Monadic Fragments of Gödel Logics: Decidability and Undecidability Results

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Abstract. The monadic fragments of first-order Gödel logics are investigated. It is shown that all finite-valued monadic Gödel logics are decidable; whereas, with the possible exception of one ($\mathbf{G}_\uparrow$), all infinite-valued monadic Gödel logics are undecidable. For the missing case $\mathbf{G}_\uparrow$ the decidability of an important sub-case, that is well motivated also from an application oriented point of view, is proven. A tight bound for the cardinality of finite models that have to be checked to guarantee validity is extracted from the proof. Moreover, monadic $\mathbf{G}_\uparrow$, like all other infinite-valued logics, is shown to be undecidable if the projection operator $\triangle$ is added, while all finite-valued monadic Gödel logics remain decidable with $\triangle$.

1 Introduction

Many-valued logics have various applications in computer science (see, e.g., [10]). They are particularly useful for modeling reasoning with graded notions and vague information. In the latter context, the family of (finite- and infinite-valued) *Gödel logics* appears as a prominent example. These are the only many-valued logics that are completely specified by the *order structure* of the underlying set of truth values. This fact characterizes Gödel logics as logics of comparative truth and renders them an important case of so-called *fuzzy logics* (see [11]).

Propositional finite-valued Gödel logics were introduced by Gödel [9] to show that intuitionistic logic does not have a characteristic finite matrix. They were generalized by Dummett [7] to an infinite set of truth values. First-order Gödel logic based on the closed unit interval $[0, 1]$ as set of truth values was introduced and axiomatized by Takeuti and Titani in [15] and called "intuitionistic fuzzy logic", there. In a more general view, the truth values for Gödel logics can be taken from any $V \subseteq [0, 1]$, that contains 0 and 1, and is closed under infima and suprema. (Gödel logic coincides with classical logic for $V = \{0, 1\}$.) In contrast to the propositional case, where there is only one infinite-valued Gödel logic with respect to validity, different sets V of truth values determine different *first-order* Gödel logics $\mathbf{G}_V$, in general. As shown in [4], $\mathbf{G}_V$ is recursively axiomatizable only when V is either finite or is order isomorphic to $[0, 1]$ or to $\{0\} \cup [\frac{1}{2}, 1]$.

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We investigate *monadic Gödel logics*, i.e. first-order $\mathbf{G}_V$ in which all predicate letters are unary (monadic). Many-valued monadic predicates can be interpreted as fuzzy sets and therefore many-valued monadic logics suffice to formalize the central concept of a *fuzzy IF-THEN rule*, like: "IF $A(x)$ and $B(x)$ THEN $C(x)$", where the predicates A, B, and C are fuzzy, i.e., they apply to x possibly only to some degree.

We show that all finite-valued monadic Gödel logics are decidable, while for infinite sets V of truth values all monadic Gödel logics are undecidable, with the possible exception of monadic $\mathbf{G}_\uparrow$, where $V = \{1 - 1/n : n \geq 1\} \cup \{1\}$. The missing case, $\mathbf{G}_\uparrow$, is interesting, since it coincides with the intersection of all monadic finite-valued Gödel logics. Its decidability status remains open. However, we prove the decidability of an important sub-case, that we call the *untangled fragment* of $\mathbf{G}_\uparrow$.

The untangled fragment of a logic consists of those (monadic) formulas in which each subformula contains at most one free variable. To appreciate the usefulness of this fragment, notice that its classical counterpart was used in [12] to formalize the knowledge base of the medical expert system CADIAG-1, represented as (classical) IF-THEN rules. This formalization made it possible to prove the decidability of the consistency checking problem in CADIAG-1 and led to a simple algorithm to actually carry out such checks.

Our decision procedure for the untangled fragment of $\mathbf{G}_\uparrow$ also provides a tight bound for the cardinality of finite models that have to be checked to guarantee validity. This bound implies a considerable gain in efficiency for the corresponding fragments of finite-valued Gödel logics (including classical logic). An elegant axiomatization for the untangled fragment of $\mathbf{G}_\uparrow$ can also be extracted from the decision procedure, contrasting the fact that $\mathbf{G}_\uparrow$ is not recursively axiomatizable [3, 4].

We also investigate monadic Gödel logics extended with the projection operator $\triangle$, see [1]. This operator maps $\triangle P$ to the distinguished truth value 1 if the value of P equals 1, and to 0 otherwise, and thus allows to recover classical reasoning inside Gödel logics. The addition of $\triangle$ does not affect the decidability of the finite-valued logics, however *all* infinite-valued monadic Gödel logics, including $\mathbf{G}_\uparrow$, turn out to be undecidable in presence of $\triangle$, even when restricted to their prenex fragments.

2 Basic facts about Gödel logics

Kurt Gödel [9] has introduced the following truth functions for conjunction, disjunction, and implication:

$$\|A \wedge B\|_\mathcal{I} = \min(\|A\|_\mathcal{I}, \|B\|_\mathcal{I}), \qquad \|A \vee B\|_\mathcal{I} = \max(\|A\|_\mathcal{I}, \|B\|_\mathcal{I}),$$

$$\|A \to B\|_\mathcal{I} = \begin{cases} 1 & \text{if } \|A\|_\mathcal{I} \leq \|B\|_\mathcal{I} \\ \|B\|_\mathcal{I} & \text{otherwise.} \end{cases}$$

Formulas are evaluated over some set V of *truth values*, where $\{0,1\} \subseteq V \subseteq [0,1]$. The propositional constant $\bot$ is semantically fixed by $\|\bot\|_\mathcal{I} = 0$. $\neg A$ abbreviates $A \to \bot$ and $A \leftrightarrow B$ abbreviates $(A \to B) \wedge (B \to A)$; therefore

$$\|\neg A\|_\mathcal{I} = \begin{cases} 1 & \text{if } \|A\|_\mathcal{I} = 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \|A \leftrightarrow B\|_\mathcal{I} = \begin{cases} 1 & \text{if } \|A\|_\mathcal{I} = \|B\|_\mathcal{I} \\ \min(\|A\|_\mathcal{I}, \|B\|_\mathcal{I}) & \text{otherwise.} \end{cases}$$

Obviously, $\|.\|_\mathcal{I}$ extends every interpretation $\mathcal{I}$, that maps propositional variables into V, uniquely to arbitrary propositional formulas. $\mathcal{I}$ *satisfies* a formula F and is called a *model* of F if $\|F\|_\mathcal{I} = 1$; F is *valid* if all interpretations are models. We identify a *logic* with its set of valid formulas.

Different choices of V in general induce different logics. The truth functions, above, imply that only the respective *order structure*, but not the particular arithmetic values of the truth values are relevant for validity or satisfiability. If $|V| = n$ $(n \geq 2)$ the set of valid formulas is called the n-valued Gödel logic. Obviously, two-valued Gödel logic is classical logic. At the propositional level there is only one infinite-valued Gödel logic $\mathbf{G}_\infty$, which is also the intersection of all finite-valued Gödel logics. Dummett [7] has shown that $\mathbf{G}_\infty$ can be axiomatized by adding the *linearity axiom*

$$(A \to B) \vee (B \to A) \tag{1}$$

to any Hilbert-style system for intuitionistic logic. Therefore $\mathbf{G}_\infty$ is sometimes also called Gödel-Dummett logic or Dummett's **LC**. More recently $\mathbf{G}_\infty$ emerged as one of the main formalizations of *fuzzy logics* (see, e.g., [11]). In this context it is very useful to enrich the logics by adding the unary operator $\triangle$ with the following meaning [1]:

$$\|\triangle A\|_\mathcal{I} = \begin{cases} 1 & \text{if } \|A\|_\mathcal{I} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The situation for infinite sets of truth values gets more interesting at the first-order level. We introduce predicates and quantifiers as follows. Instead of being propositional variables, atomic formulas are now of the form $P(t_1, \ldots, t_n)$, where P is a predicate symbol and $t_1, \ldots, t_n$ are terms, where a term, here, is either an (object) variable or a constant symbol. An interpretation $\mathcal{I}$ consists of a non-empty *domain* D and a *signature interpretation* $v_\mathcal{I}$ that maps constant symbols and object variables to elements of D. Moreover, $v_\mathcal{I}$ maps every n-ary predicate symbol P to a function from D^n into V. The truth value of an atomic formula $P(t_1, \ldots, t_n)$ is thus defined as

$$\|P(t_1, \ldots, t_n)\|_\mathcal{I} = v_\mathcal{I}(P)(v_\mathcal{I}(t_1), \ldots, v_\mathcal{I}(t_n)).$$

To fix the meaning of quantifiers we define the *distribution* of a formula A with respect to a free variable x in an interpretation $\mathcal{I}$ as $\mathrm{distr}_\mathcal{I}(A(x)) = \{\|A(x)\|_{\mathcal{I}'} \mid \mathcal{I}' \sim_x \mathcal{I}\}$, where $\mathcal{I}' \sim_x \mathcal{I}$ means that $\mathcal{I}'$ is exactly as $\mathcal{I}$ with the possible exception of the domain element assigned to x. The quantifiers correspond to the infimum and supremum, respectively, in the following sense:

$$\|(\forall x)A(x)\|_{\mathcal{I}} = \inf \mathrm{distr}_{\mathcal{I}}(A(x)) \qquad \|(\exists x)A(x)\|_{\mathcal{I}} = \sup \mathrm{distr}_{\mathcal{I}}(A(x)).$$

Note that the above definition of an interpretation as a pair $(D, v_{\mathcal{I}})$ covers also classical logic. However, to enhance clarity, we will use (in Section 4, below) $\bot$ and $\top$ instead of 0 and 1, respectively, for the classical truth values.

In the following we investigate (fragments of) first-order Gödel logics, with and without the operator $\triangle$. Every truth value set V, $\{0,1\} \subseteq V \subseteq [0,1]$, that is closed under suprema and infima induces a first-order logic $\mathbf{G}_V$ over the language without $\triangle$ and a logic $\mathbf{G}_V^{\triangle}$ if $\triangle$ is present. *Standard Gödel logic* is $\mathbf{G}_{[0,1]}$; i.e., the logic over the full real unit interval as truth value set, see, e.g., [11, 15]. We use $\mathbf{G}_n$ to denote the n-valued first-order Gödel logic for $n \geq 2$. $\mathbf{G}_\uparrow$ results from taking $V = \{1\} \cup \{1 - \frac{1}{k} \mid k \geq 1\}$ (or any other order isomorphic truth value set); $\mathbf{G}_\downarrow$ arises from $V = \{0\} \cup \{\frac{1}{k} \mid k \geq 1\}$.

Like in intuitionistic logic, also in Gödel logics (with or without $\triangle$) quantifiers cannot be shifted arbitrarily. In other words, arbitrary formulas are not equivalent to prenex formulas, in general. However, we have the following (stated in [4] without proof):

Proposition 1. *The following quantifier shift laws, where x is not free in B, and where Q denotes either $\exists$ or $\forall$ (uniformly over a formula) are valid in all Gödel logics:*

$$(\mathsf{Q}x)(A \wedge B) \leftrightarrow ((\mathsf{Q}x)A \wedge B) \tag{2}$$
$$(\mathsf{Q}x)(A \vee B) \leftrightarrow ((\mathsf{Q}x)A \vee B) \tag{3}$$
$$(\exists x)(A \to B) \to ((\forall x)A \to B) \tag{4}$$
$$(\exists x)(B \to A) \to (B \to (\exists x)A) \tag{5}$$
$$(\forall x)(A \to B) \leftrightarrow ((\exists x)A \to B) \tag{6}$$
$$(\forall x)(B \to A) \leftrightarrow (B \to (\forall x)A) \tag{7}$$

Proof. Given the truth functions for quantifiers, presented above, it suffices to note that, for all sets of reals A and all reals b the following statements hold.

- Corresponding to (2): $\inf\{\min(a,b) \mid a \in A\} = \min(\inf A, b)$ and $\sup\{\min(a,b) \mid a \in A\} = \min(\sup A, b)$.
- Corresponding to (3): $\inf\{\max(a,b) \mid a \in A\} = \max(\inf A, b)$ and $\sup\{\max(a,b) \mid a \in A\} = \max(\sup A, b)$.
- Corresponding to (4): If $a \leq b$ for some $a \in A$, then $\inf A \leq b$.
- Corresponding to (5): If $b \leq a$ for some $a \in A$, then $b \leq \sup A$.
- Corresponding to (6): $a \leq b$ for all $a \in A$ iff $\sup A \leq b$.
- Corresponding to (7): $b \leq a$ for all $a \in A$ iff $b \leq \inf A$.

(In fact almost all of these schemes are already intuitionistically valid.) $\square$

Note that the schemes that are dual to (4) and (5) are not valid in general (but are valid in $\mathbf{G}_\uparrow$ and $\mathbf{G}_n$, $n \geq 2$; see Proposition 3). Counterexamples are readily obtained for standard Gödel logic $\mathbf{G}_{[0,1]}$.

To emphasize that different sets of valid formulas result from different V, in general, consider the following formula schemes:

$$(\exists x)(A(x) \to (\forall x)A(x)) \tag{8}$$

$$(\exists x)((\exists y)A(y) \to A(x)) \tag{9}$$

Any instance of (8) is satisfied in an interpretation $\mathcal{I}$ if and only if the infimum of $\mathrm{distr}_{\mathcal{I}}(A(x))$ is a minimum, i.e., an element of $\mathrm{distr}_{\mathcal{I}}(A(x))$. Therefore (8) is valid in $\mathbf{G}_\uparrow$ and in any $\mathbf{G}_n$, but not, e.g., in $\mathbf{G}_{[0,1]}$ or in $\mathbf{G}_\downarrow$. Similarly (9) expresses that every supremum of a distribution is a maximum, with the possible exception of the value 1. Therefore (9) is valid in $\mathbf{G}_\downarrow$, in $\mathbf{G}_\uparrow$, and in all $\mathbf{G}_n$ for $n \geq 2$, but not, e.g., in $\mathbf{G}_{[0,1]}$. In fact there are infinitely many different infinite-valued first-order Gödel logics, according to [4]. The conjecture that there are just countable many different Gödel logics has recently been settled in [5]. $\mathbf{G}_{[0,1]}$ and $\mathbf{G}^{\triangle}_{[0,1]}$ are well known to be recursively axiomatizable, see, e.g., [11]. In contrast, $\mathbf{G}_\uparrow$ and $\mathbf{G}_\downarrow$ are not recursively axiomatizable, see [3, 4].

The fact that $\mathbf{G}_\uparrow = \bigcap_{n\geq 2} \mathbf{G}_n$ also holds at the first-order level, see [4]. However, this is no longer the case if we add the projection operator $\triangle$. In the enriched language, the intersection of all finite-valued Gödel logics is not a Gödel logic:

Proposition 2. $\mathbf{G}^{\triangle}_V \neq \bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$ *for every* V.

Proof. Since $\mathbf{G}^{\triangle}_n$ is a proper subset of $\mathbf{G}^{\triangle}_m$ whenever $n > m$, $\bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$ cannot coincide with any finite-valued Gödel logic. To show that $\bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$ also cannot be an infinite-valued Gödel logic, consider the formula

$$\triangle(\exists x)A(x) \to (\exists x)\triangle A(x). \tag{10}$$

It is valid in all finite-valued Gödel logics and therefore also in $\bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$. But not every interpretation $\mathcal{I}$ for $\mathbf{G}^{\triangle}_\uparrow$ satisfies all instances of (10). Take, e.g., the positive integers as domain of $\mathcal{I}$ and let $v_{\mathcal{I}}(P)(n) = 1 - \frac{1}{n}$ for some predicate symbol P. We obtain $\|(\exists x)P(x)\|_{\mathcal{I}} = \|\triangle(\exists x)P(x)\|_{\mathcal{I}} = 1$ and $\|\triangle P(x)\|_{\mathcal{I}} = \|(\exists x)\triangle P(x)\|_{\mathcal{I}} = 0$. Consequently, $\mathbf{G}^{\triangle}_\uparrow \neq \bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$. On the other hand, $\mathbf{G}_\uparrow = \bigcap_{n\geq 2} \mathbf{G}_n \subset \bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$ and therefore all instances of schemes (8) and (9) are in $\bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$. As noted above, this implies that in all interpretations of $\bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$ every infimum of a distribution is a minimum and every supremum of a distribution is either 1 or a maximum. In other words: if $\bigcap_{n\geq 2} \mathbf{G}^{\triangle}_n$ were identical with some $\mathbf{G}_V$, then its set V of truth values could not contain any accumulation point except 1. But all infinite subsets of $[0, 1]$ containing 0 and 1, that satisfy this property are order isomorphic to $\{1\} \cup \{1 - \frac{1}{k} \mid k \geq 1\}$, which is the case that we have excluded above. □

Remark 1. Note that we only had to refer to a unary predicate symbol in the above proof. I.e., Proposition 2 holds already for the monadic fragments.

3 Decidability of all finite-valued monadic Gödel logics

From now on, we will restrict our attention to *monadic* Gödel logics, i.e., all predicate symbols are unary. $\mathbf{G}_2$ is classical logic and therefore, as is well known, monadic $\mathbf{G}_2$ is decidable, whereas already a single binary predicate symbol leads to undecidability. It is straightforward to generalize this classic result to all finite-valued logics.

Theorem 1. *Monadic $\mathbf{G}_n^{\triangle}$ is decidable for all $n \geq 2$.*

Proof. Let A be any monadic formula that is not valid in $\mathbf{G}_n^{\triangle}$. Hence, there exists an interpretation $\mathcal{I}$ based on the set of truth values $V = \{\frac{j}{n-1} \mid 0 \leq j \leq n-1\}$ such that $\|A\|_{\mathcal{I}} < 1$. Let $\{P_1, \ldots, P_k\}$ be the set of different predicate symbols occurring in A. $\mathcal{I}$ induces the following equivalence relation $\equiv_{\mathcal{I}}$ on the domain D of $\mathcal{I}$:

$$c \equiv_{\mathcal{I}} d \Longleftrightarrow_{df} v_{\mathcal{I}}(P_i)(c) = v_{\mathcal{I}}(P_i)(d) \text{ for all } i \in \{1, \ldots, k\}.$$

Note that $c \equiv_{\mathcal{I}} d$ expresses that the domain elements c and d are indistinguishable with respect to the interpretation $\mathcal{I}$. Let $[c]_{\mathcal{I}}$ denote the equivalence class of the element $c \in D$, induced by $\equiv_{\mathcal{I}}$. We define a new interpretation $\mathcal{I}'$ with domain $D' = \{[c]_{\mathcal{I}} \mid c \in D\}$. D' is finite, since according to the definition of $\equiv_{\mathcal{I}}$ there can be at most n^k elements that are pairwise inequivalent. Let $v_{\mathcal{I}'}(P_i)([c]_{\mathcal{I}}) = v_{\mathcal{I}}(P_i)(c)$ for $i \in \{1, \ldots, k\}$. It is straightforward to check that $\mathcal{I}'$ is well-defined and that $\|A\|_{\mathcal{I}'} = \|A\|_{\mathcal{I}}$. This means that A is valid in $\mathbf{G}_n^{\triangle}$ iff it is satisfied in all interpretations with domain $\{1, \ldots, n^k\}$. Since there are at most $n^{k \cdot n^k}$ different such interpretations, and since evaluation of formulas over finite domains is computable, we have proved the decidability of $\mathbf{G}_n^{\triangle}$. □

Remark 2. Clearly, the 'filtration argument' of the above proof applies to the monadic fragments of arbitrary finite-valued logics, not just of Gödel logics. (In fact, the proof is probably 'folklore'. To render the paper self contained, and since there seems to be no appropriate reference in the literature, we decided to include it here.)

Remark 3. It is well known that the bound n^k for the cardinality of relevant model domains is optimal in the case $n = 2$, i.e., for classical logic. Better bounds might be achievable in general; however all such bounds seem to depend on the number of truth values n and are exponential in the number of different predicate symbols k. We show in Section 5 that much better bounds can be achieved for an interesting, non-trivial sub-case of the monadic fragments.

4 Undecidability of infinite-valued Gödel logics

We prove the undecidability of each Gödel logic $\mathbf{G}_V$, where the set V of truth values contains infinitely many values below some value that is distinct from 1. Our proof adapts and generalizes the undecidability proof sketched in [8] for monadic '**LC** with constant domains', which coincides with monadic $\mathbf{G}_{[0,1]}$. With the notable exception of $\mathbf{G}_{\uparrow}$, all infinite-valued Gödel logics satisfy the above condition on V, see Corollary 1.

We will also consider infinite-valued Gödel logics extended with the projection operator $\triangle$. Monadic prenex $\mathbf{G}^{\triangle}_{[0,1]}$ was shown to be undecidable in [2]. This result is generalized below, where we show that in fact for *all* infinite V, monadic $\mathbf{G}^{\triangle}_{V}$ is undecidable, even when restricted to prenex formulas.

Theorem 2. *Let $\mathbf{G}_V$ be a Gödel logic, where the set V of truth values satisfies the condition: $\exists p \in V, p < 1$, such that $V_p = \{y \in V \mid y \leq p\}$ is infinite. Validity in $\mathbf{G}_V$ is undecidable for monadic formulas.*

Proof. The classical theory **CE** of two equivalence relations $\equiv_1$ and $\equiv_2$ was shown to be undecidable in [14]. Let $\mathbf{G}_V$ be any Gödel logic, where V satisfies the condition: $\exists p \in V, p < 1$, such that $V_p = \{y \in V \mid y \leq p\}$ is infinite. We faithfully interpret **CE** in the monadic fragment of $\mathbf{G}_V$. The idea is to translate formulas of the form $x \equiv_i y$ into formulas $P_i(x) \leftrightarrow P_i(y)$, $i = 1, 2$, of the monadic fragment of $\mathbf{G}_V$, where P_1 and P_2 are different unary predicate symbols. Without loss of generality, we can assume formulas in **CE** to be in prenex normal form. Let S be the following formula of this kind:

$$\mathsf{Q}^* \bigwedge(\bigwedge_j x_j \equiv y_j \to \bigvee_k u_k \equiv v_k),$$

where each occurrence of $\equiv$ is either $\equiv_1$ or $\equiv_2$, and where Q^* is a string $(\mathsf{Q}_1\mathsf{z}_1)\ldots(\mathsf{Q}_n\mathsf{z}_n)$ of n quantifier occurrences. I.e., for all $i = 1, \ldots, n$, $\mathsf{Q}_i \in \{\forall, \exists\}$, and z_i denotes some variable. Let $S^\sharp$ be the following monadic formula:

$$\mathsf{Q}^* \bigwedge(\bigwedge_j (P(x_j) \leftrightarrow P(y_j)) \to [(\bigvee_k P(u_k) \leftrightarrow P(v_k)) \vee (\exists x)P_1(x) \vee (\exists x)P_2(x)]),$$

where P is P_1 or P_2, according to whether $\equiv$ is $\equiv_1$ or $\equiv_2$. We show that S is valid in **CE** if and only if $S^\sharp$ is valid in $\mathbf{G}_V$.

Let $\mathcal{M} = (D, v_{\mathcal{M}})$ be an interpretation of **CE**. By the Löwenheim-Skolem theorem we can assume D to be countable without loss of generality. We define a corresponding interpretation $\mathcal{I}(\mathcal{M}) = (D, v_{\mathcal{I}(\mathcal{M})})$ of $\mathbf{G}_{\uparrow}$ as follows. We set $v_{\mathcal{I}(\mathcal{M})}(z) = v_{\mathcal{M}}(z)$ for all variables z. (It suffices to work in a language without constant symbols.) Let us use $\equiv_i^{\mathcal{M}}$ to denote the equivalence relation $v_{\mathcal{M}}(\equiv_i)$. Note that $\equiv_i^{\mathcal{M}}$ induces a partition of the domain D into equivalence classes $E_i^c = \{d \mid d \equiv_i^{\mathcal{M}} c\}$, where $c \in D$ ($i \in \{1, 2\}$). Since $V_p = \{y \in V \mid y \leq p\}$ is infinite and D is countable, we can take some subset $W = \{w_0, w_1, \ldots\}$ of V_p as the set of (unique) indices in an enumeration $E_i^{w_0}, E_i^{w_1}, \ldots$ without repetitions of all such equivalence classes. (This enumeration is assumed to be the same for all interpretations that only differ in their variable assignments.) Referring to this enumeration of equivalence classes, we can define $v_{\mathcal{I}(\mathcal{M})}$ by

$$v_{\mathcal{I}(\mathcal{M})}(P_i)(d) = w_k \text{ if and only if } d \in E_i^{w_k},$$

where $e, d \in D$ and $i = 1, 2$.

Moreover, for each interpretation $\mathcal{I} = (D, v_\mathcal{I})$ of $\mathbf{G}_V$ we define the interpretation $\mathcal{M}(\mathcal{I}) = (D, v_{\mathcal{M}(\mathcal{I})})$ of **CE** by

$$v_{\mathcal{M}(\mathcal{I})}(\equiv_i)(d, e) = \top \text{ if and only if } v_\mathcal{I}(P_i)(d) = v_\mathcal{I}(P_i)(e).$$

for all $d, e \in D$ and $i = 1, 2$.

We prove the following claims about $\mathcal{I}(\mathcal{M})$ and $\mathcal{M}(\mathcal{I})$ by induction on the number n of quantifier occurrences in S and $S^\sharp$.

($\Rightarrow$) For every interpretation $\mathcal{M} = (D, v_\mathcal{M})$ of **CE**, where $\|S\|_\mathcal{M} = \bot$, we have $\|S^\sharp\|_{\mathcal{I}(\mathcal{M})} \leq p$.

($\Leftarrow$) For every interpretation $\mathcal{I}$ of $\mathbf{G}_V$, where $\|S^\sharp\|_\mathcal{I} < 1$, we have $\|S\|_{\mathcal{M}(\mathcal{I})} = \bot$.

Base case: $n = 0$ (i.e., there are no quantifiers).

($\Rightarrow$) Let $\|S\|_\mathcal{M} = \bot$ for some interpretation $\mathcal{M} = (D, v_\mathcal{M})$ of **CE**. By definition of $\mathcal{I}(\mathcal{M})$, we have $\|P_i(x) \leftrightarrow P_i(y)\|_{\mathcal{I}(\mathcal{M})} = 1$ if and only if $\|x \equiv_i y\|_\mathcal{M} = \top$. The exhibited conjunct of S is evaluated to $\bot$ in $\mathcal{M}$ if and only if $\| \bigwedge_j x_j \equiv y_j \|_\mathcal{M} = \top$ and $\| \bigvee_k u_k \equiv v_k \|_\mathcal{M} = \bot$. This, in turn, implies $\| \bigwedge_j P(x_j) \leftrightarrow P(y_j) \|_{\mathcal{I}(\mathcal{M})} = 1$ and $\| \bigvee_k P(u_k) \leftrightarrow P(v_k) \|_{\mathcal{I}(\mathcal{M})} = \max_k \min(\|P(u_k)\|_{\mathcal{I}(\mathcal{M})}, \|P(v_k)\|_{\mathcal{I}(\mathcal{M})}) \leq p$. Since $\|(\exists x)P_1(x) \vee (\exists x)P_2(x)\|_{\mathcal{I}(\mathcal{M})} \leq \sup(V_p) = p$, we obtain $\|S^\sharp\|_{\mathcal{I}(\mathcal{M})} \leq p$.

($\Leftarrow$) Let $\mathcal{I}$ be an interpretation of $\mathbf{G}_V$, such that $\|S^\sharp\|_\mathcal{I} < 1$. Then, for some conjunct of $S^\sharp$ (which without loss of generality we identify with the exhibited one) we have

$$\|(\bigvee_k P(u_k) \leftrightarrow P(v_k)) \vee (\exists x)P_1(x) \vee (\exists x)P_2(x)\|_\mathcal{I} < \|\bigwedge_j P(x_j) \leftrightarrow P(y_j)\|_\mathcal{I}.$$

This implies $\|\bigwedge_j P(x_j) \leftrightarrow P(y_j)\|_\mathcal{I} = 1$, since $\|\bigwedge_j P(x_j) \leftrightarrow P(y_j)\|_\mathcal{I}$ is either 1 or not greater than $\sup\{v_\mathcal{I}(P_i)(d) \mid d \in D, i = 1, 2\} = \|(\exists x)P_1(x) \vee (\exists x)P_2(x)\|_\mathcal{I}$. By the definition of $\mathcal{M}(\mathcal{I})$ we have for all variables z, z': $\|z \equiv_i z'\|_{\mathcal{M}(\mathcal{I})} = \top$ if and only if $\|P_i(z) \leftrightarrow P_i(z')\|_\mathcal{I} = 1$ ($i = 1, 2$). Therefore $\| \bigwedge_j x_j \equiv y_j \|_{\mathcal{M}(\mathcal{I})} = \top$ and $\| \bigvee_k u_k \equiv v_k \|_{\mathcal{M}(\mathcal{I})} = \bot$. Hence $\|S\|_{\mathcal{M}(\mathcal{I})} = \bot$.

Inductive case: Assuming that the claims hold for S and $S^\sharp$, we have to show that they also hold for $S_1 = (\mathsf{Q}x)S$ and $S_1^\sharp = (\mathsf{Q}x)S^\sharp$, where $\mathsf{Q} \in \{\exists, \forall\}$ and x denotes any variable.

Let S_1 be $(\exists x)S$. ($\Rightarrow$) If $\|(\exists x)S\|_\mathcal{M} = \bot$ then $\|S\|_{\mathcal{M}[^d/_x]} = \bot$ for all $d \in D$, where $\mathcal{M}[^d/_x]$ denotes an interpretation that is like $\mathcal{M}$, except for assigning the domain element d to the variable x. By the induction hypothesis we have $\|S^\sharp\|_{\mathcal{I}(\mathcal{M}[^d/x])} \leq p$, where $\mathcal{I}(\mathcal{M}[^d/x])$ is the interpretation of $\mathbf{G}_V$ corresponding to $\mathcal{M}[^d/_x]$. By definition, the interpretations $\mathcal{I}(\mathcal{M}[^d/x])$ are identical for all $d \in D$, except for the element assigned to x, since we required the underlying enumeration of equivalence classes to be the same for all $\mathcal{I}(\mathcal{M}[^d/x])$. We thus obtain $\sup_{d \in D}(\|S^\sharp\|_{\mathcal{I}(\mathcal{M}[^d/x])}) = \|(\exists x)S^\sharp\|_{\mathcal{I}(\mathcal{M})} = \|S_1^\sharp\|_{\mathcal{I}(\mathcal{M})} \leq p$, as required.

Similarly for ($\Leftarrow$): If $\|S_1^\sharp\|_\mathcal{I} = \|(\exists x)S^\sharp\|_\mathcal{I} < 1$ then $\|S^\sharp\|_{\mathcal{I}[^d/x]} < 1$ for all $d \in D$, where $\mathcal{I}[^d/x]$ denotes an interpretation for $\mathbf{G}_V$ that is like $\mathcal{I}$, except for assigning the domain element d to the variable x. By the induction hypothesis we have $\|S\|_{\mathcal{M}(\mathcal{I}[^d/x])} = \bot$, where the interpretations $\mathcal{M}(\mathcal{I}[^d/x])$ are identical for all $d \in D$, except for the element assigned to x. We thus obtain $\|(\exists x)S\|_{\mathcal{M}(\mathcal{I})} = \bot$ as required.

The case $S_1 = (\forall x)S$ is analogous. □

Corollary 1. *All infinite-valued monadic Gödel logics, with the possible exception of* $\mathbf{G}_\uparrow$*, are undecidable.*

Proof. Let V be any infinite set of reals, such that $\{0,1\} \subseteq V \subseteq [0,1]$. Suppose V does not satisfy the condition of Theorem 2. Then V contains only finitely many different elements below any given $p < 1$ for $p \in V$. It is not difficult to see that all such V are order isomorphic to $\{1\} \cup \{1 - \frac{1}{n} \mid n \geq 1\}$; i.e., to the set of truth values of $\mathbf{G}_\uparrow$. □

Theorem 2 can be strengthened as follows, if we augment the language of our logics by the projection operator $\triangle$.

Theorem 3. *Validity of monadic formulas in* $\mathbf{G}_V^\triangle$*, where* V *is infinite, is undecidable. This already holds for prenex monadic formulas.*

Proof. Similarly to the proof of Theorem 2, above, we translate classical formulas of the form $x \equiv_i y$ into formulas $\triangle(P_i(x) \leftrightarrow P_i(y))$ $(i = 1,2)$. More exactly, let S be a formula of **CE**, like in the proof of Theorem 2. Let the corresponding formula $S^\sharp_\triangle$, to be interpreted in $\mathbf{G}_V^\triangle$, be

$$\mathsf{Q}^* \bigwedge (\bigwedge_j \triangle(P(x_j) \leftrightarrow P_i(y_j)) \to \bigvee_k \triangle(P(u_k) \leftrightarrow P_i(v_k)).$$

The proof that S is valid in **CE** if and only if $S^\sharp_\triangle$ is valid in $\mathbf{G}_V^\triangle$ is analogous to that of the corresponding claim in Theorem 2. However, in defining $\mathcal{I}(\mathcal{M})$ we may now take *any* subset of (the infinite set) V as the set of indices in the underlying enumeration of equivalence classes $E_i^{w_k}$. The reason for this is that, in any interpretation $\mathcal{I}$, $\|\triangle(P(x) \leftrightarrow P(y))\|_\mathcal{I} = 0$ if $v_\mathcal{I}(P)(v_\mathcal{I}(x)) \neq v_\mathcal{I}(P)(v_\mathcal{I}(y))$, and $\|\triangle(P(x) \leftrightarrow P(y))\|_\mathcal{I} = 1$ otherwise. Hence $S^\sharp_\triangle$ itself behaves like a classical formula, i.e., it always evaluates either to 0 or to 1 Consequently, it suffices that V is infinite to be able to encode different equivalence classes by different truth values in the required way.

Finally note that, in contrast to Theorem 2, $S^\sharp_\triangle$ is a prenex formula. □

5 Efficient decidability of untangled $\mathbf{G}_\uparrow$ and $\mathbf{G}_n$

As mentioned in the introduction, application oriented investigations draw our attention to monadic formulas that exhibit a restricted form of overlap between scopes of different quantifier occurrences. We propose to view quantifier scopes as being *entangled* in general, but *untangled* in the following case.

Definition 1. *A closed monadic formula F is called* untangled *if every subformula of F contains at most one free variable.*

Example 1. $(\exists y)((\forall x)P(x) \to Q(y))$ and $(\forall y)((\exists z)((\forall x)P(x) \vee Q(z)) \to P(y))$ are untangled, but $(\exists y)(\forall x)(P(x) \to Q(y))$ and $(\exists x)(\exists y)(P(x) \wedge P(y))$ are not untangled.

The monadic fragment of classical logic was used in [12] to formalize the knowledge base of the medical expert system CADIAG-1, represented as (classical) IF-THEN rules. This formalization made it possible to prove the decidability of the consistency checking problem in CADIAG-1 and led to a simple algorithm to actually carry out such checks. An inspection of this application reveals that in fact only the untangled fragment of classical logic is needed for this purpose. In a many-valued context unary predicates are interpreted as *fuzzy sets*. This allows to formalize *fuzzy* IF-THEN rules in the untangled fragments of many-valued logics (including Gödel logics). Therefore (efficient) decision procedures for these fragments are of particular interest for fuzzy expert systems.

Remember from Proposition 1 that most quantifier shift laws are valid in all Gödel logics. For the decidability proof, below, we have to apply also the two remaining quantifier shift laws, that are not valid, e.g., in $\mathbf{G}_{[0,1]}$, but are valid in $\mathbf{G}_{\uparrow}$ and in $\mathbf{G}_n$.

Proposition 3. *The following schemes, where x is not free in B, are valid in $\mathbf{G}_{\uparrow}$ and in $\mathbf{G}_n$, for $n \geq 2$:*

$$((\forall x)A(x) \to B) \to (\exists x)(A(x) \to B) \tag{11}$$

$$(B \to (\exists x)A(x)) \to (\exists x)(B \to A(x)) \tag{12}$$

Proof. The underlying truth value set of $\mathbf{G}_{\uparrow}$ is $V = \{1\} \cup \{1 - \frac{1}{k} \mid k \geq 1\}$. Therefore, for every formula $A(x)$ and every interpretation $\mathcal{I}$ of $\mathbf{G}_{\uparrow}$ there exists an element e in the domain of $\mathcal{I}$ such that $\|A(x)\|_{\mathcal{I}[e/x]} = \inf \mathrm{distr}_{\mathcal{I}}(A(x))$, where $\mathcal{I}[e/x]$ is like $\mathcal{I}$ except (possibly) for assigning e to the variable x. Similarly, for every formula $A(x)$ and every interpretation $\mathcal{I}$ either $\|(\exists x)A(x)\|_{\mathcal{I}} = 1$ or there exists an element e in the domain of $\mathcal{I}$ such that $\|A(x)\|_{\mathcal{I}[e/x]} = \sup \mathrm{distr}_{\mathcal{I}}(A(x))$. The validity of (11) and (12) follows directly from these observations. □

Definition 2. *We define* contexts *('formulas with a place holder') inductively as follows (remember that $\neg$ and $\leftrightarrow$ are derived connectives):*

- *the empty context $[\cdot]$ is positive;*
- *if C is a positive context and F is a formula then $(C \vee F)$, $(F \vee C)$, $(C \wedge F)$, $(F \wedge C)$, $(F \to C)$, $(\forall x)C$, and $(\exists x)C$ are positive contexts, but $(C \to F)$ is a negative context;*
- *if C is a negative context and F is a formula then $(C \vee F)$, $(F \vee C)$, $(C \wedge F)$, $(F \wedge C)$, $(F \to C)$, $(\forall x)C$, and $(\exists x)C$ are negative contexts, but $(C \to F)$ is a positive context.*

The formula resulting from substituting the place holder $[\cdot]$ *in context* C *by formula* A *is denoted by* $C[A]$*. We use* $C[A]^+$ *to indicate that the exhibited* occurrence *of the subformula* A *in the formula* $C[A]$ *is* positive*, meaning that* C *is a positive context. Likewise,* $C[A]^-$ *indicates a* negative occurrence *of* A *in* $C[A]$*.*

In $C[(\exists x)A]^+$ *and in* $C[(\forall x)A]^-$ *the exhibited quantifier occurrence is called* weak*, while in* $C[(\exists x)A]^-$ *and in* $C[(\forall x)A]^+$ *it is called* strong.

Proposition 4. *In all Gödel logics without* $\triangle$ *the following principles hold:*

- *If* $A \to B$ *is valid, then also* $C[A]^+ \to C[B]^+$ *is valid;*
- *if* $A \to B$ *is valid, then also* $C[B]^- \to C[A]^-$ *is valid.*

Proof. By induction on the complexity of C.

The base case, where C is the empty context (and therefore positive) trivially holds.

We spell out two of the twelve different propositional cases of the induction step. The validity of $A \to B$ implies $\|A\|_{\mathcal{I}} \le \|B\|_{\mathcal{I}}$ for every interpretation $\mathcal{I}$. By the induction hypothesis we have $\|C[A]^+\|_{\mathcal{I}} \le \|C[B]^+\|_{\mathcal{I}}$. Thus, $\|F \wedge C[A]^+\|_{\mathcal{I}} = \min(\|F\|_{\mathcal{I}}, \|C[A]^+\|_{\mathcal{I}}) \le \min(\|F\|_{\mathcal{I}}, \|C[B]^+\|_{\mathcal{I}}) = \|F \wedge C[B]^+\|_{\mathcal{I}}$. I.e., $(F \wedge C[A]^+) \to (F \wedge C[B]^+)$ is valid.

For a context of the form $C[\cdot]^+ \to F$ we have to show that $(C[B]^+ \to F) \to (C[A]^+ \to F)$ is valid if $\|A\|_{\mathcal{I}} \le \|B\|_{\mathcal{I}}$ for all $\mathcal{I}$. (Remember that the occurrence of A is *negative* in $C^+[A] \to F$.) We distinguish two cases.

(a) $\|C[B]^+\|_{\mathcal{I}} \le \|F\|_{\mathcal{I}}$: By the induction hypothesis $\|C[A]^+\|_{\mathcal{I}} \le \|C[B]^+\|_{\mathcal{I}}$. Therefore also $\|C[A]^+\|_{\mathcal{I}} \le \|F\|_{\mathcal{I}}$ and consequently $\|C[A]^+ \to F\|_{\mathcal{I}} = 1$, which implies that $\|(C[B]^+ \to F) \to (C[A]^+ \to F)\|_{\mathcal{I}} = 1$.

(b) $\|C[B]^+\|_{\mathcal{I}} > \|F\|_{\mathcal{I}}$: this implies $\|C[B]^+ \to F\|_{\mathcal{I}} = \|F\|_{\mathcal{I}}$. If $\|C[A]^+\|_{\mathcal{I}} \le \|F\|_{\mathcal{I}}$ then $\|C[A]^+ \to F\|_{\mathcal{I}} = 1$. Otherwise $\|C[A]^+\|_{\mathcal{I}} > \|F\|_{\mathcal{I}}$ and consequently also $\|C[A]^+ \to F\|_{\mathcal{I}} = \|F\|_{\mathcal{I}}$. In both cases $\|C[B]^+ \to F\|_{\mathcal{I}} \le \|C[A]^+ \to F\|_{\mathcal{I}}$ and therefore, again, $\|(C[B]^+ \to F) \to (C[A]^+ \to F)\|_{\mathcal{I}} = 1$.

All other propositional cases are similar. The quantifier cases are straightforward, too. We just present the case for $(\forall x)C[\cdot]^-$. (The other cases are similar.) Assume that $A \to B$ is valid, i.e., $\|A\|_{\mathcal{I}} \le \|B\|_{\mathcal{I}}$ for all interpretations $\mathcal{I}$. By the induction hypothesis $\|C[B]^-\|_{\mathcal{I}} \le \|C[A]^-\|_{\mathcal{I}}$ for all $\mathcal{I}$. But this implies that $\inf \mathrm{distr}_{\mathcal{I}}(C[B(x)]^-) \le \inf \mathrm{distr}_{\mathcal{I}}(C[A(x)]^-)$ and therefore also $\|(\forall x)C[B]^-\|_{\mathcal{I}} \le \|(\forall x)C[A]^-\|_{\mathcal{I}}$ for all $\mathcal{I}$; i.e., $(\forall x)C[B]^- \to (\forall x)C[A]^-$ is valid, too. □

Theorem 4. *Untangled* $\mathbf{G}_\uparrow$ *is decidable.*

Proof. We first prove that untangled formulas in $\mathbf{G}_\uparrow$ remain valid if all strong quantifier occurrences are replaced by new constant symbols. To this aim it suffices to show that

- $C[(\forall x)A(x)]^+$ is valid if and only if $C[A(d)]^+$ is valid, and
- $C[(\exists x)A(x)]^-$ is valid if and only if $C[A(d)]^-$ is valid,

where d is a constant that does not occur in $C[A]$.

The 'only if' part of these claims follows directly from Proposition 4 and the validity of $(\forall x)A(x) \to A(d)$ and of $A(d) \to (\exists x)A(x)$, respectively.

For the 'if' part note that the validity of $F(d)$ implies the validity of $(\forall x)F(x)$ if d does not occur in F: any interpretation $\mathcal{I}$, where $\|(\forall x)F(x)\|_{\mathcal{I}} < 1$, can be extended to include the new d in such a way that $\|(\forall x)F(x)\|_{\mathcal{I}} = \|F(d)\|_{\mathcal{I}}$. Therefore the claims follow, if the following schemes are valid:

– $(\forall x)C[A(x)]^{+} \to C[(\forall x)A(x)]^{+}$ and
– $(\forall x)C[A(x)]^{-} \to C[(\exists x)A(x)]^{-}$,

where the only free occurrences of x in C are in $A(x)$ and $A(x)$ is not in the scope of any quantifier occurrence in $C[A(x)]$. The validity of these schemes is obtained by repeatedly applying the quantifier shift laws of Propositions 1 and 3 in combination with the context rules of Proposition 4. (This is possible only because the formulas are untangled; see Remark 4, below.)

So far, we have shown that every untangled formula F can be transformed (in linear time) into a formula F' that only contains weak quantifier occurrences, but is equivalent to F with respect to validity in $\mathbf{G}_{\uparrow}$ (and $\mathbf{G}_n$, see Remark 5, below). Let us call such a formula *weak*. To obtain a decision procedure, we finally prove that every weak formula G is valid if and only if it is satisfied by all interpretations, where the size of the domain is bounded by the number of constant symbols occurring in G.

Let $\mathcal{I}$ be an arbitrary interpretation and let $d_1, \ldots, d_n$ be the different constant symbols occurring in a weak formula G. Let $\mathcal{I}'$ be the interpretation that is obtained from $\mathcal{I}$ by removing from the domain D of $\mathcal{I}$ all elements except those $e \in D$, where $v_{\mathcal{I}}(d_i) = e$ for some $i \in \{1, \ldots, n\}$. It remains to check that $\|G\|_{\mathcal{I}} < 1$ implies $\|G\|_{\mathcal{I}'} < 1$. In other words: if there is a counter model for G, then there is already one with a restricted domain, as indicated. To this aim, note that the quantifier shift laws of Propositions 3 and 1 entail that every weak G is equivalent to a formula of the form $(\exists x)G'(x)$, where $G'(x)$ is weak, too. Obviously, $\mathrm{distr}_{\mathcal{I}'}(G'(x)) \subseteq \mathrm{distr}_{\mathcal{I}}(G'(x))$. Since $Y \subseteq X$ implies $\sup X \geq \sup Y$, we obtain $\|G\|_{\mathcal{I}} = \sup \mathrm{distr}_{\mathcal{I}}(G'(x)) \geq \sup \mathrm{distr}_{\mathcal{I}'}(G'(x)) = \|G'\|_{\mathcal{I}'}$. By repeating this argument for all (weak) quantifier occurrences, we obtain $\|G\|_{\mathcal{I}'} \leq \|G\|_{\mathcal{I}}$, as required.

Finally, remember that, in Gödel logics, it only depends on the relative order, but not on the absolute values of assigned truth values different from 0 and 1, whether a given interpretation satisfies a formula. This implies that the number of different interpretations with finite domain is bounded by the size of the domain and the number of relevant (unary) predicates symbols. Hence we have shown that validity for untangled $\mathbf{G}_{\uparrow}$ is decidable. □

Remark 4. Note that, in proving the validity of $(\forall x)C[A(x)]^{+} \to C[(\forall x)A(x)]^{+}$ and of $(\forall x)C[A(x)]^{-} \to C[(\exists x)A(x)]^{-}$, we had to shift quantifiers in and out (depending on the type of context). But those shifts are only over closed subformulas. This is where the defining condition for untangled formulas is used. In contrast, quantifiers cannot be moved into the scope of other quantifiers, in general. Indeed, e.g., $(\exists y)(\forall x)(P(y) \wedge Q(x))$ entails $(\exists y)(P(y) \wedge Q(d))$, which in turn entails $(\forall x)(\exists y)(P(y) \wedge Q(x))$. But the latter formula does not entail $(\exists y)(\forall x)(P(y) \wedge Q(x))$. Moreover, note that Proposition 3 only holds for $\mathbf{G}_\uparrow$ and for $\mathbf{G}_n$, $n \geq 2$, but not for other Gödel logics.

The following statement can be directly extracted from the proof of Theorem 4.

Corollary 2. *An untangled formula F is valid in $\mathbf{G}_\uparrow$ if and only if it is satisfied by all interpretations of domain size $m + c$, where m is the number of strong quantifier occurrences in F and c is the number of different constant symbols occurring in F.*

To see that the mentioned bound is tight consider a formula of the form

$$F_m = \bigwedge_{1 \leq i \leq m} (\exists x_i)(\diamond_i^1 P_1(x_i) \wedge \ldots \wedge \diamond_i^k P_k(x_i)),$$

where $\diamond_i^j$ is either $\neg$ or empty, and where the vectors $(\diamond_i^1, \ldots \diamond_i^k)$ and $(\diamond_j^1, \ldots \diamond_j^k)$ are different if $i \neq j$. Clearly, $\neg F_m$ is not valid, since F_m can be satisfied in an interpretation that assigns different elements to the m different variables. On the other hand, in any domain with less than m elements at least two conjuncts of the form $P_\ell(x_i)$ and $\neg P_\ell(x_j)$ cannot be satisfied simultaneously, which means that $\neg F_m$ is satisfied in all corresponding interpretations.

Note that our proof of Theorem 4 of implies that every untangled formula F can be translated into a propositional formula $\Pi(F)$, that is equivalent to F with respect to validity in $\mathbf{G}_\uparrow$. To this aim one replaces subformulas of F of the form $(\forall x)A(x)$ by $\bigwedge_{1 \leq i \leq m+c} A(d_i)$ and subformulas of the form $(\exists x)A(x)$ by $\bigvee_{1 \leq i \leq m+c} A(d_i)$, where each $d \in \{d_1, \ldots, d_{m+c}\}$ is either one of the c constant symbols that already occur in F or corresponds to one of the m strong quantifier occurrence in F.

It is well known that every propositional formula A is valid in $\mathbf{G}_\uparrow$ (which coincides with $\mathbf{G}_\infty$ in the propositional case) if and only if it is valid in $\mathbf{G}_{n+2}$, where n is the number of different propositional variables in A (see, e.g., [11]). Therefore one can reduce testing validity of untangled formulas in $\mathbf{G}_\uparrow$

Corollary 3. *An untangled formula F is valid in $\mathbf{G}_\uparrow$ if and only if its translation $\Pi(F)$ is valid in propositional $\mathbf{G}_{(m+c)\cdot k+2}$, where m is the number of strong quantifier occurrences, c is the number of different constant symbols occurring in F, and k the number of different predicate symbols in F.*

Testing validity is well known to be in co-NP for all finite-valued logics. Clearly, for every untangled F, the parameters m, c, and k are all linearly bounded by the size of F. Therefore Corollary 3 implies that testing validity for untangled formulas in $\mathbf{G}_\uparrow$ is in co-NP, as well. This should be contrasted with the fact that testing validity for arbitrary monadic formulas is NEXPTIME-hard already for classical logic, see [6].

Remark 5. Although we have stated Theorem 4 only for $\mathbf{G}_\uparrow$, it is clear from the proof that the untangled fragments of finite-valued logics $\mathbf{G}_n$ can be decided in the same manner. Of course, untangled $\mathbf{G}_n$ is only a subclass of monadic $\mathbf{G}_n$. However, the bounds mentioned in in Corollaries 2 and 3 do not depend on n or on the number of different predicate symbols. Therefore they are drastically better, in general, than the corresponding bounds for the unrestricted monadic fragments (cf. Remark 3 in Section 3).

6 Axiomatization of untangled $\mathbf{G}_\uparrow$

The decidability proof of Section 5 for the untangled fragment of $\mathbf{G}_\uparrow$ referred to the semantics of $\mathbf{G}_\uparrow$ at several places. However, a close inspection of the proof shows that in fact all formula schemes and rules that have been used are valid in *all* Gödel logics, except for the quantifier shift laws (11) and (12) of Proposition 3.

This observation is significant, because in [3] it has been proved that $\mathbf{G}_\uparrow$ is not recursively enumerable. In other words: (full) $\mathbf{G}_\uparrow$ cannot be recursively axiomatized. In contrast to this fact, we obtain an elegant Hilbert style axiom system that is sound and complete for untangled formulas in $\mathbf{G}_\uparrow$ from the proof of Theorem 4.

We rely on a well known axiom system for $\mathbf{G}_{[0,1]}$ (see, e.g., [11]). Remember that $\mathbf{G}_{[0,1]}$ is the intersection of all Gödel logics. The $\mathbf{G}_\uparrow$-specific laws (11) and (12) can already be derived in intuitionistic logic from the schemes (8) and (9). This leads to the following system for $\mathbf{G}_\uparrow$:

Intuitionistic axioms and rules: (any choice)
Linearity axiom:
$(A \to B) \vee (B \to A)$
General quantifier axiom (valid in $\mathbf{G}_{[0,1]}$, and thus in all $\mathbf{G}_V$):
$(\forall x)(A(x) \vee B) \to ((\forall x)A \vee B)$, where x is not free in B
Specific quantifier axioms (valid only in $\mathbf{G}_\uparrow$):
$(\exists x)(A(x) \to (\forall x)A(x))$
$(\exists x)((\exists y)A(y) \to A(x))$

According to [3], this system—like any other recursively presented proof system—cannot be complete for full $\mathbf{G}_\uparrow$. Nevertheless it is complete for *untangled* $\mathbf{G}_\uparrow$, since all laws that have been used in the proof of Theorem 4 can be derived.

7 Conclusion

We have investigated the decision problem for monadic fragments of Gödel logics. In presence of the projection operator Δ the emerging picture is clear and simple: validity is decidable for all *finite-valued* monadic Gödel logics, but is undecidable for all *infinite-valued* monadic Gödel logics. (The latter even holds for prenex formulas.) Without Δ all, but possible one, infinite-valued monadic

Gödel logics remain undecidable. (Obviously, the decidability result for finite-valued logics also carries over to the language without $\triangle$.) The missing case, $\mathbf{G}_\uparrow$, is an important and interesting logic, since it coincides with the intersection of all finite-valued logics. The (un)decidability of monadic $\mathbf{G}_\uparrow$ remains open. It is reminiscent of a long-standing open problem (see, e.g., [13]) that seems to be related: the (un)decidability of validity for monadic Łukasiewicz logic **Ł**.

Motivated by a potentially important application of many-valued logics, we have singled out a natural sub-case of monadic logic; namely, the set of *untangled* formulas. Validity in $\mathbf{G}_\uparrow$ for this fragment is shown to be decidable. In fact, efficient and tight bounds are readily extracted from our decidability proof. These bounds point to a considerable more efficient decision procedure for untangled formulas also in the case of finite-valued logics (compared to the standard decision method for the unrestricted monadic fragments). Moreover, since all quantifier shifts are valid in **Ł**, we conjecture that this decidability result can be transferred to the untangled fragment of **Ł** (and to similar logics as well).

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