Standard completeness for extensions of IMTL

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***Abstract*—We provide a standard completeness proof which uniformly applies to a large class of axiomatic extensions of Involutive Monoidal T-norm Logic (IMTL). In particular, we identify sufficient conditions on the proof calculi which ensure density elimination and then standard completeness. Our argument contrasts with all previous approaches for involutive logics which are logic-specific.**

I. Introduction

Fuzzy logics are many-valued logics that are well suited to reasoning in the context of vagueness [15]. Their intended (or *standard*) semantics is based on truth-values in the real interval $[0, 1]$, ranging from absolute falsity 0 to absolute truth 1.

Establishing whether a logic – introduced/described as an Hilbert-Frege axiomatic system – is a fuzzy logic amounts to prove that it is *standard complete*, that is complete with respect to the standard semantics. Standard completeness has been traditionally proved by algebraic methods which are typically logic-specific, e.g. [5], [12], [13], [15], [17], [21]. An alternative approach, based instead on proof-theoretic techniques, has been introduced in [18] (see also [20]); the key idea in [18] is that in a logic L the admissibility (or elimination) of the syntactic rule called *density* can lead to rational completeness for L, i.e., that a formula is derivable in L iff it is valid in all dense linearly ordered L-algebras (L-chains). Standard completeness typically follows by embedding countable dense L-chains into L-algebras with lattice reduct $[0, 1]$, using Dedekind-MacNeille completion. Introduced in [22] the density rule formalized Hilbert-style has the following form

$$\frac{(A \to p) \vee (p \to B) \vee C}{(A \to B) \vee C}\ (Density)$$

where p is an eigenvariable (propositional variable not occurring in A, B, or C). Ignoring C, this can be read contrapositively as roughly saying "if $A > B$, then $A > p$ and $p > B$ for some p"; hence the name "density" and the intuitive connection with rational completeness.

The approach in [18] led to standard completeness proofs for various logics: Godo and Esteva's Monoidal T-norm based Logic (MTL), its involutive version IMTL, the logic UL of left-continuous uninorms and some of its extensions. A different method to eliminate applications of the density rule from derivations was introduced in [8] and later generalized to provide uniform proofs of standard completeness for large classes of logics; see e.g. [2] and the bibliography therein contained. These proofs rely on: (1) the use of the algorithm in [6] to transform Hilbert-axioms into suitable proof systems (cut-free hypersequent calculi [1], [19]), (2) the identification of sufficient conditions on these calculi that allow for density elimination (and then rational completeness), and (3) the closure under Dedekind-MacNeille-style completions, shown e.g. in [7], of the algebraic structures for the considered logics.

The algebraic and the proof theoretic approaches have been most successful for proving standard completeness of non involutive logics, that is of logics in which $\neg\neg A$ is not the same as A. Establishing standard completeness for involutive logics appears to be a difficult task [19], which has been carried out only for some specific logics and with ad hoc proofs; among them for IMTL [12], [18], IMTL extended with the n-contraction axioms ($n > 2$) [5], and with (wnm) [13] (Nilpotent Minimum Logic). The absence of weakening complicates the situation even more and, for instance, it is not known whether the logic IUL, which is IMTL without weakening [18], is standard complete.

In this paper we introduce a standard completeness proof that uniformly applies to a large class of axiomatic extensions of IMTL. Starting with the cut-free hypersequent calculi for such logics introduced in [9] (and recalled in *(Section II)*):

- We identify syntactic sufficient conditions on the hypersequent rules that guarantee density elimination (*Section III*); the results in [18] ensure that the formalized logics are rational complete.
- Standard completeness then follows by showing that the corresponding classes of algebras are preserved under Dedekind-MacNeille completion. This is obtained in *(Section IV)* by adapting/extending the results in [7] to (algebras for) involutive logics.

Our proof applies to the axiomatic extensions of IMTL already known to be standard complete, provides a first syntactic proof of standard completeness for Nilpotent Minimum Logic [13], and allows for the discovery of new fuzzy logics.

II. Preliminaries

Shown in [12] (and [18]) to be the logic of involutive left continuous t-norms[1], Involutive Monoidal T-norm Logic IMTL [13] arises by adding the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$ to Involutive Full Lambek calculus with exchange and weakening $\mathrm{IFL_{ew}}$ (also known as affine Linear Logic without exponentials).

[1] T-norms are the main tool in Fuzzy Logic to combine vague information.

Here we consider formulas of IMTL to be generated from a set $V = \{a, b, c, \dots\}$ of *propositional variables*, their duals $V^{\perp} = \{a^{\perp}, b^{\perp}, c^{\perp}, \dots\}$, the constant 1 and the connectives $\wedge, \vee, \odot, \rightarrow$ and $\oplus$, where $\alpha \rightarrow \beta$ abbreviates $\alpha^{\perp} \oplus \beta$.

As shown in [6], due to the presence of the *linearity axiom*, the simplest proof system for IMTL which allows for density elimination uses hypersequents, which are disjunctive-separated sequents [1]. Following the notation in [9] the latter are written here one-sided, i.e. as multisets of formulas.

Definition 1: A *hypersequent* is a finite multiset $P_1 \mid \dots \mid P_n$ of sequents, each called a *component* of the hypersequent.

Notation. In the following, $\Gamma, \Delta \dots$ will stand for both multisets of formulas and metavariables for multisets of formulas. G, H will denote hypersequents and P, C sequents (possibly built from metavariables). Γ^k will stand for k comma-separated occurrences of Γ. We denote by $P[\Sigma_1/\Lambda_1, \dots, \Sigma_n/\Lambda_n]$ the sequent obtained by replacing any Λ_i in P with Σ_i, and by $H[\Sigma_1/\Lambda_1, \dots, \Sigma_n/\Lambda_n]$ the hypersequent obtained by applying the same substitution to each component of H. Letting $A = \{\Lambda_1, \dots, \Lambda_n\}$ and $B = \{\Sigma_1, \dots, \Sigma_n\}$, we also use the compact forms $P[B/A]$ and $H[B/A]$. The calculus HIMTL for IMTL is presented in Fig. 1. Notice that its rules are actually rule *schemes* and that the (cut) rule is redundant (eliminable, in fact). The structural[2] rule (com), which operates on different components of hypersequents, allows us to prove the prelinearity axiom. A concrete instance of a rule scheme is a rule *application*. Following standard practice, we do not always distinguish explicitly between a rule instance and a rule schema.

Following [6], an algorithm to define cut-free hypersequent calculi for axiomatic extensions of IMTL (actually, of $\mathrm{IFL_e}$) was introduced in [9]. The algorithm is based on the following classification of Hilbert axioms: the classes $\mathcal{P}_n$ and $\mathcal{N}_n$ of positive and negative formulas are defined via the following grammar: ($\mathcal{P}_0 = \mathcal{N}_0 = V \cup V^{\perp}$ and $\star \in \{\oplus, \wedge\}$)

$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \odot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1$$
$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{N}_{n+1} \star \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1}$$

As shown in [9], in presence of weakening, every $\alpha \in \mathcal{P}_3$ which in addition satisfies the *acyclicity* property, can be transformed into a finite set R_α of *analytic* structural hypersequent rules, i.e. rules satisfying the following: for each $(r) \in R_\alpha$

- (r) consists only of metavariables for multisets of formulas and hypersequent contexts.
- Each metavariable for a multiset of formulas occurs at most once in the conclusion.
- Each metavariable for a multiset of formulas occurring in the premises of (r) also occurs in the conclusion

Given a calculus $\mathcal{C}$ and a set of rules R, $\mathcal{C} + R$ will denote the calculus obtained by adding the elements of R to $\mathcal{C}$, and $\vdash_{\mathcal{C}+R}$ its derivability relation.

Theorem 1: [9] Let α be any acyclic $\mathcal{P}_3$-axiom. $\Gamma \vdash_{\mathrm{HIMTL}+\alpha} \gamma \;\Leftrightarrow\; \Gamma \vdash_{\mathrm{HIMTL}+R_\alpha} \gamma$. Moreover the calculus HIMTL $+ R_\alpha$ admits cut-elimination.

[2]Structural rules are rules that do not mention any connective.

III. Density Elimination

Consider the extensions of HIMTL $+ R$, where R is any set of analytic rules, with the *density rule* ($p \notin \Gamma, \Delta, G$):

$$\frac{G \mid \Gamma, p \mid \Delta, p^{\perp}}{G \mid \Gamma, \Delta}\ (D)$$

Note that in hypersequent calculi (D) is (syntactically) similar to (cut).

We prove that, when the rules in R have a certain shape (*Inv-SA* rules), (D) is *eliminable*, i.e. that any derivation containing applications of (D) can be transformed into a derivation of the same end-hypersequent which does not contain (D) (a (D)-free derivation). Our proof follows the idea, introduced for non involutive logics in [8] and applied to HIMTL in [19], to remove applications of (D) by suitable substitutions. Given a cut-free derivation $\mathfrak{D}$ ending in an application of (D) (below left) we replace all the occurrences of p and $p^{\perp}$ in an "asymmetric" way: with Δ and Γ, respectively. The last step of the derivation is then transformed into an application of (ec) (below right):

$$\frac{\begin{array}{c}\vdots\mathfrak{D}\\ G \mid \Gamma, p \mid \Delta, p^{\perp}\end{array}}{G \mid \Gamma, \Delta}\ (D) \qquad \frac{\begin{array}{c}\vdots\mathfrak{D}^*\\ G \mid \Gamma, \Delta \mid \Delta, \Gamma\end{array}}{G \mid \Gamma, \Delta}\ (ec)$$

However, the labeled tree $\mathfrak{D}^*$ obtained after these replacements might not be a correct derivation anymore. $\mathfrak{D}$ might indeed contain the following components:

$$\Theta, p, p^{\perp} \quad \text{and} \quad G \mid \Theta, p, p \mid \Theta', p^{\perp}, p^{\perp}$$

which are no longer derivable after the asymmetric substitution. We refer to $\Theta, p, p^{\perp}$ as a *pp-component*. A sequent P_i which contains more than one occurrence of p ($p^{\perp}$) is called *pp-complement* of a sequent P_j that contains more than one occurrence of $p^{\perp}(p)$.

Looking at the derivation $\mathfrak{D}$ bottom-up, it is clear that these "problematic" components can only originate from applications of specific structural rules. Moreover pp-complements are possible only in involutive logics (and hence they are not considered in the proofs of [2], [8]).

For HIMTL the only source of problems is the (com) rule, which is handled in the density elimination proof in [19]. Here we identify a large class of analytic structural rules extending HIMTL that are shown in Theorem 2 to allow for density elimination. Although inspired by the conditions in [2] and [3] the criteria below are stronger, and reflect the greater difficulty of dealing with calculi for involutive logics. Consider, for example, the contraction rule (c). Its addition to the calculus for MTL results in a system that admits density elimination (and in a logic – Gödel logic– that is standard complete). In contrast, the density rule is *not* eliminable in HIMTL extended with contraction. This is indeed a calculus for classical logic, which is clearly not standard complete.

To define our sufficient conditions consider any analytic rule

$$\frac{G \mid P_1 \ \cdots \ G \mid P_m}{G \mid C_1 \mid \ \cdots \ \mid C_q}\ (\mathrm{r})$$

$$\frac{}{G|\alpha,\alpha^{\perp}}\ (ax) \qquad \frac{G}{G|\Gamma}\ (ew) \qquad \frac{G|\Gamma|\Gamma}{G|\Gamma}\ (ec) \qquad \frac{G|\Gamma,\alpha \quad G|\Delta,\beta}{G|\Gamma,\Delta,\alpha\odot\beta}\ (\odot) \qquad \frac{G|\Gamma,\alpha_i}{G|\Gamma,\alpha_1\vee\alpha_2}\ (\vee)_i\ {\scriptstyle (i=1,2)} \qquad \frac{}{G|1}\ (1)$$

$$\frac{G|\Gamma}{G|\Gamma,\alpha}\ (w) \qquad \frac{G|\Gamma,\alpha \quad G|\Delta,\alpha^{\perp}}{G|\Gamma,\Delta}\ (cut) \qquad \frac{G|\Gamma_1,\Delta_2 \quad G|\Gamma_2,\Delta_1}{G|\Gamma_1,\Delta_1|\Gamma_2,\Delta_2}\ (com) \qquad \frac{G|\Gamma,\alpha,\beta}{G|\Gamma,\alpha\oplus\beta}\ (\oplus) \qquad \frac{G|\Gamma,\alpha \quad G|\Gamma,\beta}{G|\Gamma,\alpha\wedge\beta}\ (\wedge)$$

Fig. 1. Hypersequent system HIMTL

We denote by $Var(r)$ the set of metavariables occurring in its conclusion and by $Var(C_s)$ the metavariables occurring in a component C_s.

Definition 2: Let $A \subseteq Var(r)$. A is said to be *anchored* if $A \subseteq Var(C_s)$ for a component C_s of the rule conclusion. We say that $\Sigma \in Var(r)$ *anchors* A if there is a $\Lambda \in A$ such that $\{\Sigma, \Lambda\}$ is anchored. We say that $B \subseteq Var(r)$ anchors A if each $\Sigma \in B$ anchors A.

Let $A_1, A_2 \subseteq Var(r)$. (A_1, A_2) is a *unanchored pair*[3] iff there is no $\Lambda \in A_1$ such that Λ anchors A_2.

Definition 3: An analytic hyperstructural rule (r) is *semi-anchored* (*SA* for short) if for any unanchored pair (A_1, A_2) contained in a premise $G \,|\, P_i$, there is a premise of the form $G \,|\, P_i[B/A_1]$ (or $G \,|\, P_i[B/A_2]$) where $B \subseteq Var(r)$ anchors A_2 (resp. A_1).

Definition 4: An *SA* rule (r) is *involutive* (*Inv-SA* for short) if for any multiset Θ^n ($n \geq 2$) in a premise $G \,|\, \Pi, \Theta^n$ ($\Theta \notin \Pi$) there are either

1) Another premise $G \,|\, \Pi, \Theta^s, \Lambda^m$ such that $s = \{0, 1\}$, $0 < m \leq n - s$, and $\{\Theta, \Lambda\}$ is anchored, or
2) 2 other premises $G \,|\, \Pi, \Theta^s, \Lambda_1^{m_1}, G \,|\, \Pi, \Theta^s, \Lambda_2^{m_2}$ where $s \in \{0, 1\}$, $0 < m_i \leq n - s$, and $\{\Lambda_1, \Lambda_2\}$ is anchored.

Example 1: Consider the analytic rules

$$\frac{G \,|\, \Pi, \Theta_1^n, ..., G \,|\, \Pi, \Theta_{n-1}^n}{G \,|\, \Pi, \Theta_1, ..., \Theta_{n-1}}\ (c_n) \quad \frac{G \,|\, \Theta^2, \Lambda, \Pi \quad G \,|\, \Theta, \Lambda^2, \Pi}{G \,|\, \Theta, \Lambda, \Pi}\ (n)$$

$$\frac{G \,|\, \Theta_1, \Theta_2, \Pi \quad G \,|\, \Theta_1, \Theta_3, \Pi \quad G \,|\, \Theta_1, \Theta_1, \Pi \quad G \,|\, \Theta_2, \Theta_3, \Pi}{G \,|\, \Theta_1, \Pi \,|\, \Theta_2, \Theta_3}\ (wnm)$$

$$\frac{\{G \,|\, \Pi, \Theta_i^n, \Theta_{i+(2p-1)}^n\}_{n \leq i \leq (3n-2p)}^{1 \leq p \leq n} \quad \{G \,|\, \Pi, \Theta_i^n, \Theta_j^n\}_{1 \leq j \leq (3n-1)}^{1 \leq i \leq (n-1)}}{G \,|\, \Theta_n, ..., \Theta_{3n+1} \,|\, \Pi, \Theta_1, ..., \Theta_{n-1}}\ (wnm^n)$$

which correspond through the algorithm in [9] to the axioms:

- $\alpha^{n-1} \to \alpha^n$ $\quad (c_n)$
- $(\alpha^2 \odot \beta) \vee (\alpha \odot \beta^2) \oplus \alpha^{\perp} \oplus \beta^{\perp}$ $\quad (n)$
- $(\alpha \odot \beta)^{\perp} \vee ((\alpha \wedge \beta) \to (\alpha \odot \beta))$ $\quad (wnm)$
- $((\alpha \odot \beta)^n)^{\perp} \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \odot \beta)^n)$ $\quad (wnm^n)$

All of them are *Inv-SA*, except for contraction (c_2), which is *SA* but not involutive-*SA*.

The following lemma, first proved in [3] for extensions of MTL will be useful for the density elimination proof.

Lemma 1: Let $\mathfrak{D}$ be a cut-free derivation of $G \,|\, \Gamma, p \,|\, \Delta, p^{\perp}$ (with $p, p^{\perp} \notin G, \Gamma, \Delta$) in HIMTL extended with any analytic rule. Then the rule

$$\frac{G' \,|\, \Lambda, \Theta}{G \,|\, G' \,|\, \Theta, \Gamma \,|\, \Lambda, \Delta}\ (S_{\mathfrak{D}})$$

is cut-free derivable in the same system.

Proof: Follows by applying (cut) with the hypersequent $G \,|\, \Gamma, \Theta \,|\, \Delta, \Theta^{\perp}$ whose derivation is obtained by replacing p ($p^{\perp}$) in $\mathfrak{D}$ with Θ ($\Theta^{\perp}$). The (cut) thus introduced can then be removed by Theorem 1. ∎

Definition 5: A hypersequent G is p-regular if G contains neither pp-components nor pp-complements.

We show below that *Inv-SA* rules behave well with respect to the asymmetric substitutions which replace each hypersequent H with $H[\Delta/p][\Gamma/p^{\perp}]$ – henceforth denoted by H^*. Indeed, for any premise H of an *Inv-SA* rule (r) that is not p-regular, we can always find other premises of (r) from which we can derive H^* with possibly additional harmless components.

Theorem 2: Let R be any set of *Inv-SA* rules, $\mathfrak{D}$ any cut-free derivation of $G \,|\, \Gamma, p \,|\, \Delta, p^{\perp}$ ($p, p^{\perp} \notin G, \Gamma, \Delta$) in HIMTL + R. For any p-regular hypersequent H, we can find a cut-free derivation of $G \,|\, H^*$ in HIMTL + R.

Proof: By induction on the length of the derivation $\mathfrak{D}'$ of H, distinguishing cases according to the last rule (r) applied. When (r) is any rule of HIMTL the proof proceeds as in [19]. Suppose now that $\mathfrak{D}'$ ends in an *Inv-SA* rule (r)

$$\frac{\begin{matrix}\vdots \\ H' \,|\, P_1\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ H' \,|\, P_m\end{matrix}}{H' \,|\, C_1 \,|\, \dots \,|\, C_q}\ (r)$$

We show how to obtain, for any premise $H' \,|\, P_i$ of (r), a cut-free derivation of $G \,|\, {H'}^* \,|\, P_i^* \,|\, \mathcal{C}^*$, where $\mathcal{C}$ is a (possibly empty) hypersequent of the form $C_j \,|\, \ \dots \,|\, C_l$, for $C_j, \dots, C_l$ arbitrary components of the conclusion of (r). The claim then follows by (r) and (possibly many applications of) (ec). Note that, unless specified otherwise, in the following $\Theta, \Lambda, \dots$ will refer to the metavariables occurring in a rule schema, while $H', P_i, \dots$ to concrete instances of (hyper)sequents in $\mathfrak{D}'$. If a premise $H' \,|\, P_i$ is p-regular, the required derivation $G \,|\, H'^* \,|\, P_i^*$ follows by the i.h. Otherwise, we distinguish cases according to the "problematic" components contained in $H' \,|\, P_i$: (I) pp-complements and (II) pp-components.

(I) *P_i is not a pp-component, contains more than one occurrence of either p or $p^{\perp}$, and H' contains its pp-complement.* Assume P_i contains p, p (the case for $p^{\perp}$ being similar). Let P_i be of the form $\Theta_1^{n_1}, \dots, \Theta_k^{n_k}, \Pi$, where (the instantiation of) Π does not contain p or $p^{\perp}$ and (the instantiation of) each Θ_i contains p. We proceed by induction on k.

(*base case*) Only (the instantiation of) one metavariable Θ contains occurrences of p, i.e. $P_i = \Theta^n, \Pi$. By Def. 4 one of the following holds ($s \in \{0, 1\}$, $0 < m, m_1, m_2 \leq n - s$):

I.1 *There is a premise $H' \,|\, P_j$ such that $P_j = \Theta^s, \Lambda^m, \Pi$ with $\{\Theta, \Lambda\}$ anchored.*

I.2 *There are 2 premises $H' \,|\, P_{\Theta 1} = H' \,|\, \Theta^s, \Lambda_1^{m_1}, \Pi$ and $H' \,|\, P_{\Theta 2} = H' \,|\, \Theta^s, \Lambda_2^{m_2}, \Pi$, with $\{\Lambda_1, \Lambda_2\}$ anchored.*

[3]It is easy to see that (A_1, A_2) is a *unanchored pair* iff so is (A_2, A_1).

(I.1) Since $\{\Theta, \Lambda\}$ is anchored, both metavariables will occur in a component of the p-regular conclusion of (r). Hence (the instantiation of) Λ cannot contain any occurrence of p or $p^\perp$ and $H' \mid P_j$ has to be p-regular. Therefore, by i.h., we get a cut-free derivation of $G \mid H'^* \mid P_j^* = G \mid H'^* \mid \Theta^{*s}, \Lambda^m, \Pi$. Consider the derivation

$$\mathfrak{D}'' \quad \cfrac{\cfrac{G \mid H'^* \mid \Theta^{*s}, \Lambda^m, \Pi}{G \mid H'^* \mid \Theta^{*s}, \Delta, \Lambda^{m-1}, \Pi \mid \Lambda, \Gamma}\,(S_{\mathfrak{D}})}{\cfrac{G \mid H'^* \mid \Theta^{*s}, \Delta, \Lambda^{m-1}, \Pi \mid \Lambda, \Delta \mid \Gamma, \Gamma}{\vdots\,(S_{\mathfrak{D}}) \times (2m-1) \atop G \mid H'^* \mid \Theta^{*s}, \Delta^m, \Pi \mid \Lambda, \Delta \mid \Gamma, \Gamma \mid \ldots \mid \Lambda, \Delta \mid \Gamma, \Gamma}}\,(S_{\mathfrak{D}})$$

We then apply (w) to the components of the hypersequent above as follows: we turn $\Theta^{*s}, \Delta^m, \Pi$ into P_i^*; each component Λ, Δ into C_l^* for a component of the conclusion C_l (recall that $\{\Lambda, \Theta\}$ is anchored and the instantiation of Θ contains p); each component Γ, Γ into the substituted version of the pp-complement of P_i (which already occurs in H'^*). The desired derivation of $G \mid H'^* \mid P_i^* \mid C_l^*$ follows by suitable applications of (ec).

(I.2) Let the premises $H' \mid P_{\Theta 1} = H' \mid \Theta^s, \Lambda_1^{m_1}, \Pi$ and $H' \mid P_{\Theta 2} = H' \mid \Theta^s, \Lambda_2^{m_2}, \Pi$ be such that $\{\Lambda_1, \Lambda_2\}$ is anchored. We distinguish four sub-cases according to the presence of p and $p^\perp$ in (the instantiations of) Λ_1,Λ_2.

(I.2a) $p \in \Lambda_1$. Being $\{\Lambda_1, \Lambda_2\}$ anchored, Λ_2 cannot contain p or $p^\perp$, as otherwise the conclusion would not be p-regular. Then, the i.h. leads to a cut-free derivation of $G \mid H'^* \mid P_{\Theta 2}^*$. From this, we can obtain $G \mid H'^* \mid P_i^* \mid C_l^*$ by a derivation similar to $\mathfrak{D}''$ followed by applications of (w).

(I.2b) $p^\perp \in \Lambda_1, p^\perp \notin \Lambda_2$. If $s = 0$, i.e. Θ does not occur in P_{Θ_1}, the latter is p-regular and we can get $G \mid H'^* \mid P_i^* \mid C_l^*$ proceeding as in $\mathfrak{D}''$ in (I.1). Otherwise, let us consider the premise $H' \mid P_{\Theta 2}$. Since $\{\Lambda_1, \Lambda_2\}$ is anchored, Λ_2 cannot contain p. Therefore $H' \mid P_{\Theta 2}$ is p-regular and by i.h. we have a cut-free derivation of $G \mid H'^* \mid P_{\Theta 2}^* = G \mid H'^* \mid \Theta^{*s}, \Lambda_2^{m_2}, \Pi$. We can then derive $G \mid H'^* \mid P_i^* \mid C_l^*$ as follows:

$$\cfrac{G \mid H'^* \mid \Pi, \Theta^{*s}, \Lambda_2^{*m_2}}{\cfrac{G \mid H'^* \mid \Pi, \Theta^{*s}, \Delta, \Lambda_2^{m_2-1} \mid \Lambda_2, \Gamma}{\vdots\,(S_{\mathfrak{D}}) \times (m-1) \atop G \mid H'^* \mid \Pi, \Theta^{*s}, \Delta^{m_2} \mid \Lambda_2, \Gamma \mid \ldots \mid \Lambda_2, \Gamma}}\,(S_{\mathfrak{D}})$$

Since Δ occurs in Θ^* and Γ in Λ_1^*, we can use (w) to turn the sequent $\Pi, \Theta^*, \Delta^{m_2}$ into Π, Θ^n and each sequent Λ_2, Γ into Λ_1^*, Λ_2. Suitable applications of (ec) lead to $G \mid H'^* \mid P_i^* \mid C_l^*$.

(I.2c) $p^\perp \in \Lambda_1$, $p^\perp \in \Lambda_2$. If $s = 0$, as before, P_{Θ_1} is p-regular and we proceed as in $\mathfrak{D}''$ in (I.1). If $s = 1$, $(\{\Theta\}, \{\Lambda_1\})$ is a unanchored pair, as otherwise the conclusion would not be p-regular. Then, by Def. 3 there is a premise $H' \mid P_l$ such that either $P_l = P_{\Theta_1}[\Sigma/\Lambda_1]$ and Σ anchors Θ, or $P_l = P_{\Theta_1}[\Sigma/\Theta]$ and Σ anchors Λ_1. In the first case, since Σ anchors Θ, its instantiation can contain neither p nor $p^\perp$. Then, $H' \mid P_l$ is p-regular and by i.h. we have a cut-free derivation of $G \mid H'^* \mid P_l^*$, from which we can derive $G \mid H'^* \mid P_i^* \mid C_l^*$, proceeding as in $\mathfrak{D}''$ in (I.1). In case we have $P_l = P_{\Theta_1}[\Sigma/\Theta]$ with $\{\Sigma, \Lambda_1\}$ anchored, (the instantiation of) Σ cannot contain p. Hence, again, $H' \mid P_l$ is p-regular and by i.h. we have a cut-free derivation of $G \mid H'^* \mid P_l^*$. From here, by a single application of $S_{\mathfrak{D}}$, we obtain $G \mid H'^* \mid \Pi, \Theta^*, \Lambda_1^{*m_1} \mid \Sigma, \Gamma$; since Γ occurs in Λ_1^*, we can apply (w) to the sequent Σ, Γ and get Σ, Λ_1^*. Therefore, we have $G \mid H'^* \mid P_{\Theta_1}^* \mid C_l^*$, from which we can obtain $G \mid H'^* \mid P_i^* \mid \mathcal{C}^*$ proceeding as in case (I.2b).

(I.2d) Λ_1 and Λ_2 do not contain any occurrence of p or $p^\perp$. Hence, both $P_{\Theta 1}$ and $P_{\Theta 2}$ are p-regular and by i.h. we have cut-free derivations of $G \mid H'^* \mid P_{\Theta 1}^*$ and $G \mid H'^* \mid P_{\Theta 2}^*$. From these hypersequents, proceeding as in $\mathfrak{D}''$ in (I.1), we can get the premises of the following application of (com):

$$\frac{G \mid H'^* \mid P_i^* \mid \Lambda_1, \Delta \quad G \mid H'^* \mid P_i^* \mid \Lambda_2, \Delta}{G \mid H'^* \mid P_i^* \mid \Lambda_1, \Lambda_2 \mid \Delta, \Delta}\,(com)$$

Since Δ occurs in Θ^*, we can apply (w) to the sequent Δ, Δ, obtaining $\Pi, \Theta^{*n} = P_i^*$, that can be removed by (ec). Moreover, since $\{\Lambda_1, \Lambda_2\}$ is anchored, Λ_1 and Λ_2 occur in the same component C_l of the conclusion. Therefore, by an application of (w) to the sequent Λ_1, Λ_2 we get C_l^*, obtaining the desired derivation of $G \mid H'^* \mid P_i^* \mid C_l^*$.

(*inductive step*) (The instantiation of) more than one multiset in P_i contains occurrences of p, i.e. $P_i = \Theta_1^{n_1}, \ldots, \Theta_k^{n_k}, \Pi$. Consider the premise $H \mid P_i$, let $A_{\Theta_1} = \{\Theta_k\}$ and $A_{\Theta_2} = \{\Theta_1, \ldots, \Theta_{k-1}\}$. Note that no $\Theta_i \in A_{\Theta_2}$ can anchor $\{\Theta_k\}$. Hence, $(A_{\Theta_1}, A_{\Theta_2})$ has to be a unanchored pair, as otherwise the conclusion of (r) would not be p-regular. Thus, by Def. 3 there is a premise $H' \mid P_l$ such that either $H' \mid P_l = H' \mid P_i[B/A_{\Theta_1}]$ where B anchors A_{Θ_2} or $H' \mid P_l = H' \mid P_i[B/A_{\Theta_2}]$ where B anchors A_{Θ_1}. Since in both cases B cannot contain p or $p^\perp$, the induction hypothesis gives us $G \mid H'^* \mid P_l^* \mid \mathcal{C}^*$. From the latter we can easily derive $G \mid H'^* \mid P_i^* \mid \mathcal{C}^*$, proceeding as in $\mathfrak{D}''$ in (I.1).

(II) P_i *is a pp-component*.

Assume that $P_i = \Theta, \Lambda_1^{i_1}, \ldots, \Lambda_k^{i_k}, \Lambda_{k+1}^{i_{k+1}} \ldots, \Lambda_n^{i_n}$, where $A_1 = \{\Lambda_1, \ldots, \Lambda_k\}$ is the set of metavariables whose instantiation contains some ps and $A_2 = \{\Lambda_{k+1}, \ldots, \Lambda_n\}$ the set of those containing some $p^\perp$s. Note that (A_1, A_2) is a unanchored pair (no $\Lambda_i \in A_1$ can anchor a $\Lambda_j \in A_2$, as otherwise the conclusion of (r) would not be p-regular). We consider the subcases:

(II.1) $H' \mid P_i$ does not contain a pp-complement of P_i.

By Def. 3 we have either a premise $H' \mid P_j$ of the form $P_i[B_1/A_2]$, where $B_1 = \{\Sigma_{k+1}, \ldots, \Sigma_n\}$ anchors A_1, or of the form $P_i[B_2/A_1]$, where $B_2 = \{\Sigma_1, \ldots, \Sigma_k\}$ anchors A_2. Consider the first case, the other being similar. The premise $H' \mid P_j$ is p-regular, as $p^\perp$ does not occur in the instantiation of any Σ_l in B_1. We thus have a derivation of $H'^* \mid P_j^*$. Applying repeatedly $S_{\mathfrak{D}}$, we obtain

$$(*)\ H'^* \mid \Theta, \Lambda_1^{i_1}, \ldots, \Lambda_k^{i_k}, \Gamma^{i_{k+1}}, \ldots, \Gamma^{i_n} \mid \Sigma_{k+1}, \Delta \mid \ldots \mid \Sigma_n, \Delta$$

Note that any $\Sigma_l \in B_1$ anchors a $\Lambda_i \in A_1$, and that any Λ_i^* contains Δ (as any instance of $\Lambda_i \in A_1$ contains p). Hence, by applying (w) to any sequent in $(*)$ of the form Σ_l, Δ, we obtain a component C_s^* of the conclusion. By applying (w) to the component $\Theta, \Lambda_1^{i_1}, \ldots, \Lambda_k^{i_k}, \Gamma^{i_{k+1}}, \ldots, \Gamma^{i_n}$ of (*), we obtain P_i^*. Hence we get a derivation of $G \mid H'^* \mid P_i^* \mid \mathcal{C}^*$.

(II.2) H' contains a pp-complement of P_i, e.g. with $p^\perp, p^\perp$ (the other case is analogous). By Def. 3, there is a premise $H' \,|\, P_j$ such that either $P_j = P_i[B/A_2]$ and B anchors A_1, or $P_j = P_i[B/A_1]$ and B anchors A_2. In the first case, $H' \,|\, P_j$ is not immediately p-regular (recall that any instance of Λ_i in A_1 contain p and H' contains the pp-complement of P_i), but we can obtain a cut-free derivation of $G \,|\, H'^* \,|\, P_j^*$ proceeding as in (I). In the second case, $H' \,|\, P_j$ has to be p-regular and we get $G \,|\, H'^* \,|\, P_j^*$ by i.h. In both cases, the desired derivation of $G \,|\, H'^* \,|\, P_i^* \,|\, \mathcal{C}^*$ is obtained by $G \,|\, H'^* \,|\, P_j^*$ as in (II.1). ■

Remark 1: The above proof relies essentially on the presence of (w).

Corollary 1: HIMTL + (D) + *Inv-SA* rules admits density elimination.

Proof: By Th. 2 we can replace the top-most application of (D) with a cut-free, (D)-free derivation of the same hypersequent. The claim follows as shown, e.g., in [19]. ■

IV. From Density Elimination to Standard Completeness

The results on density elimination for hypersequent calculi in Section III lead to the rational completeness of the corresponding logics, i.e. the completeness w.r.t. algebras over the rationals on $[0, 1]$ (see, e.g. [2], [18], [19]). Proving that such algebras can be embedded into complete ones (in other terms, that they are preserved under completions) provides the final step towards standard completeness.

Preservation of algebras under completions is a widely investigated topic and general proofs have been provided, e.g., in [7]. The idea there is to transform suitable classes of algebraic equations into equivalent so-called analytic structural clauses and then show the preservation of the latter under Dedekind-Macneille completion (DM-completion in the following). This mirrors the transformation of axioms into analytic rules shown in [6] for non-involutive logics and in [9] for the involutive ones. However, unlike the proof-theoretic counterpart, the algebraic results in [7] have not been so far extended to involutive algebras. This issue is addressed in the following.

We assume the reader to be familiar with basic notions of universal algebra, such as the satisfaction of an equation by an algebra and the consequence relation $\models_K$ determined by a class of algebras K. For these and other unexplained concepts below we refer, e.g., to [4], [14].

It is relatively simple to provide completeness results for the logic IMTL and its extensions w.r.t. to a general class of algebraic structures, the so-called involutive FL_{ew}-chains. We recall their definition in the following.

Definition 6: An FL_{ew}-algebra is a structure $\boldsymbol{A} = (A, \wedge, \vee, \rightarrow, \odot, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid, and for any $x, y, z \in A$ the *residuation* property holds, i.e. $x \odot y \leq z \iff x \leq y \rightarrow z$. Negation in a FL_{ew}-algebra is defined as $\neg x = x \rightarrow 0$. An *involutive* FL_{ew}-algebra, IFL_{ew}-algebra for short, is an FL_{ew}-algebra satisfying $\neg\neg x = x$. An IFL_{ew}-algebra is said to be a *chain* if the lattice ordering is total.

Note that in IFL_{ew}-algebras $x \oplus y = \neg(\neg x \odot \neg y)$[4].

For any logic L which extends IMTL with a set of axioms $\alpha_1, \ldots, \alpha_n$, let us call L-algebras the corresponding class of IFL_{ew}-algebras satisfying the equations $1 \approx \alpha_1, \ldots, 1 \approx \alpha_n$. We call L-chains the L-algebras which are chains. Useful general completeness results from the literature are:

Theorem 3: For any axiomatic extension L of IMTL, calculus HL for L, and set of formulas $T \cup \gamma$ of L:

1) $T \vdash_{\text{L}} \gamma \Leftrightarrow \{1 \approx \beta\}_{\beta \in T} \models_K 1 \approx \gamma$ where K is the class of L-chains.
2) If the density rule is admissible for HL then K in 1) can be restricted to the class of *dense* L-chains.

Proof: 1. follows, e.g., from [10], 2. from [19]. ■

Let L be any axiomatic extension of IMTL with a set of acyclic $\mathcal{P}_3$ axioms $\{\alpha_1, \ldots, \alpha_n\}$. We say that L is an *Inv-SA* logic if all the rules in $R_{\alpha_1} \cup \cdots \cup R_{\alpha_n}$ obtained using the algorithm in [9] are *Inv-SA* rules. Theorems 1-3 ensure that any *Inv-SA* logic L is complete w.r.t. the corresponding class of dense L-chains. The latter are order-isomorphic to the rationals on $[0, 1]$, hence our logics are rational complete. We now show that any dense L-chain is embeddable into a complete one. Towards this aim, adapting from [7] we first introduce the notions of structural and analytic clauses, which mirror the corresponding notions for hypersequent rules.

Definition 7: A *structural clause* is a classical first-order formula of the form:

$$1 \approx t_1 \text{ and ... and } 1 \approx t_m \Rightarrow 1 \approx t_{m+1} \text{ or ... or } 1 \approx t_n \quad (q)$$

where *and, or,* $\Rightarrow$ are the classical connectives and any t_i is of the form $x_{i1} \oplus \cdots \oplus x_{ik_i}$, with x_{ij} variable or constant. (q) is a *quasiequation* if $n = m + 1$.

As usual, an algebra $\boldsymbol{A}$ *satisfies* (q) if, for any evaluation v into $\boldsymbol{A}$, if $1 = v(t_i)$ for all premises $1 \approx t_i$, then there is at least one conclusion $1 \approx t_j$ such that $1 = v(t_j)$.

Definition 8: A structural clause (q) is *analytic* if: (*Linearity*) each variable in $t_1, \ldots, t_m$ occurs exactly once in the conclusions, and (*Inclusion*) each of $t_1, \ldots, t_m$ is a $\oplus$-sum of variables in $t_{m+1}, \ldots, t_n$.

Clearly analytic clauses correspond to analytic rules. More formally, let (r) be any analytic hypersequent rule of the form

$$\frac{G \,|\, \Gamma_{11}, \ldots, \Gamma_{1k_1} \qquad G \,|\, \Gamma_{m1}, \ldots, \Gamma_{mk_m}}{G \,|\, \{\Gamma_{i1}, \ldots, \Gamma_{ik_i}\}_{i=m+1}^{n}} \ (r)$$

We call (q_r) the corresponding clause

$$\text{And } \{1 \approx x_{i1} \oplus \cdots \oplus x_{ik_i}\}_{i=1}^{m} \Rightarrow \text{ Or } \{1 \approx x_{i1} \oplus \cdots \oplus x_{ik_i}\}_{i=m+1}^{n}$$

where each (distinct) x_{ik} corresponds to the multiset metavariable Γ_{ik} in (r) and the comma is replaced by $\oplus$. It can be easily checked that (q_r) is an analytic clause.

Henceforth, we fix L to be any *Inv-SA* logic extending IMTL with axioms $\{\alpha_1, \ldots, \alpha_n\}$, $R = R_{\alpha_1} \cup \cdots \cup R_{\alpha_n}$ the set of corresponding analytic rules (see Th. 1), HL = HIMTL + R,

[4] IFL_{ew}-algebras can be equivalently defined over the signature $\{\wedge, \vee, \oplus, \neg, 0\}$.

and Q the set of corresponding clauses. We call Q_L-algebras the class of IFL_{ew}-algebras satisfying the clauses in Q.

Theorem 4: Let K be the class of L-chains and Q_K the class of Q_L-chains. We have $\models_K = \models_{Q_K}$.

Proof: By Th. 1 and 3.1, for any set of formulas $T \cup \gamma$:

$$(*) \quad T \vdash_{\text{L}} \gamma \;\Leftrightarrow\; T \vdash_{\text{HL}} \gamma \;\Leftrightarrow\; \{1 \approx \beta\}_{\beta \in T} \models_K 1 \approx \gamma$$

Hence for $\models_L \subseteq \models_{Q_L}$ it suffices to show the soundness of HL w.r.t. evaluations on Q_L algebras. This amounts to check that any rule in R is sound with respect to a clause Q, which is an easy task. For the other direction we show that any Q_L chain is an L-chain, i.e. it satisfies $1 \approx \alpha_1, \ldots, 1 \approx \alpha_n$. Note that, by $(*)$, any axiom α_i is derivable in the hypersequent calculus HL. The soundness of HL w.r.t. Q_L chains ensures that $1 \approx \alpha_1, \ldots, 1 \approx \alpha_n$ hold in any Q_L-chain. ■

By Theorem 4, showing preservation under completions for L-chains amounts at showing the same for Q_L chains. We will now show that analytic clauses are preserved under DM-completion. Recall [16] that a completion of an IFL_{ew}-algebra $\boldsymbol{A}$ is a pair $(\boldsymbol{B}, e)$ such that $\boldsymbol{B}$ is a complete IFL_{ew}-algebra and $e\colon A \to B$ is an embedding. In what follows, for simplicity we will usually identify the subalgebra $\boldsymbol{e(A)}$ of $\boldsymbol{B}$ with $\boldsymbol{A}$.

A DM completion of an algebra $\boldsymbol{A}$ is a *join-dense* and *meet-dense* completion $\boldsymbol{DM(A)}$ i.e. a completion such that for every $x \in DM(A)$, $x = \bigvee X = \bigwedge Y$ for some $X, Y \subseteq A$. It is a well known fact [16] that DM-completions are unique up to isomorphism and that the embedding $e\colon A \to DM(A)$ is regular, i.e. it preserves all existing meets and joins in $\boldsymbol{A}$. Adapting from [7], we prove the following

Theorem 5: If an IFL_{ew}-algebra $\boldsymbol{A}$ satisfies an analytic clause (q), its DM-completion $\boldsymbol{DM(A)}$ satisfies (q) as well.

Proof: Consider an evaluation v into $\boldsymbol{DM(A)}$ that satisfies the premises of (q) (cfr. Definition 7), i.e. $1 = v(t_i)$ for any $i = 1, \ldots, m$ and assume, by contradiction, that for any $j = m+1, \ldots, n$ we have $1 \neq v(t_j)$. If any t_j is of the form $x_{j1} \oplus \cdots \oplus x_{jk_j}$ we have $1 \neq v(t_j) = v(x_{j1}) \oplus \cdots \oplus v(x_{jk_j})$ for any $j = m+1, \ldots, n$. As A is meet dense in $\boldsymbol{DM(A)}$, we have that $v(x_{jl}) = \wedge X_{jl}$ for some $X_{jl} \subseteq A$. We can thus take an evaluation v' on $\boldsymbol{A}$ such that $v'(x_{jl}) \in X_{jl} \subseteq A$ for any $j = m+1, \ldots, n$ and

$$(*) \quad 1 \neq v'(x_{j1}) \oplus \cdots \oplus v'(x_{jk_j}).$$

The linearity of (q) (cfr. Definition 8) ensures that we are taking exactly one element for each X_{jl}. By the inclusion properties, v' will be defined for any variable on the left hand side of the clause. For any premise $1 \approx t_i$ we have now

$$v'(t_i) = v'(x_{i1}) \oplus \cdots \oplus v'(x_{ik_i}) \geq \bigwedge X_{i1} \oplus \cdots \oplus \bigwedge X_{ik_i} = v(t_i)$$

Since $v(t_i) = 1$, we have $v'(t_i) = 1$ for any premise of (q). Therefore one of the $1 \approx t_j$ in the conclusion will be satisfied as well by v', which contradicts (*). ■

Theorem 6: Any *Inv-SA* logic L is standard complete.

Proof: Theorems 1, 3 and Corollary 1 ensure that L is rational complete. By Theorem 4 and 5 the class of L-chains is preserved under DM-completions. It has been shown, e.g. in [19], that the DM-completion of a dense L-chain is still dense. There is thus an embedding from any dense L-chain into a complete dense one: its DM-completion. The latter is an L-algebra order-isomorphic to $[0, 1]$. Standard completeness follows by canonical arguments, e.g., as in [11]. ■

Corollary 2: IMTL extended with any of (wnm), (wnm^m) and (c_n), with $n > 2$ (see Example 1) is standard complete.

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