GL-based calculi for PCL and its deontic cousin

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Abstract. We introduce a natural sequent calculus for preferential conditional logic **PCL** via embeddings into provability logic **GL**, achieving optimal complexity and enabling countermodel extraction. Extending the method to **PCL** with reflexivity and absoluteness – corresponding to Åqvist's deontic system **F** with cautious monotony – we employ hypersequents to capture the **S5** modality; the resulting calculus subsumes the known calculi for the weaker systems **E** and **F** within Åqvist family.

1 Introduction

Conditional logics aim to capture forms of implication, $A \rightsquigarrow B$, that departs from classical implication. These logics support a range of interpretations, including the prototypical ("Typically, if A, then B"), the counterfactual ("If A were the case, then B would be"), and the deontic interpretation ("B is obligatory under condition A," usually written as $\bigcirc(B/A)$).

The most prominent systems for the *prototypical interpretation* are the so-called KLM logics [18]. Within this framework, the logic **P** of preferential reasoning has been particularly influential. **P** (and KLM logics generally) permits only shallow conditionals. An extension of **P** that accommodates nested conditionals is **PCL** (Preferential Conditional Logic), a foundational system for the *counterfactual interpretation* of conditionals. Despite its simple axiomatization, known since the work of Burgess [7], **PCL** lacks an analytic sequent calculus that balances proof-theoretic clarity with optimal complexity. Analytic calculi are characterized by the step-wise decomposition of the formula to be proved, which makes them well-suited for establishing meta-logical properties of the formalized logics and enabling automation. To our knowledge, the only analytic sequent calculus for **PCL** is in [25], which, although complexity-optimal, employs highly combinatorial rules that hinder readability and countermodel construction. Other proposals rely on the more expressive framework of labeled calculi [23,16,15], which can handle extensions of **PCL**, through neighbourhood semantics, at the cost of suboptimal computational behavior. Tableaux and resolution calculi have been proposed in [13,22], but present significant complications: [13] requires intricate blocking conditions to ensure termination, and [22] relies on a non-trivial pre-processing phase.

In this paper, we define a simple and complexity-optimal sequent calculus for **PCL** by establishing and exploiting a connection between **PCL** and Gödel-Löb logic (**GL**)[26] – the normal modal logic of arithmetic provability. The calculus enables countermodel extraction, producing the small models described in [11], for which the original paper provided only a non-constructive existence proof. The method extends to **PCLTA**, i.e, **PCL** with reflexivity and absoluteness, for which we introduce a new analytic calculus. **PCLTA** is known in the context of the *deontic interpretation* of conditionals as **F+(CM)**, that is Åqvist's dyadic deontic system **F** [1] augmented with the cautious monotonicity axiom (CM) [24].

Our calculi are based on an embedding of **PCL** into normal modal logics, extending that of its shallow counterpart, the KLM logic **P**. The well-known embedding proposed in [6], which maps **P** into **S4**, is not suitable for our purposes, as it gives rise to a different logic of nested conditionals, CT4.[1] Here we (simplify and) extend the correspondence identified in [12] between the conditional operator of **P** and the modality of **GL**, enabling the treatment of nested conditionals via bi-modal logics. We show that **PCL** naturally embeds into a combination of **GL** and **K**, while **F+(CM)** corresponds to a combination of **GL** and **S5**. These embeddings are foundational to our calculi and are formally verified within them. Similarly to previous work on modal interpretation of conditionals, *e.g.*, [12,5], we encode maximality by a unary modal operator $\mathcal{B}et$, which represents the "better" (or preferable) worlds, and is interpreted as the **GL** modality. The resulting sequent calculus for **PCL** is analytic and complexity-optimal. Due to the presence of the **S5** modality,[2] the calculus for **F+(CM)** is formulated within the hypersequent framework, a natural generalization of the sequent format that enables parallel manipulation of multiple sequents [3]. The calculus for **F+(CM)** presented here provides an alternative to the one in [10], which was based on a different semantic interpretation of the logic. Notably, our rules for the dyadic obligation coincide with those of the calculi for the weaker logics **E** and **F** in the Åqvist family [8,9], differing only in the rule for the $\mathcal{B}et$ modality. This alignment reflects the modularity of the Hilbert systems **E**, **F**, and **F+(CM)**, where **F** extends **E** with one additional axiom, and **F+(CM)** further adds (CM).

2 Preliminaries: PCL and F+(CM)

Let $\mathbb{P}$ be the set of propositional atoms. The language of **PCL** is generated by the grammar: $A ::= p \in \mathbb{P} \mid \neg A \mid A \to A \mid A \rightsquigarrow A$, with $\wedge, \vee, \leftrightarrow$ defined as usual. An axiomatization of **PCL** consists of the axioms for classical propositional logic extended with the following axioms and rules:

(CSO) $(A \rightsquigarrow B) \wedge (B \rightsquigarrow A) \to ((A \rightsquigarrow C) \leftrightarrow (B \rightsquigarrow C))$

(OR) $(A \rightsquigarrow C) \wedge (B \rightsquigarrow C) \to ((A \vee B) \rightsquigarrow C)$

(ID) $A \rightsquigarrow A$

$$\frac{A \leftrightarrow B}{(A \rightsquigarrow C) \leftrightarrow (B \rightsquigarrow C)}(RCEA)$$

$$\frac{B_1 \wedge \cdots \wedge B_m \to C}{(A \rightsquigarrow B_1) \wedge \cdots \wedge (A \rightsquigarrow B_m) \to (A \rightsquigarrow C)}(RCK)$$

(CSO) expresses that conditionally equivalent formulas have the same (conditional) consequences. It is equivalent to the pair of well-known axioms of cautious monotony (CM) $(A \rightsquigarrow B) \wedge (A \rightsquigarrow C) \to (A \wedge B \rightsquigarrow C)$ and restricted transitivity

[1] For example, $\neg(\top \rightsquigarrow \bot)$ is valid in CT4 but not in **PCL**.

[2] No cut-free sequent calculus is known for modal **S5**.

(RT) $(A \wedge B \rightsquigarrow C) \wedge (A \rightsquigarrow B) \to (A \rightsquigarrow C)$. (Id) and (OR) are the conditional versions of identity, and of the principle governing disjunctive premises. The rule (RCEA) allows substitution of equivalents in the antecedent, and (RCK) distributes conditionals over classical implication. The semantics of **PCL** is given by the following notion of *preference models* [7]. [3]

Definition 1. *A* **PCL** preference model *is a tuple* $\langle W, \{W_w\}_{w \in W}, \{\succeq_w\}_{w \in W}, V\rangle$ *where* W *is a (non-empty) finite set of worlds,* $W_w \subseteq W$ *is a set of worlds accessible from* w*,* $\succeq_w$ *is a reflexive and transitive binary relation on* W_w*, and* $V : \mathbb{P} \to \mathcal{P}(W)$ *is a valuation function. We write* $v \succ_w u$ *to denote* $v \succeq_w u$ *and* $u \not\succeq_w v$*. The satisfaction w.r.t. such models is defined as follows:*

- $(M, w) \models x$ *if* $w \in V(x)$
- $(M, w) \models A \wedge B$ *if* $(M, w) \models A$ *and* $(M, w) \models B$.
- $(M, w) \models \neg A$ *if* $(M, w) \not\models A$.
- $(M, w) \models A \rightsquigarrow B$ *if for every* $v \in Best_w(||A||)$*,* $(M, v) \models B$,.

where $||A|| = \{w \mid (M, w) \models A\}$ *and* $Best_w(X) = \{u \in X \mid \forall v \in X, v \not\succ_w u\}$.

We consider a variant of the language extended with the unary operator $\mathcal{B}et$. In the extended language, formulas are evaluated relative to a twice-pointed model (M, u, w), where u is "the point of view" (POV), whose preference relation is used and $w \in W_u$ is the world of evaluation (boolean cases are omitted):

- $(M, u, w) \models x$ if $w \in V(x)$.
- $(M, u, w) \models \mathcal{B}etA$ if for every $v \in W_u$ s.t. $v \succ_u w$, $(M, u, v) \models A$.
- $(M, u, w) \models A \rightsquigarrow B$ if for every $v \in Best_w(||A||_u)$, $(M, u, v) \models B$.

where $||A||_u = \{w \,|\, (M, u, w) \models A\}$. Notice that for formulas without $\mathcal{B}et$ the evaluation does not depend on the POV, and for such formulas satisfiability w.r.t. two-pointed models is equivalent to satisfiability w.r.t. usual pointed models (since for $(M, w) \models A$, M can be transformed into M' by adding the world u with $W_u = \{w\}$, ensuring $(M', u, w) \models A$).

The logic **F+(CM)** is the extension of **PCL** with reflexivity (i.e. $w \in W_w$) and absoluteness (i.e. $W_{w_1} = W_{w_2}$ and $\succeq_{w_1} = \succeq_{w_2}$ for any $w_1, w_2 \in W$), which arises in the context of normative reasoning [1,24]. Accordingly, an axiomatization of **F+(CM)** can be obtained by extending **PCL** with the following axioms (when referring to **F+(CM)**, we use the notation $\bigcirc(B/A)$ in place of $A \rightsquigarrow B$):

$$\text{(T)}\ A \to \neg \bigcirc (\bot/A)$$

$$(\mathrm{A}_1)\ \ \bigcirc(B/A) \to \bigcirc(\bigcirc(B/A)/C) \qquad (\mathrm{A}_2)\ \neg\bigcirc(B/A) \to \bigcirc(\neg\bigcirc(B/A)/C)$$

The semantics of **F+(CM)** simplifies that of **PCL** by avoiding accessible worlds and indexed preference relations, thanks to reflexivity and absoluteness. As a result, the satisfability in the extended language does not require POV and can be defined w.r.t. the usual pointed models.

[3] Since PCL enjoys the Finite Model Property [7], we limit our analysis to finite models, enabling a simpler truth condition for conditionals [21].

Definition 2. *An* **F+(CM)** preference model *is a tuple* $\langle W, \succeq, V \rangle$*, where* W *is a (non-empty) finite set of worlds,* $\succeq$ *is a reflexive and transitive binary relation on* W*, and* $V : \mathbb{P} \to \mathcal{P}(W)$ *is a valuation function. The satisfaction w.r.t. such models is defined as follows (boolean cases are omitted):*

- $(M, w) \models \mathcal{B}et A$ *if for every* $v \in W$ *s.t.* $v \succ w$*,* $(M, v) \models A$*.*
- $(M, w) \models \bigcirc(B/A)$ *if for every* $v \in Best(||A||)$*,* $(M, v) \models B$*.*

While **F+(CM)** typically includes an explicit **S5** modality, we omit it here as it can be defined via the conditional as: $\Box A := \bigcirc(\bot/\neg A)$.

3 A Sequent Calculus for PCL Grounded in GL

We uncover and exploit a novel connection between **PCL** and the provability logic **GL** to define **ScPCL**, an analytic sequent calculus for **PCL** which is both simple and complexity-optimal. **ScPCL** extends Gentzen's LK calculus for classical logic with the rules for $\mathcal{B}et$ and for the conditional operator. The axioms and rules of **ScPCL** are displayed in Fig. 1. The intuition behind the modal and conditional rules is as follows:

Rule $\mathcal{B}et$**:** Due to the finiteness of the models, the $\mathcal{B}et$ modality can be naturally interpreted as the **GL** modality. Indeed, **GL** is sound and complete w.r.t. finite, transitive and irreflexive Kripke frames. Due to transitivity, the truth condition for the $\mathcal{B}et$ modality can be equivalently stated as:

$$(M, u, w) \models \mathcal{B}et A \text{ if } \forall v \in W_u \text{ s.t. } v \succ_u w \text{ and } (M, u, v) \models \mathcal{B}et A,\ (M, u, v) \models A$$

This rewriting brings to the fore the fixed point property of $\mathcal{B}et$ and leads to the sequent rule below ($\mathcal{B}et A$ is the *diagonal formula*):

$$\frac{\mathcal{B}et A \Rightarrow A}{\Rightarrow \mathcal{B}et A}$$

By adding contexts to the rule, we recover the familiar rule for the **GL** calculus from [27], which we denote as $\mathcal{B}et$ in Fig. 1.

Rule $\rightsquigarrow$**:** In [12], a correspondence was established between the conditional operator in **P** and the modality $\mathcal{B}et$ of **GL**. The conditional $A \rightsquigarrow B$ in **P** was expressed as $(A \wedge \mathcal{B}et\neg A \to B) \wedge \mathcal{B}et(A \wedge \mathcal{B}et\neg A \to B)$ (denoting the **GL** modality with $\mathcal{B}et$). Here, we simplify this translation –by removing the first conjunct–, and extend it to the nested case. The extension to **PCL**, relies on the small models of [11]. There, a model satisfying any given formula is constructed as a tree-like structure consisting of separate *clusters* of worlds with a certain ordering, each child-cluster is intended to ensure the required evaluation of conditionals in some world of the parent-cluster. Since clusters are finite submodels with a transitive ordering inside, they can be seen as separate **GL**-models, while the tree structure on them can be captured by an independent accessibility relation, which suggests the following translation of the conditional into the bi-modal logic **K+GL**, leading to the $\rightsquigarrow$ rule in Fig. 1: $tr_{\mathbf{PCL}}(A \rightsquigarrow B) = \Box_{\mathbf{K}}\mathcal{B}et(A \wedge \mathcal{B}et\neg A \to B)$. The adequacy of the new embedding is proven at the end of the section.

Example 1. A derivation of cautious monotonicity (CM) in **ScPCL** is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots (A \wedge \mathcal{B}et\neg A) \rightarrow B, \neg(A \wedge B), \mathcal{B}et\neg A, A \Rightarrow}{\dots \mathcal{B}et((A \wedge \mathcal{B}et\neg A) \rightarrow B), A, B, \mathcal{B}et\neg(A \wedge B) \Rightarrow C, \mathcal{B}et\neg A}\ {\scriptstyle \mathcal{B}et,\ \mathrm{R}\neg}}{\dots \mathcal{B}et((A \wedge \mathcal{B}et\neg A) \rightarrow B), (A \wedge \mathcal{B}et\neg A) \rightarrow C, A, B, \mathcal{B}et\neg(A \wedge B) \Rightarrow C}\ {\scriptstyle \mathrm{L}\rightarrow}}{\dots \mathcal{B}et((A \wedge \mathcal{B}et\neg A) \rightarrow B), (A \wedge \mathcal{B}et\neg A) \rightarrow C \Rightarrow (A \wedge B \wedge \mathcal{B}et\neg(A \wedge B)) \rightarrow C}\ {\scriptstyle \mathrm{R}\rightarrow,\ \mathrm{L}\wedge}}{\mathcal{B}et((A \wedge \mathcal{B}et\neg A) \rightarrow B), \mathcal{B}et((A \wedge \mathcal{B}et\neg A) \rightarrow C) \Rightarrow \mathcal{B}et((A \wedge B \wedge \mathcal{B}et\neg(A \wedge B)) \rightarrow C)}\ {\scriptstyle \mathcal{B}et}}{A \rightsquigarrow B, A \rightsquigarrow C \Rightarrow A \wedge B \rightsquigarrow C}\ {\scriptstyle \rightsquigarrow}$$

Trivially derivable premises in branching rules are omitted. The topmost sequent is derivable by logical rules. Decomposing a conjunction on the left can also be done using logical rules since $A \wedge B$ abbreviates $\neg(A \rightarrow \neg B)$.

Remark 1. **ScPCL** does not strictly satisfy the subformula property. Nevertheless, there is a finitary restriction on the formulas occurring in derivations. In particular, if $\Rightarrow F$ is the root sequent, then all formulas appearing in the derivation are either subformulas of F or subformulas of $\mathcal{B}et(A \wedge \mathcal{B}et\neg A \rightarrow B)$ for some conditional $A \rightsquigarrow B$ inside F. We call this property *weak subformula property.*

The weak subformula property, together with the specific formulation of the $\mathcal{B}et$ rule, enables to define a notion of complexity for sequents that strictly decreases during bottom-up proof construction.

Definition 3. *The* complexity *of a sequent $\Gamma \Rightarrow \Delta$ is a triple (d, b, c), where d is the maximal depth of nesting of conditionals in the sequent, b is a number of $\mathcal{B}et$-formulas that appear as subformulas in $\Gamma \cup \Delta$ but do not appear as formulas in Γ, and c the total number of propositional connectives in the sequent.*

Defining a lexicographic ordering on such triples, the complexity of the premises is strictly smaller than the complexity of the conclusion for any rule application.[4] More precisely, in $\rightsquigarrow$ parameter d decreases, in $\mathcal{B}et$ parameter b decreases (if the rule application is not redundant) while parameter d decreases or remains the same, and in every logical rule parameter c decreases while both d and b either decrease or remain the same. Each parameter is polynomially bounded w.r.t. the size of the sequent. Moreover, due to the weak subformula property, the size of any sequent in a derivation is polynomially bounded by the size of the root sequent. Hence the length of any derivation branch –assuming there is no redundant rule applications– is also polynomially bounded in the size of the root sequent. Thus proof search in **ScPCL** attempting all non-redundant rule applications in any arbitrary order can be performed in polynomial space, matching the complexity of **PCL**.

Proposition 1. *Proof search in* **ScPCL** *can be performed in PSPACE.*

We now establish soundness and completeness of **ScPCL** w.r.t. preference semantics of **PCL**. As usual, we can interpret a sequent as a formula: we say that $\Gamma \Rightarrow \Delta$ is valid in **PCL** when $(\bigwedge \Gamma) \rightarrow (\bigvee \Delta)$ is valid in **PCL**.

[4] Apart from a redundant application of ($\mathcal{B}et$), which contain $\mathcal{B}etA$ on both sides in the conclusion, and therefore can be closed immediately with the axiom.

AXIOMS: $p, \Gamma \Rightarrow \Delta, p$ for $p \in \mathbb{P}$

LOGICAL RULES:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} R\neg \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} L\neg \quad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} L\to \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta} R\to$$

MODAL RULES:

$$\frac{\{\mathcal{B}et(A_i \wedge \mathcal{B}et\neg A_i \to B_i)\}_i \Rightarrow \mathcal{B}et(A \wedge \mathcal{B}et\neg A \to B)}{\Gamma, \{A_i \rightsquigarrow B_i\}_i \Rightarrow A \rightsquigarrow B, \Delta} \rightsquigarrow \quad \frac{\Gamma^{b\downarrow}, \Gamma^b, \mathcal{B}etA \Rightarrow A}{\Gamma \Rightarrow \mathcal{B}etA, \Delta} \mathcal{B}et$$

Fig. 1. The calculus ScPCL. $\Gamma^b := \{\mathcal{B}etA \mid \mathcal{B}etA \in \Gamma\}$ and $\Gamma^{b\downarrow} := \{A \mid \mathcal{B}etA \in \Gamma\}$.

Proposition 2. *If $\Gamma \Rightarrow \Delta$ is derivable in* **ScPCL**, *then it is valid in* **PCL**.

Proof. For each rule the validity of premise implies the validity of the conclusion. We only consider the cases of the rules ($\mathcal{B}et$) and ($\rightsquigarrow$), other cases being trivial.

Rule ($\mathcal{B}et$): Suppose towards a contradiction that the conclusion is not valid. Then there is a twice-pointed model (M, u, w) s.t. all the formulas in Γ are true at (M, u, w) while all formulas in Δ and $\mathcal{B}etA$ are false at (M, u, w). The latter implies that the set $X = \{v \in W_u \mid u \succ_u w \textit{ and } (M, u, v) \not\models A\}$ is not empty. Then there is some $v \in Best_u(X)$, since $\succ_u$ is a strict preorder on a finite set. (M, u, v) satisfies all formulas in $\Gamma^{b\downarrow}$ and Γ^b due to transitivity of $\succ_u$, it also falsifies A by definition of X and satisfies $\mathcal{B}etA$ since $v \in Best_u(X)$. Thus, (M, u, v) falsifies the premise, leading to a contradiction.

Rule ($\rightsquigarrow$): Suppose towards a contradiction that the conclusion is not valid. Then there are a model $M = \langle W, \{W_w\}_{w \in W}, \{\leq_w\}_{w \in W}, V\rangle$ and $u, v \in W$ s.t. $(M, u, v) \models A_i \rightsquigarrow B_i$ and $(M, u, v) \not\models A \rightsquigarrow B$. The latter implies that there is $v' \in Best_v(||A||_u^M)$ s.t. $v' \notin ||B||_u^M$. Let us consider an extended model $M' = \langle W \cup \{v'\}, \{W'_w\}_{w \in W'}, \{\succeq'_w\}_{w \in W}, V'\rangle$ that only adds a new world v'' in M with $v' \succ_u v''$ and does not change any other preference relation or valuation:

- $W'_u = W_u \cup \{v''\}$, $W'_{v''} = \emptyset$, $W'_w = W_w$ for any $w \in W \setminus \{u\}$,
- $\succeq'_w = \succeq_w$ for $w \in W \setminus \{u\}$, $\succeq'_{v''}$ is empty and $x \succeq'_u y$ if either $x \succeq_u y$ or $x = y = v''$ or $y = v''$ and $x \succeq_u v$.

$||C||_x^{M'} = ||C||_x^M$ for any $x \in W$ since v'' is never accessible during the evaluation, so $v' \notin ||B||_u^{M'}$. We observe that $(M', u, v'') \models \mathcal{B}et(A_i \wedge \mathcal{B}et\neg A_i \to B_i)$ for any $1 \leq i \leq n$ since $(M', u, v) \models A_i \rightsquigarrow B_i$. Therefore, by the validity of the premise, we get that v'' satisfies $(M, u, v'') \models \mathcal{B}et(A \wedge \mathcal{B}et\neg A \to B)$. Since $v' \succ'_u v''$ we get $v' \in ||B||^{M'}$ and thus a contradiction.

The completeness of **ScPCL** is shown by exhibiting a countermodel for any non-derivable sequent. The proof is constructive.

Proposition 3. *If $\Gamma \Rightarrow \Delta$ is valid in* **PCL**, *then it is derivable in* **ScPCL**.

Proof. The proof is by induction on the complexity of $\Gamma \Rightarrow \Delta$. If there is a negated formula or implication in $\Gamma \cup \Delta$, then a logical rule can be applied to it. At least one of the premises is a non-derivable sequent of smaller complexity, so by the inductive hypothesis there will be a countermodel falsifying it, which will also falsify the conclusion.

Now let all formulas in $\Gamma \cup \Delta$ be either a propositional variable, a conditional, or a $\mathcal{B}et$-formula. The premise of each application of ($\rightsquigarrow$) or ($\mathcal{B}et$) is an underivable sequent of smaller complexity, so by inductive hypothesis, there exists a twice-pointed model falsifying it. Suppose $(M^{+1}, u^{+1}, v^{+1}), \ldots, (M^{+n}, u^{+n}, v^{+n})$ are all such models for all possible applications of ($\mathcal{B}et$), and $(M^{-1}, u^{-1}, v^{-1}), \ldots, (M^{-m}, u^{-m}, v^{-m})$ are all such models for all possible applications of ($\rightsquigarrow$). Denote $I = \{+1, \ldots, +m, -1, \ldots, -m\}$. Let $M^i = \langle W^i, \{W^i_w\}_{w \in W^i}, \{\preceq^i_w\}_{w \in W^i}, V^i \rangle$ for $i \in I$. We will construct our countermodel for $\Gamma \Rightarrow \Delta$ as the disjoint union of all these countermodels obtained inductively, adding one special world w_0 to them. We will not modify the sets of visible worlds W^i_w and orderings $\preceq^i_w$ for any world of any submodel M^i, and select it only for the additional world w_0. Specifically, we consider the model $M = \langle W, \{W_w\}_{w \in W^i}, \{\preceq_w\}_{w \in W^i}, V \rangle$:

- $W = \{w_0\} \sqcup W^{+1} \sqcup \cdots \sqcup W^{+n} \sqcup W^{-1} \sqcup \cdots \sqcup W^{-m}$
- $W_{w_0} = \bigcup_{1 \leq i \leq m} \{w \in W^{-i} \mid w \succ^{-i}_{u^{-i}} v_i\}$
- $W_w = W^k_w$ for any $w \in W^k$ for $k \in I$
- $x \preceq_{w_0} y$ in either of the following three cases: (1) $x = y = w_0$; (2) $x = w_0$, $y \in W^{+i}$, and $v^{+i} \preceq_{u^{+i}} y$; (3) $x, y \in W^k$ for some $k \in I$ and $x \preceq_{u^k} y$.
- $\preceq_w = \preceq^k_w$ for any $w \in W^k$ for $k \in I$
- $V(p) = \begin{cases} \{w_0\} \sqcup V^{+1}(p) \sqcup \cdots \sqcup V^{+n}(p) \sqcup V^{-1}(p) \sqcup \cdots \sqcup V^{-m}(p), & \textit{if } p \in \Gamma \\ V^{+1}(p) \sqcup \cdots \sqcup V^{+n}(p) \sqcup V^{-1}(p) \sqcup \cdots \sqcup V^{-m}(p), & \textit{otherwise} \end{cases}$

First, notice that W is finite and all relations $\leq_w$ are reflexive, antisymmetric, and transitive (if the same is assumed for all submodels M^k). Also, it is straightforward to show that for every $u, w \in W^k$ for $k \in I$ $(M, u, w) \models \varphi$ iff $(M^k, u, w) \models \varphi$ (since relations and valuation within submodels did not change and the evaluation of the formula in a submodel does not move outside this submodel). Moreover, for every $w \in W^k$ for $k \in I$ $(M, w_0, w) \models \varphi$ iff $(M^k, u^k, w) \models \varphi$ (since $\preceq_{w_0}$ and $\preceq^k_{u^k}$ coincide on W^k).

We show that (M, w_0, w_0) satisfies (resp. falsifies) all formulas in Γ (resp Δ):

- for atoms due to the definition of V and the fact that the sequent can not be closed by the axiom;
- for $\mathcal{B}etA$ on the left due to presence of both A and $\mathcal{B}etA$ on the left in the premise of any $\mathcal{B}et$ application, which makes A true in v^{+i} and all y such that $v^{+i} \prec_{u^{+i}} y$ (i.e. all $w \in W^{+i}$ s.t. $w_0 \prec w$, due to antisymmetry);
- for $\mathcal{B}etA$ on the right due to possible application of $\mathcal{B}et$ to this formula with A on the right in this premise, implying that A is false in some v^{+i};
- for $A \rightsquigarrow B$ on the left due to presence $\mathcal{B}et(A \wedge \mathcal{B}et\neg A \rightarrow B)$ on the left in the premise of any $\rightsquigarrow$ application, which ensures that for every set $U_i = \{w \in W^{-i} \mid w \succ^{-i}_{u^{-i}} v_i\}$ composing W_{w_0}, $Best_{w_0}(||A||^M_{w_0}) \cap U_i = Best_{u^{-i}}(||A||^M_{u^{-i}}) \cap U_i \subseteq ||B||^{M^{-i}}_{u^{-i}} \subseteq ||B||^M_{w_0}$;
- for $A \rightsquigarrow B$ on the right due to possible application of $\rightsquigarrow$ to this formula with $\mathcal{B}et(A \wedge \mathcal{B}et\neg A \rightarrow B)$ on the right in this premise, implying there is some

world v' such that $v^{-i} \prec^{-i}_{u^{-i}} v'$ (so $v' \in W_{w_0}$) and $v' \in Best_{u^{-i}}(||A||^{M^{-i}}) \setminus ||B||^{M^{-i}} \subseteq Best_{w_0}(||A||^M) \setminus ||B||^M$.

Remark 2. The countermodel constructed in the above proof has the same structure as the small model construction for **PCL** in [11]: starting with a formula that has no $\mathcal{B}et$-modalities, the resulting model will consist of separate chains of worlds, each chain falsifies one conditional in some world of a later chain. While Friedman-Halpern small models are obtained non-constructively via selecting a finite subset of worlds from an arbitrary existing model, our calculus provides a constructive way of obtaining such small models by analyzing failed derivations.

Example 2. Consider the instance of Rational Monotony $(a \rightsquigarrow c) \Rightarrow ((a \wedge b) \rightsquigarrow c), (a \rightsquigarrow \neg b)$. The countermodel provided by the proof above is $W = \{w_0, v_0^{-1}, v_1^{-1}, v_0^{-2}, v_1^{-2}, v_2^{-2}\}$, $V(a) = \{v_1^{-1}, v_1^{-2}, v_2^{-2}\}$, $V(b) = \{v_1^{-1}, v_1^{-2}\}$, $V(c) = \{v_1^{-1}, v_2^{-2}\}$, where $W_{w_0} = \{v_1^{-1}, v_1^{-2}, v_2^{-2}\}$ includes a singleton chain of v_1^{-1} falsifying $(a \rightsquigarrow \neg b)$ and a chain $v_1^{-2} \preceq_{w_0} v_2^{-2}$ falsifying $(a \wedge b \rightsquigarrow c)$.

We use **ScPCL** to show the adequacy of translating **PCL** into **K+GL**, proving that its rules are sound and complete for the **K+GL** semantics of the translated formulas. We recall the bi-relational Kripke model semantics for **K+GL**.

Definition 4. *A* **K+GL***-model is a tuple* $M = \langle W, R_{\mathbf{K}}, R_{\mathbf{GL}}, V\rangle$*, where* W *is a* finite *set of worlds,* $R_{\mathbf{K}}$ *is arbitrary binary relation on* W*,* $R_{\mathbf{GL}}$ *is* transitive *binary relation on* W*, and* V *is valuation on* W*.* $\Box_{\mathbf{K}}$ *and* $\Box_{\mathbf{GL}}$ *modalities are evaluated in such models w.r.t. the corresponding relations:*

$$\begin{array}{lll}(M,w) \models \Box_{\mathbf{K}} A & \textit{iff} & \forall w' \in W, w R_{\mathbf{K}} w' \Rightarrow (M,w') \models A \\ (M,w) \models \Box_{\mathbf{GL}} A & \textit{iff} & \forall w' \in W, w R_{\mathbf{GL}} w' \Rightarrow (M,w') \models A\end{array}$$

Definition 5. *The translation* $tr_{\mathbf{PCL}}$ *from* **PCL** *into* **K+GL** *is as follows:*

$$\begin{array}{ll}tr_{\mathbf{PCL}}(p) & = p \\ tr_{\mathbf{PCL}}(\neg A) & = \neg tr_{\mathbf{PCL}}(A) \\ tr_{\mathbf{PCL}}(A \to B) & = tr_{\mathbf{PCL}}(A) \to tr_{\mathbf{PCL}}(B) \\ tr_{\mathbf{PCL}}(A \rightsquigarrow B) & = \Box_{\mathbf{K}}\Box_{\mathbf{GL}}(tr_{\mathbf{PCL}}(A) \wedge \Box_{\mathbf{GL}}\neg tr_{\mathbf{PCL}}(A) \to tr_{\mathbf{PCL}}(B))\end{array}$$

Lemma 1. *A is valid in* **PCL** *iff* $tr_{\mathbf{PCL}}(A)$ *is valid in* **K+GL***.*

Proof. ($\Leftarrow$): if A valid in **PCL**, by Prop.3 there exists a derivation for $\Rightarrow A$ in **ScPCL**, and we can easily show that if we apply translation $tr_{\mathbf{PCL}}$ to every formula in this derivation, each rule application will preserve validity w.r.t. **K+GL**-models (since the rule $(\rightsquigarrow)$ after translation of conditionals in the conclusion behaves exactly like the standard sequent rule for $\Box_{\mathbf{K}}$-modality).

($\Rightarrow$): if A is not valid, then $\Rightarrow A$ is underivable and we can take a preference countermodel M for A from the proof of Lemma 3 and transform[5] it into a

[5] This transformation applies only to preference models in which the relations $\succeq_{w_1}$ and $\succeq_{w_2}$ of different worlds coincide on $W_{w_1} \cap W_{w_2}$ (which permits the use of $\succ_w$ in the definition of $R_{\mathbf{GL}}$); hence we must rely on a specific kind of countermodel from the completeness proof, rather than an arbitrary one.

K+GL-countermodel for $tr_{\mathbf{PCL}}(A)$ by defining $R_{\mathbf{K}}(w) = W_w$ and $R_{\mathbf{GL}}(w) = \{u \in W \mid u \succ_w w\}$ for each world in M.

As the KLM logic **P** coincides with the Horn fragment of **PCL** the adequacy of $tr_{\mathbf{PCL}}$ also justifies the simplification of the translation from **P** into **GL** in [12]: the entailment $\{A_i \rightsquigarrow B_i\}_i \vdash (A \rightsquigarrow B)$ holds in **P** (and, so, in **PCL**) iff

$$\{\Box_{\mathbf{K}} \mathcal{B}et(A_i \wedge \mathcal{B}et\neg A_i \to B_i)\}_i \vdash (\Box_{\mathbf{K}} \mathcal{B}et(A \wedge \mathcal{B}et\neg A \to B))$$

holds in **K+GL**, which in turn holds iff

$$\{\mathcal{B}et(A_i \wedge \mathcal{B}et\neg A_i \to B_i)\}_i \vdash (\mathcal{B}et(A \wedge \mathcal{B}et\neg A \to B))$$

or, equivalently, it holds in **GL** (since there are no nested conditionals in **P**).

4 Adding absoluteness: from GL to Åqvist's F+(CM)

We introduce the calculus **HFcm** for **F+(CM)**, a deontic extension of **PCL** incorporating reflexivity and absoluteness. The rules of **HFcm** are derived by adapting the embedding of **PCL** into **GL** (see Lemma 3). We establish soundness and completeness of the system, with completeness shown relative to derivations that may include cuts. Rather than following the standard approach via cut-elimination (which could be proved similarly to [9]), in Sect. 4.1 we reformulate **HFcm** into a proof-search-oriented calculus. This transformation enables a cut-free completeness result via countermodel extraction from failed proof searches.

Incorporating absoluteness into **PCL** requires a generalization of the standard sequent framework, as no cut-free sequent calculus exists for **S5**. We use hypersequents, arguably the simplest generalization of sequents [2,4,3].

Definition 6. *A* hypersequent *is a multiset* $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$ *where, for all* $i = 1, \ldots, n$, $\Gamma_i \Rightarrow \Pi_i$ *is an ordinary sequent, called* component*.*

A hypersequent version of a sequent rule is obtained by adding a context G, representing a possibly empty hypersequent, to its premises and conclusion. The hypersequent calculus **HFcm** for **F+(CM)** consists of the (group of) rules (I)-(V) described below. **HFcm** includes (I) the *hypersequent version of the classical sequent calculus LK* as in Fig. 1, including the rules of internal weakening and contraction:

$$\frac{G \mid A, A, \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\, LC \quad \frac{G \mid \Gamma \Rightarrow \Delta, A, A}{G \mid \Gamma \Rightarrow \Delta, A}\, RC \quad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\, LW \quad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta, A}\, RW$$

(II) the *hypersequent version of the* $\mathcal{B}et$ *rule* (for the **GL** modality) from Fig. 1. To manipulate the hypersequent structure, **HFcm** includes the standard external structural rules (III) known as *ext. weakening (EW) and ext. contraction (EC)* below. These behave like weakening and contraction over whole hypersequent components. To capture the **S5** modality, we use (IV) *the rule* $s5'$ below right – a notational variant of the rule for **S5** in [19]:

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi}\ EW \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ EC \qquad \frac{G \mid \Gamma^{\bigcirc}, \Gamma' \Rightarrow \Pi'}{G \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow \Pi'}\ s5'$$

$\Gamma^{\bigcirc}$ abbreviates $\{\bigcirc(B/A) \mid \bigcirc(B/A) \in \Gamma\}$. (V) *The deontic rules* of **HFcm** are:

$$\frac{G \mid \Gamma^{\bigcirc}, A, \mathcal{B}et\neg A \Rightarrow B}{G \mid \Gamma \Rightarrow \bigcirc(B/A)}\ \mathrm{R}\bigcirc \qquad \frac{G \mid \Gamma \Rightarrow \Delta, A \qquad G \mid \Gamma \Rightarrow \Delta, \mathcal{B}et\neg A \qquad G \mid B, \Gamma \Rightarrow \Delta}{G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta}\ \mathrm{L}\bigcirc$$

Remark 3. The rules ($R\bigcirc$) and ($L\bigcirc$) are as in the calculi [8,9] for Åqvist's systems **E** and **F**. The former calculus replaces the **GL** rule for $\mathcal{B}et$ with a **K** rule; the latter uses a rule with no counterpart in normal modal logics. Remark 4 presents equivalent rules for **F+(CM)** derived starting from the rules of **ScPCL**.

In **HFcm** hypersequents are interpreted as follows: $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is valid if and only if $\bigvee_{1 \leq i \leq n} \Box(\bigwedge \Gamma_i \to \bigvee \Delta_i)$ is valid.

Lemma 2. *The calculus* **HFcm** *is sound for* **F+(CM)***.*

Proof. We prove, by induction on the height of derivations, that **HFcm** is sound with respect to the semantics in Definition 2. As an illustrative example, we consider the case of the rule R$\bigcirc$. Assume towards a contradiction that $G \mid \Gamma \Rightarrow \bigcirc(B/A)$ is not valid. Then there is a world w s.t. w satisfies the formulas in Γ and w falsifies $\bigcirc(B/A)$. Hence, there is a world u, with $u \models A$, $u \models \mathcal{B}et\neg A$ and $u \not\models B$. Since formulas $\Gamma^{\bigcirc}$ are satisfied in w, they are also satisfied in u (due to absoluteness), contradicting the validity of the premise $G \mid \Gamma^{\bigcirc}, A, \mathcal{B}et\neg A \Rightarrow B$.

Proposition 4. *The calculus* **HFcm** *is complete for* **F+(CM)** *with the cut rule:*

$$\frac{G \mid \Gamma \Rightarrow \Delta, A \qquad H \mid A, \Pi \Rightarrow \Sigma}{G \mid H \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ Cut$$

Proof. The axioms for **F+(CM)** are provable in **HFcm** and the rules can be simulated, using the cut rule for Modus Ponens. As an example we prove axiom (A_1) in **HFcm**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{G \mid A, \mathcal{B}et\neg A \Rightarrow B, A \qquad G \mid A, \mathcal{B}et\neg A \Rightarrow B, \mathcal{B}et\neg A \qquad G \mid A, \mathcal{B}et\neg A, B \Rightarrow B}{C, \mathcal{B}et\neg C \Rightarrow \ \mid \bigcirc(B/A), A, \mathcal{B}et\neg A \Rightarrow B}\ \mathrm{L}\bigcirc}{\bigcirc(B/A) \Rightarrow \ \mid C, \mathcal{B}et\neg C \Rightarrow \ \mid A, \mathcal{B}et\neg A \Rightarrow B}\ \mathrm{s5'}}{\bigcirc(B/A) \Rightarrow \ \mid C, \mathcal{B}et\neg C \Rightarrow \bigcirc(B/A)}\ \mathrm{EC, R\bigcirc, RW}}{\bigcirc(B/A) \Rightarrow \bigcirc(\bigcirc(B/A)/C)}\ \mathrm{EC, R\bigcirc, RW}}{\Rightarrow \bigcirc(B/A) \to \bigcirc(\bigcirc(B/A)/C)}\ \mathrm{R}\to$$

where G abridges $C, \mathcal{B}et\neg C \Rightarrow$.

We show that replacing $\Box_{\mathbf{K}}$ with $\Box_{\mathbf{S5}}$ and removing the outermost $\mathcal{B}et$ in the embedding $tr_{\mathbf{PCL}}$ yields an embedding of **F+(CM)** into **S5+GL**. The semantic definition of **S5+GL** is identical to that of **K+GL** (see Def. 4), except that $R_{\mathbf{K}}$ is replaced with a universal relation $R_{\mathbf{S5}}$.

Definition 7. *The translation* $tr_{\mathbf{F+(CM)}}$ *from the language of* **F+(CM)** *into the language of* **S5+GL** *is defined as* $tr_{\mathbf{PCL}}$ *apart from the conditional case:*

$$tr_{\mathbf{F+(CM)}}(\bigcirc(B/A)) = \Box_{\mathbf{S5}}(tr_{\mathbf{F+(CM)}}(A) \land \mathcal{B}et\neg tr_{\mathbf{F+(CM)}}(A) \to tr_{\mathbf{F+(CM)}}(B))$$

We chose the current translation over the $tr_{\mathbf{PCL}}$-like alternative that retains the outermost $\mathcal{B}et$ in $tr_{\mathbf{F+(CM)}}(\bigcirc(B/A))$, as it simplifies the conditional rule in **HFcm** (see Remark 4). Both translations are sound and faithful.

Lemma 3. *A is valid in* **F+(CM)** *iff $tr_{\mathbf{F+(CM)}}(A)$ is valid in* **S5+GL**.

Proof. The structure of the proof is similar to the one of Lem. 1. ($\Rightarrow$) We need to verify that application of $tr_{\mathbf{F+(CM)}}$ to every rule of **HFcm** provides a rule sound w.r.t. **S5+GL**. This verification coincides with the soundness proof in Lem. 2. ($\Leftarrow$) We argue by contraposition and we transform an arbitrary[6] **F+(CM)**-countermodel into a **S5+GL**-countermodel by keeping the set of worlds and the valuation and choosing $R_{\mathbf{GL}}(w) = \{u \in W \mid u \succ w\}$ and universal $R_{\mathbf{S5}}(w)$.

4.1 A Proof-search oriented calculus for F+(CM)

The calculus $\mathbf{HFcm}^{ps}$ for **F+(CM)**, in which all rules are invertible, contains the usual rules for boolean connectives in a cumulative form, i.e. in which formulas in the conclusion are copied in the premises, see [17], together with the following rules for the modal and conditional operator:

$$\frac{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B}{G \mid \Gamma \Rightarrow \bigcirc(B/A)}\ \mathrm{R}\bigcirc^{c} \qquad \frac{G \mid \Gamma \Rightarrow \Delta, \mathcal{B}et A \mid \Gamma^{b}, \Gamma^{b\downarrow}, \mathcal{B}et A \Rightarrow A}{G \mid \Gamma \Rightarrow \Delta, \mathcal{B}et A}\ \mathcal{B}et$$

$$\frac{G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta, A \qquad G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta, \mathcal{B}et\neg A \qquad G \mid \bigcirc(B/A), B, \Gamma \Rightarrow \Delta}{G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta}\ \mathrm{L}\bigcirc^{c}_{1}$$

$$\frac{G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, A \quad G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \mathcal{B}et\neg A \quad G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta \mid B, \Pi \Rightarrow \Sigma}{G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}\ \mathrm{L}\bigcirc^{c}_{2}$$

The rules for the conditional operator absorb the special structural rule $s5$' (for an overview of the methodology to obtain a proof search oriented calculus - there called *Kleene variant* - the reader is referred to [20]).

Lemma 4. *Every rule in* $\mathbf{HcFCM^{ps}}$ *is height-preserving invertible and the rules of weakening and contraction (internal and external) are height-preserving admissible. The rule $s5'$ is admissible.*

Proof. By induction on the height of the derivation. We present the case of the rule $s5'$. If $G \mid \Gamma^{\bigcirc}, \Gamma' \Rightarrow \Pi'$ is an axiom, so is $G \mid \Gamma \Rightarrow \ \mid \Gamma \Rightarrow \Pi'$. If the last rule applied is any rule different from $\mathrm{L}\bigcirc^{c}_{1}$ or $\mathrm{L}\bigcirc^{c}_{2}$, we invoke the induction hypothesis and reapply the rule. If the last rule is $\mathrm{L}\bigcirc^{c}_{1}$, as in

$$\frac{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), \Gamma' \Rightarrow \Pi', A \qquad G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), \Gamma' \Rightarrow \Pi', \mathcal{B}et\neg A \qquad G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), B, \Gamma' \Rightarrow \Pi'}{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), \Gamma' \Rightarrow \Pi'}\ \mathrm{L}\bigcirc^{c}_{1}$$

We proceed as follows:

[6] In **F+(CM)**, absoluteness ensures consistency between preference relations across worlds, so – unlike in **PCL**– no special form of countermodel is required.

$$\dfrac{\dfrac{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), \Gamma' \Rightarrow \Pi', A}{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A) \Rightarrow \mid \Gamma' \Rightarrow \Pi', A}\,\text{IH} \quad \dfrac{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), \Gamma' \Rightarrow \Pi', \mathcal{B}et\neg A}{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A) \Rightarrow \mid \Gamma' \Rightarrow \Pi', \mathcal{B}et\neg A}\,\text{IH} \quad \dfrac{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A), B, \Gamma' \Rightarrow \Pi'}{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A) \Rightarrow \mid B, \Gamma' \Rightarrow \Pi'}\,\text{IH}}{G \mid \Gamma^{\bigcirc'}, \bigcirc(B/A) \Rightarrow \mid \Gamma' \Rightarrow \Pi'}\,\text{L}\bigcirc_1^c$$

The case of $\text{L}\bigcirc_2^c$ is similar.

We now show that the calculi **HcFCM^{ps}** and **HFcm** are equivalent.

Lemma 5. *G is derivable in* **HFcm** *iff G is derivable in* **HcFCM$^{\mathbf{ps}}$**.

Proof. From right to left, we proceed by induction on the height of the derivation observing that the rules of **HcFCM$^{\mathbf{ps}}$** can be simulated in **HFcm** using the structural rules and $s5'$. For example, we have:

$$\dfrac{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B}{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A)}\,\text{R}\bigcirc^c \;\mapsto\; \dfrac{\dfrac{\dfrac{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B}{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A) \mid \Rightarrow \bigcirc(B/A)}\,\text{R}\bigcirc}{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A) \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A)}\,\text{LW, RW}}{G \mid \Gamma \Rightarrow \Delta, \bigcirc(B/A)}\,\text{EC}$$

From left to right, we argue by induction on the height of the derivation in **HFcm** using Lemma 4 to simulate the rules of the calculus **HFcm**.

Lemma 6. *If G is derivable in* **HcFCM$^{\mathbf{ps}}$**, *there is a derivation of the same height in which the rule $\text{R}\bigcirc^c$ is applied only once to the same formula.*

Proof. If the derivation contains more than one application of rule $(\text{R}\bigcirc^c)$ to the same formula, we have:

$$\dfrac{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A, \Theta \Rightarrow B, \Lambda \mid \ldots \mid \Pi, \Gamma_j \Rightarrow \Delta_j, \Sigma, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B}{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A, \Theta \Rightarrow B, \Lambda \mid \ldots \mid \Pi, \Gamma_j \Rightarrow \Delta_j, \Sigma, \bigcirc(B/A)}\,\text{R}\bigcirc^c$$

$$\vdots \mathcal{D}$$

$$\dfrac{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B \mid \ldots \mid \Gamma_j \Rightarrow \Delta_j, \bigcirc(B/A)}{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid \ldots \mid \Gamma_j \Rightarrow \Delta_j, \bigcirc(B/A)}\,\text{R}\bigcirc^c$$

The topmost redundant application of rule $(\text{R}\bigcirc^c)$ can be replaced by (height-preserving admissible) contraction and weakening:

$$\dfrac{\dfrac{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A, \Theta \Rightarrow B, \Lambda \mid \ldots \mid \Pi, \Gamma_j \Rightarrow \Delta_j, \Sigma, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B}{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A, \Theta \Rightarrow B, \Lambda \mid \ldots \mid \Pi, \Gamma_j \Rightarrow \Delta_j, \Sigma, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A, \Theta \Rightarrow B, \Lambda}\,\text{LW,RW}}{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A, \Theta \Rightarrow B, \Lambda \mid \ldots \mid \Pi, \Gamma_j \Rightarrow \Delta_j, \Sigma, \bigcirc(B/A)}\,\text{EC}$$

$$\vdots \mathcal{D}$$

$$\dfrac{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid A, \mathcal{B}et\neg A \Rightarrow B \mid \ldots \mid \Gamma_j \Rightarrow \Delta_j, \bigcirc(B/A)}{H \mid \Gamma_i \Rightarrow \Delta_i, \bigcirc(B/A) \mid \ldots \mid \Gamma_j \Rightarrow \Delta_j, \bigcirc(B/A)}\,\text{R}\bigcirc^c$$

To prove completeness of **HFcm**^{ps} (and thus also **HFcm**) we reconstruct a countermodel from a failed derivation using the notion of *saturated* hypersequent.

Definition 8. *A hypersequent H is saturated w.r.t. the system* **HcFCM$^{\mathbf{ps}}$** *if it is not an initial sequent and for every component $\Gamma \Rightarrow \Delta$ in H, whenever $\Gamma \Rightarrow \Delta$ contains the principal formulas in the conclusion of a rule (r), then H also contains the formulas introduced by one of the premisses of (r) for every rule (r):*

– *(R→) If* $\Gamma \Rightarrow \Delta, A \to B \in H$*, then* $A \in \Gamma$ *and* $B \in \Delta$.
– *(L→) If* $A \to B, \Gamma \Rightarrow \Delta \in H$*, then* $A \in \Delta$ *or* $B \in \Gamma$.
– *(L*$\bigcirc_1$*) If* $\bigcirc(B/A), \Gamma \Rightarrow \Delta \in H$*, then* $A \in \Delta$ *or* $\mathcal{B}et\neg A \in \Delta$ *or* $B \in \Gamma$.
– *(L*$\bigcirc_2$*) If* $\bigcirc(B/A), \Gamma \Rightarrow \Delta \in H$ *and* $\Pi \Rightarrow \Sigma \in H$*, then* $A \in \Sigma$ *or* $\mathcal{B}et\neg A \in \Sigma$ *or* $B \in \Gamma$.
– *(R*$\bigcirc$*) If* $\Gamma \Rightarrow \Delta, \bigcirc(B/A) \in H$*, then* $\Pi, A, \mathcal{B}et\neg A \Rightarrow \Sigma, B \in H$ *for some* Π, Σ.
– *(*$\mathcal{B}et$*) If* $\Gamma \Rightarrow \Delta, \mathcal{B}etA \in H$*, then* $\Pi, \Gamma^b, \Gamma^{b\downarrow}, \mathcal{B}etA \Rightarrow \Sigma, A \in H$ *for some* Π, Σ.

Theorem 1. *If* $\Rightarrow A$ *is valid in* **F+(CM)** *then it is derivable* $\mathbf{HcFCM^{ps}}$.

Proof. Assume that $\Rightarrow A$ is not derivable. The proof is in two steps. (**I**) We prove that the search for a proof terminates and that there is a saturated hypersequent. We observe that the number of components generated in any derivation $\mathcal{D}$ of $\Rightarrow A$ can be bounded. By Lem. 6 and the weak subformula property, the number of components introduced by (R$\bigcirc^c$) is bounded by the number of conditionals in A and thus is finite. By the weak subformula property, if there is an infinite bottom-up introduction of components, these are introduced by the rule $\mathcal{B}et$. Hence, since the number of possible sequents occurring in a derivation is finite, there has to be a repetition. In this case, we have met the saturation condition for the rule $\mathcal{B}et$. Thus the number of components is finite. Since we can rule out rule applications for which the saturation condition has already been met (due to the admissibility of contraction), the length of every branch of a putative derivation of $\Rightarrow A$ is bounded and the derivation is finite. Hence if A is not derivable, there is a saturated hypersequent $G^{sat} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$.

(**II**) We construct a countermodel for A on the basis of G^{sat}. We assign labels to the components $i : \Gamma_i \Rightarrow \Delta_i$ ($i \in \{1, \ldots, n\}$) and consider the model: $\mathcal{M} = \langle \{1, \ldots, n\}, \preceq, V \rangle$ with $i \in V(p)$ iff $p \in \Gamma_i$ and $i \preceq j$ iff $i = j$ or

$$i : \Gamma_i \Rightarrow \Delta_i,\ j : \Gamma_j \Rightarrow \Delta_j,\ \Gamma^b, \Gamma_i^{b\downarrow} \subseteq \Gamma_j, \text{ and } \mathcal{B}etA \in \Gamma_j \setminus \Gamma_i.$$

We have to check that the model is finite, reflexive and transitive. Finiteness and reflexivity are immediate. Assume that $i \preceq j \preceq u$ for $i \neq j \neq u$, we need to prove that $i \preceq u$. By definition, we get: $\Gamma_i^b, \Gamma_i^{b\downarrow} \subseteq \Gamma_j^b, \Gamma_j^{b\downarrow} \subseteq \Gamma_u$. By hypothesis we know that there is $\mathcal{B}etB \in \Gamma_u \setminus \Gamma_j$, therefore $\mathcal{B}etB \notin \Gamma_i$, otherwise we would get $\mathcal{B}etB \in \Gamma_j$, hence a contradiction which yields the desired conclusion.

To complete the proof, we need to establish that:

- For every i, if $B \in \Gamma_i$, then $i \models B$
- For every i, if $B \in \Delta_i$, then $i \not\models B$

This is done by induction on the complexity of B. If B is an atomic formula, the claim stems from the definition of V. If B is a compound formula, the proof follows from the use of the induction hypothesis. We deal with the case in which B is $\mathcal{B}etC$; the other cases are handled similarly. If $\mathcal{B}etC \in \Gamma_i$, suppose $i \preceq j$, then $\Gamma_i^{b\downarrow} \subseteq \Gamma_j$. So we get $C \in \Gamma_j$ and by induction hypothesis we have $j \models C$, hence the desired conclusion. If $\mathcal{B}et \in \Delta_i$, by definition of saturation w.r.t. the rule $\mathcal{B}et$, there is $\Gamma_j \Rightarrow \Delta_j, C$ with $\Gamma_i^b, \Gamma^{b\downarrow} \subseteq \Gamma_j$. Furthermore, $\mathcal{B}etC \in \Gamma_j$, but $\mathcal{B}etC \notin \Gamma_i$, otherwise the hypersequent would be derivable and therefore not saturated. So, by definition, $i \preceq j$ and $j \not\models C$ by induction hypothesis, which entails $i \not\models \mathcal{B}etC$.

As $\Rightarrow A$ is the root of the putative derivation, in the saturated hypersequent there is j s.t. $A \in \Delta_j$ and so $j \not\models A$ which yields the desired conclusion.

Remark 4. $tr_{\mathbf{F+(CM)}}$ could equivalently be defined like $tr_{\mathbf{PCL}}$ (Def. 5), preserving the outermost $\mathcal{B}et$ modality in the clause for the dyadic operator. This yields the alternative version of the rules $(L\bigcirc)$ and $(R\bigcirc)$, closer to those of **ScPCL**:

$$\frac{G \mid A \wedge \mathcal{B}et\neg A \rightarrow B, \Gamma \Rightarrow \Delta}{G \mid \bigcirc(B/A), \Gamma \Rightarrow \Delta}\ (\mathrm{L}\bigcirc^*) \qquad \frac{G \mid \Gamma^{\bigcirc} \Rightarrow \mathcal{B}et(A \wedge \mathcal{B}et\neg A \rightarrow B)}{G \mid \Gamma \Rightarrow \bigcirc(B/A), \Delta}\ (\mathrm{R}\bigcirc^*)$$

The calculus $\mathbf{HFcm}^*$ obtained from **HFcm** by replacing (L$\bigcirc$) and (R$\bigcirc$) with their starred versions, is sound and complete w.r.t. **F+(CM)**: soundness can be checked directly analogously the ($\rightsquigarrow$) case in Prop. 2, and completeness holds since (L$\bigcirc$) and (R$\bigcirc$) can be simulated in $\mathbf{HFcm}^*$ using (L$\bigcirc^*$) and (R$\bigcirc^*$).

Concluding remark

We have introduced analytic calculi for conditional logics based on the correspondence between the KLM logic **P** and the provability logic **GL**. Specifically, we have considered two extensions of **P**: **PCL**, for counterfactual reasoning, and **F+(CM)**, for deontic reasoning. The approach in this paper could be applied to obtain modular calculi for other extensions of **PCL** [14]. Our calculi are relatively simple and have good meta-logical properties. In particular, we believe that cut-elimination can be proved by adapting Valentini's strategy for **GL** [27] (see also [9]). We leave the investigation of these questions to future work.

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