \{x \leftarrow y\}$ or $\sigma = \{y \leftarrow x\}$. By the maximality condition and conditions (b) and (c) no term of form $f(x_1, \dots, x_n)$ in C can contain x or y as a subterm. Therefore such terms remain stratified.
(2) Otherwise, since $s < t$ is monadic, σ is of form $\{x \leftarrow r\}$ for some term r that is either simple or atom-like, but not a variable. Without loss of generality, we assume that x occurs in s (but not in t). Since r is not a variable, there are only the following two cases:
  (2.1) $s = x$ and $t = r$: By the maximality condition and condition (i) for $\succ$, x cannot be a proper subterm of a term in C. I.e., x is basic in C, if it occurs in C at all. Therefore $C\sigma$ is monadic.

(2.2) $s = P(x)$ and $t = P(r)$ for some $P \in \mathbf{P}$: By the maximality condition and conditions (b) and (c) for $\succ$ we have: if x occurs in C, then x is basic in C or x occurs in an atom-like term of form $P(x)$ in C. In both cases $C\sigma$ is monadic.

The case for *chaining* is analogous. E.g., consider $m = n = 1$: $E = C\sigma \cup D\sigma \cup \{u\sigma \lhd^* v\sigma\}$, where σ is mgu of s and t in the monadic clauses $C \cup \{u \lhd s\}$ and $D \cup \{t \lhd v\}$. Again, if $codom(\sigma)$ consists of variables only then E is monadic, too, by the same arguments as in (1), above. Otherwise the same case distinction as for (2), above, and analogous arguments apply. $\square$

Lemma 35 implies a bound on the *depth* of terms that occur in clauses derivable from monadic sets of clauses. This leaves open the question whether also the *length* of clauses (i.e., number of inequalities) can be bounded. However, this would contradict the following undecidability result. (We adapt a proof of Gabbay [19] for the monadic — but not prenex — fragment of G_∞.)

Theorem 36. *Validity of prenex monadic formulas of* $\mathrm{G}_\infty^\triangle$ *is undecidable.*

Proof. In [25] it has been shown that the classical theory CE of two equivalence relations is undecidable. We faithfully interpret CE in the prenex monadic fragment of $\mathrm{G}_\infty^\triangle$. In fact, already validity (and therefore also satisfiability) of a formula S of CE of form

$$Q_1 x_1 \ldots Q_n v_n (\bigwedge_j x_j \equiv y_j \supset \bigvee_k u_k \equiv v_k)$$

is undecidable, where each occurrence of $\equiv$ can be either $\equiv_1$ or $\equiv_2$. Let p_1 and p_2 be two monadic predicate symbols. We define $[x \equiv_i y]^* \stackrel{\text{def}}{=} \triangle(p_i(x) \leftrightarrow p_i(y))$, for $i = 1, 2$. Let S^* be the formula arising from S by replacing all subformulas $x \equiv_i y$ by $[x \equiv_i y]^*$.

We show that S has a CE-model $\mathcal{M} = (D; \equiv_1{}^{\mathcal{M}}, \equiv_2{}^{\mathcal{M}})$ if and only if $\mathrm{val}_{\mathcal{I}}(S^*) = 1$ for some $\mathrm{G}_\infty^\triangle$-interpretation $\mathcal{I}$. Without loss of generalization we will assume that the domain of $\mathcal{M}$ to be countable.

($\Rightarrow$) Note that each of two equivalence relations $\equiv_i{}^{\mathcal{M}}$ $(i = 1, 2)$ of the CE-model $\mathcal{M}$ induces a partition of its domain $[0, 1]$ into equivalence classes $E_i^j = \{x \mid x \equiv_i{}^{\mathcal{M}} y_j\}$, where y_j is an element of the domain D of $\mathcal{M}$ and j is some index taken from a set J. Without loss of generality will assume that the index set J is the real unit interval $[0, 1]$. (An equivalence class may have many different indices.) We define $\mathcal{I} = (D, \mathrm{val}_{\mathcal{I}})$ by setting (for $i = 1, 2$) $\mathrm{val}_{\mathcal{I}}(p_i)(d) = j$ iff $d \in E_i^j$. By straightforward induction on the complexity of S it follows that $\mathrm{val}_{\mathcal{I}}(S^*) = 1$ iff $\mathcal{M}$ a CE-model S.

($\Leftarrow$) Given a $\mathrm{G}_\infty^\triangle$-interpretation $\mathcal{I} = (D, \mathrm{val}_{\mathcal{I}})$ for S^* we define the CE-model $\mathcal{M}$ for S by taking D as its domain and setting $x \equiv_i{}^{\mathcal{M}} y$ iff $\mathrm{val}_{\mathcal{I}}(p_i)(x) = \mathrm{val}_{\mathcal{I}}(p_i)(y)$ for $x, y \in D$, $i = 1, 2$. $\square$

References

[1] A. Avron. Hypersequents, Logical Consequence and Intermediate Logics for Concurrency. *Annals of Mathematics and Artifi cial Intelligence* Vol. 4, 1991, 225–248.

[2] A. Avron. The Method of Hypersequents in Proof Theory of Propositional Non-Classical Logics. In *Logic: From Foundations to Applications. European Logic Colloquium, Keele, UK, July 20–29, 1993*. Oxford, Clarendon Press, 1996, 1–32.

[3] F. Baader, T. Nipkow. *Term Rewriting and All That*. Cambridge UP, 1998.

[4] M. Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. In *Proceedings Gödel 96. Kurt Gödel's Legacy*. LNL 6, Springer, 23–33. 1996.

[5] M. Baaz, A. Ciabattoni, R. Zach. Quantified Propositional Gödel Logics. In *Proceedings LPAR 2000*. Eds. M. Parigot, A. Voronkov LNCS 1955 Springer, 2000, 240–256

[6] M. Baaz, U. Egly, A. Leitsch. Normal Form Transformations. *Handbook of Automated Reasoning 1* Eds: A. Robinson, A. Voronkov. Elsevier, 2001, 273–333.

[7] M. Baaz, C. Fermüller. Analytic Calculi for Projective Logics. *Automated Reasoning with Analytic Tableaux and Related Methods*, TABLEAUX'99, LNAI 1617, Springer, 1999, 36–50.

[8] M. Baaz, A. Leitsch, R. Zach. Incompleteness of an infinite-valued first-order Gödel Logic and of some temporal logic of programs. In: *Proceedings of Computer Science Logic CSL'95*. Berlin, 1996. Springer.

[9] M. Baaz, H. Veith. Quantifier Elimination in Fuzzy Logic. *Computer Science Logic, 12th International Workshop, CSL'98*, LNCS 1584, 1999, 399-414.

[10] M. Baaz, H. Veith. An Axiomatization of Quantified Propositional Gödel Logic Using the Takeuti-Titani Rule. Proc. Logic Colloquium 1998, LNML, Springer, to appear.

[11] M. Baaz, H. Veith. Interpolation in fuzzy logic. *Arch. Math. Logic*, 38 (1999), 461–489.

[12] M. Baaz, R. Zach. Hypersequents and the Proof Theory of Intuitionistic Fuzzy Logic. *Proc. CSL'2000*, 2000, 187–201.

[13] L. Bachmair, H. Ganzinger. Ordered Chaining for Total Orderings. *Proc. CADE'94*, Springer LNCS 814, 1994, 435–450.

[14] L. Bachmair, H. Ganzinger. Ordered Chaining Calculi for First-Order Theories of Transitive Relations. *J. ACM* 45(6) (1998), 1007-1049.

[15] A. Degtyarev, A. Voronkov. Decidability Problems for the Prenex Fragment of Intuitionistic Logic. In *Proceedings LICS'96*. IEEE Press, 503-509. 1996.

[16] M. Dummett. A propositional calculus with denumerable matrix. *J. Symbolic Logic*, 24(1959), 97–106.

[17] J.M. Dunn, R.K. Meyer. Algebraic completeness results for Dummett's LC and its extensions. *Z. Math. Logik Grundlagen Math.*, 17 (1971), 225–230.

[18] R. Dyckhoff. A Deterministic Terminating Sequent Calculus for Gödel-Dummett Logic. *Logic Journal of the IGPL*, 7(1999), 319-326.

[19] D.M. Gabbay. Decidability of some intuitionistic predicate theories. *J. of Symbolic Logic*, vol. 37, pp. 579-587. 1972.

[20] H. Ganzinger, V. Sofronie-Stokkermans. Chaining Techniques for Automated Theorem Proving in Many-Valued Logics. In: *Proceedings 30th IEEE Symposium on Multiple-Valued Logic*. IEEE Press, 337-344. 2000.

[21] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, 1998.

[22] A. Horn. Logic with truth values in a linearly ordered Heyting algebra. *J. Symbolic Logic*, 27 (1962), 159–170.

[23] A. Leitsch. *The Resolution Calculus*. Springer, 1997. Kluwer, 1998.

[24] D. Plaisted, S. Greenbaum. A Structure-Preserving Clause Form Translation. *J. Symbolic Computation* 2 (1986), 293–304.

[25] H. Rogers. Certain logical reduction and decision problems. *Annals of Mathematics*, vol. 64, pp. 264284. 1956.

[26] G. Takeuti, T. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *J. Symbolic Logic*, 49 (1984), 851-866.