A Reduction Model for Dimensions and Incomplete Facts

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Abstract

We study models of case-based reasoning with factors and dimensions. We introduce a *reduction model* that maps reason-model cases to result-model cases by retaining only justificatory pro-reasons together with relevant opposing information. In the factor setting, the reduction is equivalent to the reason model. In the dimensional setting, it yields a systematically derived, principled model that we generalize to support reasoning with *incomplete fact situations*. Furthermore, we show how our framework can be encoded in Answer Set Programming, illustrated with a fiscal domicile scenario.

CCS Concepts

• **Theory of computation** → **Constraint and logic programming**; • **Applied computing** → *Law*.

Keywords

case-based reasoning, dimensions, reason model, ASP

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1 Introduction

Over the past years, precedential reasoning has attracted increasing attention in AI and Law. As a form of case-based reasoning (CBR), it relies on the premise that prior decisions provide guidance for resolving new cases. This principle is central to common law, where normative information is generated through prior judicial decisions rather than being only captured by statutory law.

The simplest formal model of precedential reasoning is the *result model* [Alexander 1989], which considers only case outcomes (decided for the plaintiff or the defendant) and their supporting facts. A precedent is followed when the facts of a new case are at least as favorable to the same outcome as those of the original case. A richer approach is provided by the *reason model*, which explicitly accounts for the reasons underlying outcomes. It was originally proposed by Horty and Bench-Capon [Horty and Bench-Capon 2012] and further developed in various directions over the past fifteen years (e.g. [Bench-Capon and Atkinson 2021; Canavotto 2025; Odekerken et al. 2023; Prakken 2021; van Woerkom et al. 2023]); see also [Horty 2025] for a canonical reference. One such development concerns the representation of case facts; while the original model of Horty and Bench-Capon adopts a binary representation (factors are either present or absent in a case), *dimensions* range over values, allowing for a more expressive representation of the cases. Although various extensions of the reason model to dimensions have been proposed, e.g. [Horty 2019, 2020; Prakken 2021; Rigoni 2018], none has achieved the same level of acceptance as the original model with binary factors. Another issue in this context is that case descriptions are rarely complete: judgments highlight some factual dimensions and ignore others, because, e.g. the missing facts are unknown, or not relevant to the court's reasoning. Works addressing incomplete case descriptions in the dimensional setting are, e.g. [Odekerken et al. 2023; Rigoni 2024]. However the representation of uncertain information in the form of ranges of possible values in [Odekerken et al. 2023] cannot handle scenarios in which some dimensions are completely absent from the fact situation, while [Rigoni 2024] only contains an informal discussion without formalizing it into a model.

In this paper, we propose a principled approach to formulating the reason model in a dimensional setting. We call the resulting model *reduction model* and further generalize it so that it

- applies to *incompletely specified fact situations*, and
- *infers additional precedential information* from the case base.

The reduction model takes as its starting point a reformulation of the reason model in terms of the result model, which motivates its name. This reformulation is methodological rather than foundational. In the factor-based setting, although equivalent to the original reason model it has the advantage of enabling the direct application of result-model techniques, for instance to deal with inconsistent case bases [Morello and Ciabattoni 2025; Morello et al. 2025]. In the dimensional setting we use this reformulation to define a model that remains close to the factor-based version while avoiding a collapse into the result model (important desiderata identified in [Horty 2019]). The resulting model turns out to be equivalent to a variant of [Prakken 2021] for incomplete rules, and its generalization –the reduction model– addresses cases with incomplete facts. Formalizing and extending ideas from [Rigoni 2024], we use the case base to infer additional information on the orientation of dimension values, which in turn binds a broader range of future decisions; this reflects common legal practice, where lawyers use multiple cases to show that certain facts support their decision.

Finally, we encode our framework in Answer Set Programming (ASP), one of the most successful paradigms for knowledge representation and reasoning for problem solving [Brewka et al. 2016], which enables automated consistency checking of case bases and the computation of binding outcomes.

Running Example: Tax Domicile

We illustrate the need to handle incomplete fact situations with the tax domicile scenario from [Prakken and Sartor 1998].

Example 1. *A court has to decide whether or not a defendant's fiscal domicile changed to a new domicile abroad. The cases are:*

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c_1: *the defendant has spent 18 months abroad, and the court ruled the case in her favor based on the reason that a stay longer than 12 months is sufficient for a change of domicile;*

c_2: *the defendant has earned* 85% *of her income abroad and the court ruled the case in her favor based on the fact that over* 75% *of income earned abroad is sufficient for a change of domicile.*

The precedent set by c_1 and c_2 should then force the court to decide a new case where the defendant has spent 15 months abroad and earned 80% of his income abroad in favor of the defendant as well, given that it exceeds the thresholds of both reasons for the previous decisions. This kind of reasoning is not possible in any of the precedential reasoning models mentioned before.

2 Preliminaries: Result and Reason models

We recall the basic precedential reasoning models we use, see [Horty 2019, 2020]. Decisions are binary: for the *plaintiff* π or the *defendant* δ. If s is one side, the opposite is $\bar{s}$. A set of cases is a case base.

2.1 Result and Reason model with factors

The information of a case is represented using *factors*, legally relevant aspects that can be present or absent in a case, and that always support either the plaintiff or the defendant.

Definition 1. *A* fact situation *F is a set of factors. We denote by F^s the subset of factors in F that support side s, so $F = F^s \cup F^{\bar{s}}$.*

Definition 2. *A* result model case *is a pair $\langle F, s\rangle$ with F a fact situation and $s \in \{\pi, \delta\}$.*

The central notion of the result model is that of *factual strength.*

Definition 3. *Given two fact situations F and G, F is* factually at least as strong for side s as G, *we write $G \trianglelefteq_s F$, iff (1) $G^s \subseteq F^s$, and (2) $G^{\bar{s}} \supseteq F^{\bar{s}}$. F is factually strictly stronger for s than G, $G \triangleleft_s F$ if at least one of the subset relations is strict.*

With this relation of factual strength, we can then define what it means for a case base to be consistent, and how a case base constrains the decision of a new case presented to the court. We will call such a new case a *focus case.*

Definition 4. *A case base Γ is inconsistent if and only if there exist two cases $a = \langle F_a, s\rangle$ and $b = \langle F_b, \bar{s}\rangle$ such that $F_a \trianglelefteq_s F_b$. A case base is consistent if and only if it is not inconsistent.*

We formulate the constraint imposed by the case base in terms of forced decisions, i.e. constraint determines when a focus case *must* be decided for a certain side.

Definition 5 (Result model constraint). *Let $c = \langle F, ?\rangle$ be a focus case. A consistent case base Γ* forces a decision of c for s *if and only if there exists a case $a \in \Gamma$ such that $a = \langle F_a, s\rangle$ and $F_a \trianglelefteq_s F$. The case a is then called a* precedent case for c.

The reason model adds to the case representation a *rule.* This rule provides a justification for the decision.

Definition 6. *A* reason for side s *is a set of factors supporting s. A fact situation F* satisfies *a reason U, written $F \models U$, if $U \subseteq F$.*

A rule *$r = X \rightarrow s$ has* premise *X (a reason for s) and* conclusion *$s \in \{\pi, \delta\}$. We write $Premise(r) = X$ and $Conclusion(r) = s$.*

Definition 7. *A* reason model case *$\langle F, r, s\rangle$ consists of a fact situation F, a rule r and an outcome $s \in \{\pi, \delta\}$, where*

$$F^s \models Premise(r) \text{ and } \quad Conclusion(r) = s$$

Based on a case, we can define a *preference relation* over reasons.

Definition 8. *Given a case $c = \langle F, r, s\rangle$, reason U for s is* preferred to *reason V for $\bar{s}$ according to c, written $V \prec_c U$ iff*

$$(1)\ Premise(r) \subseteq U \quad \text{and} \quad (2)\ V \subseteq F^{\bar{s}}.$$

To extend the relation to a case base Γ we say that $V \prec_\Gamma U$ if and only if there exists a case $c \in \Gamma$ such that $V \prec_c U$.

This preference relation takes the role of the factual strength relation for the definition of consistency.

Definition 9. *A case base Γ is* inconsistent *if and only if there exist two reasons U, V such that both $U \prec_\Gamma V$ and $V \prec_\Gamma U$ hold. A case base is* consistent *if and only if it is not inconsistent.*

Definition 10 (Reason model constraint). *A consistent case base Γ* forces a decision for *s of a focus case $c = \langle F, ?\rangle$ iff for all rules r with $Conclusion(r) = \bar{s}$ the case base $\Gamma \cup \langle F, r, \bar{s}\rangle$ is inconsistent.*

2.2 Result and Reason model with dimensions

The presence of factors in a given case is often a matter of degree, rather than a binary condition. Addressing this requires to consider factors with explicit values assigned. In the literature on precedential reasoning models, this is accomplished with *dimensions.*

Definition 11. *A* dimension *d is a set of values that are ordered by a total order $\leq_d^s$. A set of dimensions is called a* domain.

A dual order $\leq_d^{\bar{s}}$ can be obtained by inverting the original order $\leq_d^s$.

Definition 12. *A* fact situation *F over domain D is a set of assignments $\{\langle d_i, p_i\rangle\}$ for $1 \leq i \leq |D|$, with $d_i \in D$ and $p_i \in d_i$. We write $F(d)$ to refer to the value of dimension d in F.*

Note that in the standard dimensional setting a fact situation assigns a value to *every* dimension in the domain.

Result model cases, consistency and constraint are defined analogous to the factor-based model (Definitions 2, 4, and 5), however with a changed *factual strength relation* on fact situations.

Definition 13. *Given two fact situations F, G over domain D we say that F is* factually at least as strong for side s as G, *written $G \trianglelefteq_s F$, if and only if $G(d) \leq_d^s F(d)$ for all $d \in D$. F is factually strictly stronger for s than G written $G \triangleleft_s G$ if $G \trianglelefteq_s F$ and for at least one dimension $G(d) <_d^s F(d)$ holds.*

As before, we obtain the dimensional reason model by adding a *rule* to each case. For the dimensional setting, a rule consists of so-called *magnitude factors* that specify threshold values for dimensions and a conclusion that is one of the two sides.

Definition 14. *Let $d \in D$ be a dimension. A* magnitude factor *$M_{d,p}^s$* for side *s specifies the* threshold value *p for d. A set of magnitude factors for side s, where for every dimension d there exists at most one magnitude factor, is called a* reason *for s.*

We read a set of magnitude factors $\{M^s_{d,p}\}$ as "when each d is assigned a value greater or equal than p, then this reason supports side s". For a reason U with magnitude factor $M^s_{d,p}$, we write $U(d)$ to refer to the threshold p of the factor that mentions d. To define the preference relation, we need to redefine reason satisfaction and introduce the concepts of *reason entailment* and *negated reasons*.

Definition 15. *Let F be a fact situation and U be a reason for s. Then F* satisfies *U, written $F \models U$, if for all magnitude factors $M^s_{d,p} \in U$, $U(d) \leq^s_d F(d)$.*

Definition 16. *Let U, V be reasons for s. U* entails *V, written $U \Vdash V$, if for all fact situations F, $F \models U$ implies $F \models V$.*

Definition 17. *Let U be a reason for s. The* negated reason *$\overline{U}$ is a reason for $\bar{s}$, with $\overline{U} = \{M^{\bar{s}}_{d,p} \mid M^s_{d,p} \in U\}$.*

When it is clear from context we will write case for any kind of the considered cases. Rules and cases are defined analogous to the factor-based model (Definitions 6, and 7).

Definition 18. *Let $c = \langle F, r, s\rangle$ be a case. The* preference relation *$\prec_c$ is then defined as follows. Let U, V be reasons, then $U \prec_c V$ iff*

(1) *$V \Vdash Premise(r)$ and*
(2) *$F \models U$ or $\overline{Premise(r)} \Vdash U$.*

Let Γ be a case base, then $U \prec_\Gamma V$ iff there exists $c \in \Gamma$ s.t. $U \prec_c V$.

Reason model consistency and constraint are defined analogous to the factor-based model (Definitions 9 and 10).

3 Streamlining the reason model

We describe a *reduction* that maps the reason model to the result model, while maintaining consistency and constraints. Although the reduction is introduced as an intermediary step to handle dimensions, it remains useful in the factor-based setting, as it shows that all information relevant to the reason model can be represented at the level of fact situations alone. As a result, precedential constraints can be computed more easily, and result-model techniques (such as methods for handling inconsistent case bases [Morello and Ciabattoni 2025; Morello et al. 2025]) can be applied directly.

Let us briefly recall the main difference between the two models. In the result model, a precedential constraint is defined in terms of a relation of *factual strength* between fact situations. A fact situation is at least as strong for side s as F if it contains at least all factors favoring s in F and at most all factors favoring $\bar{s}$ in F. In the reason model a constraint is instead mediated by *reasons*. Each case justifies its outcome by a rule whose premise contains only a subset of the pro-s factors present in the fact situation. Preference between reasons is defined relative to these premises together with the opposing factors present in the case. Comparing them shows that, in the reason model, justification for the decision relies exclusively on the pro-s factors explicitly mentioned in the rule's premise, while the remaining pro-s factors play no role in determining the constraint. This observation is not new, [Liu et al. 2022] implicitly used it to encode the reason model into a modal logic, and [van Woerkom 2025] formalizes it.

Definition 19 (Reduction). *Let $c = \langle F, r, s\rangle$ be a factor-based reason-model case. The* reduction *of c is the result-model case*

$$rdct(c) := \langle X, s\rangle,$$

where $X = Premise(r) \cup F^{\bar{s}}$. We extend $rdct$ pointwise to case bases by defining $rdct(\Gamma) := \{rdct(c) \mid c \in \Gamma\}$.

Intuitively, the reduction discards those factors in the fact situation that do not contribute to the justification of the decision. Only the factors explicitly invoked by the rule in favor of s, together with the opposing factors present in the case, are retained.

Example 2. *Let $c = \langle F, r, \pi\rangle$ with $F = \{f^\pi_1, f^\pi_2, f^\delta_3, f^\delta_4\}$ and $Premise(r) = \{f^\pi_1\}$. Then $rdct(c) = \left\langle \{f^\pi_1, f^\delta_3, f^\delta_4\}, \pi\right\rangle$.*

The factor f^π_2, although present in the fact situation, is not part of the reason justifying the decision and is discarded by the reduction.

In practice, the reduction is applied as follows. To determine consistency, we first apply the reduction to a reason model case base Γ and then check consistency in the result model (Def. 4). For constraint, it suffices to test whether the reduced case base forces a decision of a focus case $c = \langle F, ?\rangle$ according to the result model (Def. 5). This follows from the observation that, in the reason model, to check whether there exists a rule for side s that is consistent with Γ, it is enough to consider the rule $r = F^s \rightarrow s$.

Importantly, this does not collapse the reason model into the result model: although constraint is computed using the result-model mechanism, it is applied to the *reduced* case base rather than the original one. We clarify this distinction in Remark 1.

For computations, when a focus case is decided, the case that is added to Γ is its reduction.

We formalize this intuition. For the remainder of this section, let Γ denote an arbitrary factor-based reason-model case base.

Proposition 1 (Faithfulness of reduction). *Γ is consistent according to the reason model iff $rdct(\Gamma)$ is consistent according to the result model.*

Proof Sketch. Γ is inconsistent if and only if there is a pair of cases $c = \langle F, r, s\rangle$ and $c' = \langle F', r', \bar{s}\rangle$ such that $F \models Premise(r')$ and $F' \models Premise(r)$ which by definition is equivalent to $Premise(r') \subseteq F^{\bar{s}}$ and $Premise(r) \subseteq F'^s$ and again by definition to $Premise(r) \cup F^{\bar{s}} \trianglelefteq_s Premise(r') \cup F'^s$. As the reduced cases $rdct(c)$ and $rdct(c')$ are $\left\langle Premise(r) \cup F^{\bar{s}}, s\right\rangle$ and $\langle Premise(r') \cup F'^s, \bar{s}\rangle$ respectively, we get that Γ is inconsistent if and only if so is $rdct(\Gamma)$. □

Corollary 1. *Let Γ be consistent, and $c = \langle F, ?\rangle$ be a focus case. Γ forces a decision for side s on c according to the reason model iff $rdct(\Gamma)$ forces a decision for s on c according to the result model.*

Remark 1. *Our reduction should not be confused with the "collapse" discussed below. Collapse means that, on the* same *case base, the reason model induces exactly the same constraints as the result model, so the stated reasons play no role. By contrast, the reduction* uses *the reasons to transform each case: it keeps the deciding premise, discards unused pro-information, and preserves countervailing information.*

4 A Dimensional Reason Model via Reduction

The reduction of the reason model to the result model will serve us as a blueprint for obtaining a reason model for dimensions. Here we recall some existing dimensional reason models and identify five desiderata that a model should satisfy.

4.1 Other dimensional reason models

We discuss three main[1] proposals for dimensional reason models.

Horty's dimensional reason model. In [Horty 2019] reasons consist of *magnitude factors* specifying threshold values for dimensions. This definition however suffered from what Horty calls *collapse*: the model ended up imposing exactly the same constraints as the dimensional result model. Hence it fails to capture the essence of the reason model: the specific reason that the judge treated as decisive in the precedent. To address the collapse, Horty refined his model in [Horty 2020] into the model we presented in Sec. 2.2. While the refinements make sure the dimensional reason model does not collapse into the result model, it still has an issue: strengthening a rule by adding arguments to the premise may change constraint in an unintuitive way. While stronger, more specific rules should lead to the precedent being applicable in fewer instances, in Horty's model this is not the case.

Rigoni. A dimensional reason model based on *complete dimensions* was proposed by [Rigoni 2018]; there each value assignment is classified as favoring either π or δ by means of fixed threshold values (possibly with intermediate regions) determined by switching points. This yields conceptual clarity and allows for a direct mapping into the binary setting, but requires to specify for each dimension, which regions of the value space favor which side, independently of the deciding case. As noted in [Prakken 2021], this is a severe limitation as such classifications are difficult to establish, and should be placed in the hands of the judge, not the model designer. Rigoni's later work [Rigoni 2024] extends this approach by avoiding explicit thresholds altogether: he models their indeterminacy as *entanglement* and proposes to keep all interpretations compatible with the case base, deriving only those constraints that hold in all of them. In the next section, we unpack this idea and use it to motivate the construction of our reduction-based model.

Prakken. [Prakken 2021] provides comparative analysis of existing dimensional reason models and alternative proposals designed to minimize *implicit normative commitments* (i.e. built-in assumptions such as fixed thresholds.) His models simplify the two discussed approaches, representing rules in the same way as fact situations: assignments of (threshold-)values to dimensions. He distinguishes between *complete-rule* approaches, where each rule assigns a threshold value to every dimension, and *incomplete-rule* variants[2], in which rules may contain only a subset of dimensions, and unmentioned dimensions are ignored when assessing constraint. Complete-rule models provide a convincing notion of constraint but impose a strong completeness requirement on rules; incomplete-rule approaches relax this requirement but result in scenarios where the resulting constraints fail to match expectations.

4.2 Motivating desiderata

The literature on dimensional reason models identifies several tensions that do not arise in the binary factor setting. Building on this discussion, we make explicit five desiderata[3] that will guide the design of our reduction-based dimensional model.

(D1) No collapse. A dimensional model should not *collapse* into the dimensional result model; see Remark 1.

(D2) Incomplete fact situations. The model should remain applicable when fact situations mention only a subset of the available dimensions. In legal datasets and judicial opinions, factual descriptions are often partial: some aspects are unknown or do not get discussed. A model that requires every case description to assign a value to every dimension is therefore of limited use.

(D3) Minimal implicit commitments. The model should avoid introducing additional normative commitments not warranted by the decided cases. In particular, it is undesirable to require global domain knowledge such as fixed threshold values, fixed switching points, or fixed priorities between a reason and its negation, unless such information is explicitly justified by the case base. This desideratum reflects the methodological goal that the binding force of a precedent should stem from what the deciding court actually relied on, rather than from externally engineered classifications.

(D4) Stability under strengthening. This concerns the behavior of constraint under *strengthening* of reasons. Intuitively, tightening a rule by increasing one of its thresholds or adding conditions should not introduce inconsistencies in a consistent case base, nor should it cause additional focus cases to become constrained. As mentioned, Horty's model fails this property, and indeed [Prakken 2021] points out that any approach that defines constraint based only on rules that may contain a subset of all dimensions will have this flaw.

(D5) Faithfulness to the binary intuition. We aim to preserve the structural asymmetry of the binary reason model. In the factor-based setting, the factors that support the courts ruling but were not cited in the reason do not contribute to constraint, whereas all factors for the opposing side are retained to constrain the possibility to distinguish the precedent. This asymmetry is essential for defeasibility. A dimensional model should mirror this asymmetry: only dimensions explicitly cited in the rule may be adjusted, while all remaining dimensions must be kept, as they bear on whether a precedent can be distinguished.

5 The Reduction Model

Having identified the desiderata for a dimensional reason model, we define our model along with a generalization that handles incomplete fact situations. To obtain an adequate notion of constraint, drawing on ideas from [Rigoni 2024], we use the case base to infer additional information about which side certain dimension values favor; thus increasing the constraining power of the case base.

To remain as close as possible to the binary reason model we proceed by first defining a reduction to fact situations, and then take this reduced structure to be the basis for precedential constraint. A key question arises. In the factor setting, the reduction retains all opposing factors present in the fact situation while discarding pro-factors not mentioned in the rule. In the dimensional setting, we do not have a clear distinction of dimensions as pro or opposing, raising the question of how to proceed with the unmentioned dimensions.

[1] We omit the approach of [Bench-Capon and Atkinson 2017], as it departs significantly from the usual reason model structure.

[2] All of Prakken's approaches still assume complete fact situations; incompleteness applies only to rules.

[3] Table 1 below compares the mentioned models with ours along these criteria.

Guided by the asymmetry in the binary model, we will retain unmentioned dimensions so that they can later prevent precedents from being distinguished too freely. In the factor-setting distinguishing on the basis of the opposing factors is only possible when an opposing factor is added. In the dimensional setting this corresponds to a value that favors the opposing side more than in the precedent. Approaches that simply discard the unused dimensions end up allowing precedents to be distinguished even when they intuitively should not be, as [Prakken 2021] observes, when he discusses the constraints of the incomplete-rule models.

We therefore retain all dimensions occurring in the fact situation. For dimensions mentioned in the rule, the rule's threshold is used; for all other dimensions, the factual value is inherited unchanged. This yields the following definition.

Definition 20 (Dimensional reduction). *Let D be a domain and let $c = \langle F, r, s\rangle$ be a dimensional reason-model case. The* reduction *of c is the result-model case $rdct(c) := \langle X, s\rangle$, where (for all $d \in D$)*

$$X(d) = \begin{cases} p & \text{if } \exists\, M^s_{d,p} \in Premise(r), \\ F(d) & \text{otherwise,} \end{cases}$$

The extension of $rdct$ to case bases is $rdct(\Gamma) = \{rdct(c) \mid c \in \Gamma\}$.

Example 3. *Let $c = \langle F, r, \delta\rangle$ with $F = \{\langle d_1, 24\rangle, \langle d_2, 10\rangle\}$ and $Premise(r) = \{M^{\delta}_{d_1,14}\}$. Then*

$$rdct(c) = \langle\{\langle d_1, 14\rangle, \langle d_2, 10\rangle\}, \delta\rangle\,.$$

The rule fixes the value for d_1, while the remaining dimensions inherit their factual values.

We call the model obtained this way the *reduction model* and it functions exactly as the reduction in the factor-based setting. Constraints are computed as in the dimensional result model, and while cases retain their conceptual structure of fact situations, reasons and decisions in the background, we will represent them only by the reduced fact situation and the decision. We therefore only need to consider the definitions of the result model (Def. 13 and 5) when generalizing the reduction model.

Remark 2. *Our model formalizes a variant of an incomplete-rule model that is discussed only informally in [Prakken 2021] and which he characterizes as "collapsing into the complete-rule model". While the constraints mechanisms of the complete-rule model and our model do indeed coincide, there are strong conceptual reasons for defining our model this way despite this "collapse": First, allowing for incomplete rules retains an important part of the information the court provides. Second, but more importantly, our reduction approach shows that the model as we defined stays closest to the intuitions of the factor-based reason model. In this sense, it provides a principled generalization of the reason model to the dimensional setting, with the added benefit of using the simple constraint mechanism of the result model which we can now generalize to deal with incomplete fact situations.*

5.1 Incomplete fact situations

So far, we assumed that fact situations assign a value to every dimension. We now drop this assumption. A fact situation may mention only a subset of dimensions, representing incomplete information.

Definition 21. *Let F be a fact situation. The domain of F, denoted $dom(F)$, is the set of dimensions assigned a value in F.*

The central question is how to define factual strength for fact situations with potentially different dimensions.

Given two fact situations F and G, their domains can be related in four possible ways: equality, inclusion in either direction, or incomparability. When $dom(F) = dom(G)$ we clearly *can* compare their factual strength, and we argue that this is in fact the only case when this comparison is generally meaningful. Clearly if neither $dom(F) \subseteq dom(G)$ nor $dom(G) \subseteq dom(F)$ their factual strength cannot be compared. The remaining cases $dom(F) \subset dom(G)$ or vice versa are more subtle. The example below illustrate why we cannot safely compare factual strength, considering w.l.o.g. the case $dom(F) \subset dom(G)$.

Example 4. *Let $F = \{\langle abroad, 4\rangle\}$, $G_1 = \{\langle abroad, 4\rangle, \langle income, 90\%\rangle\}$, and $G_2 = \{\langle abroad, 4\rangle, \langle income, 5\%\rangle\}$ be three fact situations. Clearly $dom(F) \subset dom(G_1)$ and $dom(F) \subset dom(G_2)$. We need to evaluate whether the additional dimensions in G_1 and G_2 make these fact situations stronger or weaker for a side s. In the case of G_1 the addition of the dimension $income$ with value 90% is an argument for a change of domicile, making G_1 stronger for δ than F. However, for G_2 the addition of $income$ with value 5% is a significant argument in favor of no change, and therefore G_2 is stronger for π than F, since F lacks this strong argument.*

Therefore, we will redefine the factual strength relation only over fact situations with equal domains.

Definition 22. *Let X and Y be fact situations. Then Y is at least as strong as X for side s, written $X \trianglelefteq_s Y$ if and only if*

(1) *$dom(X) = dom(Y)$ and*
(2) *$X(d) \leq^s_d Y(d)$ for all $d \in dom(X)$.*

Reduction model constraint *is defined using this relation as in Def. 5.*

Defining factual strength this way yields a conservative notion of constraint, and avoids overconstraining cases on the basis of missing information. However, the reduction model with this new notion of factual strength leads to a very weak notion of constraint that falls short of our intuition, in particular it fails to give the correct constraint for Example 1.

5.1.1 Inferring information from the case base. In dimensional models with incomplete fact situations, the challenge is to obtain a notion of precedential constraint that remains faithful to this conservative treatment of missing information while still yielding adequate constraints. We briefly recall the underlying idea and intuition of [Rigoni 2024] that motivates our approach. Rigoni suggests that even incomplete cases implicitly encode information about which ranges of values favor which side.

Two notions are central in his account: *entanglement* and *interpretation.* Entanglement arises when a decided case shows that *some* dimension–value pair in the case must support the outcome, but does not determine which one. In other words, several pairs may jointly sustain the decision, while the case itself does not identify the particular pair that carries the support. An interpretation is then one possible way of resolving this indeterminacy: it assigns to each relevant dimension–value pair a directional role, namely support for π, support for δ, or, in the more cautious reading discussed later, no determined support yet. Thus, an interpretation amounts to a coherent assignment of supporting roles to dimension–value pairs, compatible with the whole case base. Since several such assignments may be compatible with the same case base, Rigoni's proposal is not to select one arbitrarily, but to retain all compatible interpretations and derive only what holds in all of them.

Example 5. *Consider a case base that after reduction contains the cases* $c_1' = \langle\{\langle d_1, 12\rangle, \langle d_3, 8\rangle\}, \delta\rangle$ *and* $c_2' = \langle\{\langle d_1, 12\rangle\}, \pi\rangle$. c_1' *tells us that at least one of the ranges* $d_1 \geq 12$ *or* $d_3 \geq 8$ *must support* δ*; otherwise the decision for* δ *would have no supporting feature at all.* c_2' *excludes one of these possibilities, since it shows that the value* $d_1 = 12$ *occurs in a case decided for* π*. Hence every interpretation compatible with the case base must assign support for* δ *to the range* $d_3 \geq 8$*. In this sense, the case base forces the conclusion that* $\langle d_3, 8\rangle$ *supports* δ*, even though no individual case states this explicitly. This reasoning pattern would also provide the expected result for Example 1.*

We aim to make this latent directional information explicit and computable. We do so by formalizing interpretations and extracting those dimension ranges that are forced to support one side in every interpretation compatible with the case base. These universally forced ranges will then be used to construct *synthetic precedent cases*: derived cases that make the implicit support information explicit and can be used, like ordinary precedents, to check whether a focus case is constrained.

Formally connecting Rigoni to our intuition. We begin by formalizing the idea of constraint under all possible interpretations in [Rigoni 2024]. For this, we use classical propositional logic over a set of atoms *supp* that encode the directional support of a dimension-value pair. For each dimension d and value v, we introduce the atoms $\mathrm{Supp}^{\pi}(d, v)$ and $\mathrm{Supp}^{\delta}(d, v)$, intended to mean that the value v on dimension d favors the corresponding side. An interpretation is then an assignment of truth values to these atoms.[4]

However, without additional restrictions, many of these interpretations will not be plausible explanations of the case base. We therefore provide a set of logical clauses that characterizes the possible interpretations w.r.t. a case base Γ.

(1) Entanglement: For each $\langle X, s\rangle \in \Gamma$:

$$\bigvee_{d \in dom(X)} \mathrm{Supp}^{s}(d, X(d))$$

(2) Consistency: For each $\langle X_i, \pi\rangle, \langle X_j, \delta\rangle \in \Gamma$ with $X_j(d) \geq_s X_i(d)$ for $d \in dom(X_j) \cap dom(X_i)$:

$$\bigvee_{d \in dom(X_i)\setminus dom(X_j)} \mathrm{Supp}^{\pi}\big(d, X_i(d)\big) \vee \bigvee_{d \in dom(X_j)\setminus dom(X_i)} \mathrm{Supp}^{\delta}\big(d, X_j(d)\big)$$

(3) Monotonicity: For $v \leq_d^s v'$:

$$\mathrm{Supp}^{s}(d, v) \rightarrow \mathrm{Supp}^{s}(d, v')$$

Entanglement encodes the constraint that for any given case, at least one of the dimension-value pairs must favor the case's decision. Consistency encodes result model constraint: If the values of all shared dimensions of two cases with opposite decisions favor π in the case decided for δ, then for the case base to be consistent there must be a dimension-value pair in the π case that supports π, or a dimension-value pair in the δ case that supports δ. Monotonicity encodes the dimensional support relation, so that if a dimension-value pair supports side s, a dimension-value pair with a value that is stronger for s also supports s.

Definition 23. *Let* Γ *be a case base.* POSS^{Γ} *is the class of interpretations that satisfy* Entanglement, Consistency *and* Monotonicity, *called* possible interpretations with respect to Γ.

What remains is defining the constraint. Rigoni's suggestion is to constrain a focus case, "if all of the reasonable interpretations compel the same result" [Rigoni 2024]. Since this idea is not formalized there for the dimensional setting, we adopt a conservative stance and introduce as little additional structure as possible. Concretely, we define a notion of constraint that extends the result-model notion of factual strength to comparisons between distinct domains.

Definition 24. *Let* Γ *be a consistent case base and* $i \in \mathrm{POSS}^{\Gamma}$*. Then for two fact situations* X, Y *we define* $X \trianglelefteq_s^i Y$ *if and only if*

- $\forall d \in dom(X) \cap dom(Y) : X \leq_d^s Y$ *and*
- $\forall d \in dom(X), d \notin dom(Y) : \mathrm{Supp}^{\bar{s}}(d, X(d))$ *holds in* i
- $\forall d \notin dom(X), d \in dom(Y) : \mathrm{Supp}^{s}(d, Y(d))$ *holds in* i

A focus case $\langle X, ?\rangle$ *is forced for* s *under* i *if* $\langle X, ?\rangle$ *is forced for* s *according to Def. 5 using* $\trianglelefteq_s^i$ *as the relation of factual strength.*

This definition of factual strength uses only two conditions, which are both uncontroversial. First, monotonicity of dimensional support: on shared dimensions, Y must be at least as strong for s as X, and Supp is required to be monotone. Second, the factor-based factual strength. By definition, if $\mathrm{Supp}^{s}(d, v)$ holds, then $\langle d, v\rangle$ supports s, as a pro-s factor would in the binary setting. The second and third clauses of Definition 24 are therefore an adaptation of the standard result-model condition that a fact situation X is at least as strong for s as Y when X contains at least all pro-s factors and at most all pro-$\bar{s}$ factors in Y.

Definition 25. *A dimension–value pair is said to be* universally forced *for side* s *by* Γ *if* $\mathrm{Supp}^{s}(d, v)$ *holds for all* $i \in \mathrm{POSS}^{\Gamma}$.

Now we can use the universally forced dimension-value pairs to construct the set of synthetic cases.

Fix[5] Reading (R1) or (R2) and define for a case base Γ

$$\mathrm{SUPP} := \{\mathrm{Supp}^{s}(d, v) \mid \langle d, v\rangle \text{ is universally forced for } s \text{ by } \Gamma\}$$

As every possible interpretation satisfies Monotonicity, SUPP is closed under monotonicity on each dimension.

Definition 26 (s–strengthening of a decided case). *Let* $\langle X, s\rangle \in \Gamma$ *be a decided result–model case in some case base.*

An s–strengthening *of* $\langle X, s\rangle$ *is any case* $\langle X', s\rangle$ *obtained from* X *by a finite number of applications of the following two operations:*

1. s–extension: *add a new assignment* $\langle d, w\rangle$ *with* $d \notin dom(X)$ *such that there exists* v *with* $\mathrm{Supp}^{s}(d, v) \in \mathrm{SUPP}$*;*
2. $\bar{s}$–deletion: *remove an assignment* $\langle d, X(d)\rangle$ *with* $d \in dom(X)$ *such that* $\mathrm{Supp}^{\bar{s}}\big(d, X(d)\big) \in \mathrm{SUPP}$.

Clearly, both operations make the decided case stronger for s under all possible interpretations, as they amount to adding π factors or removing δ factors from a fact situation that was decided

[4] From [Rigoni 2024], different readings of these atoms are possible, depending on whether every mentioned pair must receive a side or may also remain undetermined.

[5] The distinction between (R1) and (R2) matters only at one point in the completeness proof, where consistency ensures that a failed atom yields a distinguishing dimension.

for π or vice versa. They can therefore only *weaken* the constraints that the case yields. This ensures that we are not introducing more constraint to the case base by adding the synthetic cases. The idea of the procedure we later present in detail is that a focus case is constrained if we can build a synthetic case that acts as a precedent according to Def. 22.

Before turning to the theorem showing that this procedure matches Rigoni's idea of constraint, we need to make explicit that his discussion admits two possible readings of the directional atoms. Rigoni's informal presentation suggests a binary reading, but the threshold-based perspective developed in earlier work also motivates a more cautious three-valued reading. Since both are natural, we state them separately here. We will show in the proof of the theorem that, for our purposes, the two readings are equivalent: they induce the same constrained focus cases.

Under the first reading, every mentioned dimension–value pair is assigned a definite polarity in each possible interpretation: it favors either π or δ. Uncertainty arises only across different interpretations. This is the reading most closely reflected in Rigoni's informal analysis of the running example in [Rigoni 2024] (pp. 14–15), and we refer to it as Reading (R1).

Under the second reading, possible interpretations correspond to admissible threshold configurations, and these may leave some intermediate ranges undetermined: a dimension–value pair may favor π, may favor δ, or may remain neutral. This reading is closer to the threshold-based perspective of [Rigoni 2018], and we refer to it as Reading (R2). Corollary 2 shows that they induce the same set of constrained focus cases.

The following theorem shows that this procedure is faithful to the intuition of [Rigoni 2024], as it matches our formalization of Rigoni's constraint in Definition 24.

Theorem 1 (Rigoni's closure as derived case generation). *Let Γ be a case base, and let Γ^* be obtained by adjoining to Γ all s–strengthenings of its cases. Then for every focus case $\langle X, ?\rangle$ and side s, $\langle X, ?\rangle$ is constrained to s by Γ^* according to Def. 22 if and only if for every $i \in \mathrm{POSS}^\Gamma$ there exists a case $\langle Y, s\rangle \in \Gamma$ such that $Y \trianglelefteq_s^i X$.*

Proof. *Soundness.* Assume that $\langle X, ?\rangle$ is constrained to s by Γ^*. Then there exists $\langle X', s\rangle \in \Gamma^*$ with $dom(X') = dom(X)$ and $X' \trianglelefteq_s X$ in the reduction model.

By the construction of Γ^*, there is a case $\langle X_1, s\rangle \in \Gamma$ such that X' is obtained from X_1 by a finite sequence of s–extensions and $\bar{s}$–deletions justified by SUPP. Each such operation preserves $X_1 \trianglelefteq_s^i X'$ for every $i \in \mathrm{POSS}^\Gamma$. As $X' \trianglelefteq_s X$, we get $X_1 \trianglelefteq_s^i X$ for all i.

Thus, for every interpretation i, there exists $\langle X_1, s\rangle \in \Gamma$ with $X_1 \trianglelefteq_s^i X$, so X is constrained under all interpretations.

Completeness. Suppose that $\langle X, ?\rangle$ is not constrained to s by Γ^*. Fix any $\langle X_1, s\rangle \in \Gamma$ with $X_1(d) \leq_d^s X(d)$ on all shared dimensions. Since X is not constrained by Γ^*, every attempt to derive a precedent from X_1 fails. Thus, in each attempt, one of two cases hold:

- there is $d \in dom(X) \setminus dom(X_1)$ s.t. $\mathrm{Supp}^s(d, X(d)) \notin \mathrm{SUPP}$,
- there is $e \in dom(X_1) \setminus dom(X)$ s.t. $\mathrm{Supp}^{\bar{s}}(e, X_1(e)) \notin \mathrm{SUPP}$.

By definition of SUPP, in each case there exists an $i \in \mathrm{POSS}^\Gamma$ in which the corresponding support atom is false.

Under Reading (R1), falsity directly implies polarity toward the opposite side. Under Reading (R2), falsity may correspond to "unknown", but since all constraints are disjunctive, one can modify i so that the atom becomes false while preserving satisfiability.

In both cases we obtain an interpretation i such that $X_1 \ntrianglelefteq_s^i X$. Since this holds for every $\langle X_1, s\rangle \in \Gamma$, $\langle X, ?\rangle$ is not forced for s in all interpretations. □

Corollary 2. *Readings (R1) and (R2) induce the same set of constrained focus cases.*

Proof. By the proof of Th. 1, the same construction of Γ^* captures the constraints in all interpretations under (R1) and (R2). □

Accordingly, in the remainder of the paper (and in our ASP encoding) we work with Reading (R1) only, which is simpler.

As shown below the derived case base Γ^* preserves consistency (a prerequisite to apply Definition 5).

Proposition 2 (Consistency of Γ^*). *If the original case base Γ is consistent, then the derived case base Γ^* is consistent as well.*

Proof Sketch. Let Γ be a consistent case base, and let Γ^* be obtained by adjoining to Γ all s–strengthenings of its cases. We need to show that there are no cases $c_1 = \langle F_1, s\rangle, c_2 = \langle F_2, \bar{s}\rangle \in \Gamma^*$ such that $F_1 \trianglelefteq_s F_2$ holds, where $\trianglelefteq_s$ is from Def. 22. Suppose such a pair of cases exists, so we have $dom(F_1) = dom(F_2)$ and $F_1(d) \leq_d^s F_2(d)$ for all $d \in dom(F_1)$. There are three possible configurations: (1) $c_1, c_2 \in \Gamma$: this contradicts that Γ is consistent. (2) $c_1 \in \Gamma$ and c_2 is an s–strengthening of a case $c' = \langle F', \bar{s}\rangle \in \Gamma$. By definition, we have that for all $d \in dom(F_1) \cap dom(F')$, $F_1(d) \leq_d^s F'(d)$, for all $d \in dom(F_1), d \notin dom(F')$ that $\mathrm{Supp}^{\bar{s}}(d, F_2(d)) \in \mathrm{SUPP}$ and for all $d \notin dom(F_1), d \in dom(F')$ that $\mathrm{Supp}^s(d, F'(d)) \in \mathrm{SUPP}$. Applying the logical constraints on possible interpretations yields a contradiction by the following steps: (a) $\mathrm{Supp}^{\bar{s}}(d, F_2(d)) \in \mathrm{SUPP}$ for all $d \in dom(F_1), d \notin dom(F')$ and monotonicity of support of dimensions implies that $\mathrm{Supp}^{\bar{s}}(d, F_1(d)) \in \mathrm{SUPP}$ as $F_2(d) \leq_d^{\bar{s}} F_1(d)$. (b) The entanglement constraint then implies that in every possible interpretation, for some $d \in dom(F_1) \cap dom(F')$, $\mathrm{Supp}^s(d, F_1(d))$ holds. (c) In that interpretation, by monotonicity $\mathrm{Supp}^s(d, F'(d))$ must hold. (d) However, this and $\mathrm{Supp}^s(d, F'(d)) \in \mathrm{SUPP}$ for all $d \notin dom(F_1), d \in dom(F')$ contradicts the entanglement constraint for c', as under the interpretation every dimension-value pair in F' favors s, yet c' was decided for $\bar{s}$. (3) Both c_1 and c_2 are s–strengthenings of cases $c', c'' \in \Gamma$. The case is analogous to (2). □

Our procedure. Th. 1 yields the procedure below to extract implicit constraints from a case base Γ in the reduction model:

- **(Step 1)** compute all dimension–value ranges that are forced to favor a side in every possible interpretation w.r.t. Γ, i.e. every interpretation compatible with Γ (given monotonicity, entanglement, and result–model consistency);
- **(Step 2)** when evaluating a focus case, use this information to check whether some case in Γ can be transformed by successive applications of Extension and Reduction into a derived precedent that constrains the focus case. If this precedent exists, the focus case is constrained to the corresponding side; otherwise, both outcomes remain admissible.

Example 6. *Assume that* Γ *contains the following reduced cases:*

$$c_3 = \langle\{\langle d_2, 40\rangle\}, \pi\rangle\,, \qquad c_4 = \langle\{\langle d_2, 30\rangle, \langle d_3, 7\rangle\}, \delta\rangle\,,$$
$$c_5 = \langle\{\langle d_1, 12\rangle, \langle d_2, 60\rangle\}, \delta\rangle\,.$$

Consider the focus case $c_f = \langle\{\langle d_1, 16\rangle, \langle d_2, 60\rangle, \langle d_3, 8\rangle\}, ?\rangle$.

Step 1 (universally forced ranges). *From* c_3 *we obtain the one–dimensional constraint that* $d_2 \leq 40$ *supports* π *(i.e.* $\mathsf{Supp}^{\pi}(d_2, 40) \in \mathsf{SUPP}$*). Moreover, combining* c_3 *and* c_4*, result–model consistency forces* $d_3 \geq 7$ *to favor* δ*: otherwise,* c_4 *could not be kept distinguishable from* c_3 *once their shared* d_2 *information is taken into account. Hence* $\mathsf{Supp}^{\delta}(d_3, 7) \in \mathsf{SUPP}$.

Step 2 (derived precedents and constraint). *Starting from the* δ*–precedent* c_5*, we* extend *(delta-strengthens) it with* $\langle d_3, 8\rangle$ *since* $\mathsf{Supp}^{\delta}(d_3, 7) \in \mathsf{SUPP}$ *and* $7 \leq^{\delta}_{d_3} 8$*. The resulting derived case* $c_5' = \langle\{\langle d_1, 12\rangle, \langle d_2, 60\rangle, \langle d_3, 8\rangle\}, \delta\rangle$ *is a precedent for* c_f *on the common dimension* d_3 *in the reduction model, hence* c_f *is constrained to* δ*.*

Extreme values. Rigoni additionally proposes erasing dimensions that appear with minimal values for a given side. We refrain from adopting this simplification. Even minimal values for one side may carry substantive weight in favor of that side. For example, a minimal δ–favoring value for children's schooling (e.g. zero months) already encodes a significant fact in favor of δ: the children have started school. Discarding such values amounts to assuming that minimal values for a side always favor the opposite side, an assumption we do not generally find justified. That said, Rigoni's simplification could easily be incorporated into our framework by explicitly postulating this behavior of extreme values, if such an assumption is appropriate for a given application.

The empty precedent. In Step 2 it seems natural to include the empty case, with both possible decisions, among the potential precedents. Indeed if all dimension–value pairs present in a focus case are forced to favor side s, it is reasonable to conclude s, even in the absence of any real precedent sharing those dimensions. The only objection would be that each individual factor is too weak to suffice, despite all pointing to the same direction. We regard this as an artifact of an incomplete modeling of the domain. In such situations, what is implicitly being weighed is not just the listed dimensions, but those dimensions against a background default such as a presumption of innocence (for δ), which should itself be represented explicitly as a dimension present in all cases. Once such defaults are modeled, the empty–case extension becomes conceptually unproblematic.

6 ASP implementation

To support legal argumentation and decision-making, the reduction model should be executable: given a case base and a focus case, an implementation should automatically determine if a decision is constrained. Answer Set Programming (ASP), a well known non-monotonic reasoning framework that treats non-derivable information as absent via default negation (`not`), is a natural implementation language for our framework. Indeed default negation conveniently models incomplete descriptions and supports adding or discarding dimensions when constructing synthetic cases. Moreover, SUPP corresponds to the *skeptical* consequences of the entanglement program, i.e. the atoms true in all answer sets; these can be computed directly by cautious reasoning, and then pruned by retaining only the $\leq^s_d$–minimal (undominated) thresholds.

A rule in ASP has the form

$$a \texttt{ :- } b_1, \ldots, b_k, \texttt{ not } b_{k+1}, \ldots, \texttt{ not } b_m.$$

Here, a and $b_1, \ldots, b_m$ are either positive or strongly negated predicates, as in classical logic. Intuitively, the rule states that a is true whenever all $b_1, \ldots, b_k$ are true and none of $b_{k+1}, \ldots, b_m$ can be derived. ASP builds on a logic-based rule language, interpreted under answer set semantics. Formally, for any answer set (stable model) S of a program $\mathcal{P}$ containing this rule,

$$\text{if } \{b_1, \ldots, b_k\} \subseteq S \text{ and } \{b_{k+1}, \ldots, b_m\} \cap S = \emptyset, \text{ then } a \in S.$$

Further, every $c \in S$ needs to be supported by some rule in $\mathcal{P}$, thus answer sets are subset minimal. The n-ary predicates in a and $b_1, \ldots, b_m$ are applied to terms, which are either constants or variables. Constants are written in lowercase, whereas variables begin with an uppercase letter and are implicitly universally quantified within a rule. In addition to standard rules, we also utilize choice rules, that for our purposes have the form

$$\ell \texttt{ \{ } R_1 \texttt{; } \ldots \texttt{; } R_t \texttt{ \} } u \texttt{ :- } b_1, \ldots, b_k, \texttt{ not } b_{k+1}, \ldots, \texttt{ not } b_m$$

where R_i is of the form $d : e_1 \ldots, e_j, \texttt{ not } e_{j+1} \ldots, \texttt{ not } e_n$.

Here, d and $e_1, \ldots, e_n$ are either positive or strongly negated predicates applied to terms. If $\{e_1, \ldots, e_j\} \subseteq S$ and $\{e_{j+1}, \ldots, e_n\} \cap S = \emptyset$, then d may be included in or excluded from S. A choice rule generates an arbitrary subset of the values inferred from $R_1, \ldots, R_t$. The integers ℓ and u are optional, and, if specified, provide upper and lower bounds for the size of the generated subset.

Efficient solvers to execute ASP programs exist, e.g. [Gebser et al. 2011], supporting the choice of ASP for implementing our model.

6.1 The fiscal domicile example

To outline the required implementation steps, we revisit the fiscal domicile scenario with a fresh example which contains reasons with magnitudes: Let $\Gamma = \{c_1, c_2, c_3\}$ be a case base where

$$c_1 = \left\langle\{\langle d_1, 18\rangle, \langle d_2, 60\rangle\}, \{M^{\delta}_{d_1,12}\} \to \delta, \delta\right\rangle,$$
$$c_2 = \left\langle\{\langle d_2, 30\rangle, \langle d_3, 7\rangle\}, \{M^{\delta}_{d_3,6}\} \to \delta, \delta\right\rangle,$$
$$c_3 = \left\langle\{\langle d_2, 35\rangle\}, \{M^{\pi}_{d_2,40}\} \to \pi, \pi\right\rangle.$$

A focus case $c_f = \langle\{\langle d_1, 16\rangle, \langle d_2, 70\rangle, \langle d_3, 8\rangle\}, ?\rangle$ is given, and we ask whether Γ forces a decision for either side.

We present the relevant part of the encoding below; the full encoding is available at https://github.com/cbr-asp/reduction-model.

Modeling input. For simplicity, we assume that every dimension $d \in D$ only contains integer values; in our example $\leq^{\delta}_d$ corresponds to $\leq$. We introduce the predicate `strengthens(D, S)` stating that larger values of dimension `D` favor side `S` (`plaintiff` or `defendant`). In our example, all dimensions strengthen the defendant:

```
strengthens(di, defendant).   i=1,2,3
```

Every case $\langle F, r, s\rangle \in \Gamma$ has a winning side s, value assignments in F, and magnitude factors in $Premise(r)$. We use `case(C, S)` to define a case, `val(C, D, Q)` for a value assignment, and `mf(C, D, P)` for a magnitude factor where `C` identifies a case, `S` the winning side, `D` a dimension, and `P` and `Q` values of dimension `D`. `mf` does not need to specify a side, since it can be inferred from the winning side of the case it belongs to. Focus cases are represented using the predicate `focus_case(C)` with a unique case id `C` that the value assignments link to. In our example, Γ and c_f are implemented as:

```
case(c1, defendant).
val(c1, d1, 18). val(c1, d2, 60). mf(c1, d1, 12).
case(c2, defendant).
val(c2, d2, 35). val(c2, d3, 7). mf(c2, d3, 6).
case(c3, plaintiff).
val(c3, d2, 30). mf(c3, d2, 40).
focus_case(cf).
val(cf, d1, 16). val(cf, d2, 60). val(cf, d3, 8).
```

Reducing the case base. To construct $rdct(\Gamma)$ we introduce the predicate `rval(C, D, P)` which represents the reduced value assignment replacing `val` and `mf`. Using default negation, we infer the correct value for `P`. Note that we use the token `_` to denote an anonymous variable that is fresh at each occurrence.

```
rval(C, D, P) :- mf(C, D, P).
rval(C, D, P) :- val(C, D, P), not mf(C, D, _).
```

Since this rule does not distinguish between case and focus case value assignments, the focus case is also assigned `rval`-predicates. Since a focus case does not have any magnitude factor, these are equal to the `val`-predicates. In our example, the case base $rdct(\Gamma)$ and c_f look as follows:

```
case(c1, defendant). rval(c1, d1, 12). rval(c1, d2, 60).
case(c2, defendant). rval(c2, d2, 35). rval(c2, d3, 6).
case(c3, plaintiff). rval(c3, d2, 40).
focus_case(cf).
rval(cf, d1, 16). rval(cf, d2, 60). rval(cf, d3, 8).
```

Side definitions. We introduce `side(S)` and `opposite(S, S')` for the adversarial parties `S` and `S'` in the court.

Value relation. To model $p \leq_d^s q$, we construct `ord(S, D, P, Q)` with side `S`, dimension `D`, and values `P` and `Q`. We only need to construct the relation for values that occur in `rval`. Since the natural ordering of integers already satisfies reflexivity, transitivity, and antisymmetry for the strengthened side, we merely need to infer the dual in addition to the natural ordering:

```
ord(S, D, P, Q) :-
    rval(_, D, P), rval(_, D, Q),
    strengthens(D, S), P <= Q.
ord(S', D, Q, P) :- ord(S, D, P, Q), opposite(S, S').
```

We implement the two steps from Section 5.1 in two distinct ASP programs: Step 1 computes the skeptical support information (the set SUPP), used by Step 2 to check if a focus case is constrained.

Step 1. We generate the set of $Supp^s(d, p)$ that hold in every interpretation. We use ASP's guess-and-check to generate an arbitrary admissible interpretation, in which each occurring dimension–value pair is assigned to either π or δ:

```
1 { supports(S, D, P) : side(S) } 1 :-
    case(C, _), rval(C, D, P).
```

Support is monotone along each dimension:

```
supports(S, D, Q) :- supports(S, D, P), ord(S, D, P, Q).
```

Each decided case must contain at least one value assignment supporting its outcome:

```
1 { supports(S, D, P) : rval(C, D, P) } :- case(C, S).
```

Finally, we need to ensure that cases which are not distinguishable on the shared dimensions remain distinguishable on some dimension that is not shared.

```
distinguishable(C, C') :-
    case(C, S), case(C', S'), opposite(S, S'),
    rval(C, D, P), rval(C', D, Q),
    not ord(S, D, P, Q).
1 { supports(S, D, P) :
        rval(C, D, P), not rval(C', D, _);
    supports(S', D, Q) :
        rval(C', D, Q), not rval(C, D, _) } :-
    case(C, S), case(C', S'), opposite(S, S'),
    not distinguishable(C, C').
```

Each answer set represents a possible interpretation. Their intersection (via cautious reasoning) gives the dimension thresholds universally forced for each side. In our example, this yields:

```
supports(plaintiff, d2, 40). supports(defendant, d3, 6).
```

Step 2. To compute constraints, we follow Th. 1 but avoid explicitly constructing synthetic cases. Instead, we check whether each $\langle X, s\rangle \in rdct(\Gamma)$ *can* serve as a precedent for the focus case $\langle F, ?\rangle$. Equivalently, we identify when this is impossible. There are three failure conditions that prevent $\langle X, s\rangle$ from constraining $\langle F, ?\rangle$:

(1) $X(d) \not\leq_d^s F(d)$ for some dimension d; (2) Some value $F(d)$ for dimension $d \in dom(F \setminus X)$ does not support s; and (3) Some value $X(d)$ for dimension $d \in dom(X \setminus F)$ does not support $\bar{s}$.

These conditions are translated into ASP as follows:

```
ord_failure(F, C, S) :-
    focus_case(F), case(C, S),
    rval(F, D, P), rval(C, D, Q),
    not ord(S, D, Q, P).
add_failure(F, C, S) :-
    focus_case(F), case(C, S),
    rval(F, D, P), not rval(C, D, _),
    not supports(S, D, P).
rem_failure(F, C, S) :-
    focus_case(F), case(C, S), opposite(S, S'),
    not rval(F, D, _), rval(C, D, P),
    not supports(S', D, P).
```

A synthetic precedent can be constructed from $\langle X, s\rangle$ iff none of the failure conditions applies. If such a precedent exists, the focus case is constrained:

```
precedent(F, C, S) :-
    focus_case(F), case(C, S),
    not ord_failure(F, C, S),
    not add_failure(F, C, S),
    not rem_failure(F, C, S).
constraint(F, S) :- precedent(F, C, S).
```

Case $rdct(c_1)$ can become a precedent of c_f by adding $\langle d_3, 7\rangle$ through $Supp^{\delta}(d_3, 7)$. Therefore, c_f is constrained to δ. The program contains the following output:

Table 1: Comparison of dimensional reason models w.r.t. formal properties

	No collapse	Incomplete facts	Strengthening	Close to factor	Implicit commitments
Horty (2017)	✗	✗	✗	✓	Low
Horty (2021)	✓	✗	✗	✗	Low
Rigoni (2018)	✓	✓	✓	✓	High
Prakken (2021)	✓	✗	✓	✗	Minimal
Reduction model	✓	✓	✓	✓	Low

```
precedent(cf, c1, defendant). constraint(cf, defendant).
```

7 Conclusions

We have presented the reduction model, a dimensional reason model that handles reasoning with incomplete fact situations. The model satisfies five desiderata identified in the literature, as summarized in Table 1, which compares it against the existing models discussed in this paper. Our model does not collapse into the result model, and satisfies the strengthening property. It handles incomplete fact situations while keeping the implicit commitments required for constraint low, as we avoid relying on any kind of assumptions on which side a specific value assignment favors, unless this information can be inferred from the case base itself. Finally, using the factor-based reduction as a blueprint for the dimensional model, keeps our model close to the factor-based structure and concepts.

The *reduction* provides a simplified representation on the factor side, and a principled definition of a dimensional reason model on the dimension side; this aligns with a variant of a model in [Prakken 2021], and its generalization, inspired by [Rigoni 2024], extends it to deal with incomplete facts. To strengthen constraints in such cases, we introduce an inference mechanism that derives ranges of dimension values which must favor a given side. We use this information to construct precedents that draw on multiple cases, reflecting the real-world practice in which lawyers combine aspects from several cases to form a single supporting argument.

Our model is implemented in Answer Set Programming, leveraging its power as a computational formalism for non-monotonic reasoning, and illustrated on the fiscal domicile scenario.

Future work: The first step of our procedure derives universally forced one–dimensional ranges from *full case descriptions*. A first alternative worth investigating retains the two–step architecture, but replaces full cases by their associated *reasons* when forming the entanglement constraints over directional atoms. In this variant, disjunctive clauses would involve only the dimension–value pairs explicitly cited in reasons, discarding all others. This would yield at least as much, and typically more, sided range information, since fewer dimensions would compete within each entanglement constraint. A second alternative would align more closely with Horty's factor-based account. Here, one would assume that every dimension appearing in a reason *favors the side of the decision*.

Furthermore, if feasible, a larger scale study on real-life benchmarks could give insight on what constraint mechanisms work best, and what size a case base in presence of incomplete facts needs to have to allow the inference mechanism to provide a satisfyingly strong constraint.

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