Hypersequent Calculi for some Intermediate Logics with Bounded Kripke Models

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Abstract

In this paper we define cut-free hypersequent calculi for some intermediate logics semantically characterized by bounded Kripke models. In particular we consider the logics characterized by Kripke models of bounded width Bw_k, by Kripke models of bounded cardinality Bc_k and by linearly ordered Kripke models of bounded cardinality G_k. The latter family of logics coincides with finite-valued Gödel logics. Our calculi turn out to be very simple and natural. Indeed, for each family of logics (respectively, Bw_k, Bc_k and G_k), they are defined by adding just one structural rule to a common system, namely the hypersequent calculus for Intuitionistic Logic. This structural rule reflects in a natural way the characteristic semantical features of the corresponding logic.

1 Introduction

Kripke models provide a suitable semantical characterization of propositional intermediate logics (see [8]), that is, logics including the Intuitionistic one and included in Classical Logic.

In this paper we investigate the proof theory of some intermediate logics semantically characterized by *bounded Kripke models*. More precisely, for any $k \geq 1$, we will consider the logics whose semantics are given by:

1. The class of finite trees not containing $k+1$ pairwise incomparable nodes (for short, finite trees of *width* $\leq k$);

2. The class of trees containing at most k nodes;

3. The class of trees of *width* ≤ 1 and with at most k nodes.

In the first case one gets the so called logics of *bounded width* Kripke models, Bw_k. The second case yields the logics of *bounded cardinality* Kripke models, Bc_k. Finally, in the third case we get the intermediate logics of *linear orders with at most k-elements*. These logics turn out to coincide with $(k+1)$*-valued Gödel logics* G_k.

Hilbert style axiomatizations of Bw_k and Bc_k are respectively obtained extending the familiar axiomatization of Intuitionistic Logic by the axioms $\bigvee_{i=0}^{k}(p_i \to \bigvee_{j\neq i} p_j)$

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and $p_0 \vee (p_0 \rightarrow p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{k-1} \rightarrow p_k)$. Adding to Bc_k the linearity axiom $(p \rightarrow q) \vee (q \rightarrow p)$ one gets $(k+1)$-valued Gödel logic.

The logics considered here have found applications in different areas of computer science. Indeed, Bw_1 coincides with infinite-valued Gödel logic, which is one of the most important formalizations of fuzzy logic (see [15]). Bc_1 is Classical Logic. Bc_2, also referred to as Sm [8] or logic of "here and there", has been used to analyse inference in extended logic programming (see [18]).

In this paper we define cut-free calculi for all logics Bw_k, Bc_k and G_k, with $k \geq 1$. To the best of our knowledge no analytic calculi have been provided for the first two families of logics. Things are not so bad for G_k. Indeed there exist general methods to build up analytic calculi for every finite-valued logic (see, e.g., [20, 19, 7, 14]) or projective logic ([6]), and in particular for finite-valued Gödel logics. However, the resulting calculi are just a rewriting of the truth-tables of their associated connectives. Thus making the proof theory of any such logic a purely "ad hoc" construct. In particular these calculi hide all existing relationships between finite-valued Gödel logics and the other logics considered in this paper.

In this work we will provide *cut-free hypersequent calculi* for logics Bw_k, Bc_k and G_k. Hypersequent calculi are a natural generalization of ordinary sequent calculi and turn out to be very suitable for expressing disjunctive axioms in an analytic way (see [5] for an overview). Indeed a common feature of the above mentioned logics is the fact that their properties can be expressed in a disjunctive form.

Our calculi follow a standard pattern: all calculi belonging to the same family (respectively, Bw_k, Bc_k and G_k) are uniform and are simply obtained by adding just one structural rule to the hypersequent calculus for Intuitionistic Logic. This structural rule reflects in a natural way the characteristic property of the corresponding logic.

Since hypersequent calculi are closely related to the goal-oriented proof procedures introduced in [13], our calculi might help to define similar kinds of deduction methods for Bw_k, Bc_k and G_k.

2 Preliminaries

To make the paper self contained we recall some basic notions.

Intermediate Logics and Kripke Models

The set of propositional *well formed formulas* (*wff's* for short) is defined, as usual, starting from an enumerable set of propositional variables and using the logical constants $\bot$, $\wedge$, $\vee$, $\rightarrow$. We denote with p and q, possibly with indexes, propositional variables and with A, B,... arbitrary wff's. Moreover, we use $\neg A$ as an abbreviation for $A \rightarrow \bot$.

Int and Cl will denote respectively both an arbitrary calculus for propositional Intuitionistic and Classical Logic and the set of intuitionistically and classically valid wff's.

An *intermediate propositional logic* (see, e.g., [8]) is any set L of wff's satisfying the following conditions: (i) L is consistent; (ii) Int $\subseteq$ L; (iii) L is closed under modus ponens; (iv) L is closed under propositional substitution (where a *propositional substitution* is any function mapping every propositional variable to a wff). It is well known that, for any intermediate logic L, L $\subseteq$ Cl.

If $\mathcal{A}$ is a set of axiom-schemes and L is an intermediate logic, the notation $\mathrm{L} + \mathcal{A}$ will indicate both the deductive system closed under modus ponens and arbitrary substitutions obtained by adding to L the axiom-schemes of $\mathcal{A}$, and the set of theorems of such a deductive system.

A *(propositional) Kripke model* is a structure $\underline{K} = \langle P, \leq, \Vdash \rangle$, where $\langle P, \leq \rangle$ is a *poset* (partially ordered set), and $\Vdash$ (the *forcing relation*) is a binary relation between elements of P and atomic wff's such that, for any propositional variable p, $\alpha \Vdash p$ implies $\beta \Vdash p$ for every $\beta \in P$ such that $\alpha \leq \beta$. The forcing relation is extended to arbitrary wff's as follows:

1. $\alpha \nVdash \bot$;
2. $\alpha \Vdash B \wedge C$ iff $\alpha \Vdash B$ and $\alpha \Vdash C$;
3. $\alpha \Vdash B \vee C$ iff either $\alpha \Vdash B$ or $\alpha \Vdash C$;
4. $\alpha \Vdash B \rightarrow C$ iff, for any $\beta \in P$ such that $\alpha \leq \beta$, $\beta \Vdash B$ implies $\beta \Vdash C$.

We write $\alpha \nVdash A$ to mean that $\alpha \Vdash A$ does not hold. We remark that, according to the above interpretation, $\alpha \Vdash \neg A$ iff for every $\beta \in P$ such that $\alpha \leq \beta$ we have $\beta \nVdash A$. For a finite set of wff's Γ, we write $\alpha \Vdash \Gamma$ to mean that $\alpha \Vdash A$ holds for every $A \in \Gamma$. Finally, we say that a wff A *is valid in a Kripke model* $\underline{K}$ if $\alpha \Vdash A$ for all $\alpha \in P$.

It is easy to check that the forcing relation meets the monotonicity condition, that is, for each wff A, Kripke model $\underline{K} = \langle P, \leq, \Vdash \rangle$ and element α in $\underline{K}$, if $\alpha \Vdash A$ then $\beta \Vdash A$ for every $\beta \in P$ such that $\alpha \leq \beta$.

If $\mathcal{F}$ is a non empty class of posets, we call $\{\underline{K} = \langle P, \leq, \Vdash \rangle \mid \langle P, \leq \rangle \in \mathcal{F}\}$ the *class of Kripke models built on* $\mathcal{F}$. Let $\mathcal{L}(\mathcal{F})$ be the set of wff's valid in all the Kripke models built on $\mathcal{F}$. It is well known that, for every non empty class $\mathcal{F}$ of posets, $\mathcal{L}(\mathcal{F})$ is an intermediate logic (see, e.g., [8, 12]). We say that an intermediate logic L *is characterized by the class of posets* $\mathcal{F}$ if $\mathrm{L} = \mathcal{L}(\mathcal{F})$.

Hypersequent Calculi

Hypersequent calculi are a simple and natural generalization of ordinary Gentzen calculi, see e.g. [5] for an overview.

Definition 1 A *hypersequent* is an expression of the form

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$$

where, for all $i = 1, \ldots n$, $\Gamma_i \vdash \Delta_i$ is an ordinary sequent. $\Gamma_i \vdash \Delta_i$ is called a *component* of the hypersequent. We say that a hypersequent is *single-conclusion* if, for any $i = 1, \ldots, n$, Δ_i consists of at most one wff.

The intended meaning of the symbol | is disjunctive.

For the purposes of this paper it is convenient to treat sequents and hypersequents as multisets of wff's and multisets of sequents, respectively. Moreover we only deal with single-conclusion hypersequents.

Like in ordinary sequent calculi, in a hypersequent calculus there are initial hypersequents and rules. Typically, the former are identities, that is of the form $A \vdash A$, while the latter are divided into *logical* and *structural* rules. The logical ones are essentially the same as in sequent calculi, the only difference is the presence of dummy contexts, called *side hypersequents*. We will use the symbol G to denote a side hypersequent.

The structural rules are divided into *internal* and *external rules*. The internal rules deal with wff's within components. They are the same as in ordinary sequent calculi. The external rules manipulate whole components within a hypersequent. These are external weakening (EW) and external contraction (EC).

In Table 1 we present a hypersequent calculus for Int, we call it HInt. Clearly this calculus is redundant in the sense that, if $\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ldots \mid \Gamma_n \vdash \Delta_n$ is derivable, then for some i, $\Gamma_i \vdash \Delta_i$ is derivable too.

$$\frac{}{A \vdash A}\ (\mathrm{id}) \qquad \frac{G \mid \Gamma \vdash B \qquad G' \mid \Gamma, B \vdash A}{G \mid G' \mid \Gamma \vdash A}\ (\mathrm{CUT})$$

External Structural Rules

$$\frac{G}{G \mid H}\ (\mathrm{EW}) \qquad \frac{G \mid \Gamma \vdash A \mid \Gamma \vdash A}{G \mid \Gamma \vdash A}\ (\mathrm{EC})$$

Internal Structural Rules

$$\frac{G \mid \Gamma \vdash A}{G \mid \Gamma, B \vdash A}\ (\mathrm{IW}, \mathrm{l}) \qquad \frac{G \mid \Gamma \vdash}{G \mid \Gamma \vdash A}\ (\mathrm{IW}, \mathrm{r}) \qquad \frac{G \mid \Gamma, B, B \vdash A}{G \mid \Gamma, B \vdash A}\ (\mathrm{IC})$$

Logical Rules

$$\frac{G \mid \Gamma, A_i \vdash B}{G \mid \Gamma, A_1 \wedge A_2 \vdash B}\ (\wedge, \mathrm{l_i}) \text{ for } i = 1, 2 \qquad \frac{G \mid \Gamma \vdash A \qquad G' \mid \Gamma \vdash B}{G \mid G' \mid \Gamma \vdash A \wedge B}\ (\wedge, \mathrm{r})$$

$$\frac{G \mid \Gamma, B \vdash A \qquad G' \mid \Gamma, C \vdash A}{G \mid G' \mid \Gamma, B \vee C \vdash A}\ (\vee, \mathrm{l}) \qquad \frac{G \mid \Gamma \vdash A_i}{G \mid \Gamma \vdash A_1 \vee A_2}\ (\vee, \mathrm{r_i}) \text{ for } i = 1, 2$$

$$\frac{G \mid \Gamma \vdash A \qquad G' \mid \Gamma, B \vdash C}{G \mid G' \mid \Gamma, A \rightarrow B \vdash C}\ (\rightarrow, \mathrm{l}) \qquad \frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \rightarrow B}\ (\rightarrow, \mathrm{r})$$

$$\frac{G \mid \Gamma \vdash A}{G \mid \Gamma, \neg A \vdash}\ (\neg, \mathrm{l}) \qquad \frac{G \mid \Gamma, A \vdash}{G \mid \Gamma \vdash \neg A}\ (\neg, \mathrm{r})$$

Table 1: Hypersequent calculus HInt for Intuitionistic Logic

In hypersequent calculi it is possible to define new kind of structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi with respect to ordinary sequent calculi. Effective use of this kind of rules is given by the following examples, involving intermediate logics:

- A Hilbert-style axiomatization for **LQ logic**, also known as Jankov logic [16], is obtained by extending the axioms of Int with $\neg p \vee \neg\neg p$. Semantically, LQ is characterized by the class of all finite and rooted posets with a single final element. A cut-free calculus for this logic is defined by adding the following rule to the hypersequent calculus for Int ([10])

$$\frac{G \mid \Gamma, \Gamma' \vdash}{G \mid \Gamma \vdash \ \mid \Gamma' \vdash} \text{(lq)}$$

- **Gödel logic**, also known as Dummett's LC logic [11], is given by Int + $\{(p \to q) \vee (q \to p)\}$. This logic can be seen both as an intermediate logic and as a many-valued logic. Indeed, on the one hand, it is characterized by the class of all rooted linearly ordered Kripke models. On the other hand, its connectives can be interpreted as functions over the real interval $[0,1]$ as follows: $A \wedge B = \min\{A, B\}$, $A \vee B = \max\{A, B\}$, $\neg A = 1$ if $A = 0$ ($\neg A = 0$, otherwise), $A \to B = 1$ if $A \leq B$ ($A \to B = B$, otherwise).

  By adding to HInt the following rule, called Communication rule,

$$\frac{G \mid \Gamma, \Gamma' \vdash A \qquad G' \mid \Gamma_1, \Gamma'_1 \vdash A'}{G \mid G' \mid \Gamma, \Gamma_1 \vdash A \mid \Gamma', \Gamma'_1 \vdash A'} \text{(com)}$$

  one obtains a cut-free calculus for Gödel logic, see [3].

Further examples of hypersequent calculi, ranging from modal logics to many-valued logics, can be found, e.g., in [2, 4, 5, 10, 9].

As usual, we say that a sequent $\Gamma \vdash A$ *is valid in a Kripke model* $\underline{K}$ if, for any element α in $\underline{K}$, either $\alpha \Vdash\!\!\!/\, B$ for some $B \in \Gamma$, or $\alpha \Vdash A$.

This definition can be extended to hypersequents as follows: Given a class of posets $\mathcal{F}$, we say that a hypersequent H *is valid in* $\mathcal{F}$ if for any Kripke model $\underline{K}$ built on $\mathcal{F}$ at least one of its components is valid in $\underline{K}$.

3 Logics of Bounded Width Kripke Models

This section is devoted to investigate the intermediate logics Bw_k, with $k \geq 1$ (Bw stands for "bounded width"). These logics are characterized by the class $\mathcal{F}_{w \leq k}$ of posets of width $\leq k$, that is, not containing an antichain of cardinality greater than k. A Hilbert style axiomatization of Bw_k is as follows

$$\mathrm{Int} + \{\bigvee_{i=0}^{k} (p_i \to \bigvee_{j \neq i} p_j)\}$$

In particular Bw_1 coincides with (infinite-valued) Gödel logic.

For $k \geq 1$, the hypersequent calculus HBw_k is given by adding to the hypersequent calculus for Int of Table 1 the following rule

$$\frac{G_1 \mid \Gamma_0, \Gamma_1 \vdash A_0 \ldots G_k \mid \Gamma_0, \Gamma_k \vdash A_0 \ldots G_{k^2+1} \mid \Gamma_k, \Gamma_0 \vdash A_k \ldots G_{k^2+k} \mid \Gamma_k, \Gamma_{k-1} \vdash A_k}{G_1 \mid \ \ldots \ \mid G_{k^2+k} \mid \Gamma_0 \vdash A_0 \mid \ \ldots \ \mid \Gamma_k \vdash A_k} (\mathrm{Bw}_k)$$

Remark 2 The (Bw_1) rule coincides with the following simplification, suggested by Mints, of Avron's Communication rule (see [5, p.9])

$$\frac{G \mid \Gamma_0, \Gamma_1 \vdash A_0 \qquad G' \mid \Gamma_0, \Gamma_1 \vdash A_1}{G \mid G' \mid \Gamma_0 \vdash A_0 \mid \Gamma_1 \vdash A_1}$$

We show that HBw_k is sound and complete with respect to Bw_k.

Theorem 3 (Soundness) If a hypersequent H is derivable in HBw_k, then it is valid in $\mathcal{F}_{w\leq k}$.

Proof Since all the rules of the hypersequent calculus for Int are valid in $\mathcal{F}_{w\leq k}$, it remains to show that so is the (Bw_k) rule. By way of contradiction, let us suppose that the premises of the (Bw_k) rule are valid in $\mathcal{F}_{w\leq k}$, but the conclusion is not. Thus there exists a Kripke model $\underline{K}$ built on $\mathcal{F}_{w\leq k}$ together with $k+1$ elements $\alpha_0, \dots, \alpha_k$ such that $\alpha_i \Vdash \Gamma_i$ but $\alpha_i \nVdash A_i$, for $i = 0, \dots, k$. Let us consider two different indexes $0 \leq i, j \leq k$. Since $\alpha_j \Vdash \Gamma_j$ and $\Gamma_i, \Gamma_j \vdash A_i$ is valid in $\mathcal{F}_{w\leq k}$ we have $\alpha_j \not\leq \alpha_i$. Analogously, since also $\Gamma_j, \Gamma_i \vdash A_j$ is valid in $\mathcal{F}_{w\leq k}$, $\alpha_i \not\leq \alpha_j$. Hence, $\alpha_0, \dots, \alpha_k$ are pairwise incomparable, and this contradicts the hypothesis that $\underline{K}$ is built on $\mathcal{F}_{w\leq k}$.

Theorem 4 (Completeness) If a wff $A \in \mathcal{L}(\mathcal{F}_{w\leq k})$ then $\vdash A$ is derivable in HBw_k.

Proof We rely on the completeness of the axiomatization for Bw_k (proved, e.g., in [8]), and we show that all the axioms are derivable in HBw_k. Since the axioms of Int are provable in the HInt calculus, and HBw_k contains the cut rule, it suffices to show that the axiom characterizing the intermediate logic Bw_k is derivable in the HBw_k calculus. As a matter of fact we can write

$$\dfrac{\dfrac{\dfrac{p_1 \vdash p_1}{p_0, p_1 \vdash p_1}\,(\text{IW,l})}{p_0, p_1 \vdash \bigvee_{j\neq 0} p_j}\,(\vee,\text{r}_\text{i}) \quad \dots \quad \dfrac{\dfrac{p_k \vdash p_k}{p_0, p_k \vdash p_k}\,(\text{IW,l})}{p_0, p_k \vdash \bigvee_{j\neq 0} p_j}\,(\vee,\text{r}_\text{i}) \quad \dots \quad \dfrac{\dfrac{p_0 \vdash p_0}{p_k, p_0 \vdash p_0}\,(\text{IW,l})}{p_k, p_0 \vdash \bigvee_{j\neq k} p_j}\,(\vee,\text{r}_\text{i}) \quad \dots \quad \dfrac{\dfrac{p_{k-1} \vdash p_{k-1}}{p_k, p_{k-1} \vdash p_{k-1}}\,(\text{IW,l})}{p_k, p_{k-1} \vdash \bigvee_{j\neq k} p_j}\,(\vee,\text{r}_\text{i})}{\dfrac{\dfrac{\dfrac{p_0 \vdash \bigvee_{j\neq 0} p_j \mid \dots \mid p_k \vdash \bigvee_{j\neq k} p_j}{\vdash p_0 \to \bigvee_{j\neq 0} p_j \mid \dots \mid \vdash p_k \to \bigvee_{j\neq k} p_j}\,(\to,\text{r})}{\vdash \bigvee_{i=0}^{k} (p_i \to \bigvee_{j\neq i} p_j) \mid \dots \mid \vdash \bigvee_{i=0}^{k} (p_i \to \bigvee_{j\neq i} p_j)}\,(\vee,\text{r}_\text{i})}{\vdash \bigvee_{i=0}^{k} (p_i \to \bigvee_{j\neq i} p_j)}\,(\text{EC})}\,(\text{Bw}_\text{k})$$

Remark 5 By comparing their hypersequent calculi one can immediately see that $\text{Bw}_{k+1} \subseteq \text{Bw}_k$.

By the presence of (IC), to prove the cut-elimination theorem for HBw_k, we have to consider the multi-cut rule, that is:

$$\frac{G \mid \Gamma \vdash A \qquad G' \mid \Gamma', A^n \vdash B}{G \mid G' \mid \Gamma^n, \Gamma' \vdash B}\ (\text{mcut})$$

where A^n and Γ^n stand for $A, \ldots, A$ (n times) and $\Gamma, \ldots, \Gamma$ (n times), respectively.

It is easy to see that the cut rule is a particular case of the multi-cut rule. Conversely, each application of the multi-cut rule can be replaced by repeated applications of the cut rule.

Theorem 6 (Cut-elimination) If a hypersequent H is derivable in HBw_k then it is derivable in HBw_k without using the cut rule.

Proof Cut-elimination for hypersequent calculi works essentially in the same way as for the corresponding sequent calculi.

It is enough to show that if $\mathcal{P}$ is a proof in HBw_k of a hypersequent H' containing only one multi-cut rule which occurs as the last inference of $\mathcal{P}$, then H' is derivable in HBw_k without the multi-cut rule.

As noticed in [9], a simple way to make the inductive argument work in the presence of the (EC) rule is to consider the number of applications of this rule in a given derivation as an independent parameter. Let r be the number of the applications of the (EC) rule in the proofs of the premises of the multi-cut rule, c be the complexity of the multi-cut wff, and h be the sum of the length of the proofs of the premises of the multi-cut rule.

The proof will proceed by induction on lexicographically ordered triple (r, c, h).

We shall argue by cases according to which inference rule is being applied immediately before the application of the multi-cut rule:

1. either $G \mid \Gamma \vdash A$ or $G' \mid \Gamma', A^n \vdash B$ is an initial hypersequent;
2. either $G \mid \Gamma \vdash A$ or $G' \mid \Gamma', A^n \vdash B$ is obtained by a structural rule;
3. both $G \mid \Gamma \vdash A$ and $G' \mid \Gamma', A^n \vdash B$ are lower sequents of some logical rules such that the principal formulas of both rules are just the multi-cut wff;
4. either $G \mid \Gamma \vdash A$ or $G' \mid \Gamma', A^n \vdash B$ is a lower sequent of a logical rule whose principal formula is not the multi-cut wff.

We will give here a proof for some relevant cases, omitting the side hypersequents that are not involved in the derivation.

Suppose that the last inference in the proof of one premise of the multi-cut is the (EC) rule, and the proof ends as follows:

$$\frac{\dfrac{\Gamma \vdash A \mid \Gamma \vdash A}{\Gamma \vdash A}\,(\text{EC}) \qquad A^n, \Gamma' \vdash B}{\Gamma', \Gamma^n \vdash B}\ (\text{mcut})$$

Let r' be the number of applications of the (EC) rule in the above proof. This proof can be replaced by

$$\dfrac{\dfrac{\Gamma \vdash A \mid \Gamma \vdash A \quad A^n, \Gamma' \vdash B}{\Gamma^n, \Gamma' \vdash B \mid \Gamma \vdash A}\,{\scriptstyle(\text{mcut})} \qquad A^n, \Gamma' \vdash B}{\dfrac{\Gamma', \Gamma^n \vdash B \mid \Gamma', \Gamma^n \vdash B}{\Gamma', \Gamma^n \vdash B}\,{\scriptstyle(\text{EC})}}\,{\scriptstyle(\text{mcut})}$$

which contains two multi-cuts with $r' - 1$ applications of the (EC) rule. Then, all these multi-cuts can be eliminated by induction hypothesis.

We show how to eliminate a multi-cut involving the (Bw_k) rule. Suppose, for instance, that the proof $\mathcal{P}$ ends as follows:

$$\dfrac{\dfrac{\Gamma_0, \Gamma_1 \vdash A_0 \ldots \Gamma_0, \Gamma_k \vdash A_0 \ \ldots \ \Gamma_k, \Gamma_0 \vdash A_k \ldots \Gamma_k, \Gamma_{k-1} \vdash A_k}{\Gamma_0 \vdash A_0 \mid \ \ldots \ \mid \Gamma_k \vdash A_k}\,{\scriptstyle(\mathrm{Bw_k})} \qquad \Sigma, A_0^n \vdash B}{\Gamma_0^n, \Sigma \vdash B \mid \ \ldots \ \mid \Gamma_k \vdash A_k}\,{\scriptstyle(\text{mcut})}$$

This proof can be replaced by

$$\dfrac{\dfrac{\Gamma_0, \Gamma_1 \vdash A_0 \quad \Sigma, A_0^n \vdash B}{\Gamma_0^n, \Sigma, \Gamma_1 \vdash B}\,{\scriptstyle(\text{mcut})} \quad \ldots \quad \dfrac{\Gamma_0, \Gamma_k \vdash A_0 \quad \Sigma, A_0^n \vdash B}{\Gamma_0^n, \Sigma, \Gamma_k \vdash B}\,{\scriptstyle(\text{mcut})} \quad \ldots \Gamma_k, \Gamma_{k-1} \vdash A_k}{\Gamma_0^n, \Sigma \vdash B \mid \ \ldots \ \mid \Gamma_k \vdash A_k}\,{\scriptstyle(\mathrm{Bw_k})}$$

in which there are k multi-cuts having the same r and c as $\mathcal{P}$, while the sum of the length of the proofs of the premises is smaller than the one of the multi-cut in $\mathcal{P}$. Then these multi-cuts can be eliminated by induction hypothesis.

The cases involving the logical rules, are essentially treated as in the cut-elimination proof of the LJ sequent calculus for Int (see, e.g., [21]).

For instance, suppose that in both the premises of the multi-cut rule the last inference is the rule for $\to$. Let us also assume that the proof $\mathcal{P}$ ends as follows:

$$\dfrac{\dfrac{\Gamma, B \vdash C}{\Gamma \vdash B \to C}\,{\scriptstyle(\to,\text{r})} \qquad \dfrac{\Gamma', (B \to C)^n \vdash B \quad \Gamma', (B \to C)^n, C \vdash D}{\Gamma', B \to C, (B \to C)^n \vdash D}\,{\scriptstyle(\to,\text{l})}}{\Gamma', \Gamma^{n+1} \vdash D}\,{\scriptstyle(\text{mcut})}$$

This proof can be replaced by

$$\dfrac{\dfrac{\dfrac{\Gamma', (B \to C)^n, C \vdash D \quad \Gamma \vdash B \to C}{\Gamma', \Gamma^n, C \vdash D}\,{\scriptstyle(\text{mcut})} \qquad \dfrac{\dfrac{\Gamma', (B \to C)^n \vdash B \quad \Gamma \vdash B \to C}{\Gamma', \Gamma^n \vdash B}\,{\scriptstyle(\text{mcut})} \quad \Gamma, B \vdash C}{\Gamma', \Gamma^{n+1} \vdash C}\,{\scriptstyle(\text{mcut})}}{\Gamma', \Gamma', \Gamma^{2n+1} \vdash D}\,{\scriptstyle(\text{mcut})}}{\Gamma', \Gamma^{n+1} \vdash D}\,{\scriptstyle(\text{IC})}$$

in which four multi-cuts occur. In two of them, all numbers r and c are as in $\mathcal{P}$, while the sum of the length of the proofs of the premises is smaller than that of multi-cut in $\mathcal{P}$. In the two remaining multi-cuts, r is the same as in $\mathcal{P}$ and the complexity of the multi-cut wff's is strictly smaller than the one of the multi-cut in $\mathcal{P}$. Then all these multi-cuts can be eliminated by induction hypothesis.

4 Logics of Bounded Cardinality Kripke Models

In this section we define cut-free calculi for logics Bc_k, with $k \geq 1$ (Bc stands for "bounded cardinality"). These logics are characterized by the class $\mathcal{F}_{c\leq k}$ of rooted posets of cardinality $\leq k$, in symbols, $|P| \leq k$. A Hilbert style axiomatization of Bc_k is given by

$$\mathrm{Int} + \{p_0 \vee (p_0 \to p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{k-1} \to p_k)\}$$

Bc_1 and Bc_2 coincide with Cl and Sm ([8]), respectively. The latter, like LQ and Gödel logic, is one of the seven propositional logics satisfying Craig's Interpolation Theorem (see [17]). In [1] it was defined a cut-free sequent calculus for Sm by translating a suitable duplication-free tableau calculus originating from a semantical framework. However in this calculus the rules for connectives are quite complex and the subformula property does not hold.

For $k \geq 1$, the hypersequent calculus HBc_k is given by adding to the hypersequent calculus for Int the following rule

$$\frac{\ldots \quad G_{i,j} \mid \Gamma_i, \Gamma_j \vdash A_i \quad \ldots}{G_{0,1} \mid \ldots \mid G_{k-1,k} \mid \Gamma_0 \vdash A_0 \mid \ldots \mid \Gamma_{k-1} \vdash A_{k-1} \mid \Gamma_k \vdash} \; (\mathrm{Bc}_k)$$

for every i, j such that $0 \leq i \leq k-1$ and $i+1 \leq j \leq k$.

Remark 7 The (Bc_1) rule coincides with the (cl) rule introduced in [10]. Therefore HBc_1 is a (single-conclusion) hypersequent calculus for Cl.

We show that HBc_k is sound and complete with respect to Bc_k.

Theorem 8 (Soundness) If a hypersequent H is derivable in HBc_k, then it is valid in $\mathcal{F}_{c\leq k}$.

Proof Similar to the proof of Theorem 3. By way of contradiction, let us suppose that the premises of the (Bc_k) rule are valid in $\mathcal{F}_{c\leq k}$, but the conclusion is not. Thus there exists a Kripke model $\underline{K}$ built on $\mathcal{F}_{c\leq k}$ together with $k+1$ elements $\alpha_0, \ldots, \alpha_k$ such that $\alpha_h \Vdash \Gamma_h$ but $\alpha_h \nVdash A_h$ for $h = 0, \ldots, k-1$ and $\alpha_k \Vdash \Gamma_k$. Let the indexes i and j satisfy the conditions $0 \leq i \leq k-1$, $1 \leq j \leq k$ and $i \leq j$. Since $\Gamma_i, \Gamma_j \vdash A_i$ is valid in $\underline{K}$, and $\alpha_j \Vdash \Gamma_j$, by the monotonicity property of $\Vdash$ we get that $\alpha_j \not\leq \alpha_i$ and hence $\alpha_i \neq \alpha_j$. Then, $\alpha_0, \ldots, \alpha_k$ are distinct elements and $\underline{K}$ has at least $k+1$ elements, a contradiction.

Theorem 9 (Completeness) If a wff $A \in \mathcal{L}(\mathcal{F}_{c\leq k})$ then $\vdash A$ is derivable in HBc_k.

Proof As in the proof of Theorem 4, we only have to prove that the axiom characterizing the intermediate logic Bc_k is derivable in the HBc_k calculus. To this purpose let us write

$$
\dfrac{
  p_0 \vdash p_0 \quad \ldots \quad
  \dfrac{p_0 \vdash p_0}{p_0 \wedge \ldots \wedge p_{k-1} \vdash p_0}\,(\wedge,\mathrm{l_i})
  \quad \ldots \quad
  \dfrac{\dfrac{p_{k-1} \vdash p_{k-1}}{p_0 \wedge \ldots \wedge p_{k-2}, p_{k-1} \vdash p_{k-1}}\,(\mathrm{IW,l})}{p_0 \wedge \ldots \wedge p_{k-2}, p_0 \wedge \ldots \wedge p_{k-1} \vdash p_{k-1}}\,(\wedge,\mathrm{l_i})
}{
  \dfrac{
    \dfrac{
      \dfrac{\vdash p_0 \mid p_0 \vdash p_1 \mid \ldots \mid p_0 \wedge \ldots \wedge p_{k-1} \vdash}{\vdash p_0 \mid p_0 \vdash p_1 \mid \ldots \mid p_0 \wedge \ldots \wedge p_{k-1} \vdash p_k}\,(\mathrm{IW,r})
    }{\vdash p_0 \mid \;\vdash p_0 \to p_1 \mid \ldots \mid \;\vdash p_0 \wedge \ldots \wedge p_{k-1} \to p_k}\,(\to,\mathrm{r})
  }{
    \dfrac{\ldots \mid \;\vdash p_0 \vee (p_0 \to p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{k-1} \to p_k) \mid \ldots}{\vdash p_0 \vee (p_0 \to p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{k-1} \to p_k)}\,(\mathrm{EC})
  }\,(\vee,\mathrm{r_i})
}\,(\mathrm{Bc_k})
$$

Theorem 10 (Cut-elimination) If a hypersequent H is derivable in HBc_k then it is derivable in HBc_k without using the cut rule.

Proof The proof proceeds as in Theorem 6.

Remark 11 By comparing their hypersequent calculi it is not hard to see that, for each $k \geq 1$, both Bc_{k+1} and Bw_k are included in Bc_k.

As an example, we show how to prove the linearity axiom $(p \to q) \vee (q \to p)$ in the hypersequent calculus HBc_2 for Sm logic.

$$
\dfrac{
  \dfrac{\dfrac{\dfrac{p \vdash p}{p, q \vdash p}\,(\mathrm{IW,l})}{p \vdash q \to p}\,(\to,\mathrm{r})}{p \vdash (p \to q) \vee (q \to p)}\,(\vee,\mathrm{r_i})
  \quad
  \dfrac{\dfrac{\dfrac{q \vdash q}{q, p \vdash q}\,(\mathrm{IW,l})}{q \vdash p \to q}\,(\to,\mathrm{r})}{q \vdash (p \to q) \vee (q \to p)}\,(\vee,\mathrm{r_i})
  \quad
  \dfrac{q \vdash q}{q, p \vdash q}\,(\mathrm{IW,l})
}{
  \dfrac{
    \dfrac{
      \dfrac{\vdash (p \to q) \vee (q \to p) \mid p \vdash q \mid q \vdash}{\vdash (p \to q) \vee (q \to p) \mid p \vdash q \mid q \vdash p}\,(\mathrm{IW,r})
    }{\vdash (p \to q) \vee (q \to p) \mid \;\vdash p \to q \mid \;\vdash q \to p}\,(\to,\mathrm{r})
  }{
    \dfrac{\vdash (p \to q) \vee (q \to p) \mid \;\vdash (p \to q) \vee (q \to p) \mid \;\vdash (p \to q) \vee (q \to p)}{\vdash (p \to q) \vee (q \to p)}\,(\mathrm{EC})
  }\,(\vee,\mathrm{r_i})
}\,(\mathrm{Bc_2})
$$

5 Finite-Valued Gödel Logics

Gödel's family of finite-valued propositional logics G_k, with $k \geq 1$, was introduced in the context of an investigation aimed to understand Intuitionistic Logic. Accordingly, the connectives of these logics were defined as functions over the set $\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\}$, as already described in Section 2.

An alternative characterization of their semantics in terms of Kripke models is given by the class $\mathcal{F}_{k\text{-lo}}$ of linearly ordered posets with at most k-elements.
A Hilbert style axiomatization of G_k is

$$
\mathrm{Int} + \{(p \to q) \vee (q \to p), p_0 \vee (p_0 \to p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{k-1} \to p_k)\}
$$

In the literature one can find sequent calculi (or dually, tableau systems) for every finite-valued logic, and in particular for finite-valued Gödel logics, see e.g. [20, 19, 7, 14]. In these calculi the many-valued semantics is strongly built into the proof theory. Indeed in any n-valued logic, sequents (or tableaux) are split into n or $n-1$ parts, each corresponding, respectively, to a different truth-value of the logic ([20, 19, 7]), or to a set of truth-values ([14]). The rules for connectives are directly drawn from their truth-tables. Therefore these calculi are far from traditional calculi and, more importantly, their structure makes it impossible to relate the various many-valued systems via their corresponding calculi. In particular these calculi hide all existing relationships between finite-valued Gödel logics and the other logics considered in this paper.

In this section we define cut-free hypersequent calculi for finite-valued Gödel logics. These calculi are obtained by simply adding to the calculus for Int some suitable structural rules.

A first method to define a hypersequent calculus for G_k is to add both the rules (Bw_1) and (Bc_k) to the hypersequent calculus for Int of Table 1. From a semantical point of view, the (Bw_1) rule forces the states of the model to be linearly ordered, while the (Bc_k) rule establishes that these states can be at most k. Soundness and completeness of this calculus with respect to G_k directly follow from the results proved in the previous sections.

An alternative cut-free calculus for G_k can be defined replacing the (Bw_1) and (Bc_k) rules by the following single rule which semantically combine the meaning of (Bw_1) and (Bc_k)

$$\frac{G_1 \mid \Gamma_0, \Gamma_1 \vdash A_0 \quad G_2 \mid \Gamma_1, \Gamma_2 \vdash A_1 \quad \ldots \quad G_k \mid \Gamma_{k-1}, \Gamma_k \vdash A_{k-1}}{G_1 \mid \ldots \mid G_k \mid \Gamma_0 \vdash A_0 \mid \ldots \mid \Gamma_{k-1} \vdash A_{k-1} \mid \Gamma_k \vdash} \; (G_k)$$

For $k \geq 1$, let HG_k be the hypersequent calculus obtained by adding to HInt the (G_k) rule.

Remark 12 HG_1 and HG_2 are respectively a (single-conclusion) hypersequent calculus for Cl and a hypersequent calculus for Sm logic. Notice that HG_1 coincides with HBc_1, while HG_2 is simpler than HBc_2.

We show that HG_k is sound and complete with respect to $(k+1)$-valued Gödel logic.

Theorem 13 (Soundness) If a hypersequent H is derivable in HG_k, then it is valid in $\mathcal{F}_{k\text{-lo}}$.

Proof The proof proceeds as in Theorem 3. By way of contradiction, let us suppose that the premises of the (G_k) rule are valid in $\mathcal{F}_{k\text{-lo}}$, but the conclusion is not. Then there exists a Kripke model $\underline{K}$ built on $\mathcal{F}_{k\text{-lo}}$ together with elements $\alpha_0, \ldots, \alpha_k$ such that $\alpha_i \Vdash \Gamma_i$ but $\alpha_i \not\Vdash A_i$ for $i = 0, \ldots, k-1$ and $\alpha_k \Vdash \Gamma_k$. Let $0 \leq i \leq k-1$. Since $\alpha_{i+1} \Vdash \Gamma_{i+1}$ and $\Gamma_i, \Gamma_{i+1} \vdash A_i$ is valid, we must have $\alpha_{i+1} \not\leq \alpha_i$. Being $\underline{K}$ linearly ordered, we must have $\alpha_0 < \alpha_1 < \ldots < \alpha_k$. Therefore $\underline{K}$ is built on a linearly ordered poset with at least $k+1$ distinct elements, which is impossible.

Theorem 14 (Completeness) If a wff $A \in \mathcal{L}(\mathcal{F}_{k\text{-lo}})$ then $\vdash A$ is derivable in HG_k.

Proof It suffices to observe that both the (Bw_1) and the (Bc_k) rules are derivable in HG_k. This is trivial for the latter rule since its premises are strictly included in the premises of the (G_k) rule. Let k be an odd number. To derive the (Bw_1) rule in the HG_k calculus let us write

$$\dfrac{\dfrac{\dfrac{G \mid \Gamma_0, \Gamma_1 \vdash A_0 \qquad G' \mid \Gamma_0, \Gamma_1 \vdash A_1 \qquad \dots \qquad G \mid \Gamma_0, \Gamma_1 \vdash A_0}{G \mid G' \mid \dots \mid G \mid \Gamma_0 \vdash A_0 \mid \Gamma_1 \vdash A_1 \mid \dots \mid \Gamma_0 \vdash A_0 \mid \Gamma_1 \vdash}\ (\mathrm{G_k})}{G \mid G' \mid \dots \mid G \mid \Gamma_0 \vdash A_0 \mid \Gamma_1 \vdash A_1 \mid \dots \mid \Gamma_0 \vdash A_0 \mid \Gamma_1 \vdash A_1}\ (\mathrm{IW,r})}{G \mid G' \mid \Gamma_0 \vdash A_0 \mid \Gamma_1 \vdash A_1}\ (\mathrm{EC})$$

The derivation of the (Bw_1) rule for k being even is similar. Therefore the claim follows by Theorems 4 and 9.

Theorem 15 (Cut-elimination) If a hypersequent H is derivable in HG_k then it is derivable in HG_k without using the cut rule.

Proof Similar to the proof of Theorem 6.

Remark 16 By comparing their hypersequent calculi one can immediately see that $G_{k+1} \subseteq G_k$. Moreover it follows that for each $n \geq k$, both Bw_n and Bc_n are included in $(k+1)$-valued Gödel logic.

References

[1] A. Avellone, M. Ferrari, and P. Miglioli. Duplication-free tableau calculi and related cut-free sequent calculi for the interpolable propositional intermediate logics. *Logic Journal of the IGPL*, 7(4):447–480, 1999.

[2] A. Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52:939–951, 1987.

[3] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals for Mathematics and Artificial Intelligence*, 4:225–248, 1991.

[4] A. Avron. Natural 3-valued logics characterization and proof theory. *Journal of Symbolic Logic*, 56:276–294, 1991.

[5] A. Avron. The method of hypersequents in the proof theory of propositional nonclassical logics. In *Logic: from Foundations to Applications, European Logic Colloquium*, pages 1–32. Oxford Science Publications. Clarendon Press. Oxford, 1996.

[6] M. Baaz and C. Fermüller. Analytic calculi for projective logics. In *Proceedings of International Conference TABLEAUX'99, Analytic Tableaux and Related Methods*, volume 1617 of *LNAI*, pages 36–51. Springer-Verlag, 1999.

[7] W.A. Carnielli. On sequents and tableaux for many-valued logics. *Journal of Non-Classical Logic*, 8:59–76, 1991.

[8] A. Chagrov and M. Zakharyaschev. *Modal Logic.* Oxford University Press, 1997.

[9] A. Ciabattoni. Bounded contraction in systems with linearity. In *Proceedings of International Conference TABLEAUX'99, Analytic Tableaux and Related Methods*, volume 1617 of *LNAI*, pages 113–128. Springer-Verlag, 1999.

[10] A. Ciabattoni, D.M. Gabbay, and N. Olivetti. Cut-free proof systems for logics of weak excluded middle. *Soft Computing*, 2(4):147–156, 1998.

[11] M. Dummett. A propositional calculus with a denumerable matrix. *Journal of Symbolic Logic*, 24:96–107, 1959.

[12] M. Ferrari and P. Miglioli. A method to single out maximal propositional logics with the disjunction property I. *Annals of Pure and Applied Logic*, 76:1–46, 1995.

[13] D.M. Gabbay and N.Olivetti. *Goal-Directed Proof Theory.* Kluwer, 2000.

[14] R. Hähnle. *Automated Deduction in Multiple-Valued Logics.* Oxford University Press, 1993.

[15] P. Hájek. *Metamathematics of fuzzy logic.* Kluwer, 1998.

[16] V.A. Jankov. The calculus of the weak "law of excluded middle". *Mathematics of the USSR*, 8:648–650, 1968.

[17] L.L. Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras. *Algebra and Logic*, 16:427–455, 1977.

[18] D. Pearce. Stable inference as intuitionistic validity. *Journal of Logic Programming*, 38(1):79–91, 1999.

[19] G. Rousseau. Sequents in many valued logics I. *Fund. Math.*, 60:23–33, 1967.

[20] M. Takahashi. Many-valued logics of extended Gentzen style I. *Science Reports of the Tokyo Kyoiku Daigaku*, 9:95–116, 1967.

[21] G. Takeuti. *Proof Theory.* North-Holland, 1975.