Normative reasoning in Mīmāṃsā: a deontic logic approach

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Chapter 1

Introduction

This thesis investigates connections and synergies between the Mīmāṃsā school of Indian philosophy and deontic logic, the branch on formal logic that is concerned with obligation, permission, and related concepts.

Originated in India in the last centuries BCE and developed nearly up to the present day, Mīmāṃsā school is among the earliest and most important schools of Indian philosophy. The school focused specifically on the interpretation and systematization of the prescriptive portions of the Vedas. These are the texts considered as sacred and authoritative by most Indian religious and philosophical traditions. Their prescriptive parts contain ritual exhortations and descriptions of the sacrificial methods, including all the acts that should or should not be performed in connection to them.

In their endeavour to interpret the prescriptive parts of the Vedas, Mīmāṃsā authors have laid out a vast body of theories pertaining to the analysis of normative concepts. Such theories, identifying and clarifying all the relevant prescriptions and the relations among them, enable the readers to understand the Vedic passages as consistent collections of information about duty, so that it is possible to derive "what has to be done" only from the Vedas interpreted through Mīmāṃsā "lens", independently of any (human or divine) authority.

The theories developed by Mīmāṃsā authors have been highly influential on almost every aspect of Indian thought, including literary theory and theology, as well as historical and contemporary Indian law.

Why investigating Mīmāṃsā using logic, the study of reasoning structures that has been developed in European tradition? The answer lies in the fact that Mīmāṃsā theory of inference (*anumāna*) and its approach to Vedic prescriptive texts share many tracts with formal logic. Indeed they are both founded on a rigorous method, according to which correct conclusions are obtained when derived from reliable premisses via inferential steps that are, in turn, verifiable and based on fixed general principles.

Being sacred texts, the Vedas are assumed not to contain contradictions. A contradiction, indeed, would require to cancel or set aside a normative statement: if two Vedic commands seem conflicting —that is complying with one of them implies violating the other one— there must be a mistake or a deficiency in their interpretation. The Mīmāṃsakas (philosophers belonging to the Mīmāṃsā school) discuss general laws for the correct interpretation of the normative statements in the sacred texts as a consistent set, often starting from the analysis of concrete cases of commands that appear to be conflicting with each other. For an "alleged" conflict, Mīmāṃsā texts show that the contradiction is only apparent —i.e. there can be a situation where the commands at stake are enforced and none of them is violated— or that it is the result of an error in the understanding of the normative statements, which then need to be reinterpreted.

Key for this operation is the use of *nyāya*s, comprehensive interpretative principles meant to explain the nature and the context of application of Vedic injunctions. Although the *nyāya*s are introduced in connection with the discussion of concrete issues and controversies, they are intended to be general and abstract principles that can apply in all the circumstances that satisfy certain requirements. Hence, once a *nyāya* has been formulated in relation to some specific passage of the Vedas, Mīmāṃsā texts recall it when dealing with similar problematic normative statements.

Depending on the issues they are meant to clarify, there are different kinds of *nyāya*s, classified in [47] as hermeneutic principles, linguistic and *"deontic"* ones. The latter are used for analysing the different types of normative statements in the Vedas, by identifying their essential structures, their abstract characteristics and the general features of their behaviour, independently of the specific content of each injunction.

As dealing with seemingly conflicting statements is one of the main concerns of Mīmāṃsā authors, some *nyāya*s are introduced with the specific aim of resolving conflicts. This can be achieved by giving priority to one command over others. Principles with such a function are called *bādha*s (*bādha* in Sanskrit means *"suspension"* or *"blockage"*): if in a given situation two contradictory commands seem to apply, the principle suspends the effectiveness of one command (prioritizing the other one) for that specific situation. Prominent examples of *bādha*s, analysed in this thesis, are the prioritization based on the specificity of the applicability conditions (a norm which applies in a more specific case overrules a more general norm) and the prioritization based on the hierarchy of the sources of duty recognized as reliable.

The first principle, known in contemporary logic and Artificial Intelligence as *specificity principle*, has been actually used by Mīmāṃsā authors for centuries (called *Guṇapradhāna* or *Sāmānya-viśeṣa*) and it has been "inherited" by *Dharmaśāstra*, Indian jurisprudence. For what concerns the hierarchy of sources, Mīmāṃsā authors recognized three other sources of duty, besides the explicit normative statements in the sacred texts. These are, in descending order, the "recollected texts", based on the Vedas, the behaviour of people who studied the sacred texts, and their inner feeling of approval. However, this does not contradict the previous statement about the Vedas representing the only reliable source about the correct behaviour. Indeed, the authority of any of the other three sources is based on the authority of the one above it, so that, finally, they are all based on the Vedas. For this reason, in case of conflict between two commands found in different sources, the one found in the source higher in the hierarchy overrules the other one. Although the optimal solution in case of apparently conflicting normative statements consists in cancelling the conflict at any level, i.e. interpreting the commands in such a way that they cannot overlap, the use of *bādha*s still represents an acceptable approach. Indeed, an overridden norm is not eliminated, but only suspended, e.g., the fact that in a specific situation a command takes priority over another one just updates the readers' knowledge with an exception to the overruled command.

When avoiding the overlapping of conflicting commands is impossible and there are no characteristics which can determine the priority of one command over the others, Mīmāṃsā scholars applied the principle called *vikalpa*. This consists in considering the conflicting commands as optional. *vikalpa* represents the last resort for Mīmāṃsā authors, who seem to use it only in cases of plain contradictions, e.g. in presence of explicit statements of the form *"you must do X"* and *"you must do not X"*. However, though the arbitrary choice among conflicting norms seems to cancel the commands which are not chosen, *vikalpa* is ultimately acceptable. This happens because the choice is not a final decision and the commands remain optional: when facing again the same conflict, one can make a different choice. As it express the requirement to comply at least with one of the conflicting commands, *vikalpa* corresponds to the principle known in deontic logic as *disjunctive response*.

It is important to note that considering conflicting commands as optional, as well as reasoning with exceptions and priorities among Vedic norms, means that those norms are deemed to be *defeasible*, i.e. their effectiveness can be "defeated" by the effectiveness of another command under certain circumstances. This implies that Mīmāṃsā authors did not regard all the Vedic normative statements as static and complete instructions that explicitly describe every possible situation in which a command is enforced. Rather, from Mīmāṃsā point of view, some of the norms in the sacred texts are general and flexible enough to admit exceptions and to be adaptable even to situations that are not explicitly mentioned.

The expression *defeasible reasoning* usually refers to "a kind of reasoning that is rationally compelling, though not deductively valid" ([77]). This means, for example, that premisses are not assumed to be always valid, but they represent generalizations, that hold in most cases under ordinary conditions, but which can be corrected or cancelled if an exception occurs. Though extensively researched only after the middle of the twentieth century, the phenomena that fall under the concept of *defeasibility* have been recognised and discussed in philosophy and epistemology since ancient times. As noted, e.g. by Ferrer Beltrán and Ratti in [15], a distinction between deductive reasoning and the inferences that characterize everyday life can already be found in Aristotle's Topics. Deductive reasoning allows valid consequences to be derived from premisses which are assumed to hold under any circumstance; however, in everyday life, we tend to rely upon common sense hypotheses, that hold "normally", but can be revised in view of new information. In recent times, a systematic treatment of defeasible reasoning has been carried out in the field of Artificial Intelligence. With the aim of formalizing and implementing the kind of inferences which constitute the main part of our actual reasoning, based common sense, new formal logics have been developed, often falling under the broad category of *non-monotonic* logics. Such systems, indeed, allow to reason without *monotonicity*: this means that a conclusion following from a set of premisses does not necessarily follow from a new set of premisses which includes the first one. Non monotonic reasoning is suitable for capturing common sense, as it enables the reasoner to draw provisional conclusions in presence of incomplete information and to formulate weakened universal statements which hold normally but are subject to exceptions. As stressed e.g. in [86], in everyday life —and therefore in moral and legal reasoning— we need to make valid inferences from not absolutely reliable, incomplete, or inconsistent informations, while Classical Logic, which is instead monotonic, is meant to capture the concept of deduction as used in axiomatic systems like pure mathematics. Hence, monotonicity is not necessarily a desirable feature of a logic which aims to represent the ordinary inferences that usually are assumed to be at the base of ethical and normative reasoning (see e.g. [62]).

In this respect, Mīmāṃsā approach results particularly modern, as, to some extent, it recognises and "regulates the use" of defeasible norms with a rigorous and rational method of analysis, implicitly allowing non-monotonic inferences. Despite this, and despite the importance of Mīmāṃsā school in the whole Indian tradition, the investigation of Mīmāṃsā reasoning using the tools provided by formal logic is very recent. In contrast with the situation for other South Asian philosophical schools, namely *Navya Nyāya*, *Vyākaraṇa* and Buddhist epistemology, the study of Mīmāṃsā principles using logical methods has been undertaken only in the last few years, in the context of the research project *"Reasoning Tools for Deontic logic and Applications to Indian Sacred Texts"*. Before this project, one of the few attempts —if not the only one— to analyse Mīmāṃsā texts using formal logic is constituted by Laurence R. Horn's [66]. However, here the formal notation of logic is only employed to distinguish between the prohibition to perform an action (*"you must not do X"*) and the prescription to refrain from doing it, (*"you must bring about the situation (not)X"*); hence Horn's work is far from being an analysis of Mīmāṃsā reasoning through logical methods. The main reason for the delay of this study is that most scholars working on Mīmāṃsā are not trained in logic, while the mostly untranslated Sanskrit texts are inaccessible to logicians. Hence, only a close cooperation between experts from very different fields made possible the work carried out in this thesis, which contains some of the various results obtained in the context of the mentioned interdisciplinary research project, focusing in particular on the resolution of normative conflicts in Mīmāṃsā.

The starting point of our investigation have been the *nyāya*s, found and "extracted" from the discussions in the Sanskrit texts. Before attempting a formalization, they have to be translated, interpreted and abstracted, as they are often expressed in metaphorical language; the last step is the transformation of the abstracted *nyāya*s into logical formulas or reasoning methods. An English presentation of some *nyāya*s can be found in [46], while the appendix in [75] contains a list of translated *nyāya*s, not systematically explained or organized. The search for more *nyāya*s is subject of ongoing research by the Sanskritists.

Considering the rigorous and systematic nature of Mīmāṃsā approach, Mīmāṃsā school can be rightfully considered to have contributed to the early history of deontic logic. Launched as an active academic area of study around the 1950s, by the work of G.H. Von Wright ([140]), deontic logic is nowadays considered as a part of the wider field of modal logic. Indeed, the formal systems (deontic logics) introduced to capture and formalize the characteristics of norms extend a base logic (mostly classical or intuitionistic logic) with deontic operators, that express modalities that qualify the statements to which they are applied as obligatory, permitted, or forbidden.

In line with the research project *"Reasoning Tools for Deontic logic and Applications to Indian Sacred Texts"* mentioned above, the purpose of this thesis is to formally analyse the examples of reasoning found in Mīmāṃsā texts. Indeed, the tendency towards systematization of the analysis of Vedic commands, based on the use of *nyāya*s, together with the strict rules concerning the structure of arguments, makes Mīmāṃsā normative reasoning essentially already a *non-formal* deontic logic. For this reason, we do not "impose" an existent deontic logic to analyse the examples of reasoning found in the texts. Rather, we aim at "extracting" the principles (*nyāya*s) developed in the context of Mīmāṃsā normative reasoning and formalize them to define a formal deontic logic which corresponds to the non-formal one that Mīmāṃsā scholars were implicitly using.

A very common way to describe or introduce a logic, consists in using *Hilbert systems*, which are deductive systems constituted by sets of axioms and a small number of natural inference rules. Such systems are particularly appropriate in our case, as the axioms which define the properties of the deontic operators represent the most intuitive translations of some important *nyāya*s. More specifically, we translate some of the *nyāya*s, made available to logicians by the work of Sanskritists and experts in Indian philosophy, into axioms and add the sole *Modus Ponens* inference rule. As it states that, if a conditional premiss *"if α is true, then β is true"* and its antecedent *"α is true"* hold, then also its consequent *"β is true"* holds, the Modus Ponens is a very natural rule that can be found in almost all systems of reasoning based on a proper theory of inference.

First applied in [29], the strategy for defining a deontic logic which mimics Mīmāṃsā reasoning consists in a bottom-up step-by-step method from the Sanskrit texts to formal logic, that in principle proceeds by trial and error. This means that the logic resulting from the translation of *nyāya*s into axioms was checked for inner consistency (impossibility to derive the absurd consequence $\perp$) and consistency with respect to the examples of reasoning found in Mīmāṃsā texts: in case those conditions were not guaranteed, the interpretative principles were analysed again and transformed into different formulas. The logic thus introduced in [29], called *basic Mīmāṃsā Deontic Logic* (bMDL), though modelling only the concept of obligation, has been successfully used to formalize concrete cases discussed by Mīmāṃsā authors, analysing their solutions and the applications of *nyāya*s. The same method has been employed for introducing the logic MD+, described in this thesis, that formalizes with new needed operators the other deontic concepts used in Mīmāṃsā. Indeed, according to the Sanskrit and Indian philosophy experts, the different kinds of Vedic commands, as interpreted by Mīmāṃsakas, cannot be reduced to the sole concept of "obligation".

When introducing a logic, two important aspects need to be considered: its semantics and proof theory. The former is based on the use of structures that give meanings to the formulas of the logic, making possible to say, for every (formalized) statement in the logic, whether the statement is true or false. Such structures allow us to consider all the possible states of affairs where some given Vedic commands hold, identify and analyse the states in which those commands are complied with or violated. Hence, for instance, if two norms are truly conflicting, i.e. complying with one entails violating the other one, then there is no state where they are both respected. Conversely, when two deontic statements are not incompatible, a semantic analysis can identify and characterize the state(s) where both the commands at stake hold and are complied with, thus providing an explanation of why they are not contradictory.

When it comes to reasoning (e.g., checking whether a conclusion follows from certain premisses), it is preferable to use a proof-theoretic approach. Hilbert systems are cumbersome for finding derivations and the task requires *analytic calculi* instead, i.e., deductive systems in which proofs consist only of concepts already contained in the result (in Leibniz's words *praedicatum inest subjecto*). Analytic calculi are indeed very useful to reason within the logics (e.g. derivability from premises, inconsistency of sentences) but also to prove metalogical properties about them (e.g. consistency and decidability); moreover they are key for developing automated reasoning methods. Since its introduction by Gentzen (in [52]), the sequent calculus has been the favourite framework to define analytic calculi, and it is the one which is employed in this thesis. Sequents are derivability assertions of the form $\alpha_1, \cdots, \alpha_n \Rightarrow \beta_1 \cdots, \beta_m$, meaning that the conjunction of all the formulas on the left hand side implies the disjunction of all the formulas on the right hand side (" if all the formulas $\alpha_1, \cdots, \alpha_n$ are simultaneously true, then at least one formula among $\beta_1 \cdots, \beta_m$ is true"). Specifically, an immediate translation of certain Hilbert systems into the corresponding sequent calculi is possible thanks to the methods developed in [80]. The use of a sequent calculus, which employs more complicated structures with respect to a Hilbert system, allows us to prove that the use of Modus Ponens (and equivalent rules of inference) can be avoided. This key rule of Hilbert systems, indeed, introduces formulas in a derivation which cannot be found in its conclusion, thus determining a loss of information from the premisses to the conclusion.

However, not all the *nyāya*s can be simply converted into Hilbert axioms and consequently into sequent rules. This is the case of the *bādha*s, which express mechanisms of conflict resolution that are applied only in cases where avoiding any form of conflict is impossible; by their very nature, they make sense only in connection with the use of formalized Vedic commands which are assumed to hold. In order to formalize the *bādha* which prioritizes norms with more specific conditions, we introduce new "special" sequent-style rules that derive a command enforceable under certain circumstances from a set of Vedic injunctions, only if there is no more specific conflicting command that applies under those circumstances.

A similar mechanism allows to extend those rules for including the prioritization based on the hierarchy of sources: a command enforceable under certain circumstances can be derived from a set of Vedic injunctions only if there is no conflicting command, which is more specific or found in a more important source, that applies under those circumstances.

Our investigation proves to be beneficial for both logicians and scholars of Indian philosophy. Indeed, connecting Mīmāṃsā reasoning with formal logic provides inspirations and new motivations for the development of technical tools, e.g., for modelling controversies originally in natural language; the proposed solution could be potentially applied in other fields, as Law or Artificial Intelligence. Moreover, formalizing Mīmāṃsā reasoning allows a comparison with other kinds of normative reasoning, by analysing the similarities and differences between the corresponding formal systems. On the other hand, from the perspective of the research on Mīmāṃsā in Indian philosophy, a formalization brings a high level of abstraction, which makes it possible to analyse the structure of arguments independently from their specific contents and hence clarify their characteristics. Furthermore, it contributes to the debate on some important topics and controversies discussed by philosophers and Sanskritists. Indeed, we have used the introduced formal methods for analysing concrete examples found in Mīmāṃsā texts (e.g. the controversy around the *Śyena* sacrifice, described in Ch.4), thus shedding light on assumptions and reasoning steps which are only implicitly used by Mīmāṃsā authors. Finally, a formalization provides new stimuli for research in Mīmāṃsā as it poses new questions like, e.g., about the relative strength of deontic operators or about the preferred principles applied for resolving apparent deontic conflicts.

Overview

The thesis is organized as follows.

Chapter 2 intends to briefly introduce the Mīmāṃsā school from a historical and philosophical point of view, focusing on the features of normative reasoning in the context of this school.

After a brief outline on the main aspects and authors of the this school, we present the different kinds of *nyāya*s, with particular attention to the one classified as deontic principles. These represent the base for discussing the most important characteristics that Mīmāṃsā scholars attributed to Vedic injunctions, specifically concerning the difference between the deontic concepts used in Mīmāṃsā and their interactions and relative strengths.

Finally we focus on how apparent conflicts among Vedic commands are dealt with in Mīmāṃsā, and on the *bādha*s (34 of which are listed in the conclusions of this thesis), developed with the primary aim of resolving deontic conflicts by prioritizing a Vedic rule over another one. As mentioned above, *bādha*s are particularly interesting both from the philosophical and logical perspectives because they introduce defeasible reasoning. From the formal point of view, using commands that admit exceptions means reasoning in a way that is not completely "deductive": one cannot infer that a prescription which is *generally* valid in all situations *necessarily* holds in a given circumstance, as this circumstance could be an exception to the prescription. Hence (in many cases) Vedic norms —as modern laws— are abstract enough to include all the concrete cases and admit exceptions, instead of giving an explicit complete description of all the cases where they are enforceable, which will necessarily restrict their adaptability. Moreover, from the formal point of view, using commands that admit exceptions means reasoning in a way that is not completely "deductive": one cannot infer that a prescription which is *generally* valid in all situations *necessarily* holds in a given circumstance, as this circumstance could be an exception to the prescription.

Chapter 3 focuses on the logics that have been progressively "extracted" from Mīmāṃsā deontic reasoning. First, we present the non-normal dyadic deontic logic bMDL (basic Mīmāṃsā deontic logic), defined in [29] by applying the step-by-step bottom-up methodology mentioned above. bMDL extends classical propositional logic with the alethic operator $\Box$ —expressing that a formula is true at any possible state— and one deontic operator for obligations. For this logic we present the cut-free sequent calculus and the neighbourhood-style semantics introduced in [29]. In Section 3.2 we make a brief comparison between the logic bMDL and some of the most relevant related works on logics suitable to express conditional obligations.

As the logic bMDL makes use of the alethic operator $\Box$, which does not have any corresponding element in Mīmāṃsā reasoning, in Section 3.3 we present the $\Box$-free fragment of this system, that turns out to be the dyadic version of the known logic MD (see [27]).

The experts of Indian philosophy seem to agree on the fact that the different kinds of Vedic commands, as interpreted by Mīmāṃsā authors, cannot be reduced to the sole concept of "obligation". For this reason, in Section 3.4, the dyadic logic MD is extended with new operators for prohibitions and recommendations, whose properties are again meant to reflect the ones attributed to the corresponding Vedic commands by Mīmāṃsā authors. Also for this logic, called $\mathsf{MD+}$, we provide a cut-free sequent calculus and a semantic characterization, with the proofs of soundness and completeness that are unpublished results. In the last section of this chapter, the proof theory and semantics of MD+ are used to analyse the controversy around the *Śyena* sacrifice, widely debated by Mīmāṃsā authors. Such an analysis allows us to compare the different interpretations of the commands given by Mīmāṃsakas and identify the states of affairs consistent with those interpretations.

Chapter 4 presents a formal analysis of some of the *bādha*s, the principles for resolving normative conflicts by giving priority to one command over the others. We introduce sequent-style rules for reasoning in presence of deontic assumptions (representing the Vedic commands) and for applying the specificity principle, according to which a norm with more specific conditions overrules a more general one.

In order to clarify the idea behind the formal mechanism capturing the specificity principle, in section 4.2 we extend first the sequent calculus for the obligations-only logic MD with new sequent-style rules that allow to derive enforceable obligations from a list of (possibly conflicting) prescriptions, corresponding to the ones found in the Vedas. Then we extend with similar rules the full calculus for MD+ and present the technical properties (cut-elimination and decidability) of the resulting system.

As mentioned, the specificity principle is not the only method for resolving conflicts among norms: another important principle for choosing between two rules in a given situation involves considering the texts where they are originally stated. Mīmāṃsā authors recognize, beside the Vedas themselves, other three different sources —in principle all based on the sacred texts— hierarchically ordered. Hence, we also show how the sequent-style rules previously introduced can be adapted to capture also the hierarchy of sources, enabling us to derive a command only if no conflicting norm is found in a more important source.

Finally, we show how the formal system we developed naturally implements the *vikalpa* principle mentioned above. We use this feature, together with the fact that *vikalpa* is the least preferred option for dealing with a conflict, for modelling concrete controversies found in Mīmāṃsā texts. We compare the different interpretations of the conflicting norms according to the principle of minimizing the number of applications of *vikalpa*. This allows to mimic the reasoning of Mīmāṃsā authors, thus clarifying some of their choices, e.g. their preference for an interpretation of a specific norm over another one.

Chapter 5 contains some final considerations on the mechanisms presented in this thesis to formalize Mīmāṃsā reasoning. Moreover, we present some of the possible research directions for future work. Those include e.g. further developments of the system for capturing new *nyāya*s, the formalization of other *bādha*s, and the application of formal methods to compare Mīmāṃsā reasoning with (Indian and European) jurisprudence.

Appendix consists in a list of 34 (translated and commented) *bādha*s found in one of the main works by Mīmāṃsā author Kumārila. The list here has been translated by the Sanskritists working on the project *Reasoning Tools for Deontic Logic and Applications to Indian Sacred Texts*, with comments and examples resulting from the discussion and close cooperation between the logicians and the experts in Sanskrit and Indian philosophy involved in this project.

Sources

The core of this thesis is based on the published paper [31], the accepted paper [32] and various unpublished material, including:

- the bi-neighbourhood semantics for bMDL in the second part of section 3.1;
- the semantics of MD+ (section 3.4.2);
- the new analysis of the controversy concerning the Śyena sacrifice (section 3.4.3);
- the superiority relation among Vedic norms (section 4.3.2);
- the list of *bādha*s in the Appendix.

Abbreviations

PMS *Pūrva Mīmāṃsā Sūtra*[1]

ŚBh *Śābarabhāṣya*

ŚV *Ślokavārttika*

TV *Tantravārttika*

[1] Pūrva Mīmāṃsā Sūtra is constituted by twelve books (called *adhyāya*), each with 4 or 8 chapters (*pāda*) divided in *adhikaraṇa*s (sections), each containing one or more *sūtra*s (aphorisms). The notation of the references will indicate the book, chapter, and aphorism in that order; e.g. PMS 1.2.3. will denote the third aphorism in the second chapter of the first book.

Chapter 2

Deontic reasoning in Mīmāṃsā

Originated in India in the last centuries BCE and developed almost to the present day, the philosophical system called *Mīmāṃsā* represents the earliest Indian philosophical school whose texts are still extant. It is regarded as one of the fundamental schools of Indian philosophy and it is the first school in India with main focus on normative reasoning. Mīmāṃsā arises from a tradition of exegesis of the Vedas, i.e. the sacred texts for the plethora of philosophical and religious schools and movements that nowadays are gathered under the name of *Hinduism*. The Mīmāṃsā school focuses on prescriptions an prohibitions found in the Vedas, and in particular those related to sacrifices and ritual actions. Through this focus and the systematic and analytic methods adopted by its scholars, Mīmāṃsā had and still has a profound influence on many aspects of Indian thought. For instance, though originally elaborated for dealing with the sacred texts, its principles came to be used for analysing argumentations in Sanskrit grammar, Sanskrit poetics and rhetoric, philosophy, and law. In particular, recently there has been a revival of interest in the application of Mīmāṃsā rules of interpretation to contemporary Indian jurisprudence, as attested by "selected judgements" in K.L. Sarkar's [116].

In this chapter we provide a general presentation of Mīmāṃsā school, focusing on its systematic deontic reflections. In Section 2.1, we outline the historical and philosophical features of this school that are most interesting from the perspective of modern deontic reasoning[1], in particular the anti-mysticism and the systematic approach to the analysis of

[1] A detailed historical and philosophical introduction to Mīmāṃsā school, not restricted to deontic reasoning, is beyond the scope of the present work. For a very short introduction to Mīmāṃsā see [45], for a more detailed one, see [73] and [72] (on the subschool of *Prābhākara Mīmāṃsā*). Further key references are the articles and books by Kei Kataoka (see the bibliography in [76]), John Taber (see, e.g., [125]) and Larry McCrea (e.g., [94] on Mīmāṃsā epistemology).

prescriptive statements in the Vedas.

Key for Mīmāṃsā systematic approach is the elaboration of interpretative principles (*nyāya*s), intended to enable the readers to understand and apply the prescriptions in the Vedas, and to solve apparent conflicts among those prescriptions. A short characterization of those principles will be presented in Section 2.2, paying particular attention to the investigation of *nyāya*s that determine the properties of commands' formal representations.

In Section 2.3, the concept of proper deontic statement as defined by Mīmāṃsā scholars will be introduced: we will focus on the general features attributed by those philosophers to Vedic commands, identifying the elements which determine the categorization of different kinds of injunctions. Based on those characteristics, a classification of the various types of duties discussed in Mīmāṃsā texts will be provided; this will constitute the basis for the definition in Section 3.4 of the deontic operators in the formal system.

Finally, in Section 2.4 we will discuss some of the mechanisms employed by Mīmāṃsā authors to deal with seemingly conflicting commands in the sacred texts. Specifically, we will focus on principles used to solve conflicts among prescriptions by prioritizing them, in particular the principle according to which preference is given to commands with narrower conditions.

2.1 Basic outline

The Sanskrit word "mīmāṃsā" means "reflection" or "critical investigation". Emerging from a long history of scriptural exegesis, the aim of this school is to interpret and systematise the prescriptive portions of the Vedas, the sacred texts of what is now known as Hinduism.

The Vedas are a large body of Sanskrit texts, dated in a period from around the middle of 2nd to the middle of 1st millennium BCE and orally transmitted for centuries. These texts are traditionally divided in four collections: the *Ṛgveda*, the *Yajurveda*, the *Sāmaveda* and the *Atharvaveda*. Each collection is further subdivided in four classes or sections, distinguished according to their topics: the *Saṃhitā*s (consisting in hymns, prayers and benedictions), the *Brāhmaṇa*s (containing ritual exhortations and prose commentaries on the sacrificial methods, which explain the meaning and the symbolic import of their components), the *Āraṇyaka*s (on methods for ceremonies that should be performed by a specific category of people, identified approximately as hermits), and the *Upaniṣad*s (discussing meditation and spiritual philosophy).

Mīmāṃsā scholars (*Mīmāṃsaka*s) focus mainly on the *Brāhmaṇas*, as they intend to provide an analysis of Vedic texts relative to sacrificial rules. They consider the *Saṃhitās* as consisting mostly of *mantras* (sacred formulas, similar to prayers or invocations) to be uttered during sacrifices, while the *Āraṇyakas* and the *Upaniṣads*, together with the parts of the *Brāhmaṇas* not prescribing rituals, are regarded by Mīmāṃsakas as subordinate to the parts prescribing duties. In other words, all the non-prescriptive portions of the Vedas, including myths, legends and philosophical speculations, are considered to contain ancillary statements, to be interpreted as complements meant to help refining the meaning of Vedic commands.

Because of this main interest in analysing Vedic statements enjoining an agent to perform a ritual act (*karman*, or "action", being used as the technical term for these ritual actions), Mīmāṃsā school has been also called *Karma Mīmāṃsā* ("study of actions"). On the other hand, on the basis of the specific analysed portions of the Vedas, Mīmāṃsā is also referred to as *Pūrva Mīmāṃsā* ("prior study"), as opposed to the *Uttara Mīmāṃsā*, which indicates the *Vedānta* school, focused on the parts of the Vedas (the Upaniṣads) that are subsequent to the ones (the Brāhmaṇas) investigated by Mīmāṃsā.

Since Mīmāṃsā is rooted in a tradition of Vedic exegesis focused on ritual prescriptions, one might expect a reference to some deity or higher power for guaranteeing the correspondence between what is obligatory and what is morally good. Instead, a very interesting characteristic of Mīmāṃsā thought, which is shared by most of its scholars, is represented by a form of *atheism*, which constitutes the conceptual basis of their reflection.

This feature constitutes the foundation of Mīmāṃsā scholars' understanding of the Vedic commands more as a code of "practical conduct", than as an "ethical system" as traditionally understood in Western European philosophy. Without appealing to any divine authority or metaphysical intuition of a greater good, there are no "superior principles" on the basis of which the rules in the Vedas could be reinterpreted, modified or ordered. Hence, the Vedic commands remain as the only authority and the only valid criterion for interpreting them is the preservation of the internal consistency of the sacred texts.

Even if the names of some gods appear in the analysed Vedas, Mīmāṃsā authors consider them as mere linguistic entities and reject gods as a metaphysical pillar that holds up the structure of duty. As a matter of fact, the sacred texts, which represent the fundamental means of accessing to the knowledge (*pramāṇa*) about duty, are considered to be *"authorless"*. This concept, expressed by the Sanskrit word *apauruṣeya* —literally translated as "not created by personal beings"— has, in this case, several meanings: not only the Vedas are not considered to be a message transmitted by gods to human beings, but also they could not have been created by personified deities, with traits, like desires, which they share with humans.

The rules in the sacred texts do not even need to be legitimated and enforced by supernatural beings, as the *karman* —seen almost as a natural principle of causality— guarantees that any ritual action has the result described in the Vedas.

Essentially, Mīmāṃsā authors do not appear concerned with investigating the metaphysical reality behind the Vedas, or with an explicit definition of the moral values independent from the sacred texts. This feature distinguishes Mīmāṃsā reflection from most of the religious and moral traditions, as it constitutes a *"deontology without ethics"* ([48]).

The reason why deontic concepts are not invoked in connection with any definition of moral values lies again in the role of sacred texts in Mīmāṃsā school. As the Vedas are considered not to contain any redundancy, Mīmāṃsā philosophers interpret the very presence of an explicit injunction in the sacred texts as the fact that this injunction cannot be rationally derived from any idea of "righteousness" independent or even abstracted from the texts. Hence, the prescriptive statements in Vedas represent the main source for knowing *Dharma*, which is understood by Mīmāṃsakas as "duty", or "what one *must* do".

In Mīmāṃsā texts this concept is intrinsically linked to the idea of being the content of a Vedic command: for instance, the conventional meaning of this word within the school's foundational text is stipulated by the principle *"the Dharma is a purpose* (or goal) *characterized by an injunction"*[2], meaning that Dharma is *"that which is made known by a Vedic injunction"* ([128]).

The fact that the Vedas are considered the only epistemic authority for what concerns the duty does not exclude the use of common experience as guide for the domain of "facts", accessible through observation of the world and reasoning. Knowledge and beliefs about reality, derived by common experience, can direct the interpretation of a Vedic command, but an interpretation cannot depart from the texts in the name of a more reasonable hypothesis. As proof of this, it appears that the internal consistency of the sacred texts remains in principle a criterion of interpretation more important than being easily acceptable from the point of view of common experience.

However, in a sense, the operation of interpreting Vedic commands connects the domain of *Dharma* —whose only reliable source is constituted by the Vedas— with the domain of what can be known through the sole use of senses and intellect, and whose main source is common experience. As Mīmāṃsā is a school of textual exegesis, which means that their focus is

[2] PMS 1.1.2 in [128].

precisely the interpretation of texts about sacrifices and ritual actions, the considerations above could be a key to make sense of the famous claim of one of the most important Mīmāṃsā authors, Kumārila Bhaṭṭa, stating *"by the Mīmāṃsakas [...] nothing is accepted except what is commonly experienced"*[3]. Indeed, as will be seen later, common experience is also the base of interpretative principles employed by Mīmāṃsā authors, which are explicitly stated not to derive from the Vedas: exegetical rules in Mīmāṃsā come from human rationality and reflection on worldly experience and common use of language, hence it seems they have much in common with the contemporary thought regarding positive laws.

The school's foundational text, the *Pūrva Mīmāṃsā Sūtra* (PMS from now on), essentially an example of Vedic hermeneutics, is attributed to the author Jaimini (ca. 4th to 2nd century BCE). Hence, the rise of Mīmāṃsā school is contemporary to the emergence of Buddhism: the two represent in a sense opposite movements, as the former aims to provide the philosophical instruments for making sense of the Vedic rituals and the latter, compared to the Vedic schools of Hinduism, tends to prioritise the meditation over the tradition of ritual sacrifices [137].

Already in PMS, the analysis of Vedic commands consists in the elaboration of general principles (*nyāya*s in Sanskrit) for understanding and interpreting the Vedic prescriptive statements and determining the conditions and modalities of their applicability[4].

The most ancient commentary on PMS available to us is the *Śābarabhāṣya* (ŚBh from now on), composed by the author known as Śabara in a period between the 3rd and the 5th century CE[5]. Together with Jaimini's text, it constitutes what we might call *"common Mīmāṃsā"*, as it is accepted by all later Mīmāṃsakas.

While remaining on the pattern laid down by the PMS, later authors introduce slightly different perspectives and interpretations —in particular regarding the deontic issues— without, however, developing explicitly novel philosophical systems. Indeed, unlike the trend in modern European philosophy after the work of Descartes, novelty is generally not perceived as a value in South Asian philosophy.

In particular for schools based on the interpretation of the sacred texts, the role of a philosopher consists in justifying, explaining and using what has been stated by the authorities of the school in order to make sense of the Vedic passages. Hence, though some of the most important authors' views significantly differ, they maintain a common base constituted

[3]ŚV codanā 98d–99ab in [76].

[4]For a short introduction to Mīmāṃsā in general, see [45]. For injunctions and ritual duties as conceived in Mīmāṃsā see sections 3—6 of [43] and [102].

[5]The dating is just tentative cfr E. Freschi in [45].

by foundational authorities and basic ideas. In this respect, it is important to note that Mīmāṃsā texts reveal a divergence of views more on "why" a norm should be interpreted in a specific way, than on "what" is the correct interpretation of the norm. In other words, differences arise in justifying the interpretation of a Vedic command, while, looking only at the conclusions of arguments —concerning what the sacred texts actually tell agents to do— Mīmāṃsā authors tend to agree on the same interpretations.

Main Mīmāṃsā authors and sub-schools

Specifically, the period between 6th and the 8th centuries CE give rise to some of the leading figures of this school, namely **Kumārila Bhaṭṭa** (ca. 7th century CE), **Prabhākara Miśra** (ca. 7th century CE), and **Maṇḍana Miśra** (ca. 8th century CE). In this period Mīmāṃsā is indeed considered to reach one of its highest points from the perspective of the influence on Indian philosophy, with the flourishing of its most representative "sub-schools".

Kumārila[6], founder of one of these sub-school, the *Bhāṭṭa*, is the author of four (sub)commentaries on different portions of the ŚBh. In particular, the present work will make use of some examples of reasoning taken from two of Kumārila's (sub)commentaries, called *Ślokavārttika* (henceforth ŚV) and *Tantravārttika* (TV from now on). Kumārila is recognized as one of the most influential thinkers in the whole history of Indian philosophy, the best-known among his significant contributions being the development of his epistemological doctrine[7]. For what concerns his interpretation of normative statements, Kumārila seems to reduce any ritual act to three components: a (desired) state of affairs to be brought about (*bhāvya*), an instrument (*karaṇa*) for "producing" this state (the prescribed action), and a procedure (*itikartavyatā*) for actualizing it.

Let us consider, for example, a statement prescribing the performance of a given sacrifice to the agent who desires happiness: following Kumārila's interpretation, happiness is the *bhāvya*, the given sacrifice is the *karaṇa*, and all the actions required for the performance of the sacrifice constitute the *itikartavyatā*. The "obligation" to perform the sacrifice is not completely excluded, as the command is binding for everyone who desires happiness, but the deontic content (the "obligatoriness") of the given sacrifice is not independent from its result.

This view is slightly in contrast with the thought of **Prabhākara**[8], founder of the other main sub-school of Mīmāṃsā, called *Prābhākara*. Author of a (sub)commentary on the ŚBh called *Bṛhatī*, Prabhākara claims that what is conveyed by an injunction is not just an

[6]For an introduction to Kumārila's thought, see [64, 125, 8].

[7]Many interesting aspects of Kumārila's epistemology are analysed e.g. in [7, 42, 23]

[8]For an introduction to Prabhākara's thought, see [72]

instrument to a desired end, but something that has to be done (*kārya*), i.e. a proper duty.

Let us look at the previous example of a statement prescribing the performance of a given sacrifice to the agent who desires happiness: following Prabhākara's interpretation, the given sacrifice is something that should be performed, independently from its results, and the desire for happiness is meant to identify the addressees of the duty. The fact that happiness is the result of completing the sacrifice does not affect the deontic content of the ritual act itself. For better understanding this interpretation of commands, let us consider the example, close to our everyday life, of paying taxes. People live in organized communities because they want to have access to public services like healthcare and education, and everyone who lives in an organized community has to pay taxes. The obligation to pay taxes is addressed to people living in organized communities and in the end it guarantees the access to public services to everyone in the community. However, the reason why people pay taxes is the fact that, independently from their desires and from the long-term results of this behaviour, it is a legal obligation.

Finally, the position of **Maṇḍana** on the nature of Vedic commands is almost diametrically opposed to Prabhākara's perspective. Maṇḍana Miśra —author of various treatises on, among other topics, the nature of prescriptions— takes Kumārila's point of view and goes even further, claiming that an injunction is nothing more than the description of the cause-effect relation between two states of affairs. The statement *"the one who desires happiness should perform the given sacrifice"* does not express any duty to perform the sacrifice or to bring about the state of happiness, it only represents a factual statement about a given sacrifice being an instrument for realizing the state of happiness.

As the thought of Maṇḍana is considered to depart from Mīmāṃsā tradition in some respects, and his analysis of commands is more about the concept of *instrumentality* than about duty, in general we do not use here examples of reasoning taken from his work.

2.2 Mīmāṃsā methods: *nyāya*s

In order to analyse Mīmāṃsā's normative reasoning with the tools of mathematical logic, first, account should be taken of the method employed by the philosophers of this school for making sense of the Vedic precepts. Indeed, the rigour and regularity of such methods make Mīmāṃsā normative reasoning particularly suitable for an analysis based on modern formal logic. By its very nature, logic is based on the idea that correct reasoning is characterized by conclusions following from reliable premisses via precisely defined (and verifiable) inferential steps. The same idea underlies Mīmāṃsā authors' reasoning, as evidenced by their theory of inference (*anumāna*) and by their systematic approach to normative reasoning. According to such perspective, any flaw in the premisses, in the chain of transmission, or concerning their compatibility makes the conclusions unreliable. The classical example emphasising such strict requirements is that of *"a chain of truthful blind people transmitting information about colours"* (*andhaparamparānyāya*, TV on PMS 1.3.27).

As already mentioned, Mīmāṃsā authors aimed not only at clarifying how single rituals should be performed, but to provide general rules for consistently explaining the way all prescriptions in the sacred texts should be understood. From a modern perspective, it appears that the idea behind Mīmāṃsā method is to build a rigorous system of reasoning such that from true premisses about commands it is always possible to derive the correct interpretation or solution, by using general and abstract rules.

This *theoretical* interest is reflected in the structure of argumentations in Mīmāṃsā texts. The analysis of a controversy constitutes a dialectical process, usually expressed by a discourse between several different views (one or more opponents and one or more respondents, who might or might not be identical with the upholder of the final view) providing general reasons that support a thesis or attacks the opponent's one. This way, the upholder of the finally established view (*siddhāntin* in Sanskrit) can refine his premises, including refutation of objector's theses as arguments supporting his claims; hence, it is ensured that the only assumptions made are the ones accepted as true by everyone[9].

The structure of arguments, as exposed in ŚBh, is composed by five steps:

(1) enunciation of the topic (*viṣaya*)

(2) enunciation of the dilemma (*saṃśaya*)

(3) first thesis, i.e. preliminary view of the dilemma (*pūrvapakṣa*)

(4) antithesis to the preliminary view (*uttarapakṣa*)

(5) conclusive view (*siddhānta*)

The third and fourth steps (which can be repeated in case of complex controversies) and the fifth one make use of general and abstract reasons for legitimating any thesis. Those general interpretative rules, involved in the justification of any hypothesis, are called *nyāya*s: such rules have the key purpose of guiding the readers through a textual passage and enable their understanding of the text independently of any authorial intention.

[9]The structure of argumentations seems to have many similarities with the dialectical method developed in European tradition, in particular with the *quaestiones* and *disputationes* typical of Medieval Scholasticism. Comparative studies on the methods for argumentation used in European and Indian philosophical traditions include e.g. [63, 113].

Nyāya s make it possible to understand all the significant injunctions in a textual passage as a coherent corpus. Moreover, they are —at least in principle— independent from the specific controversy they are developed for, meant to be general and applicable in any case. Evidence of Mīmāṃsā *nyāya*s' general and abstract nature is provided by the fact that they have been subsequently used also in the field of Indian jurisprudence (*Dharmaśāstra*).

A *nyāya* is a metarule, i.e. "a rule ruling other rules" ([46]): this means that *nyāya*s are principles concerning the interpretation of the whole system of sacrificial rules in the *Brāhmaṇas*.

*Nyāya*s can even be principles regulating the usage of Sanskrit terms in formulating other Mīmāṃsā *nyāya*s; the latter group, much smaller than the other, includes *nyāya*s that in most cases cannot be expressed in the context of formal deontic logic. Indeed, not only they mainly belong to the group of linguistic *nyāya*s, but they also pose the problem of representing a second level of abstraction, being essentially meta-metarules.

As already mentioned, *nyāya*s are not extracted from the Vedic texts, but rather they represent an application of rationality and common sense to the interpretation of the Vedas; as noted again in [46], it seems that they are all based on very general principles of reasoning, *a posteriori* identified as the following:

(i) *Economy*: when possible, the simplest option should be preferred. This is valid also on a practical level, e.g. when a ritual prescribes the offering of multiple items of the same kind, without specifying their number, Mīmāṃsā rules tend to recommend the use of the lowest possible number of items.

    The same general law of economy is applied to actions and to interpretative rules themselves: for instance, once a principle has been found, which solves a certain controversy, the same principle should be applied in all the similar cases.

(ii) A sentence should be understood as conveying a *single duty*: this principle could seem strange considering, for example, injunctions prescribing many things together. This rule appears to deal with the difficulty of extracting single commands from the Vedic texts, which, unless organized according to Mīmāṃsā principles, may lack a clear structure. Accordingly, a command which seems to prescribe multiple things is interpreted as prescribing indeed a single core duty.

(iii) Reference to the *common experience*: when no exception is indicated, the simplest solution —the one inspired by experience and common sense— should be preferred. As already mentioned, the comparison with worldly experience is a key principle of Mīmāṃsā thought.

(iv) The Vedas are assumed to be the only instrument of knowledge for what concerns the duty. This means that they should be *understandable*, *useful* (communicating informations that one would not have known without them) and *consistent* (not containing conflicting informations).

Based on those general principles, the essential *nyāya*s, representing the foundation of Mīmāṃsā interpretative structure, can be included under one or more of the three groups of *hermeneutic*, *linguistic* and *deontic* metarules (see [47]). However, it is important to recall that the classification of *nyāya*s represents an *a posteriori* analysis, which could be imprecise and slightly influenced by modern distinctions: the three categories above are not defined and distinguished by Mīmāṃsā authors, who mostly introduced the principles of interpretation when the argumentation required them. It should be also noted that the classification of *nyāya*s is not rigid: as interpretative principles for analysing textual passages, in a sense they can all be considered hermeneutic; moreover, as the analysed texts mostly concern duty, many *nyāya*s fall under the category of deontic *nyāya*s. The principles that lend themselves better to a formal representation in the context of deontic logic are clearly expected to be the deontic ones, however, many of the principle we will use in the next chapters can be classified as hermeneutical *nyāya*s as well. In particular, though deontic *nyāya*s are the only ones that can be translated into axioms, hermeneneutic *nyāya*s represent the theoretical basis of many features of the logical system and, specifically, of the rules for resolving deontic conflicts.

On the other hand, *linguistic nyāya*s, as they concern specific characteristics of the Sanskrit language, are less likely to be translated in formal logic.

Linguistic *nyāya*s constitute quite technical metarules, concerning grammatical and syntactical aspects of the Vedic prescriptive passages.

They are mainly needed in order to discuss the interpretation of linguistic peculiarities of the Sanskrit form of the various prescriptions. For instance, linguistic *nyāya*s include:

- delimitation of sentences (i.e. detecting distinct prescriptions within the continuous text of the prescriptive portion of the Vedas);
- textual linguistics (i.e. establishing the correct meaning of words, according to the principle that the meaning of a specific word in the Veda should be as much as possible similar to its meaning in ordinary language);
- functioning of morphology (grammatical rules concerning the variations of the terms according to their functions in the proposition).

As they are used, for example, to single out and classify statements and to define the conditions for the interpretation of metaphorical passages, linguistic *nyāya*s may be considered a subset of hermeneutic principles ([47]). However, linguistic *nyāya*s slightly differ from the hermeneutic ones, as they often consider single linguistic elements instead of the entire structure of a prescription and they are based on the analysis of the uses of Sanskrit language in various contexts, not limited to the Vedic injunctions.

Hermeneutic *nyāya*s have a fundamental role in Mīmāṃsā, as they have the purpose of identifying, in the sacred texts, the single injunctions and the duties they convey, by distinguishing the prescriptive parts from other Vedic passages that do not contain commands of any kind. The latter are regarded, by Mīmāṃsā authors, just as ancillary to the prescriptive portions, serving as tools for understanding or completing the injunctions. Given their importance in Mīmāṃsā school —primarily focused on the hermeneutics of prescriptive Vedic texts— hermeneutic *nyāya*s are the most numerous among metarules: this is also reflected by the fact that, in general, identifying them in the texts is easier than for other *nyāya*s.

In order to enable the readers to recognize the prescriptions and analyse their contents, hermeneutic *nyāya*s characterize the Vedic injunctions through syntactic features and by determining the typical traits shared by the conveyed duties.

An example of hermeneutic *nyāya*s regarding the identification of prescriptive sentences is constituted by the *postulation of result* (ŚBh on PMS 4.3.10), stating that any prescribed action must have a result.

Hence, for instance, if something which, for some reasons, is considered a prescription does not mention any positive outcome, happiness is to be postulated as its general result. This principle also means that, when a positive result of an action is mentioned, the sentence should, in most cases, be interpreted as a prescription.

Following [47], the most relevant hermeneutic *nyāya*s, in addition to the *postulation of result*, are briefly stated below:

(i) *On instruments of knowledge*

What is stated directly by the texts is an instrument of knowledge more powerful than what is inferred from the context, the implicit connections, the syntactical aspects, etc. (PMS 6.1.51–52)

(ii) *On the distinction between topic and comment*

Only what is intended is part of the prescription, and whether something is intended or not is determined by its (syntactical) link to the principal duty in the sentence. If a term constitutes the *topic* of the sentence, it has the purpose to recall something already known and its linguistic peculiarities —e.g. being singular or plural— do not count; by contrast, if a term is part of the *comment*, which says something new about the topic (in this case about the prescribed duty), its peculiarities should be considered.

The hermeneutic *nyāya*s analysing the characteristics of the duties conveyed by Vedic prescriptions differ from the deontic ones in that their focus is not specifically on the form of commands, but on the interpretation of the injunctions' content.

Deontic *nyāya*s concern specifically the form of sacrificial prescriptions in the sacred texts and define the constitutive elements of commands, distinguishing their different kinds and determining most of the characteristics of their formal representations. However, Mīmāṃsakas seem to consider the *nyāya*s we place under this category as secondary to the main hermeneutic purpose. Not only, similarly to hermeneutic and linguistic metarules, they are not the subjects of systematic descriptions, but their nature and application are also, in general, less debated than the ones of other *nyāya*s; for instance, no specific section (*adhikaraṇa*) has been found that is dedicated exclusively to one of the metarules we categorize only as deontic *nyāya*s.

However, as deontic *nyāya*s are the only ones suitable to be translated into axioms, they represent the core of our method for "extracting" a logic from Mīmāṃsā texts.

As an example we can consider the principle that is interpreted, abstracted and transformed in the first axiom of bMDL in Section 3.1:

> When the various (requirements of a given duty), beginning with the origination [of a new duty], are not established by other distinct prescriptions, then [the only prescription available] itself creates the other four prescriptions that are related to it.

This meta-rule affirms that if the content of an obligation has some necessary conditions that are not prescribed separately, then the obligation also prescribes all the necessary conditions of its content. Hence all obligations have the characteristic of prescribing all the necessary consequences of what they enjoin.

Though necessary for defining the formal properties of prescriptions, the deontic *nyāya*s are not only difficult to find and recognize in the texts, but also require a high level of interpretation and abstraction to be properly understood. Indeed, they first require a translation from Sanskrit, an interpretation and afterwards an abstraction, before being transformed into logical formulas.

Following again [47], some of the main principles that can be considered both as hermeneutic and deontic *nyāya*s, as they concern what Vedic commands express, are listed below:

(i) *Meaningfulness*: a command in the sacred texts cannot be meaningless or inapplicable: if it appears to be unenforceable, either it does not represent an injunction, or there is an error in the interpretation (PMS 1.2.23).

(ii) *Novelty*: since no Vedic rule can be useless, there are no injunctions which just repeat an already mentioned content, but each command should convey something new (*apūrva*) (PMS 1.2.19). This means also that, since the Vedas are the only valid instrument of knowledge about duties, if an obligation (or a prohibition) seems to convey something that the agent is already inclined to do (resp., to avoid), or that can be known directly or indirectly on the basis of sense perceptual data, the command should be interpreted as conveying something different.

(iii) *Duty as action*: the content conveyed by a Vedic command is always an action (that can be intentionally performed or avoided) (PMS 1.2.1). This means that, from a Mīmāṃsā perspective, Vedic commands do not —at least explicitly— rule the emotive, psychological, or moral states of their addressees.

    At this point it is important to note that, slightly in contrast with the previous *nyāya*, in the formal system the prescribed or prohibited actions will be translated as declarative sentences which can be either true or false. Hence, for instance an obligation like "the one who desires happiness should perform the given sacrifice" will be read as: "it is obligatory that the sentence *"the one who desires happiness performs the given sacrifice"* is true". Thus, in the formal system, we do not really talk about actions (like *"performing the given sacrifice"*), but about states of affairs (e.g. the state where the sentence *"the one who desires happiness performs the given sacrifice"* is true). This gives the technical advantage of allowing us to use propositional logic, but it implies that in the formal language there is no distinction between actions and states, hence, the requirements expressed by the *Duty as action nyāya* do not correspond to any aspect in the logic.

(iv) *Singleness*: an injunction can convey only one piece of deontic information: the prescribed act can be composed by many subsidiary actions, but the content of a prescription is always one ritual action serving one purpose (ŚBh on PMS 2.1.12).

(v) *Auxiliarity*: each prescriptive portion of the sacred texts contains a principal prescription, and all the other actions prescribed in that portion are subsidiary to that action conveyed by the principal prescription (PMS 1.2.7).

Hermeneutic and linguistic principles, necessary for identifying and interpreting the injunctions, determine some implicit characteristic of prescriptions' formal representations; e.g. the *Meaningfulness* and the *Novelty nyāya*s are the conceptual basis for a mechanism —like the one developed here— which solves conflicts between injunctions without completely invalidating one of them. However, *nyāya*s of that sort are seldom expressible explicitly in logical terms, i.e., except in rare cases, they do not correspond to a specific axiom or formal characteristic of the operators, but they are rather represented as semantic features of the whole system.

2.3 General features of injunctions

Before approaching the translation of deontic (and hermeneutic) *nyāya*s into formal principles and mechanisms, we introduce the characteristics of Vedic commands as understood by Mīmāṃsā authors.

For any command, Mīmāṃsakas seem to identify at least the following elements: (a) the proper content, i.e. what should be brought about for respecting the injunction, (b) the motivations for following the commands, (c) the consequence of obeying and not obeying them, and (d) the conditions (*adhikāra*) determining the addressees of a given command. On the basis of those, the deontic concepts in Mīmāṃsā can be classified in different ways.

With regard to the *proper content of Vedic rules*, it is necessary to clarify some aspects of our approach to Mīmāṃsā analysis.

As mentioned above, for the Mīmāṃsā school, the Vedas are the only valid instrument of knowledge on the cause-effect relationship between (human) actions and positive or negative results (in terms of karman). Considering that, from the perspective of Mīmāṃsā authors, Vedic rules necessarily concern actions, the positive commands —the ones prescribing to perform (not avoid) actions— have reference to ritual actions (sacrifices). Mīmāṃsakas distinguish between *"main"* (or *"primary"*) actions, i.e. the acts of choosing to undertake sacrifices (e.g. the sacrifice *Agniṣṭoma*, the praise of the fire deity), and *"subsidiary"* rites that one has to perform to complete the sacrifice (for instance the rites called *pravargya* and *upasad*, to be performed on the second day of *Agniṣṭoma*); those, in turn, can entail many separate activities (such as, for the *pravargya*, preparing the hot milk mixed with boiling ghee and pouring it).

Remark 2.3.1 *For the sake of the present work, the relations between the levels of this* *hierarchy of ritual actions are not taken into consideration; in particular, we consider the inferior levels to "inherit" the same kind of deontic property characterizing the superior ones. For instance, as will be shown in the next section, on the basis of the classification of sacrifices, we can abstract two kinds of deontic properties: a primary ritual action can be "strongly obligatory" if the consequences of the failure to perform it are in general undesirable, or "weakly obligatory" (or "recommended") if one can omit the enjoined act without any consequence. Here we consider the subsidiary rites of a strongly obligatory sacrifice to be strongly obligatory and the subsidiaries of a recommended sacrifice to be weakly obligatory. This choice has the advantage of avoiding the use of new logical operators for subsidiary actions. Moreover, it captures the relations between primary actions and their subsidiaries from the point of view of their consequences: if one gets a positive result by performing a sacrifice, then each of its subsidiary acts is necessary to obtain the positive result.*

However, this choice of formalization leaves out some aspects of the relations between main actions and subsidiary ones as conceived by Mīmāṃsā authors, specifically concerning the way in which subsidiary actions should be performed. In particular, once the choice to undertake a sacrifice has been made, the subsidiaries of a strongly obligatory rite should be performed, again according to Mīmāṃsā authors, at the best of one's possibilities, while, by contrast, the subsidiaries of a recommended sacrifice should be performed exactly as prescribed in the sacred texts, otherwise the sacrifice is invalid. This phenomenon has been described as "deontic reversal" and formalized in [48]; however, the complex formal language in that work makes a proof theoretic approach to the system very difficult.

For what concerns the *motivations for following commands* and the *consequences of obeying and not obeying them*, the distinction between those two elements is more complex and highlights the differences among Mīmāṃsā authors.

According to what we called *"common Mīmāṃsā"* (PMS and ŚBh), every prescribed or prohibited action is said to have some result (*phala*). For prescriptions (as opposed to prohibitions), depending on the type of prescribed sacrifice, the result could be a specific desired state, or a general positive result identified with "heaven" (*svarga*) or "happiness" (*prīti*)[10]. The desire for heaven or happiness is assumed to be common to all living beings, so that it is considered to be the consequence of all sacrifices in which no other result is explicitly mentioned. As already noted, the desired result of a ritual actions is often considered to be also the trigger or the reason for accomplishing a duty: this is true e.g. for the author

[10]Heaven is explicitly said to correspond to happiness, e.g., in ŚBh 6.1.1 (*prītisādhane svargaśabda iti*) and 6.1.2 (*prītiḥ svarga iti*).

Maṇḍana and his followers, who interpret injunctions as devoid of any deontic content, such that the only reason for respecting them is the prospect of achieving the final goal. However, we have already mentioned that the two concepts of desire for heaven and motive for obeying the rules seem to be slightly different from the perspective of Prabhākara's school. Authors of this school appear to consider ritual norms not only as instructions for obtaining a desired consequence[11], but as moral obligations, that eventually will bring about the promised result, but should be obeyed only because of their very nature. In this, Prabhākara's school seems to anticipate, under some respects, the idea of duty for duty's sake, typical of Kant's categorical imperative[12].

The other central concept in Mīmāṃsā interpretation of injunctions is the *eligibility* (*adhikāra*): it represent the set of conditions necessary to identify the addressee of a duty, who will obtain a positive (or negative) result from the performance of a ritual action (resp. of a forbidden act).

Being eligible (*adhikārin*) firstly entails being able to complete the sacrifice: this includes, in addition to a thorough knowledge of the Vedas, both physical and economic ability. Those abilities should be understood as permanent conditions generally characterizing the eligible person, therefore, someone who temporarily loses (or gains) an economic or physical ability, in principle does not lose (or gain) the *adhikāra* for performing a sacrifice. On a practical level, this is clear if we consider that many rituals are complex, to be repeated throughout one's life, and such that a single performance can last for days; for instance, the condition of being given enough money to perform a sacrifice for one day could not enable to complete or repeat the ritual.

While it is very unlikely that someone who lacks the *adhikāra* for performing a ritual gains it, the possibility of loosing the *adhikāra* is discussed in Mīmāṃsā texts, but only as the consequence of some traumatic experience or permanent change in one's conditions.

Moreover, the *adhikārin* should be both enjoined to perform the sacrifice and entitled to its result. This means satisfying requirements on the gender (both women and men can be eligible for a sacrifice, but women usually can participate only jointly with their husbands), the caste of birth (commonly only the upper three castes are involved in ritual actions), and

[11]Some authors ([97]) even go so far as to say that Prabhākara denied the connection of some types of sacrifices with a result; however this thesis appears not to be supported by the extant texts of the school and most of the scholars prefer a more moderate position ([48]).

[12]The similarity between Prabhākara and Kant has been discussed by scholars in the field of comparative philosophy: although it should not be pushed too far, it highlights the differences between Prabhākara and other Mīmāṃsakas, in particular Maṇḍana, who seems to refuse the very concept of duty. For further details on the analogies between Prabhākara and Kant, see e.g. [44, 126].

on the desires.

This last condition, expressed by the deontic *nyāya* "*each action is prescribed in relation to a responsible person who is identified because of their desire*' (cf. PMS 6.1.1—3), is particularly interesting because it identifies the individual as capable of taking (ritual) actions with the one having desires. The concept of "subject" —which in European philosophical tradition is mostly understood as the unique consciousness underlying all the perceptions, actions, and intentions— seems to emerge in Mīmāṃsā primarily as agent of sacrifices, hence appearing indivisible from the notion of having desires.

This could suggests a conflict between Mīmāṃsakas' thought and the idea, common to many Hindu schools of philosophy and to Buddhism, that the state of liberation can be reached only through the extinction of desires. As suggested in [41], this discrepancy between the common view regarding liberation and the role of desires in Mīmāṃsā can be seen as a proper difference among Hindu schools' perspectives, or as a specific interest of Mīmāṃsā scholars in providing a systematic deontic theory for the subject in its mundane status, leaving out the one who already achieved liberation and transcends this status.

The problem of considering the desires as part of the eligibility conditions is also connected with the contemporary debate about the attribution of responsibility to artificial intelligences, "subjects" which represent "agents" (they can take action), but lack the elements classically related to the moral responsibility, e.g. intentions, desires, and free will[13].

2.3.1 Deontic concepts: prescriptions and prohibitions

According to the properties of commands mentioned above, it is possible to classify the distinct deontic concepts used by Mīmāṃsakas and identify similarities and differences between them.

A first distinction concerns the difference between the two concepts of prescription (*vidhi*) and prohibition (*niṣedha*). According to Mīmāṃsā authors, the two are not symmetrical. This means that proper bans ("it is forbidden to undertake this action") are not equivalent to *negative obligations* ("it is obligatory not to undertake this action"). Hence, in principle, both negative obligations and *"positive prohibitions"* ("it is forbidden not to undertake this action") could be formulated.

To confirm the presence of a strong distinction between prescriptions and prohibitions,

[13] *Machine ethics* is one of the most studied fields of the last years, in particular due to the technical advancements in Computer Science and Artificial Intelligence, which led, for instance, to the introduction of autonomous cars. The topic has been investigated from computer scientists (see e.g. [25]), philosophers (see e.g. [142] for an overview of the different approaches), and, in the last few years, also by legal scholars, as the number of states introducing legislation e.g. for autonomous vehicles increases every year.

not necessarily dependent on the grammatical (positive or negative) form, in commentaries on PMS 6.2.19, it has been debated whether injunction not to eat *kalañja* (probably a variety of garlic) represents a proper prohibition, or instead an obligation to refrain from consuming that product.

If such a command is a prohibition, it means that the agents have the duty not to eat *kalañja*, independently from their desires and intentions; such a command could be violated, at least in principle, even by accidentally consuming this product. Conversely, in the case of a negative obligation, the command should be read as conveying the act of choosing to refrain from eating *kalañja*; hence, theoretically, if the agents decide to eat that vegetable, they are not compliant with the obligation, even if an external contingency prevents them from realizing their intentions. For prescriptions the conveyed idea (*buddhi*) is indeed *"activation"* or *"being impelled to act"*, while in case of prohibitions it is *"inhibition"*, or *"being prevented from taking an action"*. For this reason, though the most evident distinction between them regards the results of compliance and non-compliance, prescriptions and prohibitions should be considered as genuine deontic concepts and cannot be deprived of their deontic content and reduced to instructions for obtaining desirable results or avoiding sanctions. Such an interpretation —that could be formally represented in deontic logic by a Kanger-Andersonian reduction— would be closer to the instrumental reading of commands given by Maṇḍana Miśra.

For what concerns the results of obeying and disrespecting prescriptions and prohibitions, complying with the Vedic obligations —i.e. performing prescribed sacrifices— leads to a positive desired result. The nature of the result, as we will see in the next section, varies depending on the kind of sacrifice: for the ones we previously called "strongly obligatory", the intended result is heaven/happiness, while, for the "weakly obligatory" ones, there are many possible positive results, usually less important than happiness and not necessarily desired by everyone.

On the other hand, complying with a prohibition does not give any result, but it only maintains a neutral state, preventing an infraction or offence (*pratyavāya*), which eventually would result in a sanction, i.e. the accumulation of negative karman.

The situation is not completely symmetrical in case of non-compliance with commands: we would expect that, as performing a prohibited action gives a sanction and avoiding it gives no result, obligations behave in a similar way, bringing positive results if obeyed and no result if disrespected. However, it seems that the failure to perform a "strongly obligatory" sacrifice also gives an undesirable result. It is not clear if Mīmāṃsakas even address the issue of the difference between the absence of heaven/happiness and the presence of negative karman as an additional sanction. Hence, contemporary scholars are not certain whether to interpret the bad result of not complying with a "strong obligation" as the absence of the good effect of the ritual, or as a consequence which is negative in itself, comparable to the result of disobeying a prohibition.

The considerations about positive and negative results of complying or not complying with commands also suggest that the concept of *adhikāra* is defined in different ways for prescriptions and prohibitions. Indeed, as mentioned above, people who are eligible for a ritual action are mainly identified because of their desire, while prohibitions in principle apply independently from people's desires.

Nonetheless, because of the *nyāya*s about meaningfulness and novelty, there cannot be a command forbidding something that an agent has no reason to do. Hence, any prohibition presupposes that the forbidden act has been previously established through a prescription (*śāstraprāpta* prohibition), or, in most cases, that it has been established on the basis of one's natural inclination to perform the very same action (*rāgaprāpta* prohibition). This implies that, in general, the element of desire is not entirely excluded from the selection of the addressees of prohibitions.

A further distinction between different kinds of prohibitions is based on the context of application: a prohibition can be relative to the person (*puruṣārtha*), applying throughout the life of the responsible agent, or relative to the sacrifice (*kratvartha*), applying only in a specific ritual context. Again, it is not completely clear if the violation of a prohibition relative to a specific ritual results in something which is intrinsically negative, or only causes the failure of the sacrifice, being then symmetrical with respect to an obligation for a subsidiary rite. However, as already said, the system developed in the present work aims to apply only to injunctions involving "main" actions, therefore both prohibitions and prescriptions which are conditioned by another command exceed the aim of this discussion.

2.3.2 Types of sacrifices: obligations and recommendations

As mentioned in the previous paragraph, while prohibitions can be independent from a ritual context, the prescriptions that Mīmāṃsā focus on are ritual exhortations and sacrificial methods. However, not all the sacrifices are prescribed in the same way: for different types of ritual actions there are different conditions and motivations for agents to act and different kinds of consequences following the compliance (or non-compliace) with the Vedic command. Hence, by analysing different types of prescribed ritual actions, it is possible to recognise the different kinds of duty they convey.

The primary ritual actions enjoined by Vedic prescriptions fall into the three categories of "fixed sacrifices" (*nitya-karman*), "occasional sacrifices" (*naimittika-karman*), and "optional or elective sacrifices" (*kāmya-karman*).

(i) *Nitya-karman* sacrifices represent ritual actions which the eligible agent should perform recurrently throughout his life, e.g. the thrice-daily worship, *sandhyāvandana*, is a daily sacrifice consisting in a ritual that should be performed at every dawn, noon, and twilight.

   Sacrifices of this kind are usually obligatory for the largest set of agents including all the "twice-born men" (*dvija*s), i.e. men belonging to the higher three castes, who have undergone the sacred thread ceremony within the prescribed time-frame, studied the Vedas in the school of a Brahmin, returned home after duly finishing the studies, and married.

   As it is meant to include everyone (who is allowed to study the Vedas), the *adhikāra* for this type of rituals is assumed to involve the most general desire, for heaven/happiness, which is attributable to any living being.

   For this reason, injunctions prescribing *nitya-karman* sacrifices are associated with the deontic concept of "strong obligation": this means not only that violating them is considered as a *fault* (*doṣa*), but also that they all give instructions only for the ideal state, i.e. it must be possible to follow all the obligation of this kind.

(ii) *Naimittika-karman* sacrifices are those which the responsible agent should perform when specific circumstances occur; *nimitta* means indeed "occasion". Occasions are one-time states or events happening during the agent's life, like the birth of a son (*jāteṣṭi* sacrifice) or a solar eclipse (*grahaṇaśrāddha* sacrifice), which require the performance of a ritual, that should not be regularly repeated, unless the occasion triggering it occurs again.

   An occasional sacrifice has, in principle, the same *adhikārin* of a fixed one, with the addition of the conditions expressing the occasion: this means that also *naimittika-karman* sacrifices are supposed to bring about the state of heaven/happiness if duly performed, and an undesirable consequence if the responsible agent fails to complete the sacrifice.

   The similarity between occasional and fixed sacrifices often cause disagreement among Mīmāṃsā authors, about whether a given ritual belongs to one or the other class. In particular, sacrifices which should not be repeated every day, but only at a specific time in the calendar year, can be controversial; in those cases the predictability of the occasion is assumed to be the decisive factor: the occurrence of conditions for fixed sacrifices are typically known from the beginning of agents' life, while occasions triggering *naimittika-karman* sacrifices are not certain in advance.

For instance, the full-moon and new-moon sacrifices (*Darśapūrṇamāsa*) are triggered by occasions (the lunar phases), but, as their occurrences are predictable and they should be repeated throughout one's life, they are generally considered to be fixed sacrifices.

Since they have the same results and *adhikāra* as fixed sacrifices, the prescription enjoining *naimittika-karman* sacrifices are also considered examples of "strong" obligations.

(iii) *Kāmya-karman* sacrifices, finally, are performed only in order to obtain the specified result mentioned in the statements enjoining them, in the "Vedic way"; for instance, "one who desires cattle should sacrifice with the *citrā*" or "one who desires to kill his enemy should sacrifice bewitching with the *Śyena*".

Elective sacrifices appear to have a deontic content different from the other types of sacrifice, as they are binding only for the agents who desire the mentioned specific result. Moreover, it seems that there are no explicit mentions of bad results or punishments for the ones who fail to perform the *kāmya-karman* ritual, except for not obtaining the desired specific result. However, the interpretation of the deontic content of such sacrifices divides authors like Kumārila and Maṇḍana from the scholars belonging to *Prābhākara* (sub-)school, who apparently tend to read all Vedic commands as binding. For Kumārila's and, even more, for Maṇḍana's disciples, *kāmya-karman* sacrifices seem to be essentially instructions for achieving one's objectives: even the individual who has the proper *adhikāra* —in this case almost identifiable with the desire for the specific result— is not properly compelled to perform a *kāmya-karman* ritual. Choosing not to perform such a sacrifice does not lead, indeed, to any result, other than not reaching the desired new state of affairs (but choosing to undertake one of those ritual actions, without completing all of its subsidiary rites exactly as stated, is considered a fault). On the other hand, from the point of view of Prabhākara, it seems that the duty to perform a *kāmya-karman* sacrifice is not dissimilar from the kind of deontic content conveyed by the commands to perform *Nitya-karman* and *naimittika-karman* rituals. The person who has the *adhikāra* for a *kāmya-karman* sacrifice and fails to perform it will get negative karman, as it is assumed that this person will (at least try to) obtain the object of desire in a "non-Vedic" way. Indeed, the desire for a specific result, representing the *adhikāra* of a *kāmya-karman* ritual, is seen as the decision already made to obtain that result. Hence, for Prabhākara, the command to perform an elective sacrifice is not less binding then the others, but it usually applies to the smaller group of people who share the desire for a particular result, while the desire for heaven/happiness characterizes (almost) all human beings. On the other hand, from Kumārila's perspective, the injunctions to perform elective sacrifices are considered "weak obligations", or *"recommendations"*: they are instructions for doing something the agent already desires, in the best possible way (the Vedic way). This means that, unlike strong obligations, they can conflict in the sense that two different sacrifices can be prescribed for the same purpose, and the eligible agent is allowed to choose the preferred one: it is not necessarily possible to be compliant with all the commands of this kind (recommendations) in the sacred texts.

Remark 2.3.2 *Note that it might be possible for something to be prescribed both as a* nitya-karman *or* naimittika-karman *sacrifice and as a* kāmya-karman *ritual. For instance, the sequence of actions constituting the* Agnihotra *ritual represents the content both of a fixed sacrifice and of an elective one: it seems that, if the agents perform the ritual perfectly, according to the stricter rules governing elective sacrifices, they are compliant both with the* kāmya-karman *ritual and with the* nitya-karman *one.*

The weakened deontic content in Kumārila's interpretation of elective sacrifices is also interesting in view of the fact that *kāmya-karman* sacrifices are the only ones which admit, as preconditions, states which are morally "sub-ideal", e.g. implying the desire of something forbidden, or a state where a ("strong") prescription has already been violated.

For what concerns fixed sacrifice, since they are strongly obligatory for everyone (who is allowed to study the Vedas) and they should be repeatable, in general it is safe to assume that their *adhikāra*s cannot involve any violation. Generally, the same could be stated about occasional sacrifices, because it seems that someone who is in a state of infraction and desires to perform the rituals for achieving the result of heaven, first needs to (desire to) emerge from this negative state.

Accordingly, a very special case of sacrifices are the expiatory rites (*prāyaścitta*): those have as their addressee a person who committed a violation, typically making some errors during a sacrifice, e.g. "having dropped the barley, one offers the expiatory oblation" (ŚBh on PMS 6.5.45). The nature of *prāyaścitta* sacrifices represents a controversial matter: though they are classified as elective rites[14], the eligible agents for them appears not to be selected because of their desire (whether it is the general desire for happiness or a specific new state of affairs), but only because they "did something wrong". Differently from typical *kāmya-karman* sacrifices, the expiatory ones only purge the sin, while no other desirable result is involved. Hence, it could seem that they are understood as *naimittika-karman* sacrifices, but, unlike occasional rituals, there is no further negative result due to the non-performance of any *prāyaścitta* ritual. Accordingly, if one does not perform a *prāyaścitta*, one only obtains the bad result of the bad violation which remained unexpiated; no additional bad result will accrue to one.

Regarding *prāyaścitta* sacrifices, some authors, e.g. Rāmānujācārya (15th–16th c. CE), claim that also expiations are performed to achieve something desired —possibly the elimination of the effects of the violation.

2.3.3 Permissions

Finally, we should mention the deontic concept of permission. This concept seems not to be treated in the texts we called "common Mīmāṃsā" as a separate notion: as proof of this, unlike for prescription (*vidhi*) and prohibition (*niṣedha*), these texts do not refer to "permission" with a specific term. In some passages (e.g. ŚBh on PMS 5.3.2 and 7.2.13) which appear to discuss what could be classified as a permission, the latter is expressed by words like *nyāyya* (regular) —meaning that an action under certain circumstances is not bearing its usual deontic content— or by statements about the occurrence of an action under certain circumstances. In this case the statements are interpreted as permissions because in general they refer to actions which people are naturally inclined to do, hence they would not have any significant meaning if they did not express that those actions are permitted under some given circumstances (and not under different circumstances).

In other cases, the presence of a permission is indicated by the use of the optative suffix: this, however, is not a decisive element, as the optative suffix could denote both permissions and prescriptions.

Therefore, the choice to interpret a statement as a permission or as a prescription is usually based on the nature of its content; as mentioned above, if the statement is relative to an action for which there can be a natural inclination, it is unlikely to be a command, but it should probably be interpreted as the permission to perform this action only under some specific conditions. Indeed, permissions are not understood as independent deontic

[14]This discussion is based on a section of the 12th chapter of the PMS and its commentaries.

statements, but as complementary to other deontic concepts.

This means that, on one hand, what is not forbidden or denied by an obligation is in general permitted. On the other hand, given a permission to do (or abstain from performing) an act under certain conditions, it is possible to suppose that the Vedas contain a command which forbids (or makes obligatory) to perform the same act in the absence of the specific conditions indicated by the permission.

One of the typical examples of the connection between permissions and other deontic statements is the discussion (in ŚBh on PMS 6.8.18) about the legitimacy of a second marriage: the permission to take a second wife if one's wife is not virtuous and fertile is inferred from the prohibition to remarry in case that the spouse does not lack those qualities.

Considering the opposite direction (a command inferred from a permission), the statement "the five five-nailed (animals[15]) are to be eaten" (*pañca pañcanakhā bhakṣyāḥ* in, among others, Śabara on PMS 10.7.28), is interpreted as the permission "one can eat meat if it is the meat of one of the five five-nailed animals" implying the presence of a general prohibition "one should not eat meat".

However, even when it is agreed that a statement represents a permission, it is not always clear whether it is an exclusion of an implicit prohibition or of an implicit obligation. For instance, considering the previous example, if there was some mention of a result, the permission to eat the five five-nailed animals could have been alternatively interpreted as implying the general obligation to refrain from eating meat; in this case the consequences of eating the meat of the five five-nailed animals would not have changed, but avoiding to eat the meat of other animals would have given a positive result. Since no result is mentioned, though, it is more likely that the implicit command is a prohibition, and not a negative obligation.

2.4 Deontic conflicts

Mīmāṃsā texts discuss cases where the interpretation of some deontic statements in the Vedas is far from being simple. Since some of those cases involve apparently conflicting deontic statements found in the sacred texts —which are assumed to be internally consistent—, Mīmāṃsakas develop strategies for dealing with the conflicts, reinterpreting the contradicting statements and deciding what commands are enforceable in a given situation.

[15]It is not completely clear what are the five five-nailed animals to which the statement refers, however it seems that the set include some species of wild rodents, wild boars, lizards, hares, and turtles.

Before briefly presenting those strategies, we introduce deontic conflicts as they appear in Mīmāṃsā arguments. First, a deontic conflict is represented by a state of affairs such that two (or more) commands which are not mutually compatible apply. As already observed, two Vedic commands are considered to be incompatible if they can apply under the same circumstances, they cannot be both complied with and, in addition, failing to comply with any of them results in a negative consequence. This means that there are "proper" deontic conflicts only among commands which can be represented as strong obligations (corresponding to the injunctions to perform *nitya-karman* and *naimittika-karman* sacrifices) or prohibitions.

Obligations to perform *kāmya-karman* sacrifices cannot really be involved in a conflict, as choosing not to perform them does not give any negative result, except for failing to obtain a desired consequence.

Example 2.4.1. Let us consider, as an example, the controversy arising in connection with the *Śyena* sacrifice (see [21]), a *kāmya-karman* ritual that has the purpose of killing someone and is prescribed for those who have the desire of killing an enemy. This controversy is broadly discussed by Mīmāṃsā authors. A first attempt of a formal account can be found in [29] and [47]:

(i) "One should not harm any living being";

(ii) "One who desires to kill his enemy should sacrifice bewitching with the *Śyena*".

All Mīmāṃsā authors agree that even if both of the commands are found in the sacred texts and the first one (a prohibition) forbids violence in general while the second enjoins it, they do not give rise to a contradiction. Their solution, is however different. The Bhāṭṭa school (headed by Kumārila) explains that, as the second deontic statement concerns a *kāmya-karman* sacrifice, it is meant to *recommend* how to obtain the desired result, but it is not intended to give any restriction to the agents' free will to take action for obtaining this result or not. In fact, the result itself is forbidden, hence the desire for it could never be the adhikāra of a strong obligation. Maṇḍana, who here amplifies Kumārila's tenets, explains that a rational agent would just not perform the Śyena because such a performance would imply obtaining a limited good in exchange for the much bigger and undesirable consequence of transgressing a prohibition.

A different viewpoint is taken by *Prābhākara* (sub-)school, in particular by Salikanatha (ca. 9th century CE) that instead insists that the Vedic command prescribing the Śyena is only addressed at people who are already in the forbidden condition of being willing to kill their enemy. This means, in particular, that performing the *Śyena* sacrifice does not exclude the negative result given by the violation of the prohibition (i); rather, it guarantees that one obtains the desired (evil) result in the best possible way. From a modern point of view, we could say that performing the *Śyena* sacrifice represents "the lesser of two evils", as it implies harming a living being (killing an enemy), but it is still better than obtaining this result in a different (non-Vedic) way. For this reason, the *Śyena* controversy and specifically the solution proposed by *Prābhākara* school has many similarities with the so-called *"gentle murder paradox"* (introduced in [40]), well known in contemporary deontic logic and broadly discussed as an example of "contrary-to-duty" obligation (see Ch.3 Section 3.2). Prabhākara's and Kumārila's solutions for the Śyena controversy will be formally analysed in section 3.4.3. ■

The deontic statements belonging to the "special category" of permissions do not give rise to conflicts defined as states where two commands apply, which cannot be both complied with and such that failing to comply with any of them results in a negative consequence. Hence, permissions are also assumed not to generate conflicts. However, in case we have a permission incompatible with a prescription (or a prohibition), the occurrence of a conflict is not avoided only because it is safe to keep following the prescription (or prohibition) ignoring the exemption, but also because, in most cases, permissions can be construed as a single deontic statement with their complementary command. A permission is indeed supposed to "complete" the conditions of application of its complementary command, indicating in which cases it is not enforced: going back to the example in the previous section, the permission *"one can eat the meat of the five five-nailed animals"* and its complementary prohibition *"one should not eat meat"* could be interpreted as the single enforceable command *"one should not eat meat if it is not the meat of one of the five five-nailed animals"*.

Therefore, from the perspective of Mīmāṃsā, deontic conflicts arise in states that meet the conditions of application of (at least) two commands such that each one of them is an obligation for a *nitya-karman* or *naimittika-karman* sacrifice (strong obligation), or it is a prohibition, and the *adhikārin* cannot be compliant with the (at least) two applicable commands. Specifically, it should be noted that Mīmāṃsakas seem to rarely discuss cases of deontic conflicts with more than two incompatible commands, e.g. three obligations that cannot be fulfilled together but such that any two of them are compatible.

2.4.1 Resolving deontic conflicts

For Mīmāṃsā authors, the line which one cannot cross is that of the Veda being meaningless or purposeless (*nirarthaka*). Every interpretation leading to the Veda being meaningless or purposeless should be a priori rejected. In order to deal with deontic conflicts while avoiding to consider one of the involved commands as meaningless or purposeless, they use different mechanisms, included in the broad categories of (a) *reinterpretation* of one or both commands, (b) *prioritization* of one command over the other, and (c) *optionality* of the two commands, in this preference order.

Solution (c) is the *vikalpa* principle and represents the least preferred alternative, as it allows the agents to arbitrarily choose one of the two commands to comply with. In simple terms, the principle states that, when there is a conflict between two (or more) commands and there is no way to chose the applicable norm, one should comply at least with one of the original commands.

Vikalpa (optionality of the two commands) constitutes the very last resort, because it implies making a Vedic command temporarily inapplicable: though it is not purposeless as it can be followed at another time, the command which is (temporarily) not fulfilled does not hold. Hence, the agent who chooses to comply with one command does not get the bad result from not fulfilling the other. Moreover the choice between the two commands is arbitrary, which means that it cannot be justified, explained, or repeated on the basis of general guiding principles, but it is completely left to the (fallible) agent.

As it is the least desirable solution, the *vikalpa* principle is reserved to the limit case of contrasting commands that apply exactly under the same conditions, are exactly at the same level (usually, only if they are both enjoining subsidiary ritual acts) and have exactly the same purpose (i.e. they are, in principle, interchangeable). For instance, if, in the context of a sacrifice, a (single) cake should be offered and two different commands prescribe to use different ingredients for the same cake (e.g. rice and barley), the two are optional. Indeed, by applying *vikalpa* nothing is done which is not stated in the Vedas, whereas making two cakes would imply repeating the ritual twice, which is not prescribed in the sacred texts; similarly, mixing the ingredients would violate both the prescriptions, as the Vedas do not mention any ritual action that should be done by means of both the ingredients.

By contrast, when two contrasting commands (e.g., prescribing two different pre-sacrificial procedures) are prescribed for different purposes, one should perform both of them. The principle ruling this course of action is called “accumulation” (*samuccaya*).

Solution (b) can be applied when the conflicting deontic statements are not at the same hierarchical level or their conditions of application stated in the texts are not exactly the same. In those cases Mīmāṃsā authors use principles of prioritization for choosing the enforceable command: this is intermediate in the scale of preferences according to which the possible solutions are ordered. The principles of prioritization identify the "invalidating elements" (*bādha*s), through which, in case of conflict, a deontic statement *suspends* the application of another one. As already mentioned, the term *bādha* is then used to indicate a principle for resolving (apparent) normative conflicts by giving priority to one command over the others.

Before introducing the most important kinds of *bādha*s, we need to clarify the concept of *"suspension"* of commands' applications.

Let us consider a very general scheme:

(i) "One should perform the action α under the condition β";

(ii) "One should perform the action δ under the condition γ".

Given that the two actions α and δ are incompatible, there is a conflict in states which meet both the conditions β and γ.

If there is an invalidating element (*bādha*) of one of the two commands that makes it preferable to the other one, the enforceability of the non-preferred command is suspended in the situation which meets both the conditions of application of the two deontic statements. Assuming that there is a *bādha* which makes the command (i) overruling (ii), we can infer that:

- if a state meets the condition β, the command to perform α is enforced;
- if a state meets the condition γ, *in general* the command to perform δ is still enforced;
- if a state meets both the conditions β and γ, the command to perform α is enforced and the command to perform δ is not.

Therefore, not only the command (ii) is overruled only in the circumstances where both the conditions β and γ are verified and remains enforceable in every other situation that verifies γ and not β, but the reader is also allowed to retain the general command (ii) as the *default rule* for future undetermined instances. As suggested by the expression "in general", this means that the command (ii) remains the standard applicable rule in all states of affairs where the condition γ is true, without the need to check all the statements that are true or false in those states —in particular without the need to verify that the condition β is false—, but (ii) admits exceptions. In other words, if one knows that γ is true in a given state and does not know anything about β, then (ii) applies; if in the same situation one finds out that also β is true, then (ii) is suspended and (i) applies.

The interpretation of Vedic commands as general norms that admit exceptions seems to support the thesis according to which the logic underlying Mīmāṃsā deontic reasoning does not follow the rule of inference whereby the set of premisses of any derived conclusion can be extended without any restriction. In view of the fact that this rule corresponds to what is nowadays called *"monotonicity of entailment"*(see Section 4.5), it is understandable that the work of some Mīmāṃsā scholars —in particular Kumārila— has been included in the debate on non-monotonic reasoning at the root of ancient Indian theory of inference[16].

However, at least from a philosophical point of view, it seems that Mīmāṃsakas are not ready to consider the norms in the sacred texts as default (hence fallible) rules. From Mīmāṃsā perspective, apparently, when a command is overruled and its effectiveness is suspended for the circumstances which generated the conflict, the suspension is not read as an "amendment" to the sacred texts. Rather, the way the readers understand the commands in the Vedas is updated: a Vedic command always had the same meaning and conditions of application which included the exceptions, but the readers did not understand it until they encountered the conflict. Hence, the commands in the sacred texts can be considered not as default norms, but as a completely described system of rules, at least in principle, for a (virtual) "perfect reader", who knows and understands the Vedas in their entirety. Only the (necessarily limited) actual readers' understanding of a command can be a default rule.

A list of the most important elements (*bādha*s) that determine the principles of prioritization among commands is found in Kumārila's Tantravārttika on PMS 3.3.14 (see their discussion in Appendix). We (tentatively) classify those principles as instances of few methods of prioritizing commands; among such few methods, the following are the most relevant for a formal analysis.

- *"No empty rule"*.

  This is the prioritization principle corresponding to the *meaningfulness nyāya*: every command should be enforceable at least under some circumstances (otherwise it would be meaningless), hence, if the only way for applying a norm is to suspend another one (which is not, in turn, applicable only as long as the first is blocked), the latter should be suspended.
- *"Economicity Principle"*.

  This principle essentially states that a Vedic norm invalidating as few commands as possible takes priority over one invalidating many of them.
- *"Hierarchy of sources"*.

  This principle states that a command from a more important source suspends a conflicting one from a lower source.

  In Kumārila's TV four sources of duty are mentioned: in order of importance, *śruti*

[16]A thorough investigation on Indian logic is beyond the scope of the present work; for an overview on Indian logic see [51, 50]. For the discussion about non-monotonic reasoning in Indian logic see, in particular, [101, 127]

(the sacred texts, i.e. the Vedas), *smṛti* (the "recollected texts", that are supposed to integrate the sacred texts or even reconstruct missing passages of the Vedas), *sadācāra* (norms based on the imitation of the behaviour of people learned in the Vedas) and *ātmatuṣṭi* (the inner feeling of approval by people who are learned in the Vedas). Those sources are naturally ordered, as the inferior one is always meant to be based on the superior one and to supplement it, so that in the end everything has its foundation in the Vedas. For this reason, the authority of one source is greater than the sum of less valuable sources' authorities and the the authority of the *śruti* stands uncontradicted and uncontradictable.

It should be observed that the conflicts discussed in Mīmāṃsā texts mainly involve the first two sources and, much less often, *sadācāra*. Indeed, the connection of *ātmatuṣṭi* with the Vedas is the weakest and, as already mentioned, inner feelings cannot be considered a valid instrument of knowledge for what concerns the duty. Hence, *ātmatuṣṭi* represents the last resort, to be considered only in situations where no other source gives instruction on how to behave: this makes conflicts between *ātmatuṣṭi* and the superior sources impossible by definition.

However, even the conflicts between *śruti* and *smṛti*, which are discussed in Mīmāṃsā texts, are conceptually problematic, as the "recollected texts" are valid only insofar as they are based on the Vedas. *Smṛti* are indeed supposed to clarify and complement the sacred texts, sometimes expressing duties contained in scattered branches of the Vedas, branches that might have not been found or studied yet, passages which have fallen into oblivion, or even passages that have never existed explicitly (*nityānumeya*).

It seems that Mīmāṃsakas explain the origin of conflicts between *śruti* and *smṛti* as caused by errors in the transmission from the Vedas to the *smṛti*s, or by the intervention of someone in the chain of transmission, who stands to gain from the wrong command. For this reason, Mīmāṃsā authors recommend not to follow a command from *smṛti* if the connection with the sacred texts is not clear and one can see a motive in it.

- *"Specificity principle"* (*Guṇapradhāna*, or *Sāmānya-viśeṣa*).

  This principle states that commands which are meant to apply in more specific conditions overrule commands with more general conditions of application. *Guṇapradhāna*/specificity represents a very natural and fundamental principle of normative reasoning, as it allows to reason with exceptions; indeed, it has been used also in western European jurisprudence at least from the Middle Ages, expressed by the brocard *"Lex specialis derogat legi generali"*. Given its essential role in reasoning with generalized default rules, the same principle is widely used nowadays, e.g., in Artificial Intelligence, and, as it will be pointed out later, it represents one of the bases of non-monotonic logics.

Considering the observations in the previous sections about the conditions of application of Vedic commands, there are many different ways in which a Vedic norm can be more specific than another one. In particular, the most natural situations in which the specificity principle applies include conflicts between obligations for occasional and fixed sacrifices: as they are triggered by a specific occasion, *naimittika-karman* sacrifices take priority over *nitya-karman* ones.

Other circumstances where the specificity principle is used by Mīmāṃsā authors are represented by conflicts between norms such that the set of addressees of a command includes the set of addressees of the other. For instance, let us consider the case of *Sāmidhenī* hymn, used for kindling the sacrificial fire[17]. The general command stating that the *Sāmidhenī* hymn with 15 verses should be recited is overruled by the more specific norm stating that the *Sāmidhenī* hymn with 17 verses should be recited if the sacrificing agent belongs to the *Vaiśya* caste.

However, it is important to point out that for many authors desires are usually not considered among the conditions of application of a command which can make it more specific than another one. This could seem not relevant as the obligations to perform *kāmya-karman* sacrifices do not generally give rise to conflicts, but it becomes important when comparing (strong) obligations and prohibitions. Indeed, while the desire for heaven/happiness is always explicit or implied for strong obligations, prohibitions are not assumed to be always triggered by a desire, but this does not make obligations always more specific.

Other *bādha*s concern the relations between primary ritual act and its subsidiaries, the position in the ritual sequence, the purposes of commands, and many other aspects of Vedic prescriptions. However, the application of prioritization principles is generally not the most preferred solution for dealing with deontic conflicts, because it implies the temporary suspension of a command and hence a form of departure from what is explicitly stated in the sacred texts.

Solution (a) consists in explaining the conflict as due to a failure in understanding the commands, and rephrasing one or both of the seemingly conflicting commands in such a way

[17] For a description of the ritual including bibliographical references, see e.g. [112]

that the conflict disappears. This is the solution preferred by Mīmāṃsā authors as it cancels the conflict instead of resolving it, avoiding the problem of understanding the nature of conflicts and how they can appear in the sacred texts. However, in some (less desirable) cases, reinterpretation can be applied also in order to make the commands suitable for prioritization and avoid *vikalpa*.

The possible reinterpretations aimed to prevent deontic conflicts are manifold and heavily dependent on the specific characteristics of the conflict and on the linguistic forms of the commands involved. Nevertheless, a "standard" reformulation of a pair of conflicting obligation and prohibition can be identified, which corresponds to rephrasing both the conflicting norms as one command with an embedded exception. According to this method, two commands (i)"One should perform the action α under the condition β" and (ii)"One should not perform the action α under the condition γ" can be rephrased as the norm (iii)"One should perform the action α under all the conditions β but γ". Hence, the command (ii) is not a prohibition, but an exclusion (*paryudāsa*), which is not an independent command, but it represents the exception embedded in (i), that is then updated and becomes (iii). To illustrate how exclusions are different both from conflicting commands and from permissions, let us take a simple example.

Example 2.4.2. Consider two deontic statements: the first one concerns the duty to perform a certain ritual action α under a condition β; the second one involves the same ritual action α under some more specific conditions β and γ, but contains a negation. Assuming that the first statement is understood as an obligation and that there are no clear evidences supporting a specific interpretation of the second one, we may have the following different readings.

- Obligation/Prohibition: *"one has the duty to perform the action α under the condition β"* and *"one has the prohibition to perform the action α under the conditions β and γ"*.
- Obligation/Obligation: *"one has the duty to perform the action α under the condition β"* and *"one has the duty not to perform the action α under the conditions β and γ"*.
- Obligation/Permission: *"one has the duty to perform the action α under the condition β"* and *"one has not the duty to perform the action α under the conditions β and γ"*.
- Obligation/Exclusion: *"one has the duty to perform the action α under the condition β and not γ"*.

■

The device of *paryudāsa* seems to be reserved for conflicts between obligations and prohibitions; for what concerns conflicts between two obligations, sometimes it is possible to re-interpret an obligation as a recommendation or as a permission, but the application of those transformations are so heavily dependent on all the peculiarities of the single cases, that they cannot be generalized. Some formalized examples of such cases, showing the intricacy of reinterpreting commands, will be presented in Example4.4.13 in Ch.4. On the other hand, it seems that conflicts between two prohibitions are more rare and there is no “standard” way for dealing with them.

Chapter 3

Mīmāṃsā Deontic Logic

On the basis of the considerations in the previous chapter, it seems clear that the rigour of Mīmāṃsā scholars' analysis and their search for general principles for legitimizing each step of their reasoning justify a formal approach to the study of this school.

The use of a deontic logic is suggested not only by the explicit purpose of the Mīmāṃsā school to analyse the prescriptions in the Vedas, but also by its methods. Indeed, in describing the characteristics of the different kinds of duties connected to the various commands, Mīmāṃsā scholars outline a classification of normative statements, which can intuitively serve as a basis for the definition of deontic operators.

The first step towards a formal representation of the Mīmāṃsakas' analysis of Vedic commands is represented by the non-normal dyadic deontic logic bMDL (basic Mīmāṃsā Deontic Logic), introduced in [29]. bMDL, only containing a deontic operator for obligations, resulted from the examination of over 50 deontic *nyāya*s. This logic has been defined by applying the step-by-step bottom-up method already mentioned in the introduction to this thesis: the constitutive elements of Mīmāṃsā reasoning, expressed by *nyāya*s or inferred by Indologists from examples in the texts, are mapped into Hilbert axioms thus defining the characteristics of the modal operators.

A further analysis of the deontic reasoning in Mīmāṃsā texts (see, e.g., [47, 48]) pointed out that the different kinds of commands in the Sacred Texts cannot be reduced to the notion of obligation. For this reason, we extend the logic with the additional deontic concepts of *recommendation* and *prohibition*.

Thanks to the flexibility and modularity of the system, the logic $\mathsf{MD+}$ we propose for formalising Mīmāṃsā deontic reasoning is suitable to be progressively extended and modified as new *nyāya*s and examples are found, translated and interpreted.

In this chapter we describe the logic so far "extracted" from Mīmāṃsā, illustrating the main steps of its development, i.e. the logics bMDL and MD+. They are characterized both from the proof theoretic and semantic point of view, by providing, for each system, a cut-free sequent calculus and a neighbourhood semantics.

First, the logic bMDL ([29]) will be presented. The axioms of bMDL and the corresponding *nyāya*s will be shown, together with the cut-free sequent calculus, and the neighbourhood-style semantics for this basic logic; regarding the latter, we will also correct a mistake in [29] in the definition of countermodels.

In the second part, following our own work in [32], the $\Box$-free fragment of bMDL, which is shown to be the dyadic version of non-normal deontic logic MD introduced in [27] (cf. [48]), will be extended with new operators for prohibitions and recommendations, whose properties reflect Mīmāṃsā principles. The resulting logic MD+ is then used to provide a formal analysis of the mentioned Śyena controversy, according to the interpretations of Mīmāṃsā authors Kumārila and Prabhākara.

3.1 The Base Logic bMDL

Basic Mīmāṃsā Deontic Logic bMDL -the first logical system based on Mīmāṃsā- extends the alethic system S4 with the deontic operator $\mathcal{O}(A/B)$, which is intuitively read as "under the condition B, it ought to be the case that A". The properties of this operator are expressed by three Hilbert axioms that formalize appropriate deontic *nyāya*s.

The choice of a dyadic operator for obligations represents a standard method to deal with conflicting commands (see, e.g., [54]): indeed it allows to give incompatible commands to different addressees without generating conflicts. From a more philosophical point of view, the use of the dyadic operator $\mathcal{O}(A/B)$ is substantiated by the key role of the adhikāra (see Ch.2 Section 2.3). Since an obligation does not apply to someone who lacks the requirements or abilities to fulfil it, or to a hypothetical being with no desire for happiness, a command should always be interpreted as being valid under some conditions.

The language $\mathcal{L}_{\mathsf{bMDL}}$ of Basic Mīmāṃsā Deontic Logic is then an extension of the language of classical propositional logic, with the alethic operator $\Box$ and the dyadic deontic operator $\mathcal{O}(A/B)$.

Definition 3.1.1 *Given a set* Var $= \{p, q, r, ...\}$ *of propositional variables, the well formed* *formulas of the language $\mathcal{L}_{\mathsf{bMDL}}$ are generated by:*

$$\varphi ::= \bot \mid p \mid \varphi \to \varphi \mid \Box\,\varphi \mid \mathcal{O}(\varphi/\varphi)$$

where $p \in \mathsf{Var}$ and $\bot$ represents an arbitrary proposition that is logically false in classical propositional logic. The other propositional connectives $\neg, \vee, \wedge, \leftrightarrow$ are defined by $\{\bot, \to\}$ in the usual way, i.e. $\neg\varphi := \varphi \to \bot$, $\varphi \vee \psi := \neg\varphi \to \psi$, $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$, and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Before presenting the axioms for the deontic operator, it is necessary to explain all the choices made. First, classical logic as base system has been preferred to intuitionistic logic, used instead in [1], one of the works which inspired our approach. The reason for this choice is that Mīmāṃsā authors seem to implicitly assume the legitimacy of double negation and reductio ad absurdum. Indeed, it can be observed in the rules of grammar that a command which seems to forbid to refrain from an action ("you must *not* do *not X*") is often interpreted as a "positive" command to do the action, if no sanctions or other peculiarities of prohibitions are mentioned. Moreover, an example of reductio ad absurdum can be found in the book *Nyāyamañjarī* by Mīmāṃsā author Jayanta Bhatta (c. 9th Century CE): *"When there is a contradiction* (φ and not φ), *at the denial of one (alternative), the other is known (to be true)"*; this means that deriving falsity from the assumption of $\neg\varphi$ amounts to proving φ, which represents an instance of the reductio ad absurdum argument.

The characteristics of the dyadic deontic operator $\mathcal{O}(\cdot/\cdot)$ are defined by the axioms in Def.3.1.2, each of which represents the formal translation of one or more *nyāya*s explicitly stated in Mīmāṃsā texts or "extracted" by expert Sanskritists from Mīmāṃsakas' arguments. Because of the more "natural" translation of *nyāya*s into axioms, the logic is defined Hilbert-style; however, in order to prove the meta-logical properties of bMDL (e.g. decidabiliy, consistency), later we will use the axioms of bMDL transformed into sequent rules, using the method developed in [80].

Definition 3.1.2 (Axioms of bMDL [29])

1. $(\Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\theta)) \to \mathcal{O}(\psi/\theta)$
2. $\Box(\psi \to \neg\varphi) \to \neg(\mathcal{O}(\varphi/\theta) \wedge \mathcal{O}(\psi/\theta))$
3. $(\Box((\psi \to \theta) \wedge (\theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi)) \to \mathcal{O}(\varphi/\theta)$.

The axioms above, in addition to the ones characterizing the alethic necessity operator $\Box$

of $\mathsf{S4}$ and the base system of classical propositional logic, define the *basic Mīmāṃsā Deontic Logic* bMDL. If a formula φ can be deduced from a set of assumptions Γ, by using the inference rules of Modus Ponens and Necessitation applied to instantiations of the axioms of bMDL or assumptions in Γ, we write $\Gamma \vdash_{\mathsf{bMDL}} \varphi$.

Remark 3.1.3 *The alethic necessity operator* $\Box$ *of* $\mathsf{S4}$ *is used in* bMDL *for characterizing propositional assumptions, i.e. the boxed formulas in* bMDL *are assumed to be valid in every possible state. However, while the properties of the deontic operator are "extracted" from Mīmāṃsā texts, it is not the same for the alethic operator* $\Box$*. Indeed Mīmāṃsā scholars did not develop explicit principles representing the properties of necessity, as they did not distinguish this concept from epistemic certainty. The two best known necessity operators are that for* $\mathsf{S4}$ *and for* $\mathsf{S5}$*; in [29] the alethic necessity operator* $\Box$ *of* $\mathsf{S4}$ *has been chosen in order to keep the (proof theory of the) system as simple as possible.*

Note that the deontic *nyāya*s used in [29] as the basis of the axioms above resulted from the analysis of many examples of reasoning in Mīmāṃsā texts and more than two hundred *nyāya*s, translated and explained by experts in Sanskrit and Indian philosophy.

Axiom 1 arises from various examples, summarized by the following statement, which constitutes the reformulation (and abstraction) of the Sanskrit *nyāya*s in the *Tantrarahasya* (IV.4.3.3), composed by the Mīmāṃsā author *Rāmānujācārya* (15th-17th c. CE) (see [43]).

> If the accomplishment of X presupposes the accomplishment of Y, the obligation to perform X prescribes also Y.

In axiom 1, the fact that the accomplishment of X (indicated by φ) depends on the performance of Y (indicated by ψ) is expressed by the formula $\Box(\varphi \to \psi)$, where the use of $\Box$, as in the other two axioms, ensures that the interdependence is assumed to be necessary and not just contingent. Moreover, considering what was already said about eligibility conditions, the principle is interpreted as referring to obligations which apply under the same circumstances to the same group of people, represented in the axiom by θ.

Axiom 2 constitutes the formal representation of a principle, applied by Mīmāṃsā authors in many discussions, which states that an agent cannot be required to act in contradiction with himself on some object. The following statement is an abstract formulation of this principle as found in TV on PMS 1.3.3:

> Given that purposes Y and Z exclude each other, if one should use item X for the purpose Y, then it cannot be the case that one should use it at the same time for the purpose Z.

Generalizing, axiom 2 ensures that if two actions are such that performing one consumes the resources and makes the agent ineligible for the other one (i.e. they always exclude each other), then they cannot be prescribed simultaneously to the same group of eligible people under the same conditions. The formulation of this axiom suggests another important characteristic of the logic bMDL: it appears that two obligations $\mathcal{O}(\varphi/\theta)$ and $\mathcal{O}(\psi/\theta)$ with the same formula θ in the second argument are supposed to be fulfilled at the same time; this means that eligibility conditions, occasion, and "time slot" for performing an obligatory action are all indicated by the second argument of the deontic operator, and that the correct interpretation of the conjunction $\wedge$ in the logic implies simultaneity.

Finally, axiom 3 results from the considerations in ŚBh on PMS 6.1.25 on adhikāra: the text suggests that, if two different sets of eligibility conditions always identify the same group of people, then what is obligatory under the conditions in one of those sets is also obligatory under the conditions in the other set. An abstract form of this *nyāya* is expressed by the following statement:

> If conditions X and Y are always equivalent, given the duty to perform Z under the condition X, the same duty applies under Y.

It is important to stress the fact that deriving an obligation $\mathcal{O}(\varphi/\theta)$ from another one $\mathcal{O}(\varphi/\psi)$ on the basis of their conditions is only possible when the formulas in their second argument are equivalent, i.e. $\Box((\psi \to \theta) \wedge (\theta \to \psi))$. This means that the obligation $\mathcal{O}(\varphi/\theta)$ (e.g. *"the Satī sacrifice is obligatory for all widows with children"*) cannot be derived from an obligation $\mathcal{O}(\varphi/\psi)$ (*"the Satī sacrifice is obligatory for all widows*) and the fact that $\Box(\psi \to \theta)$ (*being a widow with children implies being a widow*). As will be further discussed later, this is tantamount to saying that the deontic operator $\mathcal{O}(\cdot/\cdot)$ is not (downwards) monotonic in its second argument, in the following sense: an operator $\star$ is *downward monotonic* if its argument can be substituted by a more specific one, i.e. when the property expressed by $\star$ can be attributed to φ, then it is attributable to $\varphi \wedge \psi$ too. On the other hand, $\star$ is said to be *upward monotonic* if its argument can be substituted by a weaker, less specific one, i.e. if the property expressed by $\star$ can be attributed to $\varphi \wedge \psi$, then it can be attributed to φ.

3.1.1 Proof theory

This section summarises the results in [29], where bMDL was introduced and a cut-free sequent calculus was provided for this logic. The calculus is obtained by transforming the three Hilbert axioms "extracted" from the *nyāya*s into sequent rules that preserve cut-elimination, using the methods developed in [80, 79].

Hilbert systems, consisting of a set of axioms and few inference rules, are typically used to describe or introduce new logics. A key feature of such systems is modularity: small differences between logics correspond to small differences in their sets of axioms, which represent the most intuitive translations of generic principles of reasoning as they mirror syntactic or semantic properties of the corresponding operators.

Hilbert style calculi are, however, extremely cumbersome when it comes to reasoning or to proving important meta-logical properties of the formalized logics (e.g. decidabiliy, consistency). These tasks call instead for analytic calculi, i.e. systems in which proof search proceeds by step-wise decomposition of the formula to be proved. Since its introduction by Gerhard Gentzen in 1930s ([52]), sequent calculus has been one of the preferred frameworks to define analytic proof systems.

Definition 3.1.4 (Sequents, derivations in sequent calculi, height of a derivation) *A* sequent *is defined as a tuple* $\Gamma \Rightarrow \Delta$ *of multisets of formulas, interpreted as* $\bigwedge\Gamma \rightarrow \bigvee\Delta$.

The schemes of inference

$$\frac{\Gamma, \varphi_1 \Rightarrow \psi_1, \Delta \quad \cdots \quad \Gamma, \varphi_n \Rightarrow \psi_n, \Delta}{\Gamma, \varphi \Rightarrow \Delta}\, r_L \qquad \frac{\Gamma, \varphi_1 \Rightarrow \psi_1, \Delta \quad \cdots \quad \Gamma, \varphi_n \Rightarrow \psi_n, \Delta}{\Gamma \Rightarrow \psi, \Delta}\, r_R$$

are, respectively, a left and a right sequent rule with n premisses; the new formula in the conclusion, not occurring in the premisses is called principal formula *and the formulas in the premisses from which the principal formula derives are called* active formulas*.*

A derivation *in a sequent calculus is a finite tree where each node is labelled with a sequent such that the labels of a node follow from the labels of its children using the rules of the calculus, and the leaves are labelled with initial sequents (the conclusions of zero-premisses rules) (see also [130]). The zero-premisses rules do not introduce new connectives or operators, but convey basic properties of the relation expressed by* $\Rightarrow$*, i.e. the identity* $\varphi \Rightarrow \varphi$ *and the principle* $\bot \Rightarrow$ *(also known as* "ex falso sequitur quodlibet"*). The height of a derivation represents the number of nodes in the longest branch from the root to the leaves: we write* $\mathcal{S}' \vdash^n_{\mathsf{G}} \mathcal{S}$ *for* "there is a derivation of $\mathcal{S}$ from the set of assumptions $\mathcal{S}'$ in the calculus G and the height of this derivation is equal to n"*.*

In sequent calculi, the more general cut-rule

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Sigma, \varphi \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\, \mathsf{Cut}$$

is used to replace the rule of *Modus Ponens*

$$\frac{\vdash \varphi \quad \vdash \varphi \to \psi}{\psi}\ \mathsf{MP}$$

of Hilbert systems.

As can easily be seen, in both the applications of (Cut) and (MP) some information (instantiated by the formula φ) gets lost; this means a calculus which makes use of the cut-rule does not satisfy the *subformula property*[1] and makes it hard to build a bottom-up proof search.

A main challenge in sequent calculi is to show that inference rules that result in loss of information are *admissible*.

Definition 3.1.5 *An inference rule is said to be* admissible *in a formal system if any formula or structure (e.g. sequent) that can be derived using that rule is derivable in the system without that rule.*

If an inference rule is admissible in a formal system, intuitively, the set of theorems of the system does not change when that rule is added or removed, therefore the rule is redundant.

The elimination of the cut rule (cut-elimination theorem[2]) in well-designed sequent calculi (aka cut-free sequent calculi) shows that any application of the cut rule in a derivation can be removed and substituted by other rule applications, maintaining the same conclusion; this result entails that the cut rule is admissible and that the calculi are analytic.

In the last years some methods to extract analytic sequent calculi from Hilbert systems of a certain form have been introduced (see e.g. [30, 34, 80, 79]). The method in [80], that allows the transformation of modal axioms into suitable sequent rules, has been used in [29] to transform the Hilbert system for bMDL into a cut-free sequent calculus.

The resulting sequent calculus $\mathsf{G}_{\mathsf{bMDL}}$ is composed of the modal rules[3] in Fig. 3.2, together with the standard propositional rules for implication of the $\mathsf{G3}$-calculus [130], and the standard left rule for $\bot$ (in Fig.3.1). Note that the usual sequent rules for $\neg, \vee, \wedge$ are derivable using the definitions of those connectives in terms of $\bot, \to$.

We write $\vdash_{\mathsf{G}_{\mathsf{bMDL}}} \Gamma \Rightarrow \Delta$ if there is a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}_{\mathsf{bMDL}}$.

[1] A property of certain logical calculi, in which all the derivations are such that any formula at any node of the derivation is a subformula of the formulas occurring in its conclusion.

[2] The cut-elimination theorem was first proved by Gerhard Gentzen in [52] for the sequent calculus LK and its single-conclusion version LJ, for classical and intuitionistic logic respectively.

[3] The notation $\Gamma^{\Box}$ in the rules in Fig. 3.2 indicates the multiset Γ in which all formulas not of the form $\Box A$ are deleted.

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\ \mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L$$

$$\frac{\Gamma, \varphi \to \psi, \psi \Rightarrow \Delta \quad \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

Figure 3.1: The propositional rules of $\mathsf{G_{bMDL}}$

$$\frac{\Gamma^{\Box} \Rightarrow \varphi}{\Gamma \Rightarrow \Box\varphi, \Delta}\ \mathsf{4} \qquad \frac{\Gamma, \Box\varphi, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathsf{T} \qquad \frac{\Gamma^{\Box}, \varphi \Rightarrow \theta \quad \Gamma^{\Box}, \psi \Rightarrow \chi \quad \Gamma^{\Box}, \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta}\ \mathsf{Mon}$$

$$\frac{\Gamma^{\Box}, \varphi \Rightarrow}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{D_1} \qquad \frac{\Gamma^{\Box}, \varphi, \theta \Rightarrow \quad \Gamma^{\Box}, \psi \Rightarrow \chi \quad \Gamma^{\Box}, \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D_2}$$

Figure 3.2: The modal rules of $\mathsf{G_{bMDL}}$

Let us assume that the calculus $\mathsf{G_{bMDL}}$ contains the rules in Fig. 3.1 and Fig. 3.2, plus the rule Cut from Fig. 3.3. Extensions of $\mathsf{G_{bMDL}}$ with the structural rules from Fig. 3.3 (except for Cut) are indicated by simply affixing the names of the rules to the name of the calculus; e.g. $\mathsf{G_{bMDL}ConW}$ designates $\mathsf{G_{bMDL}}$ extended with the rules Con_L of Contraction on the left hand side, Con_R of Contraction on the right hand side, W_L of Weakening on the left hand side, and W_R of Weakening on the right hand side.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{Con}_R$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Sigma, \varphi \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}$$

Figure 3.3: Standard structural rules

From the construction in [80], the results of Thm.3.1.6 and Lem.4.3.22 follow; however, here we only present a sketch of the proofs and results which have been introduced and explained in detail in [29].

Theorem 3.1.6 (Cut-elimination - Thm.1 Sec.3 [29]) *The rule* Cut *is eliminable in* $\mathsf{G_{bMDL}ConW}$.

Proof. By construction the rules of $\mathsf{G}'_{\mathsf{bMDL}}\mathsf{ConW}$ in Fig.3.4, in which the principal formulas of the propositional rules and the rule T are not copied into the premisses, satisfy the general sufficient criteria introduced in [80] for cut-elimination. Moreover, by using the rules Con_L, Con_R, and W, the calculus $\mathsf{G}'_{\mathsf{bMDL}}\mathsf{ConW}$ is proved to be equivalent to $\mathsf{G}_{\mathsf{bMDL}}\mathsf{ConW}$; therefore cut-free derivations of $\mathsf{G}'_{\mathsf{bMDL}}\mathsf{ConW}$ can be transformed into cut-free derivations of $\mathsf{G}_{\mathsf{bMDL}}\mathsf{ConW}$ applying those structural rules. □

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\ \mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Sigma, \varphi \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}$$

$$\frac{\Gamma\psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

$$\frac{\Gamma^{\Box} \Rightarrow \varphi}{\Gamma \Rightarrow \Box\varphi, \Delta}\ 4 \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathsf{T} \qquad \frac{\Gamma^{\Box}, \varphi \Rightarrow \theta \quad \Gamma^{\Box}, \psi \Rightarrow \chi \quad \Gamma^{\Box}, \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta}\ \mathsf{Mon}$$

$$\frac{\Gamma^{\Box}, \varphi \Rightarrow}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{D}_1 \qquad \frac{\Gamma^{\Box}, \varphi, \theta \Rightarrow \quad \Gamma^{\Box}, \psi \Rightarrow \chi \quad \Gamma^{\Box}, \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_2$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{Con}_R$$

Figure 3.4: The calculus $\mathsf{G}'_{\mathsf{bMDL}}\mathsf{ConW}$

Lemma 3.1.7 (Lem.1 Sec.3 [29]) *The rules* Con_L*,* Con_R*,* W_L*, and* W_L *are admissible in* $\mathsf{G}_{\mathsf{bMDL}}$ *(without* Cut*).*

Proof. Admissibility of contraction rules follows from the general criteria in [80, Thm. 16], while that of weakening can be proved by induction on the height of the derivation: considering the last applied rule before the application of W_L or W_R, we apply the induction hypothesis to its premiss(es), followed by an application of the same rule. □

Hence, the Cut rule is admissible also in $\mathsf{G_{bMDL}}$ and the calculus satisfies the subformula property, and therefore it is analytic.

Moreover, since the rules Con_L, Con_R, and W are admissible in $\mathsf{G_{bMDL}}$ (Lem.4.3.22), from now on in this section on bMDL the simpler case of *set-based sequents* will be analysed, i.e. the rules of $\mathsf{G_{bMDL}}$ will be adapted to have tuples of sets of formulas, instead of multisets, as premisses and conclusions.

Proof search procedure

Now we describe the explicit proof search procedure for $\mathsf{G_{bMDL}}$, that will be used later for proving the completeness of this calculus and the decidability of the logic.

In order to avoid getting caught in a loop, the proof search algorithm [Alg. 1] below makes use of *histories*:

Definition 3.1.8 (Histories - Def.2 Sec.3 [29]) *A* history $\mathcal{H}$ *is a finite sequence* $[\Gamma_1 \Rightarrow \Delta_1; \ldots; \Gamma_n \Rightarrow \Delta_n]$ *of (set-based) sequents, where we write* $\mathsf{last}_L(\mathcal{H})$ *(resp.* $\mathsf{last}_R(\mathcal{H})$*) for* Γ_n *(resp.* Δ_n*) and* $\mathsf{last}(\mathcal{H})$ *for* $\mathsf{last}_L(\mathcal{H}) \Rightarrow \mathsf{last}_R(\mathcal{H})$*.*

Given two histories $\mathcal{H} = [\Gamma_1 \Rightarrow \Delta_1; \ldots; \Gamma_n \Rightarrow \Delta_n]$ *and* $\mathcal{H}' = [\Sigma_1 \Rightarrow \Pi_1; \ldots; \Sigma_m \Rightarrow \Pi_m]$ *with* $n \leq m$ *we write* $\mathcal{H} \preccurlyeq \mathcal{H}'$ *if for all* $i \leq n$ *we have* $\Gamma_i = \Sigma_i$ *and* $\Delta_i = \Pi_i$*; moreover we write* $\mathcal{H} \mathbin{+\!\!+} \mathcal{H}'$ *for the concatenation of the two histories.*

Following [56], the rules $\mathsf{Mon}, \mathsf{4}, \mathsf{D}_1, \mathsf{D}_2$ are called *transitional* or *dynamic*, while the non-modal rules and the rule T are called *static*; given a history $\mathcal{H}$ as input, the algorithm Alg.1 first saturates the last sequent under the static rules with one premiss, and, if the result is not an initial sequent, checks if it has been derived by a two-premiss static or transitional rule. Then, for each possible premiss, the proof search procedure is applied again on the input constituted by the concatenation of the given history and this premiss.

The check at step 11 of Alg.1 makes clear how histories are used in order to avoid analysing the same sequent multiple times.

The following theorem proves that the algorithm actually describes the proof search procedure for $\mathsf{G_{bMDL}}$, i.e. a history $\mathcal{H}$ is accepted as input if and only if $\mathsf{last}(\mathcal{H})$ is derivable in $\mathsf{G_{bMDL}}$; furthermore it is proved that the proof search procedure terminates in a finite time, hence, for any sequent, it is always possible to decide whether it is derivable in $\mathsf{G_{bMDL}}$ or not.

Lemma 3.1.9 (Termination - Lem.2 Sec.3 [29]) *The procedure in algorithm 1 terminates.*

Algorithm 1: The proof search procedure for $\mathsf{G_{bMDL}}$ (Sec.3 [29])

Input: A history $\mathcal{H}$
Output: Is $\mathsf{last}(\mathcal{H})$ derivable in $\mathsf{G_{bMDL}}$ given the history $\mathcal{H}$?

1 Saturate $\mathsf{last}(\mathcal{H})$ under the one-premiss static rules;
2 **if** $\mathsf{last}(\mathcal{H})$ *is an initial sequent* **then**
3   accept the history
4 **else**
5   **for** *every possible application of a two-premiss static rule to* $\mathsf{last}(\mathcal{H})$ **do**
6   **for** *every premiss* $\Sigma \Rightarrow \Pi$ *of this application* **do**
7   recursively call the proof search procedure with input $\mathcal{H} +\!\!+ [\Sigma \Rightarrow \Pi]$;
8   accept the application if each of these calls accepts
9   **for** *every possible application of a transitional rule to* $\mathsf{last}(\mathcal{H})$ **do**
10   **for** *every premiss* $\Sigma \Rightarrow \Pi$ *of this application* **do**
11   **if** *there is an* $\mathcal{H}' \preccurlyeq \mathcal{H}$ *with* $\Sigma \subseteq \mathsf{last}_L(\mathcal{H}')$ *and* $\Pi \subseteq \mathsf{last}_R(\mathcal{H}')$ **then**
12   reject the premiss
13   **else**
14   call the proof search procedure with input $\mathcal{H} +\!\!+ [\Sigma \Rightarrow \Pi]$;
15   accept the premiss if this call accepts
16   accept the rule application if each of the premisses is accepted
17   accept the history if at least one of the possible applications is accepted

Proof. For any history $\mathcal{H}$, from the subformulas of the sequent $\mathsf{last}(\mathcal{H})$ it is possible to construct a number N of different (set-based) sequents which is exponential in the size of $\mathsf{last}(\mathcal{H})$; therefore the proof search procedure can be recursively called at most N times before every rule application is rejected. Moreover, there are only finitely many possible (backwards) applications of a rule of $\mathsf{G_{bMDL}}$ for each sequent, hence the subroutine (steps 5 to 16 of Alg.1) can be executed only a finite number of times. $\square$

Theorem 3.1.10 (Prop.1 Sec.3 [29]) $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$ *iff the procedure accepts as input the history* $[\Gamma \Rightarrow \Delta]$.

Proof. If the procedure accepts the input $[\Gamma \Rightarrow \Delta]$, then the derivation tree of $\Gamma \Rightarrow \Delta$ in $\mathsf{G_{bMDL}}$ is built by labelling the nodes with each sequent $\mathsf{last}(\mathcal{H})$ such that the history $\mathcal{H}$ is given as input to the recursive calls of the algorithm, following the order of the accepted backwards applications of rules.

For the other direction, since the weakening rule is admissible in $\mathsf{G_{bMDL}}$ (Lem.4.3.22), if $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$, then there is a *minimal* derivation of it, i.e. a derivation such that, if $\Omega \subseteq \Sigma$ and $\Theta \subseteq \Pi$, then, in a branch containing the sequents $\Sigma \Rightarrow \Pi$ and $\Omega \Rightarrow \Theta$, $\Sigma \Rightarrow \Pi$ cannot occur on the path between $\Omega \Rightarrow \Theta$ and the root. Hence, by induction on the height of such a minimal derivation, it can be seen that the procedure accepts the input $[\Gamma \Rightarrow \Delta]$. $\square$

As already mentioned, the modal rules of $\mathsf{G_{bMDL}}$ have been constructed on the basis of the Hilbert axioms defining the logic bMDL, by using the technique of [80]. Therefore, the equivalence of the sequent calculus and the axiomatic one follows by construction (Thm. 3.1.12).

From this result, together with the fact that the calculus $\mathsf{G_{bMDL}}$ satisfies the subformula property, the consistency of the logic bMDL follows, i.e. $\bot$ is not a theorem of bMDL:

Corollary 3.1.11 *The logic* bMDL *is consistent.*

Theorem 3.1.12 (Equivalence of sequent and axiomatic calculus - Thm.2 Sec.3.1 [29]) *For all sets $\mathcal{S}$ of sequents and sequents $\Gamma \Rightarrow \Delta$:*

$$\mathcal{S} \vdash_{\mathsf{G_{bMDL}Cut}} \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \{\bigwedge \Sigma \to \bigvee \Pi \mid \Sigma \Rightarrow \Pi \in \mathcal{S}\} \vdash_{\mathsf{bMDL}} \bigwedge \Gamma \to \bigvee \Delta \ .$$

Proof. The standard results for propositional calculi are automatically valid for the equivalence of the axiomatic system bMDL and the sequent calculus G3 with a zero-premiss rule $\overline{\Rightarrow \theta}$ for each modal axiom schema θ of bMDL.

The result then follows from interderivability, guaranteed by construction in [80], of these zero-premiss rules with the rules of $\mathsf{G_{bMDL}}$ for the deontic operator. □

Corollary 3.1.13 *The logic* bMDL *is decidable.*

3.1.2 Semantics

The proof-theoretic approach in a sense identifies the meaning of formulas with their roles in inferences, as the sequent rules determine the inferential relations between formulas.

On the other hand, the semantics associates the meaning of formulas with (sets of) states, and relations among them. Intuitively, the *neighbourhood* function identifies a set of "deontically acceptable" sets of (accessible) worlds for some specific situations.

The semantic approach provides insights on Mīmāṃsakas' interpretations of the commands: by observing which situations are compatible with the respect of Vedic norms, it allows us to explain why Mīmāṃsā authors chose specific solution to the deontic controversies they analysed. In [29] the semantics for bMDL has been successfully used for analysing the well known controversy concerning the malefic sacrifice called *Śyena*, showing that, since there are states of affairs where all the rules are complied with, seemingly conflicting commands are not actually contradictory.

The semantics for bMDL is constructed in [29] on the basis of the standard semantics for the modal logic $\mathsf{S4}$, i.e. Kripke-frames with transitive and reflexive accessibility relation, see e.g. [16]. The semantic counterpart of the deontic modality $\mathcal{O}$ is described by using a neighbourhood semantics [27], modified to take into account only accessible worlds. Intuitively, for a given state of affairs (i.e. a possible world), representing the current state of the subject, the neighbourhood function selects, among accessible worlds, sets of worlds (representing the performance of obligatory rituals) which are deontically adequate for other sets of possible states (the *adhikāra* conditions).

In order to simplify the representation of the semantics, the standard notation is adopted: given a relation $R \subseteq W \times W$ and a possible world $w \in W$, $R[w]$ denotes the set $\{v \in W \mid wRv\}$; furthermore, X^c denotes the complement of a set X (relative to an implicitly given set).

Definition 3.1.14 (m-frame, m-model - Def.5 Sec.4 [29]) *A* Mīmāṃsā-frame *(or* m-frame*) is a triple* $(W, R, \mathcal{N})$ *consisting of a non-empty set* W *of* possible worlds *or* states*, an* accessibility relation $R \subseteq W \times W$ *and a map* $\mathcal{N} : W \to \wp(\wp(W) \times \wp(W))$ *such that:*

1. *R is transitive and reflexive;*
2. *if $(X, Y) \in \mathcal{N}(w)$, then $X \subseteq R[w]$ and $Y \subseteq R[w]$;*
3. *if $(X, Z) \in \mathcal{N}(w)$ and $X \subseteq Y \subseteq R[w]$, then also $(Y, Z) \in \mathcal{N}(w)$;*
4. *if $(X, Y) \in \mathcal{N}(w)$, then $(X^c \cap R[w], Y) \notin \mathcal{N}(w)$;*
5. *$(\varnothing, X) \notin \mathcal{N}(w)$ (this condition, derivable from the previous ones, is explicitly stated for simplicity).*

A Mīmāṃsā-model *(or* m-model*) is a m-frame with a* valuation *function* $\sigma : \mathsf{Var} \to \wp(W)$*.*

The required properties of the accessibility relation and of the neighbourhood map in Def. 3.1.14 intuitively mirror the properties defined by the axioms of bMDL: the first condition corresponds to the axioms $(\mathsf{4})$ and (T) of $\mathsf{S4}$ for the alethic modality $\Box$, while the condition 2 corresponds to the deontic axioms 1 and 3 of bMDL and guarantees that the truth of a deontic formula $\mathcal{O}(\varphi/\psi)$ cannot depend on possible worlds which are not R-accessible; also the third condition expresses a property given by axiom 1, while the last two conditions are related to the axiom 2 of bMDL.

Definition 3.1.15 (Satisfaction, truth set - Def.6 Sec.4 [29]) *Let* $\mathfrak{M} = (W, R, \mathcal{N}, \sigma)$ *be a m-model. The* truth set $[\![\varphi]\!]_{\mathfrak{M}}$ *of a formula* φ *in* $\mathfrak{M}$ *is defined recursively by the following clauses*

1. $[\![p]\!]_{\mathfrak{M}} := \sigma(p)$
2. $[\![\Box\varphi]\!]_{\mathfrak{M}} := \{w \in M \mid R[w] \subseteq [\![\varphi]\!]_{\mathfrak{M}}\}$
3. $[\![\mathcal{O}(\varphi/\psi)]\!]_{\mathfrak{M}} := \{w \in W \mid ([\![\varphi]\!]_{\mathfrak{M}} \cap R[w], [\![\psi]\!]_{\mathfrak{M}} \cap R[w]) \in \mathcal{N}(w)\}$

plus the standard clauses for the boolean connectives. $\mathfrak{M}, w \Vdash \varphi$ *indicates that* $w \in [\![\varphi]\!]_{\mathfrak{M}}$*, i.e. for the m-model* $\mathfrak{M}$*, the formula* φ *is satisfied in* w*. A formula* φ *is said to be* valid *in a m-model* $\mathfrak{M}$ *if for all worlds* w *of* $\mathfrak{M}$ *we have* $\mathfrak{M}, w \Vdash \varphi$*.*

Remark 3.1.16 *Notice that when* R *is a universal relation (i.e.,* $R = W \times W$*),* $\mathfrak{M}, w \Vdash \mathcal{O}(\varphi/\psi)$ *iff* $([\![\varphi]\!]_{\mathfrak{M}}, [\![\psi]\!]_{\mathfrak{M}}) \in \mathcal{N}(w)$*.*

The fact that the standard approach of neighbourhood semantics is modified to take into account only accessible worlds appears immediately clear by looking at clause 3, where the definition is restricted to worlds accessible from the current world.

Given the definition of validity in a m-model, the rules of $\mathsf{G_{bMDL}}$ are now shown to preserve validity; from this result the soundness of the calculus $\mathsf{G_{bMDL}}$ with respect to m-models also follows.

Lemma 3.1.17 *For any rule of* $\mathsf{G_{bMDL}}$*, if the interpretations of its premisses are valid in all m-models, then so is the interpretation of its conclusion.*

Proof. The claim is proved by contraposition, showing that, if the negated interpretation of the conclusion is satisfiable in a m-model, then so is the negated interpretation of (at least) one of the premisses. The proof is standard for the rules $\mathsf{4}, \mathsf{T}$ of $\mathsf{S4}$ and for the non modal rules. Following again the proof in [29], for the rules introducing the deontic operator, only the case of $\mathsf{D_2}$ will be shown, as the proofs for other cases are similar.

Suppose that for the m-model $\mathfrak{M} = (W, R, \mathcal{N}, \sigma)$ the negation of the conclusion is satisfied in $w \in W$, i.e. $\mathfrak{M}, \sigma, w \Vdash \bigwedge\Gamma \wedge \mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\theta/\chi) \wedge \neg(\bigwedge\Delta)$. Hence, by Cond.3 in Def. 3.1.15, $([\![\varphi]\!] \cap R[w], [\![\psi]\!] \cap R[w]) \in \mathcal{N}(w)$ and $([\![\theta]\!] \cap R[w], [\![\chi]\!] \cap R[w]) \in \mathcal{N}(w)$. From Cond. 4 in Def. 3.1.14, $([\![\varphi]\!]^c \cap R[w], [\![\psi]\!] \cap R[w]) \notin \mathcal{N}(w)$ follows, therefore $[\![\theta]\!] \cap R[w] \neq [\![\varphi]\!]^c \cap R[w]$ or $[\![\psi]\!] \cap R[w] \neq [\![\chi]\!] \cap R[w]$.

If $[\![\psi]\!] \cap R[w] \neq [\![\chi]\!] \cap R[w]$ holds, then one of the following is also true: $[\![\psi]\!] \cap [\![\chi]\!]^c \cap R[w] \neq \varnothing$ or $[\![\chi]\!] \cap [\![\psi]\!]^c \cap R[w] \neq \varnothing$. If $[\![\psi]\!] \cap [\![\chi]\!]^c \cap R[w] \neq \varnothing$ holds, we find a world $v \in [\![\psi]\!] \cap [\![\chi]\!]^c \cap R[w] \neq \varnothing$ and, by transitivity, we obtain $\mathfrak{M}, \sigma, v \Vdash \bigwedge\Gamma^{\Box} \wedge \psi \wedge \neg\chi$, satisfying the negation of the second premiss of the rule $\mathsf{D_2}$; the other case is analogous and satisfies the negation of the third premiss of the rule $\mathsf{D_2}$.

If $[\![\psi]\!] \cap R[w] \neq [\![\chi]\!] \cap R[w]$ does not hold, then $[\![\theta]\!] \cap R[w] \neq [\![\varphi]\!]^c \cap R[w]$ is true. Then, by Cond. 3 in Def. 3.1.14, we have $[\![\varphi]\!]^c \cap R[w] \nsubseteq [\![\theta]\!] \cap R[w]$ and hence we find a world $v \in [\![\varphi]\!] \cap [\![\theta]\!] \cap R[w]$ such that, by transitivity, we obtain $\mathfrak{M}, \sigma, v \Vdash \bigwedge \Gamma^{\Box} \wedge \varphi \wedge \theta$, satisfying the negation of the first premiss of the rule. $\Box$

Corollary 3.1.18 (Soundness of $\mathsf{G_{bMDL}}$) *For every sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$, then $\bigwedge \Gamma \to \bigvee \Delta$ is valid in all m-models.*

Proof. Since it has already been proved (Lem. 3.1.17) that the rules of $\mathsf{G_{bMDL}}$ preserve validity in m-models, the claim follows by induction on the height of the derivation. $\Box$

To prove completeness of the system $\mathsf{G_{bMDL}}$ with respect to m-models, we need to construct a countermodel for a given sequent starting from a failed proof search for it.

This means that, given a sequent $\Gamma \Rightarrow \Delta$ not derivable in $\mathsf{G_{bMDL}}$, starting from a rejecting application of Alg. 1 on input $[\Gamma \Rightarrow \Delta]$, it is shown how to build a m-model $\mathfrak{M}_{\Gamma \Rightarrow \Delta} = (W, R, \mathcal{N}, \sigma^{\mathcal{H}})$ such that $\bigwedge \Gamma \wedge \bigwedge \neg \Delta$ is satisfied in a world of $\mathfrak{M}_{\Gamma \Rightarrow \Delta}$.

First we present the countermodel construction (Def. 3.1.19) as it has been introduced in [29]; we show that it presents some problems which make it possible to find an example (ex.3.1.20) of a case in which it is impossible to determine whether a world is in the truth set of a formula. Therefore, another construction will be presented, which will be used for proving completeness of the calculus $\mathsf{G_{bMDL}}$ with respect to m-models.

Definition 3.1.19 (countermodel from a rejecting application of Alg.1 - Sec.4 [29]) *Given a sequent $\Gamma \Rightarrow \Delta$ rejected by an application of Alg. 1, the elements of the m-model $\mathfrak{M}_{\Gamma \Rightarrow \Delta} = (W, R, \mathcal{N}, \sigma^{\mathcal{H}})$ are defined as follows:*

- *The set of possible worlds W is the set of all histories occurring in the run of the procedure.*
- *The accessibility relation R is defined as the reflexive and transitive closure of the intermediate relation R', such that $\mathcal{H} R' \mathcal{H}'$ iff (1) $\mathcal{H} \preccurlyeq \mathcal{H}'$ or (2) $\mathcal{H}' \preccurlyeq \mathcal{H}$ and there is an application of a transitional rule with conclusion $\mathsf{last}(\mathcal{H})$ and a premiss $\Sigma \Rightarrow \Pi$ such that $\Sigma \subseteq \mathsf{last}_L(\mathcal{H}')$ and $\Pi \subseteq \mathsf{last}_R(\mathcal{H}')$ (the condition (2) is intuitively meant to take into account the loops which have been detected by the procedure).*
- *The valuation σ is characterized by defining for every variable p:*

$$\sigma(p) := \{\mathcal{H} \in W \mid p \in \mathsf{last}_L(\mathcal{H})\}.$$

. *The function* $\mathcal{N} : W \to \mathcal{P}(\mathcal{P}(W) \times \mathcal{P}(W))$ *is characterized by defining for every history* $\mathcal{H}$ *in* W*:*

$$\mathcal{N}(\mathcal{H}) := \left\{ (X,Y) \in \mathcal{P}(R[\mathcal{H}])^2 \;\middle|\; \begin{array}{l} \textit{for some formula } \mathcal{O}(\varphi/\psi) \in \mathsf{last}_L(\mathcal{H}) : \\ [\![\varphi]\!] \cap R[\mathcal{H}] \subseteq X \textit{ and } [\![\psi]\!] \cap R[\mathcal{H}] = Y \end{array} \right\}$$

However, the definition turns out to be circular, in the sense that the fact that a world (history) belongs to the truth set of a modal formula depends on the truth sets of the modal formulas which appear in the last sequent of the same history. The following example shows that this definition does not allow to check, for any formula φ and any world $\mathcal{H} \in W$, if $\mathcal{H}$ is in φ's truth set.

Example 3.1.20. Let the set of worlds W be $\{\mathcal{H}', \mathcal{H}''\}$, such that $\mathcal{H}'' \preccurlyeq \mathcal{H}'$, $\mathsf{last}(\mathcal{H}') = \mathcal{O}(p/q) \Rightarrow$, and $\mathsf{last}(\mathcal{H}'') = \mathcal{O}(\mathcal{O}(p/q)/r), p \Rightarrow$, with $p, q, r \in \mathsf{Var}$.

Now let us check whether $\mathcal{H}''$ is in the truth set $[\![\mathcal{O}(p/q)]\!]$ of the formula $\mathcal{O}(p/q)$.

By definition of truth sets (Def.3.1.15), $[\![\mathcal{O}(p/q)]\!] = \{w \in W \mid ([\![p]\!]_{\mathfrak{M}} \cap R[w], [\![q]\!]_{\mathfrak{M}} \cap R[w]) \in \mathcal{N}(w)\}$, therefore we need to check if $([\![p]\!]_{\mathfrak{M}} \cap R[\mathcal{H}''], [\![q]\!]_{\mathfrak{M}} \cap R[\mathcal{H}'']) \in \mathcal{N}(\mathcal{H}'')$.

By Def.3.1.19, $([\![p]\!] \cap R[\mathcal{H}''], [\![q]\!] \cap R[\mathcal{H}'']) \in \mathcal{N}(\mathcal{H}'')$ if there is a formula $\mathcal{O}(\varphi/\psi) \in \mathsf{last}_L(\mathcal{H}'')$ such that $[\![\varphi]\!] \cap R[\mathcal{H}''] \subseteq [\![p]\!] \cap R[\mathcal{H}'']$ and $[\![\psi]\!] \cap R[\mathcal{H}''] = [\![q]\!] \cap R[\mathcal{H}'']$. Since the only modal formula in $\mathsf{last}_L(\mathcal{H}'')$ is $\mathcal{O}(\mathcal{O}(p/q)/r)$, we need to check if $[\![\mathcal{O}(p/q)]\!] \cap R[\mathcal{H}''] \subseteq [\![p]\!] \cap R[\mathcal{H}'']$ and $[\![r]\!] \cap R[\mathcal{H}''] = [\![q]\!] \cap R[\mathcal{H}'']$. The second condition clearly holds, as $[\![r]\!] \cap R[\mathcal{H}''] = \varnothing = [\![q]\!] \cap R[\mathcal{H}'']$. For what concerns the first one, on the other hand, we are caught in a loop, as $[\![p]\!] \cap R[\mathcal{H}''] = \{\mathcal{H}''\}$, but, in order to find out if $\mathcal{H}''$ is in $[\![\mathcal{O}(p/q)]\!]$ (which was the initial question) we need to know if $\mathcal{H}''$ is in $[\![\mathcal{O}(p/q)]\!]$. ■

Hence we need to define the neighbourhood function in such a way that checking whether a world is in a modal formula's truth set does not involve analysing the truth sets of all the modal formulas in this world: we will do so, fixing the mistake in [29].

However, as shown in the next example Ex.3.1.21, a syntactic definition may not capture the needed properties. This is due to the fact that the rule of (downward) monotonicity does not hold for the second argument of the deontic operator, i.e. $(\Box(\chi \to \theta) \wedge \mathcal{O}(\varphi/\theta)) \to \mathcal{O}(\phi/\chi)$ does not hold.

Example 3.1.21. Let us consider the following definition of the neighbourhood function:

$$\mathcal{N}(\mathcal{H}) \coloneqq \left\{ (X,Y) \in \mathcal{P}(R[\mathcal{H}])^2 \;\middle|\; \begin{array}{l} \text{there is a formula } \mathcal{O}(\varphi/\psi) \in \mathsf{last}_L(\mathcal{H}) \text{ such that} \\ \{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq X \text{ and} \\ \{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_L(\mathcal{H}')\} = Y \end{array} \right\}$$

and let the set of worlds W be $\{\mathcal{H}', \mathcal{H}''\}$, such that $\mathcal{H}' \preccurlyeq \mathcal{H}''$, $\mathsf{last}(\mathcal{H}') = \mathcal{O}(r \wedge q/q \wedge r) \Rightarrow$, $\mathsf{last}(\mathcal{H}'') = r \wedge q, q, r \Rightarrow$, with $q, r \in \mathsf{Var}$.

Since $[\![q \wedge r]\!] = [\![r \wedge q]\!]$, we would expect that $\mathcal{H}' \in [\![\mathcal{O}(r \wedge q/r \wedge q)]\!]$. However, since the only modal formula in $\mathcal{H}'$ is $\mathcal{O}(r \wedge q/q \wedge r)$, $\mathcal{H}' \in [\![\mathcal{O}(r \wedge q/r \wedge q)]\!]$ if $\{\mathcal{H}^* \in R[\mathcal{H}'] \mid r \wedge q \in \mathsf{last}_L(\mathcal{H}^*)\} \subseteq [\![r \wedge q]\!]$ and $\{\mathcal{H}^* \in R[\mathcal{H}'] \mid q \wedge r \in \mathsf{last}_L(\mathcal{H}^*)\} = [\![r \wedge q]\!]$; but $\{\mathcal{H}^* \in R[\mathcal{H}'] \mid q \wedge r \in \mathsf{last}_L(\mathcal{H}^*)\} = \varnothing$ and $[\![r \wedge q]\!] = \mathcal{H}''$, therefore we have the counterintuitive result $\mathcal{H}' \notin [\![\mathcal{O}(r \wedge q/r \wedge q)]\!]$. ∎

The highlighted problems can be solved by using the *bi-neighbourhood semantics*, introduced in [37] and already employed for the construction of countermodels in [36]. The idea behind this approach is to associate to each world a pair of neighbourhoods, each one providing independent positive or negative support for a modal formula. This is particularly useful in the absence of monotonicity, whereas, as we can understand from the previous example, in case of monotonic modal operators, one of the two neighbourhoods of a pair is the empty set.

Def.3.1.22 and the following proofs represent a new result, as they deviate from the ones in [29] and correct the flaws of Def.3.1.19, by using the approach of bi-neighbourhood semantics for defining the neighbourhood function.

Definition 3.1.22 (countermodel from a rejecting application of Alg.1) *The set of possible worlds W, the accessibility relation R, and the valuation function σ of the structure $\mathfrak{M}_{\Gamma\Rightarrow\Delta} = (W, R, \mathcal{N}, \sigma^{\mathcal{H}})$ are defined as in Def.3.1.19.*

The neighbourhood function $\mathcal{N}$ is defined by the following condition:

$$\mathcal{N}(\mathcal{H}) \coloneqq \left\{ (X,Y) \in \mathcal{P}(R[\mathcal{H}])^2 \;\middle|\; \begin{array}{l} \textit{there is a formula } \mathcal{O}(\varphi/\psi) \in \mathsf{last}_L(\mathcal{H}) \textit{ such that} \\ \{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq X \textit{ and} \\ \{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq Y \textit{ and} \\ \{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq Y^c \end{array} \right\}$$

Now we show that the structure in the previous definition is a m-model.

Lemma 3.1.23 (model lemma) *The structure $\mathfrak{M}_{\Gamma\Rightarrow\Delta} = (W, R, \mathcal{N}, \sigma^{\mathcal{H}})$ in Def.3.1.22 is* *a m-model.*

Proof. We show that the structure $\mathfrak{M}_{\Gamma\Rightarrow\Delta} = (W, R, \mathcal{N}, \sigma^{\mathcal{H}})$ in Def.3.1.22 satisfies all the properties in Def.3.1.14.

- (Properties (1) and (2) of Def.3.1.14)

  The accessibility relation R is transitive and reflexive by definition, and the definition of the neighbourhood function guarantees that any couple (X, Y) of subsets of W in $\mathcal{N}(\mathcal{H})$ is such that $X \subseteq R[\mathcal{H}]$ and $Y \subseteq R[\mathcal{H}]$.

- (Property (3) of Def.3.1.14)

  If $(X, Z) \in \mathcal{N}(\mathcal{H})$, then there is a formula $\mathcal{O}(\varphi/\psi) \in \mathsf{last}_L(\mathcal{H})$ such that $\{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq X$ and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq Z$ and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq Z^c$. Therefore, if $X \subseteq Y \subseteq R[\mathcal{H}]$, then $\{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq Y$, hence also $(Y, Z) \in \mathcal{N}(\mathcal{H})$.

- (Property (4) of Def.3.1.14)

  We show that $(X^c \cap R[w], Y) \notin \mathcal{N}(\mathcal{H})$ if $(X, Y) \in \mathcal{N}(\mathcal{H})$ by contradiction.

  Let us assume $(X, Y) \in \mathcal{N}(\mathcal{H})$ and $(X^c \cap R[w], Y) \in \mathcal{N}(\mathcal{H})$. Then there are two formulas $\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi)$ in $\mathsf{last}_L(\mathcal{H})$ such that both the following conditions (i) and (ii) hold:

  (i) $\{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq X$ and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \theta \in \mathsf{last}_L(\mathcal{H}')\} \subseteq X^c$ (which, in particular, means that $\{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \bigcap \{\mathcal{H}' \in R[\mathcal{H}] \mid \theta \in \mathsf{last}_L(\mathcal{H}')\} = \varnothing$);

  (ii) $\{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq Y$, $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq Y$, $\{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq Y^c$, and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq Y^c$ (which, in particular, means that $\{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_L(\mathcal{H}')\} \bigcap \{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_R(\mathcal{H}')\} = \varnothing$ and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_L(\mathcal{H}')\} \bigcap \{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_R(\mathcal{H}')\} = \varnothing$).

  By definition of the proof search procedure 1, if there is a history $\mathcal{H}$ such that $\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \in \mathsf{last}_L(\mathcal{H})$, then there is another history $\mathcal{H}^*$ occurring in the run of the procedure such that (a) $\psi \in \mathsf{last}_L(\mathcal{H}^*)$ and $\chi \in \mathsf{last}_R(\mathcal{H}^*)$, or (b) $\chi \in \mathsf{last}_L(\mathcal{H}^*)$ and $\psi \in \mathsf{last}_R(\mathcal{H}^*)$, or (c) $\varphi, \theta \in \mathsf{last}_L(\mathcal{H}^*)$; moreover $\mathcal{H}^*$ is such that $\mathcal{H} \preccurlyeq \mathcal{H}^*$ or $\mathcal{H}^* \preccurlyeq \mathcal{H}$ and there is an application of D_2 with conclusion $\mathsf{last}(\mathcal{H})$ and a premiss $\Sigma \Rightarrow \Pi$ such that $\Sigma \subseteq \mathsf{last}_L(\mathcal{H}')$ and $\Pi \subseteq \mathsf{last}_R(\mathcal{H}')$, therefore, in any case, $\mathcal{H}^* \in R[\mathcal{H}]$.

  But this means that in cases (a) $\mathcal{H}^* \in \{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_L(\mathcal{H}')\} \bigcap \{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_R(\mathcal{H}')\}$, in case (b) $\mathcal{H}^* \in \{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_L(\mathcal{H}')\} \bigcap \{\mathcal{H}' \in R[\mathcal{H}] \mid \psi \in \mathsf{last}_R(\mathcal{H}')\}$, and in case (c) $\mathcal{H}^* \in \{\mathcal{H}' \in R[\mathcal{H}] \mid \varphi \in \mathsf{last}_L(\mathcal{H}')\} \bigcap \{\mathcal{H}' \in R[\mathcal{H}] \mid \theta \in \mathsf{last}_L(\mathcal{H}')\}$, contradicting the conditions (i) and (ii).

- (Property (5) of Def.3.1.14)

  Property (5) can be derived from properties (3) and (4).

$\square$

Since $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma^{\mathcal{H}}$ is constructed from a rejecting run of Alg. 1 on input $[\Gamma \Rightarrow \Delta]$, considering any history $\mathcal{H} \in W$ as the actual world, it is possible to verify that the structure provides a countermodel also to $\mathsf{last}(\mathcal{H})$.

The following lemma proves that if a sequent is not derivable in the calculus $\mathsf{G_{bMDL}}$ (and therefore, by Thm.3.1.10, it is rejected by the proof search algorithm), then there is a countermodel for it, i.e. a m-model which satisfies its formula-interpretation negated. This represents the fundamental step for proving that, for every formula valid in all m-models, the corresponding sequent is derivable in $\mathsf{G_{bMDL}}$. This is proved by contraposition, showing that, if a sequent is not derivable in $\mathsf{G_{bMDL}}$, then it is not valid in all m-models.

Lemma 3.1.24 (truth lemma) *For every history $\mathcal{H} \in W$: (i) if $\varphi \in \mathsf{last}_L(\mathcal{H})$, then $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma, \mathcal{H} \Vdash \varphi$ and (ii) if $\psi \in \mathsf{last}_R(\mathcal{H})$, then $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma, \mathcal{H} \Vdash \neg\psi$.*

Proof. The statements (i) and (ii) are proved together, by induction on the complexity of φ (resp. ψ).

Since the algorithm Alg. 1 saturates each sequent under propositional rules and T, and the transitional rules copy the boxed formulas in the antecedent into the premisses, the cases where $\varphi = p$ with $p \in \mathsf{Var}$, $\varphi = \bot$, $\psi = \bot$, and the main connective of φ (resp. ψ) is $\Box$ or a propositional one are trivial.

Let us consider the case where $\varphi = \mathcal{O}(\theta/\chi)$. By induction hypotheses $\{\mathcal{H}' \in R[\mathcal{H}] \mid \theta \in \mathsf{last}_L(\mathcal{H}')\} \subseteq [\![\theta]\!]$, $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq [\![\chi]\!]$, and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq [\![\chi]\!]^c$; therefore $([\![\theta]\!] \cap R[\mathcal{H}], [\![\chi]\!] \cap \mathcal{H}) \in \mathcal{N}(\mathcal{H})$, which means $\mathcal{H} \in [\![\mathcal{O}(\theta/\chi)]\!]$ and, by definition of satisfaction (Def.3.1.15), $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma, \mathcal{H} \Vdash \mathcal{O}(\theta/\chi)$.

For verifying the case of $\psi = \mathcal{O}(\theta/\chi)$, we show that $([\![\theta]\!] \cap R[\mathcal{H}], [\![\chi]\!] \cap \mathcal{H}) \notin \mathcal{N}(\mathcal{H})$, i.e. there is no $\mathcal{O}(\zeta/\xi) \in \mathsf{last}_L(\mathcal{H})$ such that (i) $\{\mathcal{H}' \in R[\mathcal{H}] \mid \zeta \in \mathsf{last}_L(\mathcal{H}')\} \subseteq [\![\theta]\!]$ and (ii) $\{\mathcal{H}' \in R[\mathcal{H}] \mid \xi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq [\![\chi]\!]$ and (iii) $\{\mathcal{H}' \in R[\mathcal{H}] \mid \xi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq [\![\chi]\!]^c$.

This is trivial if there are no formulas of the form $\mathcal{O}(\zeta/\xi)$ in $\mathsf{last}_L(\mathcal{H})$; if $\mathsf{last}_L(\mathcal{H})$ contains such a $\mathcal{O}(\zeta/\xi)$, then the rule Mon can be applied backwards to $\mathsf{last}(\mathcal{H})$. Hence there is another history $\mathcal{H}^*$ occurring in the run of the procedure such that $\mathcal{H}^* \in R[\mathcal{H}]$

and (a) $\zeta \in \mathsf{last}_L(\mathcal{H}^*)$ and $\theta \in \mathsf{last}_R(\mathcal{H}^*)$, or (b) $\chi \in \mathsf{last}_L(\mathcal{H}^*)$ and $\xi \in \mathsf{last}_R(\mathcal{H}^*)$, or (c) $\xi \in \mathsf{last}_L(\mathcal{H}^*)$ and $\chi \in \mathsf{last}_R(\mathcal{H}^*)$.

By induction hypothesis $\{\mathcal{H}' \in R[\mathcal{H}] \mid \theta \in \mathsf{last}_R(\mathcal{H}')\} \subseteq [\![\theta]\!]^c$, therefore in case (a) $\mathcal{H}^* \in \{\mathcal{H}' \in R[\mathcal{H}] \mid \zeta \in \mathsf{last}_L(\mathcal{H}')\}$ and $\mathcal{H}^* \in [\![\theta]\!]^c$, falsifying (i). Again by induction hypotheses $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_L(\mathcal{H}')\} \subseteq [\![\chi]\!]$ and $\{\mathcal{H}' \in R[\mathcal{H}] \mid \chi \in \mathsf{last}_R(\mathcal{H}')\} \subseteq [\![\chi]\!]^c$, therefore in case (b) the condition (ii) is contradicted, and case (c) falsifies (iii). □

Using this lemma, it is possible to prove the completeness of the sequent calculus with respect to the m-models.

Theorem 3.1.25 (Completeness) *For every sequent $\Gamma \Rightarrow \Delta$, if $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid in every m-model, then $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$.*

Proof. Reasoning by contraposition, if $\nvdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$, then by Lem. 3.1.9 and Thm. 3.1.10, the procedure in Alg. 1 terminates and rejects the input $[\Gamma \Rightarrow \Delta]$. Therefore, by Lem. 3.1.24, $\mathfrak{M}_{\Gamma \Rightarrow \Delta}, [\Gamma \Rightarrow \Delta] \Vdash \bigwedge \Gamma \wedge \neg \bigvee \Delta$, which means that $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is not valid in every m-model. □

3.2 Related Works

The fact that the properties of Mīmāṃsā logic bMDL are solely extracted from Mīmāṃsā texts, renders this logic weaker than most known deontic logics, e.g., the logics considered in [138, 139, 65, 136, 110, 55]. As it will be shown later (Sec.3.3), the $\Box$-free fragment of bMDL corresponds indeed to the dyadic version of the logic MD, the smallest system of deontic logic with monotonicity and the D axiom (see[27]).

As the relation between the second and the first argument of bMDL's deontic operator can be seen as a conditional weaker than material implication, the logic bMDL, in some respects, appears to be close to conditional logics (see e.g. [26, 99]) or logics of counterfactuals (see [81, 98]). However, bMDL lacks some of the principles which characterize even the minimal and least complex systems of conditional logic (e.g. the system CC in [27]). Among those axioms and rules, the most common one is the so-called *aggregation* principle, whose "deontic version" can be formalised as $\mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\theta/\psi) \rightarrow \mathcal{O}(\varphi \wedge \theta/\psi)$. This is instead present in some of the first and most significant dyadic deontic logics e.g. the system introduced in [6] and examined in [104], and in the logics with deontic conditionals described in [138, 65, 136, 82]. The fact that no mention of *nyāya*s corresponding to the deontic aggregation principle has been found in the texts is most likely due to the *singleness nyāya*, according to which the content of an injunction can be only one action with one specific objective. Hence, a formula like $\mathcal{O}(\varphi \wedge \theta/\psi)$ means that the two actions represented by φ and θ are components of the same main ritual, and such an information cannot be inferred from two commands, though having the same conditions. As no mention of a (meta-)rule analogous to the deontic aggregation principle has been found in Mīmāṃsā texts, no axiom or rule corresponding to this principle is in the logic bMDL. Since the logic is too weak for comparing obligations with different conditions, it seems that adding this principle to bMDL would not give rise to immediate contradictions. However, contradictions arise when we consider more than two obligations that are incompatible as a group. Indeed, the D axiom allows us to compare obligations only two by two, so we could have the assumptions $\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\psi), \mathcal{O}(\chi/\psi)$ such that $\Box\neg(\varphi \wedge \theta \wedge \chi)$: hence, two applications of the aggregation principles would give the contradictory obligation $\mathcal{O}(\varphi \wedge \theta \wedge \chi/\psi)$.

Moreover, most conditional logics are characterized by the *identity* axiom, which in our system could be formalised as $\mathcal{O}(\varphi/\varphi)$; this principle has been widely discussed in deontic logic, among others, in [110, 103], as it express that "under the condition that a statement is true, it ought to be the case that it is true", with the risk of giving deontic content to all facts. Though not contradictory in bMDL, it would allow us to derive counter-intuitive formulas, like $\mathcal{O}(\mathtt{kill}/\mathtt{kill})$ ("under the condition that you are killing your enemy, it ought to be the case that you kill your enemy"), and it would rule out the possibility to say that a command is in force even when violated (e.g. $\mathcal{O}(\neg\mathtt{kill}/\mathtt{kill})$, "even under the condition that you are killing your enemy, it ought to be the case that you do not kill your enemy"). In Mīmāṃsā the identity axiom would also, in a sense, go against the *novelty* nyāya, according to which Vedic commands do not convey something that is already true or obvious, otherwise they would be considered useless and re-interpreted.

Finally, it should be observed that Mīmāṃsā texts seem not to refer to commands as separated from their conditions, as reflected in the logic by the lack of the rule of Factual Detachment (which allows to derive the conclusion $\mathcal{O}(\varphi/\top)$ from the premisses $\mathcal{O}(\varphi/\psi)$ and ψ) and the rule of Deontic Detachment (which allows to derive the conclusion $\mathcal{O}(\varphi/\top)$ from the premisses $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\psi/\top)$). The philosophical reasons behind this characteristic seem to lie in Mīmāṃsaka's understanding of Vedic normative statements: as the Vedas hold independently from any ethical system or "greater good", the validity of Vedic commands is not based on contingent circumstances or on other duties, but only on the sacred texts and on the way injunctions are stated in them. In this sense, even Prabhākara, one of the authors who seemed to interpret Vedic injunctions as morally binding, could write: "*A prescription regards what has to be done. But it does not say that it has to be done*" (*Bṛhatī*, [124]). This is interpreted as stating that (conditional) Vedic commands are themselves the only authority and explain what is a duty under certain conditions; hence, even if these conditions hold, the content of a Vedic command could never become unconditionally obligatory, or it would "overcome" the sacred texts.

The reasons for not considering the Deontic Detachment principle could be the particular interpretation of conditional commands in Mīmāṃsā texts. One of the classical critics (see e.g. [135]) to this principle in modern deontic logic regards the fact that a prescription like $\mathcal{O}(\varphi/\top)$ holds in ideal circumstances, while $\mathcal{O}(\psi/\varphi)$ concerns the actual world, where things are not necessarily ideal. Therefore an unconditional command of the first kind, that is meant for ideal states, cannot follow from conditional one of the second kind, which applies in sub-ideal circumstances. However, it seems that Mīmāṃsā author had also a different motivation for the absence of such a principle: as already mentioned in Ch.2, the content of a command (of the kind we are focusing on) is a state where the subject has decided to undertake a certain (ritual) action or to refrain from it; conversely, the eligibility conditions in the second argument of the deontic operator necessarily include an element of desire, which obviously cannot be obligatory. Hence, for primary ritual actions, the two elements of a command are relatively independent and it is never even taken into account that an obligation can enjoin the conditions for identifying the addressee of another duty. This feature of Mīmāṃsā reasoning also prevents the applicability of a transitivity rule, representing a generalised form of Deontic Detachment, according to which one can derive $\mathcal{O}(\varphi/\psi)$ from $\mathcal{O}(\varphi/\theta)$ and $\mathcal{O}(\theta/\psi)$. This is consistent with the idea that prescriptions are conditioned by the occurrence of the facts, events and desires in their eligibility conditions, while the other properties of such facts, events and desires, different from their "happening", do not influence the commands.

For what concerns the Factual Detachment principle, its absence highlights the fact that, for Mīmāṃsā authors, the sets of states that are "deontically acceptable", from the perspective of a specific state of affairs, do not depend on the contingent facts (formalised as propositional formulas) which are true in that specific state. For better understanding this position, let us consider again the problematic scenario[4] around the Śyena controversy (see Section 3.4.3) analysed in [29]:

(1) $\mathcal{O}(\neg\mathtt{harm}/\top)$ for "one should not perform violence on any living being"

(2) $\mathcal{O}(\mathtt{\acute{S}y/des\_kill})$ for "someone who desires to kill an enemy should sacrifice with the

[4]The example corresponds to only one of the many possible interpretations of the statements regarding the Śyena sacrifice, different, e.g. from the ones analysed in Section 3.4.3. As this sacrifice has been the subject of a long controversy which involved authors from all the main Mīmāṃsā sub-schools, many solutions have been proposed and the statements themselves have been read in different ways.

Śyena"

(3) $\Box(\texttt{Śy} \to \texttt{kill})$ for "performing the Śyena entails an enemy"

(4) $\Box(\texttt{kill} \to \texttt{harm})$ for "killing the enemy entails harming a living being".

From (2), (3) and (4), by the first axiom of bMDL, we can derive $\mathcal{O}(\texttt{harm}/\texttt{des\_kill})$; hence, in a state where the person desires to harm his enemy ($\texttt{des\_kill}$), using the Factual Detachment principle it would be easy to derive $\mathcal{O}(\texttt{harm}/\top)$ and, by using the (1) and the axiom $(\Box((\psi \to \theta) \wedge (\theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi)) \to \mathcal{O}(\varphi/\theta)$ (the third axiom of bMDL), a contradiction.

The application of the Factual Detachment principle is blocked because the facts that are contingently true in our state, e.g. the evil desire of harming an enemy, do not change the nature of an obligation, e.g. performing the Śyena. A command like $\mathcal{O}(\neg\texttt{harm}/\top)$, which is meant to apply to all who desire positive karman (i.e. everyone), is intrinsically different from an obligation with conditions like $\mathcal{O}(\texttt{Śy}/\texttt{des\_kill})$.

However, the restricted form of Factual Detachment which allows to derive the conclusion $\mathcal{O}(\varphi/\top)$ from the premisses $\mathcal{O}(\varphi/\psi)$ and $\Box\psi$, proposed in [65] and more explicitly in [61], would not give rise to inconsistencies in bMDL. Indeed, as will be more easily observed in the section dedicated to the semantics for bMDL, a possible state of affairs cannot be "deontically adequate" from the point of view of another state w, if it is not coherent with what is necessarily true in w. Hence, the reason why we do not add this principle to the system bMDL depends again in the choice of extracting all the characteristics of the logic from Mīmāṃsā texts, trying to avoid the inclusion of elements we did not find in those texts.

The principles of Factual and Deontic Detachment have been extensively discussed in deontic logic literature (e.g., among others, in [2, 106, 111]), as they are involved in the debate on dilemmas (e.g. in [28] and [40]) caused by the so-called Contrary-To-Duty obligations; these are prescriptions which are dependent on the violation of other injunctions. Indeed, the controversial issue, as for the previous formulation of the Śyena controversy, arises only when the obligations are considered "at the moment of their application", as detached from their conditions.

The position of the Mīmāṃsā school with respect to the problem of Contrary-To-Duty obligations will be briefly recalled in Section 4.5, with reference to the extensive discussion in the deontic literature about the difference between Contrary-To-Duty obligations (that do not cancel the original "ideal-state" obligations) and prioritized obligations (which properly overrule the weaker ones)[5].

[5] See e.g. [133, 134, 11] and [14, 120] for the problem of prioritizing rules in presence of Factual Detachment

3.3 The logic MD: □-free Fragment of bMDL

Here we show how to avoid the use of □ for characterizing assumptions in bMDL (see Rmk.3.1.3) by using (global) propositional assumptions for representing the propositional facts. Semantically, this means that instead of considering, for each w in the set W of all possible worlds, the set $R[w]$ of states accessible from w, we give some statements (the propositional assumptions) which are true in every possible world in W. From a general perspective, using assumptions rather than boxed formulas is tantamount to reasoning about factual reality instead of the more complex concept of necessary truth. Indeed, this is not only a welcomed simplification of the system (no need of axioms and rules for the alethic modality □), but it is also consistent with Mīmāṃsā authors' thought (see Rmk.3.1.3), as they did not consider the concept of logical necessity in the context of their deontic reasoning.

The □-free fragment of bMDL has been proved in [48] to coincide with the dyadic version of the logic MD (the smallest classical system of modal logic with a rule for monotonicity and the axiom D, called EMD in [27]), which is defined by extending a suitable axiomatic basis for classical propositional logic with the following modal axioms and rule:

$$(\mathsf{Mon}_{\mathcal{O}})\ \ \mathcal{O}(\varphi \wedge \psi/\theta) \to \mathcal{O}(\varphi/\theta) \qquad (\mathsf{D}_{\mathcal{O}})\ \ \neg(\mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\neg\varphi/\psi)) \qquad \frac{\varphi \leftrightarrow \theta \quad \psi \leftrightarrow \chi}{\mathcal{O}(\varphi/\psi) \to \mathcal{O}(\theta/\chi)}\ \mathsf{Cg}$$

The proof in [48] uses the logic's semantics; we present below a simpler proof which uses proof theory.

The calculus for the dyadic version of the deontic logic MD is given below in Fig.3.5.

$$\frac{}{p \Rightarrow p}\ \mathsf{init} \qquad \frac{}{\bot \Rightarrow}\ \bot_L \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{Con}_R \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R$$

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi)}\ \mathsf{Mon}_{\mathcal{O}} \qquad \frac{\varphi, \theta \Rightarrow \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow}\ \mathsf{D}_{\mathcal{O}} \qquad \frac{\varphi \Rightarrow}{\mathcal{O}(\varphi/\psi) \Rightarrow}\ \mathsf{P}_{\mathcal{O}}$$

Figure 3.5: The sequent calculus G_{MD} for MD.

Soundness and completeness of the calculus G_{MD} with respect to the logic MD follow by the method in [80] for extracting cut-free sequent calculi from Hilbert axioms.

The following theorem from [32] shows that a sequent which does not contain any boxed formula is derivable in bMDL iff it is derivable in MD. Hence (dyadic) MD turns out to be the $\Box$-free fragment of bMDL.

Theorem 3.3.1 *Given a sequent $\Gamma \Rightarrow \Delta$ which does not contain $\Box$, $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathsf{G_{MD}}} \Gamma \Rightarrow \Delta$.*

Proof. The right-to-left direction is trivial as the rules of $\mathsf{G_{MD}}$ are already instances of $\mathsf{G_{bMDL}}$ rules, except for Weakening and Contraction rules, which have been shown to be admissible in $\mathsf{G_{bMDL}}$ (4.3.22). The other direction follows since any application of a rule of $\mathsf{G_{bMDL}}$ whose main formula does not contain $\Box$ can be simulated by the corresponding rules of $\mathsf{G_{MD}}$ and the Weakening rules. $\square$

The semantic proof of the equivalence of the $\Box$-free fragment of bMDL and (the dyadic version of) MD in [48] considers the following structures:

Definition 3.3.2 (Model for MD) *A frame for the system MD is a couple $(W, \mathcal{N}_{\mathsf{MD}})$, where W is a non-empty set of possible worlds and $\mathcal{N}_{\mathsf{MD}} : W \to \wp(\wp(W) \times \wp(W))$ is a neighbourhood function such that*

1. *if $(X, Z) \in \mathcal{N}_{\mathsf{MD}}(w)$ and $X \subseteq Y \subseteq W$, then also $(Y, Z) \in \mathcal{N}_{\mathsf{MD}}(w)$;*
2. *if $(X, Y) \in \mathcal{N}_{\mathsf{MD}}(w)$, then $(X^c, Y) \notin \mathcal{N}_{\mathsf{MD}}(w)$ (where X^c is the relative complement of the set X with respect to W);*

A model over such a frame is a structure $\mathfrak{M} = (W, \mathcal{N}_{\mathsf{MD}}, \sigma)$, where $\sigma : \mathsf{Var} \to \wp(W)$ is a valuation function.

As shown in [27] and [48], the dyadic version of MD is complete with respect to the class of neighbourhood frames in the previous definition.

Definition 3.3.3 (Truth sets) *Let $\mathfrak{M} = (W, \mathcal{N}_{\mathsf{MD}}, \sigma)$ be a model as defined in Def.3.3.2. For any formula φ, its truth set $[\![\varphi]\!]_{\mathfrak{M}}$ in $\mathfrak{M}$ is defined (recursively) by the conditions*

1. $[\![p]\!]_{\mathfrak{M}} := \sigma(p)$
2. $[\![\mathcal{O}(\varphi/\psi)]\!]_{\mathfrak{M}} := \{w \in W \mid ([\![\varphi]\!]_{\mathfrak{M}}, [\![\psi]\!]_{\mathfrak{M}}) \in \mathcal{N}_{\mathsf{MD}}(w)\}$

together with the standard conditions for propositional connectives. In symbols, a formula φ is satisfied in a world w ($\mathfrak{M}, w \Vdash \varphi$) iff $w \in [\![\varphi]\!]_{\mathfrak{M}}$, and φ is valid in a model $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash \varphi$ for all worlds w of $\mathfrak{M}$.

In order to prove that a model over a frame for MD can be converted in a m-model, we need to define a translation function that maps each formula of the language $\mathcal{L}_{\mathsf{MD}}$ for MD to a formula of $\mathcal{L}_{\mathsf{bMDL}}$. For this purpose we use the simple *identity* function $I(.)$, such that

$I(p) = p$ for any propositional variable $p \in \mathsf{Var}$;

$I(\bot) = \bot$;

$I(\varphi \to \psi) = I(\varphi) \to I(\psi)$;

$I(\mathcal{O}(\varphi/\psi)) = \mathcal{O}(I(\varphi)/I(\psi))$.

Since for any formula φ, $I(\varphi) = \varphi$, we could avoid indicating the application of the identity function; however, in what follows the notation $I(\varphi)$ is maintained with the aim of making it easier to distinguish the logic we are referring to (MD or bMDL).

Given a model $\mathfrak{M} = (W, \mathcal{N}_{\mathsf{MD}}, \sigma)$ over a frame for MD, let us define a model $\mathfrak{M}^* = (W^*, R^*, \mathcal{N}^*, \sigma^*)$ where $W^* = W$, $R^* = W^* \times W^*$, $\sigma^* = \sigma$, and, for any world $w \in W^*$, $\mathcal{N}^*(w)$ is the set of all pairs $([\![\varphi]\!]_{\mathfrak{M}^*}, [\![\psi]\!]_{\mathfrak{M}^*})$ such that $\varphi, \psi \in \mathcal{L}_{\mathsf{bMDL}}$, and there are $\theta, \xi \in \mathcal{L}_{\mathsf{MD}}$ s.t. $([\![\theta]\!]_{\mathfrak{M}}, [\![\xi]\!]_{\mathfrak{M}}) \in \mathcal{N}_{\mathsf{MD}}(w)$, $[\![I(\theta)]\!]_{\mathfrak{M}^*} \subseteq [\![\varphi]\!]_{\mathfrak{M}^*}$ and $[\![I(\xi)]\!] = [\![\psi]\!]$.

Following [48], it is now shown that a model $\mathfrak{M} = (W, \mathcal{N}_{\mathsf{MD}}, \sigma)$ as defined in Def.3.3.2 can always be converted in a m-model as in Def.3.1.14.

In particular, this will prove that, if there is a model over a frame for the system MD which falsifies a formula φ, then a corresponding formula $I(\varphi)$ from the language of bMDL is falsifiable in a m-model.

First, we prove that the model $\mathfrak{M}^*$ defined above is a m-model (Def.3.1.14).

Theorem 3.3.4 $\mathfrak{M}^* = (W^*, R^*, \mathcal{N}^*, \sigma^*)$ *is a m-model.*

Proof. We prove the claim by showing that the model $\mathfrak{M}^* = (W^*, R^*, \mathcal{N}^*, \sigma^*)$ satisfies the properties in (Def.3.1.14).

- (Property 1 of Def.3.1.14)

  Being R^* the universal relation, it is transitive and reflexive.

- (Property 2 of Def.3.1.14)

  For any $w \in W^*$ the set $R^*[w] = \{v \in W^* \mid wR^*v\}$ is equal to W^*, therefore, since $\mathcal{N}^*$ is defined on W^*, the property 2 of Def.3.1.14 is trivial, and for any $X \subseteq W$ the additional condition $X \subseteq R^*[w]$ can be ignored.

- (Property 3 of Def.3.1.14)

The fact that, if $(X,Y) \in \mathcal{N}^*(w)$ and $X \subseteq Z$, then also $(Z,Y) \in \mathcal{N}^*(w)$ is proved by contraposition. Let us assume $(Z,Y) \notin \mathcal{N}^*(w)$ and $X \subseteq Z$. Thus, we have $(U,Q) \notin \mathcal{N}_{\mathsf{MD}}(w)$ for any couple of sets U,Q such that $Q = Y$ and $U \subseteq Z$; then in particular, consider $U = X$. Hence, by definition of $\mathcal{N}^*$, $(U,Q) \notin \mathcal{N}^*(w)$, therefore $(X,Y) \notin \mathcal{N}^*(w)$.

- (Property 4 of Def.3.1.14)

  Let us assume $(X,Y) \in \mathcal{N}^*(w)$, then $(Z,Q) \in \mathcal{N}_{\mathsf{MD}}(w)$, for some truth sets Z,Q such that $Q = Y$, and $Z \subseteq X$. Therefore, by 2 of Def.3.3.2, $(Z^c,Q) \notin \mathcal{N}_{\mathsf{MD}}(w)$. Then, since $X^c \subseteq Z^c$ follows from $Z \subseteq X$, by the properties proved above, $(X^c,Y) \notin \mathcal{N}^*(w)$.

- (Property 5 of Def.3.1.14)

  $(\varnothing,Y) \notin \mathcal{N}^*(w)$ can be proved by using a *reductio ad absurdum* argument: let us assume $(\varnothing,Y) \in \mathcal{N}^*(w)$, then $(\varnothing,Y) \in \mathcal{N}_{\mathsf{MD}}(w)$. From this, by condition 2 of Def.3.3.2, $(\varnothing^c,Y) \notin \mathcal{N}_{\mathsf{MD}}(w)$, i.e. $(W,Y) \notin \mathcal{N}_{\mathsf{MD}}(w)$. But, by condition 1 of Def.3.3.2, $(X,Y) \in \mathcal{N}_{\mathsf{MD}}(w)$ for any X such that $\varnothing \subseteq X$, so in particular $(W,Y) \in \mathcal{N}_{\mathsf{MD}}(w)$. Hence, the two consequences $(W,Y) \notin \mathcal{N}_{\mathsf{MD}}(w)$ and $(W,Y) \in \mathcal{N}_{\mathsf{MD}}(w)$ give a contradiction and $(\varnothing,Y) \notin \mathcal{N}^*(w)$ is shown to be true.

$\square$

It is now shown that, for any formula $\varphi \in \mathcal{L}_{\mathsf{MD}}$, its evaluation in the model $\mathfrak{M}$ is the same as the evaluation of its translation $I(\varphi) \in \mathcal{L}_{\mathsf{bMDL}}$ in the model $\mathfrak{M}^*$. This means that, if a formula $\varphi \in \mathcal{L}_{\mathsf{MD}}$ is true (resp. false) at some world w in $\mathfrak{M}$, then $I(\varphi)$ is true (resp. false) at the same world in $\mathfrak{M}^*$.

Theorem 3.3.5 $\mathfrak{M}, w \Vdash \varphi$ *iff* $\mathfrak{M}^*, w \Vdash I(\varphi)$ *for any formula* $\varphi \in \mathcal{L}_{\mathsf{MD}}$ *and world* $w \in W$.

Proof. The theorem is proved by induction on the complexity of the formula $\varphi \in \mathcal{L}_{\mathsf{MD}}$. Since $\sigma^* = \sigma$, the case of $\varphi = p$ with $p \in \mathsf{Var}$ is straightforward; moreover, given the standard conditions of validity for connectives, the only non-trivial case is $\varphi = \mathcal{O}(\chi/\psi)$. Given $\mathfrak{M}, w \Vdash \mathcal{O}(\chi/\psi)$, i.e. $([\![\chi]\!]_{\mathfrak{M}}, [\![\psi]\!]_{\mathfrak{M}}) \in \mathcal{N}_{\mathsf{MD}}(w)$, $([\![I(\chi)]\!]_{\mathfrak{M}^*}, [\![I(\psi)]\!]_{\mathfrak{M}^*}) \in \mathcal{N}^*(w)$ follows by definition of $\mathcal{N}^*(w)$. Hence, $\mathfrak{M}^*, w \Vdash \mathcal{O}(I(\chi)/I(\psi))$ and, by the properties of the translation function t, $\mathfrak{M}^*, w \Vdash I(\mathcal{O}(\chi/\psi))$. For the other direction, let us assume $\mathfrak{M}, w \nVdash \mathcal{O}(\chi/\psi)$, i.e. $([\![\chi]\!]_{\mathfrak{M}}, [\![\psi]\!]_{\mathfrak{M}}) \notin \mathcal{N}_{\mathsf{MD}}(w)$. Hence, by condition 1 of Def.3.3.2, for any $\xi \in \mathcal{L}_{\mathsf{MD}}$ s.t. $[\![\xi]\!]_{\mathfrak{M}} \subseteq [\![\chi]\!]_{\mathfrak{M}}$, $([\![\xi]\!]_{\mathfrak{M}}, [\![\psi]\!]_{\mathfrak{M}}) \notin \mathcal{N}_{\mathsf{MD}}(w)$. Then, by definition of $\mathcal{N}^*(w)$, $([\![I(\chi)]\!]_{\mathfrak{M}^*}, [\![I(\psi)]\!]_{\mathfrak{M}^*}) \notin \mathcal{N}^*(w)$, i.e. $\mathfrak{M}^*, w \nVdash \mathcal{O}(I(\chi)/I(\psi))$ and therefore $\mathfrak{M}^*, w \nVdash I(\mathcal{O}(\chi/\psi))$. $\square$

Theorem 3.3.5 shows that a model $\mathfrak{M} = (W, \mathcal{N}_{\mathsf{MD}}, \sigma)$ over a frame for the logic MD can be converted in $\mathfrak{M}^* = (W^*, R^*, \mathcal{N}^*, \sigma^*)$ previously defined. Hence, together with Thm.3.3.4, it shows that a model over a frame for the system MD which satisfies (or falsifies) a formula φ can always be converted into a m-model satisfying (resp. falsifying) the corresponding formula $I(\varphi)$. This represents the semantic version of the (syntactic) theorem 3.3.1, proving that the models for MD constitute a proper subset of the m-models, in particular models for MD are equivalent to those m-models where $R = W \times W$.

To sum up, the equivalence between the dyadic version of the logic MD and the $\Box$-free fragment of bMDL has been proved both syntactically (Thm.3.3.1) and semantically (Lem.3.3.4, Thm.3.3.5).

3.4 The System MD+

The logic bMDL represents only a first step towards the formal representation of Mīmāṃsaka's interpretation of Vedic commands. As mentioned in Ch.2 Section 2.3, in Mīmāṃsā deontic concepts like prohibitions and recommendations are not derived notions, i.e., similarly to the situation in Talmudic logic as investigated in [1], they cannot be defined on the basis of obligations. On a "meta-logical" level, they differ in strength, expected sanctions and rewards, and triggering factors. Even if those aspects do not correspond to specific formal elements in the logic, they play a key role in defining the interactions between different kinds of commands; for instance, a ritual action and its opposite cannot be obligatory under the same conditions, but it is possible that one is obligatory and the other just recommended (prescribed as an elective sacrifice). The operators presented here for prohibitions and recommendations are specifically targeted at formalizing Mīmāṃsā deontic reasoning; however, operators with similar properties could be applied in other contexts of normative reasoning. For instance, let us consider the use of deontic operators for obligations and prohibitions in comparing moral and legal duties (see Ex.4.4.11), in line with the argument for using deontic notions in the formalization of legal texts (see [74, 115]). For this reason, while the properties of the operator $\mathcal{O}$ of MD make it suitable for describing the deontic concept connected with fixed and occasional sacrifices, different operators are needed for expressing proper prohibitions and recommendations –the weaker form of obligation at the basis of *kāmya-karman* rituals, from the point of view of Kumārila's scholars. In order to give an account of the differences between those deontic concepts, we improve the formal analysis of Mīmāṃsā reasoning by extending the logic MD (i.e. the $\Box$-free fragment of bMDL) with new operators for recommendations and prohibitions; the logic thus obtained is called $\mathsf{MD+}$.

The new deontic operators $\mathcal{F}(\cdot/\cdot)$ for prohibitions and $\mathcal{R}(\cdot/\cdot)$ for recommendations are characterized by the following axioms and rules, which, as in the case of operator $\mathcal{O}$, reflect as much as possible the principles found in Mīmāṃsā texts.

($\mathsf{D}_{\mathcal{F}}$) $\neg(\mathcal{F}(\varphi/\psi) \wedge \mathcal{F}(\neg\varphi/\psi))$

($\mathsf{Mon}_{\mathcal{F}}$) *if* $\vdash_{\mathsf{MD+}} \theta \rightarrow \varphi$ *and* $\vdash_{\mathsf{MD+}} \psi \leftrightarrow \chi$ *then* $\vdash_{\mathsf{MD+}} \mathcal{F}(\varphi/\psi) \rightarrow \mathcal{F}(\theta/\chi)$

($\mathsf{D}_{\mathcal{O}\mathcal{F}}$) $\neg(\mathcal{O}(\varphi/\psi) \wedge \mathcal{F}(\varphi/\psi))$

($\mathsf{Mon}_{\mathcal{R}}$) *if* $\vdash_{\mathsf{MD+}} \varphi \rightarrow \theta$ *and* $\vdash_{\mathsf{MD+}} \psi \leftrightarrow \chi$ *then* $\vdash_{\mathsf{MD+}} \mathcal{R}(\varphi/\psi) \rightarrow \mathcal{R}(\theta/\chi)$

($\mathsf{P}_{\mathcal{R}}$) $\neg\mathcal{R}(\bot/\varphi)$

Before giving a brief description of the axioms, it should be noted that the concepts of recommendations and prohibitions are better expressed by dyadic operators. In the case of recommendations, the reason is immediately clear, as the eligibility conditions for elective sacrifices include not only possible occasions and a generic desire for happiness, but also a specific desire for a particular new state of affairs. For what concerns $\mathcal{F}(\cdot/\cdot)$ the choice could seem to have weaker justifications, as prohibitions do not depend on the concept of *adhikāra* and their targets are not identified on the base of desires. However, as already noted, a sentence which seems to forbid an action is not interpreted as a prohibition unless there is a *prāpti*, i.e. the already obtained cognition of a reason to act in the opposite way; then the second argument ψ in and $\mathcal{F}(\varphi/\psi)$ is needed for expressing the *prāpti* and a specific situation in the case of contextual (*kratvartha*) prohibitions.

The first axiom ($\mathsf{D}_{\mathcal{F}}$) ($\neg(\mathcal{F}(\varphi/\psi) \wedge \mathcal{F}(\neg\varphi/\psi))$) then expresses the principle that two prohibitions having the same *prāpti* cannot forbid an action and its negation in the same situation. This axiom formalizes the *meaningfulness* nyāya stating that no command in the Sacred Texts can be unenforceable; since having the prohibition of an action and its negation under the same conditions would necessarily entail a sanction, this situation would make the compliance with one of the rules ineffective, as it does not avoid the accumulation of negative *karman*.

The rule ($\mathsf{Mon}_{\mathcal{F}}$) (if $\vdash_{\mathsf{MD+}} \theta \rightarrow \varphi$ and $\vdash_{\mathsf{MD+}} \psi \leftrightarrow \chi$ then $\vdash_{\mathsf{MD+}} \mathcal{F}(\varphi/\psi) \rightarrow \mathcal{F}(\theta/\chi)$), corresponding to the property of downward monotonicity, is motivated by many examples of reasoning in Mīmāṃsā texts, e.g. the argument in Medhātithi's Manubhāṣya, (c. 825–1000), which seems to use the general prohibition to commit violence, together with the fact that suicide is a form of violence (suicide → self-violence and self-violence → violence), for justifying the derived prohibition to commit suicide.

Remark 3.4.1 *It should be noted that all operators are monotonic in their first arguments, as the possibility of involving or being composed by other actions represents an intrinsic property of ritual acts; the same cannot be said for the second arguments, which, if monotonic, would exclude any chance of admitting exceptions to the rules.*

The third axiom ($\mathsf{D}_{\mathcal{OF}}$) expresses another consequence of the *meaningfulness* nyāya together with the so-called "ought implies can principle", discussed in sections about the concept of adhikāra in Jaimini's texts, and stating that, when an agent is compelled to follow a prescription, he is supposed to have the capacities and the possibility to do so in practice without unwelcome consequences, like damages or sanctions. As a consequence, carrying out an obligatory action cannot give as a result a sanction, i.e. the same act cannot be forbidden. Moreover, from the argumentations in PMS 10.8.6, the consideration arises that if something in the texts seems to be both the object of a prohibition and of a prescription with the same conditions, the commands should be reinterpreted or, in choosing what actually "has to be done", the act should be considered neither forbidden nor obligatory.

($\mathsf{Mon}_{\mathcal{R}}$), expresses the property of (upward) monotonicity for the operator $\mathcal{R}$, which, as mentioned, represents the deontic concept connected with elective sacrifices. This characteristic is justified by considerations similar to the one cited as explanation for the axiom 1 of bMDL. These considerations also adapt to the deontic concept of recommendation because, as noticed in [48], they are more about Mīmāṃsakas' conception of relations among ritual actions than about a specific kind of command.

The last axiom ($\mathsf{P}_{\mathcal{R}}$) ensures that recommendations cannot be self-contradictory: this represents the minimal requirement for Vedic commands and another form of the *meaningfulness* nyāya, as prescribing an absurd (elective) ritual would make the recommendation impossible to follow, therefore useless and meaningless. $\mathsf{P}_{\mathcal{R}}$ constitutes a much weaker requirement with respect to the axioms $\mathsf{D}_{\mathcal{O}}$ and $\mathsf{D}_{\mathcal{F}}$, which guarantee that there are no conflicts in the whole sets of obligations and prohibitions. However, this choice is motivated by the fact that, in contrast to the other kinds of commands, there are recommendations for different elective rituals meant to give the same desired result and such that the performance of one excludes the other, e.g. the case of *kāriri* and *twelve-nights* rituals, both for obtaining the rain (see [48]). In those cases Mīmāṃsakas seem to assume that the recommendations are both in force and the agent can choose, as performing one sacrifice is sufficient for the achievement of the desired goal.

Finally, it should be noted that the concept of permission, introduced in Ch.2, does not correspond to an explicit operator in MD+: this reflects the already mentioned idea that permissions, linguistically similar to prescriptions, but conveying actions that the agents are naturally inclined to do, should be read only as exception to other commands. However, it will be shown that they are formalized on a different level, where, unlike the formulas of MD+, they do not represent "what an agent actually should do", but they express (interpreted) explicit commands in the Vedas.

3.4.1 Proof Theory

The sequent calculus $\mathsf{G}_{\mathsf{MD+}}$ (in Fig. 3.4.1) for the logic MD extended with the axioms for the new operators $\mathcal{F}(\cdot/\cdot)$ and $\mathcal{R}(\cdot/\cdot)$ is again obtained by using the method in [80] for transforming the modal axioms into sequent rules.

$$\frac{}{p \Rightarrow p}\ \mathsf{init} \qquad \frac{}{\bot \Rightarrow}\ \bot_L \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi)}\ \mathsf{Mon}_{\mathcal{O}} \qquad \frac{\varphi, \theta \Rightarrow \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow}\ \mathsf{D}_{\mathcal{O}}$$

$$\frac{\theta \Rightarrow \varphi \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{F}(\varphi/\psi) \Rightarrow \mathcal{F}(\theta/\chi)}\ \mathsf{Mon}_{\mathcal{F}} \qquad \frac{\Rightarrow \varphi, \psi \qquad \theta \Rightarrow \chi \quad \chi \Rightarrow \theta}{\mathcal{F}(\varphi/\theta), \mathcal{F}(\psi/\chi) \Rightarrow}\ \mathsf{D}_{\mathcal{F}}$$

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\theta/\chi)}\ \mathsf{Mon}_{\mathcal{R}} \qquad \frac{\varphi \Rightarrow}{\mathcal{R}(\varphi/\psi) \Rightarrow}\ \mathsf{P}_{\mathcal{R}} \qquad \frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow}\ \mathsf{D}_{\mathcal{OF}}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{Con}_R$$

Figure 3.6: The calculus $\mathsf{G}_{\mathsf{MD+}}$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ \neg_L \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ \neg_R$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \vee_L \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ \vee_R \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \wedge_L \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ \wedge_R$$

Figure 3.7: The derived rules for the $\neg, \vee, \wedge$ according to the definition of those connectives in terms of $\bot, \to$: $(\neg\varphi = \varphi \to \bot)$, $(\varphi \vee \psi = \neg\varphi \to \psi)$, $(\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi))$.

Note that the resulting sequent calculus $\mathsf{G}_{\mathsf{MD+}}$ admits cut-elimination by construction (see [80]); therefore the subformula property holds for $\mathsf{MD+}$ and this shows that the logic is consistent.

However, to build later a proof search procedure for $\mathsf{G}_{\mathsf{MD+}}$, which will be used in proving semantic completeness, the calculus needs to be modified in such a way that it supports backwards proof search and, in particular, countermodel construction. For this purpose, we adopt the hypersequent (Def.3.4.2) version $\mathsf{G}^*_{\mathsf{MD+}}$ (in Fig.3.8) of the calculus $\mathsf{G}_{\mathsf{MD+}}$. As this calculus is introduced to simplify later the countermodel construction, it has not been used before in [32], where the logic $\mathsf{MD+}$ is presented only syntactically.

The *Hypersequent* framework ([9, 10, 108]) is an extension of the sequent one, which allows us to express properties not captured in the sequent framework.

Definition 3.4.2 (Hypersequents) *A hypersequent* $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | ... | \Gamma_n \Rightarrow \Delta_n$ *is a finite multiset of classical sequents, called* components *of* $\mathcal{G}$*: intuitively* $\mathcal{G}$ *is read as the disjunction of its components.*

Given the following instance of a hypersequent rule R

$$\frac{\mathcal{G} \mid \mathcal{S}_1 \quad \mathcal{G} \mid \mathcal{S}_2}{\mathcal{G} \mid \mathcal{C}}\ \mathsf{R}$$

we call the sequents $\mathcal{S}_1, \mathcal{S}_2$ *and* $\mathcal{C}$ *the* active *components (of the premisses and of the conclusion, respectively) and we call and the sequents in* $\mathcal{G}$ *the* context *components.*

In our case, the use of hypersequents is not intended to capture systems of logic whose characteristics cannot be expressed by sequent calculi. Instead, following [36], we use a hypersequent calculus in building an algorithm for the proof search procedure and in countermodel construction. Indeed, though proof search can be performed using sequents, the use of a hypersequent framework allows the proof search procedure to more easily keep track of the analysed sequents, avoiding loops and infinite branches in the search tree.

Note that the structural rules of (internal) weakening and (internal) contraction are absorbed into the rules of the calculus: besides the addition of the context in the zero-premisses rules and the changes in the propositional rules ($\rightarrow_L$) and ($\rightarrow_R$), the rules $\mathsf{P}_{\mathcal{O}}$ and $\mathsf{P}_{\mathcal{F}}$ have been included for absorbing the contraction of the principal formulas of $\mathsf{D}_{\mathcal{O}}$ and $\mathsf{D}_{\mathcal{F}}$, respectively.

For proving that the calculus $\mathsf{G}^*_{\mathsf{MD+}}$ (in Fig.3.8) is equivalent to $\mathsf{G}_{\mathsf{MD+}}$, we first show that the structural rules for internal and external weakening and contraction (see Fig.3.4.1) are admissible in this system; intuitively, this also guarantees that we can restrict the proof

$$\frac{}{\mathcal{G} \mid \Gamma, p \Rightarrow p, \Delta}\ \mathsf{init} \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}\ \bot_L$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi \to \psi, \psi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

$$\frac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta \mid \varphi \Rightarrow \theta \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta \mid \psi \Rightarrow \chi \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta \mid \chi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{O}} \qquad \frac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta \mid \varphi, \theta \Rightarrow \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta \mid \psi \Rightarrow \chi \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta \mid \chi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{O}}$$

$$\frac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \mathcal{F}(\theta/\chi), \Delta \mid \theta \Rightarrow \varphi \\ \mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \mathcal{F}(\theta/\chi), \Delta \mid \psi \Rightarrow \chi \\ \mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \mathcal{F}(\theta/\chi), \Delta \mid \chi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \mathcal{F}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{F}} \qquad \frac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta \mid \ \Rightarrow \varphi, \theta \\ \mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta \mid \psi \Rightarrow \chi \\ \mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta \mid \chi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{F}}$$

$$\frac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta \mid \varphi \Rightarrow \theta \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta \mid \psi \Rightarrow \chi \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta \mid \chi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{OF}} \qquad \frac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\theta/\chi), \Delta \mid \varphi \Rightarrow \theta \\ \mathcal{G} \mid \Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\theta/\chi), \Delta \mid \psi \Rightarrow \chi \\ \mathcal{G} \mid \Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\theta/\chi), \Delta \mid \chi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{R}}$$

$$\frac{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \varphi \Rightarrow}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{O}} \qquad \frac{\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \Delta \mid \ \Rightarrow \varphi}{\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{F}} \qquad \frac{\mathcal{G} \mid \Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \Delta \mid \varphi \Rightarrow}{\mathcal{G} \mid \Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{R}}$$

Figure 3.8: The calculus $\mathsf{G}^*_{\mathsf{MD+}}$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R \quad \frac{\mathcal{G} \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \varphi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R$$

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ E\mathsf{W} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ E\mathsf{Con}$$

Figure 3.9: Hypersequent rules for internal and external weakening and contraction

search to *sets* of *set-based* sequents.

Lemma 3.4.3 (Admissibility of internal weakening) *The (hypersequent version of the) rules* (W_L) *and* (W_R) *for weakening are height-preserving admissible in* $\mathsf{G}^*_{\mathsf{MD+}}$*, i.e. if a hypersequent* $\mathcal{G}$ *is derivable in* $\mathsf{G}^*_{\mathsf{MD+}}$ *with the rules for internal weakening, then it is derivable in the same number of steps without those rules.*

Proof. We want to prove that for any derivation

$$\dfrac{\mathfrak{D}\left\{\dfrac{\vdots}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\right.}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \quad \text{or} \quad \dfrac{\mathfrak{D}\left\{\dfrac{\vdots}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\right.}{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R$$

there is a derivation

$$\mathfrak{D}'\left\{\dfrac{\vdots}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\right. \qquad \text{or} \qquad \mathfrak{D}'\left\{\dfrac{\vdots}{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}\right.$$

such that $\mathfrak{D}'$ has at most the same height as $\mathfrak{D}$.

This is proved by induction on the height of the derivation. For the base cases, if the last applied rule is (init) or $(\bot_L)$, since (W_L) and (W_R) are absorbed into the initial rules, the application of (W_L) or (W_R) can be removed, without adding any new rule's application, as in the following example:

$$\dfrac{\overline{\mathcal{G} \mid \Gamma, p \Rightarrow p, \Delta}\ \mathsf{init}}{\mathcal{G} \mid \Gamma, \varphi, p \Rightarrow p, \Delta}\ \mathsf{W}_L \quad \text{is transformed in} \quad \overline{\mathcal{G} \mid \Gamma, \varphi, p \Rightarrow p, \Delta}\ \mathsf{init}$$

For all the other rules we proceed by applying the induction hypothesis to the premiss(es) of last applied rule, followed by an application of the same rule. □

Lemma 3.4.4 (Admissibility of internal contraction) *The (hypersequent version of* *the) rules* (Con_L) *and* (Con_R) *for contraction are height-preserving admissible in* $\mathsf{G}^*_{\mathsf{MD+}}$.

Proof. For admissibility of contraction rules, again we proceed by induction on the height of the derivation. The base cases where the last applied rule is (init) or $(\bot)$ are trivial. If the last applied rule is $(\to_L)$ or $(\to_R)$ we apply the induction hypothesis to the premiss(es) of the rule, followed by an application of the same rule. If the last applied rule is a modal rule, we distinguish two cases: if at most one of the contracted formulas is principal, we proceed as above; if both the contracted formulas are principal -i.e. the last applied rule is $(\mathsf{D}_{\mathcal{O}})$ or $(\mathsf{D}_{\mathcal{F}})$- we have a derivation ending in:

$$\dfrac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \varphi, \varphi \Rightarrow \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \psi \Rightarrow \psi \\ \mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \psi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{O}} \quad \text{or} \quad \dfrac{\begin{array}{c}\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\varphi/\psi) \Rightarrow \Delta \mid \ \Rightarrow \varphi, \varphi \\ \mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\varphi/\psi) \Rightarrow \Delta \mid \psi \Rightarrow \psi \\ \mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\varphi/\psi) \Rightarrow \Delta \mid \psi \Rightarrow \psi\end{array}}{\mathcal{G} \mid \Gamma, \mathcal{F}(\varphi/\psi), \mathcal{F}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{F}}$$

In the first case (on the left), we apply the induction hypothesis twice to the first premiss, followed by an application of the rule $(\mathsf{P}_{\mathcal{O}})$:

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \varphi, \varphi \Rightarrow}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \varphi, \varphi \Rightarrow}\ I.H.}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \varphi, \Rightarrow}\ I.H.}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{O}}$$

The case of $(\mathsf{D}_{\mathcal{F}})$ is analogous. □

Lemma 3.4.5 (Admissibility of external weakening and external contraction) *The rules* $(E\mathsf{W})$ *for external weakening and* $(E\mathsf{Con})$ *for external contraction are height-preserving admissible in* $\mathsf{G}^*_{\mathsf{MD+}}$.

Proof. Again, we prove the lemma by induction on the height of the derivation. In the case of $(E\mathsf{W})$ the claim is simply proved by applying the induction hypothesis to the premiss(es) of the last applied rule, followed by an application of the same rule.

For proving the admissibility of external contraction rule, we distinguish two cases, depending on the last applied rule before the application of $(E\mathsf{Con})$. If the last applied rule is a modal rule, again we apply the induction hypothesis to the premiss(es) of this modal rule, followed by an application of the same modal rule.

In case the last applied rule is $(\to_L)$, we have a derivation ending in:

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta \mid \Gamma, \varphi \to \psi, \psi \Rightarrow \Delta \qquad \mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta \mid \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \to_L}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} E\mathsf{Con}$$

Therefore, we need to use the (already proved height-preserving admissible) rules (W_L) and (W_R) as follows:

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \Gamma, \varphi \to \psi, \psi \Rightarrow \Delta \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi, \psi \Rightarrow \Delta \mid \Gamma, \varphi \to \psi, \psi \Rightarrow \Delta} \mathsf{W}_L}{\mathcal{G} \mid \Gamma, \varphi \to \psi, \psi \Rightarrow \Delta} IH \qquad \dfrac{\dfrac{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta \mid \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta} \mathsf{W}_R}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta} IH}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \to_L$$

The case for $(\to_R)$ is similar:

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta \mid \Gamma \Rightarrow \varphi \to \psi, \Delta} \to_R}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta} E\mathsf{Con}$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta \mid \Gamma, \varphi \Rightarrow \varphi \to \psi, \Delta} \mathsf{W}_L}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta} \mathsf{W}_R}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \varphi \to \psi, \Delta} IH}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta} \to_R$$

□

Moreover, as shown by the following lemma, the rules for internal and external weakening and contraction can be used for proving that the rules of $\mathsf{G}^*_{\mathsf{MD+}}$ are *invertible*, i.e. if the conclusion of a rule's application is derivable, then the premiss(es) of the same application are derivable. Intuitively, this property guarantees that the order in which the rules are applied in a derivation does not affect the derivability of the sequent at the root.

Lemma 3.4.6 (The rules of $\mathsf{G}^*_{\mathsf{MD+}}$ are invertible) *For any rule of $\mathsf{G}^*_{\mathsf{MD+}}$, if a hypersequent instantiating the conclusion of a rule is derivable, then the hypersequents instantiating* *its premiss(es) are derivable.*

Proof. The claim of the lemma can be proved by using the admissible rules (W_L), (W_R), $(E\mathsf{W})$ (Lem.3.4.3, Lem.3.4.4).

For example, from the conclusion of $(\mathsf{P}_\mathcal{O})$, its premiss can be obtained by $(E\mathsf{W})$, and from the conclusion of $(\rightarrow_R)$, its premiss can be obtained by (W_L), (W_R):

$$\dfrac{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta \mid \varphi \Rightarrow}\ E\mathsf{W} \qquad \dfrac{\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi \rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \varphi \rightarrow \psi, \Delta}\ \mathsf{W}_L}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \varphi \rightarrow \psi, \Delta}\ \mathsf{W}_R$$

□

Using the previous lemmas we can show that the intuitive translation of a hypersequent as the disjunction of its components corresponds to the formal interpretation in the sense that, when a hypersequent is derivable at least one of its component is separately and independently derivable.

Lemma 3.4.7 *If* $\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m$*, then there is a derivation of* $\Gamma_i \Rightarrow \Delta_i$ *for some* $1 \leq i \leq m$ *in the calculus* $\mathsf{G}^*_{\mathsf{MD+}}$ *and the height of this derivation is at most equal to* n.

Proof. We prove the claim by induction on the height of the derivation.

First, consider that all the rules in $\mathsf{G}^*_{\mathsf{MD+}}$ have only one active component in the conclusion and at most two active components in the premisses; moreover, when there is a second active sequent in a premiss (in case of modal rules), it represents a copy of the active component in the conclusion. Hence, since the order of the components does not count, let us write

$$\dfrac{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G} \mid \Gamma_m \Rightarrow \Delta_m \mid \Sigma^1 \Rightarrow \Pi^1 \quad \ldots \quad \vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G} \mid \Gamma_m \Rightarrow \Delta_m \mid \Sigma^\ell \Rightarrow \Pi^\ell}{\vdash^{n+1}_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G} \mid \Gamma_m \Rightarrow \Delta_m}\ R$$

for the last applied rule R with ℓ premisses, where $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_{m-1} \Rightarrow \Delta_{m-1}$ is the context, $\Gamma_m \Rightarrow \Delta_m$ is the active sequent in the conclusion, and $\Sigma^j \Rightarrow \Pi^j$ for each $1 \leq j \leq \ell$, possibly together with the copy $\Gamma_m \Rightarrow \Delta_m$, are the active components in the premisses.

For the base case, if R is init or $\bot_L$, the active component $\Gamma_m \Rightarrow \Delta_m$ of the conclusion is such that $\Gamma_m \bigcap \Delta_m \neq \varnothing$ or $\bot \in \Gamma_m$ and can be derived separately by applying the same zero-premisses rule.

If R is a rule with premisses, then, by induction hypothesis, for any premiss $\mathcal{G} \mid \Gamma_m \Rightarrow \Delta_m \mid \Sigma^j \Rightarrow \Pi^j$ with $1 \leq j \leq \ell$, there is one component which is independently derivable (with a derivation of height at most equal to n). If the derivable sequent in the premisses of R is in the context $\mathcal{G}$, or it is the copy $\Gamma_m \Rightarrow \Delta_m$, then the same derivable component is in the conclusion, verifying the claim.

If R is a propositional rule and no sequent in the context of the premisses is derivable, then (by induction hypothesis) for each premiss the active component is derivable; from $\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1 \ \ldots \ \vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell$, the principal component of the consequence is derivable (in $n+1$ steps) by applying the same propositional rule.

If R is a modal rule and the derivable sequent in the premisses of R is not in the context $\mathcal{G}$ and it is not the copy of the active sequent of the conclusion, then, by induction hypothesis, for each premiss the active component different from $\Gamma_m \Rightarrow \Delta_m$ is derivable. Therefore, we need to use the rule of External Weakening (proved height-preserving admissible in Lem.3.4.5):

$$
\dfrac{\dfrac{\dfrac{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G} \mid \Gamma_m \Rightarrow \Delta_m \mid \Sigma^1 \Rightarrow \Pi^1}{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1}\ I.H.}{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma_m \Rightarrow \Delta_m \mid \Sigma^1 \Rightarrow \Pi^1}\ E\mathsf{W} \quad \ldots \quad \dfrac{\dfrac{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G} \mid \Gamma_m \Rightarrow \Delta_m \mid \Sigma^\ell \Rightarrow \Pi^\ell}{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell}\ I.H.}{\vdash^n_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma_m \Rightarrow \Delta_m \mid \Sigma^\ell \Rightarrow \Pi^\ell}\ E\mathsf{W}}{\vdash^{n+1}_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma_m \Rightarrow \Delta_m}\ R
$$

□

Thanks to the previous lemma, it is now possible to prove that the calculi $\mathsf{G}^*_{\mathsf{MD+}}$ and $\mathsf{G}_{\mathsf{MD+}}$ have the same set of derivable sequents.

Lemma 3.4.8 (equivalence $\mathsf{G}^*_{\mathsf{MD+}}$ - $\mathsf{G}_{\mathsf{MD+}}$) $\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$

Proof. We prove the claim by induction on the height of the derivations.

For the right-to-left direction (if $\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$), let us consider a derivation in $\mathsf{G}_{\mathsf{MD+}}$ ending with the following rule application:

$$
\dfrac{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1 \quad \ldots \quad \vdash_{\mathsf{G}_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell}{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta}\ R_{\mathsf{G}_{\mathsf{MD+}}}
$$

By induction hypothesis and, if required, the already proved height-preserving admissible (Lem.3.4.3, 3.4.5) rules W_L, W_R and $E\mathsf{W}$, the desired conclusion can be derived by using the rule $R_{\mathsf{G}^*_{\mathsf{MD+}}}$ of $\mathsf{G}^*_{\mathsf{MD+}}$ corresponding to $R_{\mathsf{G}_{\mathsf{MD+}}}$:

$$
\dfrac{\dfrac{\dfrac{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1}\ I.H.}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta \mid \Sigma'^1 \Rightarrow \Pi'^1}\ \mathsf{W} \quad \dots \quad \dfrac{\dfrac{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell}\ I.H.}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta \mid \Sigma'^\ell \Rightarrow \Pi'^\ell}\ \mathsf{W}}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta}\ R_{\mathsf{G}^*_{\mathsf{MD+}}}
$$

Where W indicates possible applications of W_L, W_R (if $R_{\mathsf{G}^*_{\mathsf{MD+}}}$ is a propositional rule), or $E\mathsf{W}$ (if $R_{\mathsf{G}^*_{\mathsf{MD+}}}$ is a modal rule), and $\Sigma'^i \Rightarrow \Pi'^i$ for $1 \leq i \leq \ell$ is the sequent $\Sigma^i \Rightarrow \Pi^i$ where the possibly needed formulas have been added.

For the other direction, let us consider a derivation ending with the following rule application:

$$
\dfrac{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta \mid \Sigma^1 \Rightarrow \Pi^1 \quad \dots \quad \vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta \mid \Sigma^\ell \Rightarrow \Pi^\ell}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta}\ R_{\mathsf{G}^*_{\mathsf{MD+}}}
$$

For each premiss which is the conclusion of a derivation of height n, there is a component which is separately derivable in n steps (Lem.3.4.7). If $R_{\mathsf{G}^*_{\mathsf{MD+}}}$ is a modal rule and this component is the copy of the active sequent in the conclusion, we have $\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$ and the desired result immediately follows by inductive hypothesis. Otherwise, for every premiss the active component which is not the copy of the active sequent in the conclusion is derivable; hence we have:

$$
\dfrac{\dfrac{\dfrac{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta \mid \Sigma^1 \Rightarrow \Pi^1}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1}\ \text{Lem.3.4.7}}{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Sigma^1 \Rightarrow \Pi^1}\ I.H. \quad \dots \quad \dfrac{\dfrac{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta \mid \Sigma^\ell \Rightarrow \Pi^\ell}{\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell}\ \text{Lem.3.4.7}}{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Sigma^\ell \Rightarrow \Pi^\ell}\ I.H.}{\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma' \Rightarrow \Delta'}\ R_{\mathsf{G}_{\mathsf{MD+}}}
$$

possibly followed by applications of (Con_L) or (Con_R), if $R_{\mathsf{G}^*_{\mathsf{MD+}}}$ is $(\rightarrow_L)$ or $(\rightarrow_R)$ and the premiss(es) contain copies of the active formula in the conclusion. □

Before defining the proof search procedure for the calculus $\mathsf{G}^*_{\mathsf{MD+}}$, let us introduce the definition of *saturation*, which intuitively gives the conditions for avoiding multiple applications of the same rule to the same principal formula(s) of a sequent in a hypersequent.

Definition 3.4.9 (Saturation) *A hypersequent $\mathcal{G}$ is saturated iff each sequent $\Gamma \Rightarrow \Delta$ in $\mathcal{G}$ is saturated in $\mathcal{G}$, i.e. $\bot \notin \Gamma$, $\Gamma \cap \Delta = \varnothing$, and:*

- *if $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$ or $\varphi \in \Delta$ ($\Gamma \Rightarrow \Delta$ is saturated in $\mathcal{G}$ under the rule $(\rightarrow_L)$);*
- *if $\varphi \rightarrow \psi \in \Delta$, then $\varphi \in \Gamma$ and $\psi \in \Delta$ ($\Gamma \Rightarrow \Delta$ is saturated in $\mathcal{G}$ under the rule $(\rightarrow_R)$);*

- *if* $\mathcal{O}(\varphi/\psi) \in \Gamma$ *and* $\mathcal{O}(\theta/\chi) \in \Delta$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi \in \Gamma'$ *and* $\theta \in \Delta'$, *or* $\psi \in \Gamma'$ *and* $\chi \in \Delta'$, *or* $\chi \in \Gamma'$ *and* $\psi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{Mon}_{\mathcal{O}})$ *);*
- *if* $\mathcal{O}(\varphi/\psi) \in \Gamma$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi \in \Gamma'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{P}_{\mathcal{O}})$ *);*
- *if* $\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \in \Gamma$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi, \theta \in \Gamma'$, *or* $\psi \in \Gamma'$ *and* $\chi \in \Delta'$, *or* $\chi \in \Gamma'$ *and* $\psi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{D}_{\mathcal{O}})$ *);*
- $\mathcal{F}(\varphi/\psi) \in \Gamma$ *and* $\mathcal{F}(\theta/\chi) \in \Delta$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\theta \in \Gamma'$ *and* $\varphi \in \Delta'$, *or* $\psi \in \Gamma'$ *and* $\chi \in \Delta'$, *or* $\chi \in \Gamma'$ *and* $\psi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{Mon}_{\mathcal{F}})$ *);*
- *if* $\mathcal{F}(\varphi/\psi) \in \Gamma$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{P}_{\mathcal{F}})$ *);*
- $\mathcal{F}(\varphi/\psi), \mathcal{F}(\theta/\chi) \in \Gamma$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi, \theta \in \Delta'$, *or* $\psi \in \Gamma'$ *and* $\chi \in \Delta'$, *or* $\chi \in \Gamma'$ *and* $\psi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{D}_{\mathcal{F}})$ *);*
- *if* $\mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \in \Gamma$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi \in \Gamma'$ *and* $\theta \in \Delta'$, *or* $\psi \in \Gamma'$ *and* $\chi \in \Delta'$, *or* $\chi \in \Gamma'$ *and* $\psi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{D}_{\mathcal{OF}})$*);*
- *if* $\mathcal{R}(\varphi/\psi) \in \Gamma$ *and* $\mathcal{R}(\theta/\chi) \in \Delta$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi \in \Gamma'$ *and* $\theta \in \Delta'$, *or* $\psi \in \Gamma'$ *and* $\chi \in \Delta'$, *or* $\chi \in \Gamma'$ *and* $\psi \in \Delta'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{Mon}_{\mathcal{R}})$*);*
- *if* $\mathcal{R}(\varphi/\psi) \in \Gamma$, *then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *in* $\mathcal{G}$ *such that* $\varphi \in \Gamma'$ *(*$\Gamma \Rightarrow \Delta$ *is saturated in* $\mathcal{G}$ *under the rule* $(\mathsf{P}_{\mathcal{R}})$ *).*

The proof search procedure for the calculus $\mathsf{G}^*_{\mathsf{MD+}}$ is given in Fig.2.

Remark 3.4.10 *Note that for any* $\Gamma \Rightarrow \Delta$ *in a saturated hypersequent* $\mathcal{G}$ *the conditions* $\bot \notin \Gamma$ *and* $\Gamma \bigcap \Delta = \varnothing$ *hold. Hence, every saturated hypersequent is rejected by the proof search procedure.*

The following theorem shows that the proof search procedure terminates in a finite number of steps and that it is sound and complete with respect to the calculus, i.e. $\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G}$ iff the algorithm accepts $\mathcal{G}$ as an input.

Theorem 3.4.11 (Termination, soundness and completeness) *The procedure in Alg.2 terminates and accepts the input* $\mathcal{G}$ *iff* $\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G}$

Algorithm 2: The proof search procedure for $\mathsf{G}^*_{\mathsf{MD+}}$

Input: A hypersequent $\mathcal{G}$
Output: Is $\mathcal{G}$ derivable in $\mathsf{G}^*_{\mathsf{MD+}}$?

if *there is at least one sequent* $\Gamma \Rightarrow \Delta$ *in* $\mathcal{G}$ *such that* $\bot \in \Gamma$ *or* $\Gamma \bigcap \Delta \neq \varnothing$ **then**
  accept the input.
else
  if $\mathcal{G}$ *is saturated* **then**
    reject the input.
  else
    arbitrarily choose a non-saturated sequent $\Gamma \Rightarrow \Delta$ in $\mathcal{G}$, and
    if $\Gamma \Rightarrow \Delta$ *is not saturated under* $(\rightarrow_L)$ *and/or* $(\rightarrow_R)$ **then**
      choose an application of a propositional rule under which $\Gamma \Rightarrow \Delta$ is not saturated with a matching principal formula, and
      for *every premiss* $\mathcal{G}'$ *of this application* **do**
        recursively call the proof search procedure with input $\mathcal{G}'$.
        Accept if all these calls accept, else reject
    else
      choose an application of a modal rule under which $\Gamma \Rightarrow \Delta$ is not saturated with matching principal formulas, and
      for *every premiss* $\mathcal{G}'$ *of this application* **do**
        recursively call the proof search procedure with input $\mathcal{G}'$.
        Accept if all these calls accept, else reject.

Proof. Let us consider the input $\mathcal{G} = \{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n\}$. Given the definition of the proof search procedure, a rule is never applied (backwards) to a component $\Gamma_i \Rightarrow \Delta_i$ if this is saturated under it. Therefore the algorithm does not create more than once the same new formula (on the same side of the same component) from the same rule, nor the same component. This means that from a formula of any component it is possible to construct only a finite number of different new sequents. Together with the fact that any formula(s) can be the principal one(s) only in a finite number of rules, this bounds the maximal number of possible recursive calls of the proof search procedure. Hence after a finite number of steps (dependent on the number of subformulas of the formulas in each sequent, and on the number n of different components in $\mathcal{G}$) the hypersequent is accepted or the procedure has generated its saturated version and rejects the input.

Now let us prove the second claim of the lemma, i.e. that the procedure in Alg.2 accepts the input $\mathcal{G}$ iff $\vdash_{\mathsf{G}^*_{\mathsf{MD+}}} \mathcal{G}$.

If the proof search procedure accepts the input $\mathcal{G}$, it means that the derivation tree of $\mathcal{G}$ in the calculus $\mathsf{G}^*_{\mathsf{MD+}}$ can be built by labelling each node with each hypersequent given as an input to the recursive calls of the algorithm, following the order of the accepted backwards applications of rules.

For the other direction, let us consider a derivation $\mathfrak{D}$ for the hypersequent $\mathcal{G}$ in the calculus $\mathsf{G}^*_{\mathsf{MD+}}$ with no applications of the (already proved admissible) rules of internal and external weakening and contraction, and minimal, i.e. such that the same rule is not applied twice with the same main formula. Using the admissibility of internal and external weakening and contraction, indeed, it can be proved that for any hypersequent derivable in $\mathsf{G}^*_{\mathsf{MD+}}$ there is a minimal derivation in the same calculus.

If the last applied rule in $\mathfrak{D}$ has zero premisses, by Lem.3.4.7 there is a component $\Gamma_i \Rightarrow \Delta_i$ which is derivable independently by using a zero-premisses rule, i.e. such that $\bot \in \Gamma_i$ or $\Gamma_i \cap \Delta_i \neq \varnothing$; therefore the algorithm accepts $\mathcal{G}$. If the derivable hypersequent $\mathcal{G}$ does not contain initial sequents, there are recursive calls of the procedure which apply (backwards) the rules applied in $\mathfrak{D}$ (notice that the rules of $\mathsf{G}^*_{\mathsf{MD+}}$ are invertible, therefore the order in which they are applied does not matter). Hence, by induction on the height of the derivation $\mathfrak{D}$, it can be observed that the procedure accepts $\mathcal{G}$. □

As the proof search procedure terminates, we have that for any sequent (corresponding to a formula), it is possible to determine whether this sequent is derivable in the logic $\mathsf{MD+}$ or not.

Corollary 3.4.12 *The logic* $\mathsf{MD+}$ *is decidable.*

3.4.2 Semantics

As in the case of bMDL, a semantic characterization of the logic $\mathsf{MD+}$, will be used (in Section 3.4.3) for analysing an example of seemingly conflicting commands and comparing the solutions resulting from different interpretations of the involved norms.

The semantics for $\mathsf{MD+}$ is an extension of the one for the dyadic version of MD. The deontic operator of MD is now interpreted as the operator for proper obligations and two new neighbourhood functions are defined for the operators of prohibition and recommendation.

Definition 3.4.13 (model for $\mathsf{MD+}$) *A* $\mathsf{MD+}$ -frame *is a tuple* $(W, \mathcal{N}_{\mathcal{O}}, \mathcal{N}_{\mathcal{F}}, \mathcal{N}_{\mathcal{R}})$ *where* W *is a set of possible worlds and each of* $\mathcal{N}_{\mathcal{O}}, \mathcal{N}_{\mathcal{F}}, \mathcal{N}_{\mathcal{R}}$ *is a function* $W \to \wp(\wp(W) \times \wp(W))$ *such that:*

1. *If* $(X, Z) \in \mathcal{N}_{\mathcal{O}}(w)$ *and* $X \subseteq Y \subseteq W$, *then* $(Y, Z) \in \mathcal{N}_{\mathcal{O}}(w)$ *(corresponding to* $\mathsf{Mon}_{\mathcal{O}}$*);*

2. *If* $(X, Y) \in \mathcal{N}_{\mathcal{O}}(w)$, *then* $(X^c, Y) \notin \mathcal{N}_{\mathcal{O}}(w)$ *(corr.* $\mathsf{D}_{\mathcal{O}}$*);*

3. *If* $(X,Z) \in \mathcal{N}_{\mathcal{F}}(w)$ *and* $Y \subseteq X \subseteq W$*, then* $(Y,Z) \in \mathcal{N}_{\mathcal{F}}(w)$ *(corr.* $\mathsf{Mon}_{\mathcal{F}}$*);*

4. *If* $(X,Y) \in \mathcal{N}_{\mathcal{F}}(w)$*, then* $(X^c,Y) \notin \mathcal{N}_{\mathcal{F}}(w)$ *(corr.* $\mathsf{D}_{\mathcal{F}}$*);*

5. *It cannot be the case that* $(X,Z) \in \mathcal{N}_{\mathcal{O}}(w)$ *and* $(X,Z) \in \mathcal{N}_{\mathcal{F}}(w)$ *(corr.* $\mathsf{D}_{\mathcal{O}\mathcal{F}}$*);*

6. *If* $(X,Z) \in \mathcal{N}_{\mathcal{R}}(w)$ *and* $X \subseteq Y \subseteq W$*, then* $(Y,Z) \in \mathcal{N}_{\mathcal{R}}(w)$ *(corr.* $\mathsf{Mon}_{\mathcal{R}}$*);*

7. *If* $(X,Y) \in \mathcal{N}_{\mathcal{R}}(w)$*, then* $X \neq \varnothing$ *(corr.* $\mathsf{P}_{\mathcal{R}}$*).*

A MD+-model $\mathfrak{M}$ *extends an* MD+*-frame* $(W,\mathcal{N}_{\mathcal{O}},\mathcal{N}_{\mathcal{F}},\mathcal{N}_{\mathcal{R}})$ *by a* valuation *function* σ *which associates to each* p *in the set* Var *of propositional variables a subset of* W*.*

Definition 3.4.14 (truth sets) *The* truth set $[\![A]\!]$ *of a formula* A *in a* MD+*-model* $\mathfrak{M} = (W,\mathcal{N}_{\mathcal{O}},\mathcal{N}_{\mathcal{F}},\mathcal{N}_{\mathcal{R}},\sigma)$ *is defined via:*

. $[\![p]\!] = \sigma(p)$ *for* $p \in \mathsf{Var}$

. $[\![\bot]\!] = \varnothing$

. $[\![A \to B]\!] = [\![A]\!]^c \cup [\![B]\!]$

. $[\![\mathcal{O}(A/B)]\!] = \{w \in W \mid ([\![A]\!],[\![B]\!]) \in \mathcal{N}_{\mathcal{O}}(w)\}$

. $[\![\mathcal{F}(A/B)]\!] = \{w \in W \mid ([\![A]\!],[\![B]\!]) \in \mathcal{N}_{\mathcal{F}}(w)\}$

. $[\![\mathcal{R}(A/B)]\!] = \{w \in W \mid ([\![A]\!],[\![B]\!]) \in \mathcal{N}_{\mathcal{R}}(w)\}$

For $w \in [\![A]\!]$ *we also write* $\mathfrak{M},w \Vdash A$*. A formula* A *is* valid *in a* MD+*-model* $\mathfrak{M} = (W,\mathcal{N}_{\mathcal{O}},\mathcal{N}_{\mathcal{F}},\mathcal{N}_{\mathcal{R}},\sigma)$ *iff* $[\![A]\!] = W$*.*

First, we prove that the sequent calculus $\mathsf{G}_{\mathsf{MD+}}$ is sound with respect to the MD+-models in the previous definition. For this proof we do not use the more complex hypersequent version of the calculus, as a precise characterization of countermodels is not needed. However, soundness of the hypersequent calculus with respect to the MD+-models then follows from soundness of $\mathsf{G}_{\mathsf{MD+}}$ with the equivalence of the two calculi (Lem.3.4.8).

Lemma 3.4.15 (Soundness) *If* $\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$*, then the interpretation* $\bigwedge \Gamma \to \bigvee \Delta$ *of the sequent* $\Gamma \Rightarrow \Delta$ *is* valid *in every* MD+*-model (*$\mathfrak{M} = (W,\mathcal{N}_{\mathcal{O}},\mathcal{N}_{\mathcal{F}},\mathcal{N}_{\mathcal{R}},\sigma)$ *as defined above).*

Proof. The lemma is proved by showing that, for any rule of $\mathsf{G}_{\mathsf{MD+}}$, if there is a countermodel for the (formula interpretation of the) conclusion of this rule, then there is a countermodel for (the formula interpretation of) at least one of its premisses.

First, notice that there are no countermodels for the (formula interpretation of) conclusions of zero-premisses rules. Considering the rule $(\bot)$, if $\mathfrak{M},w \Vdash \neg(\bigwedge \Gamma \wedge \bot \to \bigvee \Delta)$ then w is in the set $\bigcap[\![\Gamma]\!] \cap [\![\bot]\!] \bigcap[\![\Delta]\!]^c$; but, since $[\![\bot]\!] = \varnothing$, also $\bigcap[\![\Gamma]\!] \cap [\![\bot]\!] \bigcap[\![\Delta]\!]^c = \varnothing$, hence $\mathfrak{M}, w \Vdash \neg(\bigwedge\Gamma \wedge p \to \bigvee\Delta \vee p)$ does not hold for any possible world $w \in W$.

For the rule (init), if $\mathfrak{M}, w \Vdash \neg(\bigwedge\Gamma \wedge p \to \bigvee\Delta \vee p)$ then $w \in \bigcap[\![\Gamma]\!] \cap [\![p]\!] \bigcap[\![\Delta]\!]^c \cap [\![p]\!]^c$, but $[\![p]\!] \cap [\![p]\!]^c = \varnothing$, hence also $w \in \bigcap[\![\Gamma]\!] \cap [\![p]\!] \bigcap[\![\Delta]\!]^c \cap [\![p]\!]^c = \varnothing$, therefore there is no $w \in W$ such that $\mathfrak{M}, w \Vdash \neg(\bigwedge\Gamma \wedge \bot \to \bigvee\Delta)$.

The cases of (W_L), (W_R), (Con_L), (Con_R) are trivial due to the properties of the membership relation $\in$ and the operations $\bigcup, \bigcap$ on $\wp(W)$.

For the other rules, let us consider the contexts empty, i.e., if the sequent $\Gamma'\varphi \Rightarrow \psi\Delta'$ is the conclusion of a rule which introduced φ and ψ, then $\Gamma' = \Delta' = \varnothing$; since the context formulas are copied into the premisses, they add the same conditions to the countermodels for the conclusion and for the premisses, therefore they can be ignored in this proof.

($\to_L$) Assume that there is a model $\mathfrak{M} = (W, \mathcal{N}_\mathcal{O}, \mathcal{N}_\mathcal{F}, \mathcal{N}_\mathcal{R}, \sigma)$ such that, for a $w \in W$, $\mathfrak{M}, w \Vdash \varphi \to \psi$; by Def.3.4.13, $w \in [\![\varphi]\!]^c \bigcup [\![\psi]\!]$, which means $\mathfrak{M}, w \Vdash \neg\varphi$ or $\mathfrak{M}, w \Vdash \psi$.

($\to_R$) If $\mathfrak{M}, w \Vdash \neg(\varphi \to \psi)$, by Def.3.4.13, $w \notin [\![\varphi]\!]^c \bigcup [\![\psi]\!]$, hence $w \in [\![\varphi]\!] \bigcap [\![\psi]\!]^c$, that is $\mathfrak{M}, w \Vdash \varphi \wedge \neg\psi$.

($\mathsf{Mon}_\mathcal{O}$) $\mathfrak{M}, w \Vdash \mathcal{O}(\varphi/\psi) \wedge \neg\mathcal{O}(\theta/\chi)$ means that $([\![\varphi]\!], [\![\psi]\!]) \in \mathcal{N}_\mathcal{O}(w)$ and $([\![\theta]\!], [\![\chi]\!]) \notin \mathcal{N}_\mathcal{O}(w)$, then, for the conditions in Def.3.4.13, $[\![\varphi]\!] \nsubseteq [\![\theta]\!]$ or $[\![\psi]\!] \neq [\![\chi]\!]$. In the first case there is a world v such that $v \in [\![\varphi]\!] \bigcap [\![\theta]\!]^c$, i.e. $\mathfrak{M}, v \Vdash \varphi \wedge \neg\theta$, in the second there is a world u such that $\mathfrak{M}, u \Vdash \psi \wedge \neg\chi$ or $\mathfrak{M}, u \Vdash \chi \wedge \neg\psi$; therefore, in any case we have a countermodel to at least one of the premisses of the application of ($\mathsf{Mon}_\mathcal{O}$) with conclusion $\mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi)$.

($\mathsf{P}_\mathcal{O}$) Given a model such that, for a $w \in W$, $([\![\varphi]\!], [\![\psi]\!]) \in \mathcal{N}_\mathcal{O}(w)$, by Def.3.4.13, we know that $[\![\varphi]\!] \neq \varnothing$. Otherwise, if $[\![\varphi]\!] = \varnothing$, since the empty set is contained in its complement, we would obtain by condition 1 in Def.3.4.13 $([\![\varphi]\!]^c, [\![\psi]\!]) \in \mathcal{N}_\mathcal{O}(w)$, in contrast with condition 2 in Def.3.4.13, stating that, if $([\![\varphi]\!], [\![\psi]\!]) \in \mathcal{N}_\mathcal{O}(w)$, then $([\![\varphi]\!]^c, [\![\psi]\!]) \notin \mathcal{N}_\mathcal{O}(w)$. Hence, as $[\![\varphi]\!] \neq \varnothing$, there is a world v in $[\![\varphi]\!]$, i.e. $\mathfrak{M}, v \Vdash \varphi$.

($\mathsf{D}_\mathcal{O}$) If $\mathfrak{M}, w \Vdash \mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\theta/\chi)$, both $([\![\varphi]\!], [\![\psi]\!])$ and $([\![\theta]\!], [\![\chi]\!])$ are in $\mathcal{N}_\mathcal{O}(w)$, hence, by condition 2 in Def.3.4.13, if $[\![\psi]\!] = [\![\chi]\!]$, then $[\![\varphi]\!] \nsubseteq [\![\theta]\!]^c$. Therefore there is a world v such that $v \in [\![\varphi]\!] \bigcap [\![\theta]\!]$, i.e. $\mathfrak{M}, v \Vdash \varphi \wedge \theta$, or there is a world u such that $\mathfrak{M}, u \Vdash \psi \wedge \neg\chi$ or $\mathfrak{M}, u \Vdash \chi \wedge \neg\psi$.

($\mathsf{Mon}_\mathcal{F}$) Given a model and a world such that $\mathfrak{M}, w \Vdash \mathcal{F}(\varphi/\psi) \wedge \neg\mathcal{F}(\theta/\chi)$, we have $([\![\varphi]\!], [\![\psi]\!]) \in \mathcal{N}_\mathcal{F}(w)$ and $([\![\theta]\!], [\![\chi]\!]) \notin \mathcal{N}_\mathcal{F}(w)$ therefore, by condition 3 in Def.3.4.13, $[\![\theta]\!] \nsubseteq [\![\varphi]\!]$, so there is a world v such that $\mathfrak{M}, v \Vdash \varphi \wedge \neg\theta$, or $[\![\psi]\!] \neq [\![\chi]\!]$, so there is a world u such that $\mathfrak{M}, u \Vdash \psi \wedge \neg\chi$ or $\mathfrak{M}, u \Vdash \chi \wedge \neg\psi$.

($\mathsf{P}_{\mathcal{F}}$) Given a model such that, for a $w \in W$, $(\llbracket\varphi\rrbracket, \llbracket\psi\rrbracket) \in \mathcal{N}_{\mathcal{F}}(w)$, we have $\llbracket\varphi\rrbracket \neq W$. Indeed, if $\llbracket\varphi\rrbracket = W$, we would obtain that the complement of W, i.e. the empty set, is both included in $\mathcal{N}_{\mathcal{F}}(w)$, by condition 3 in Def.3.4.13, and not included in it, by condition 4 in Def.3.4.13. Since $\llbracket\varphi\rrbracket \neq W$, there is a world v in $\llbracket\varphi\rrbracket^c$, hence $\mathfrak{M}, v \Vdash \neg\varphi$.

($\mathsf{D}_{\mathcal{F}}$) Given that $\mathfrak{M}, w \Vdash \mathcal{F}(\varphi/\psi) \wedge \mathcal{F}(\theta/\chi)$, $(\llbracket\varphi\rrbracket, \llbracket\psi\rrbracket) \in \mathcal{N}_{\mathcal{F}}(w)$ and $(\llbracket\theta\rrbracket, \llbracket\chi\rrbracket) \in \mathcal{N}_{\mathcal{F}}(w)$. Therefore, if $\llbracket\psi\rrbracket = \llbracket\chi\rrbracket$, then $\llbracket\varphi\rrbracket^c \nsubseteq \llbracket\theta\rrbracket$, which means there is a world v such that $v \in \llbracket\varphi\rrbracket^c \bigcap \llbracket\theta\rrbracket^c$, i.e. $\mathfrak{M}, v \Vdash \neg\varphi \wedge \neg\theta$, or there is a world u such that $\mathfrak{M}, u \Vdash \psi \wedge \neg\chi$ or $\mathfrak{M}, u \Vdash \chi \wedge \neg\psi$.

($\mathsf{D}_{\mathcal{OF}}$) If $\mathfrak{M}, w \Vdash \mathcal{O}(\varphi/\psi) \wedge \mathcal{F}(\theta/\chi)$, then $(\llbracket\varphi\rrbracket, \llbracket\psi\rrbracket) \in \mathcal{N}_{\mathcal{O}}(w)$ and $(\llbracket\theta\rrbracket, \llbracket\chi\rrbracket) \in \mathcal{N}_{\mathcal{F}}(w)$, which means, by condition 5 in Def.3.4.13, that $\llbracket\psi\rrbracket \neq \llbracket\chi\rrbracket$ or $\llbracket\varphi\rrbracket \nsubseteq \llbracket\theta\rrbracket$. Hence, there is a world v such that $v \in \llbracket\varphi\rrbracket \bigcap \llbracket\theta\rrbracket^c$, and so $\mathfrak{M}, v \Vdash \varphi \wedge \neg\theta$, or there is a world u such that $\mathfrak{M}, u \Vdash \psi \wedge \neg\chi$ or $\mathfrak{M}, u \Vdash \chi \wedge \neg\psi$.

($\mathsf{Mon}_{\mathcal{R}}$), ($\mathsf{P}_{\mathcal{R}}$) The proofs for the rules ($\mathsf{Mon}_{\mathcal{R}}$) and ($\mathsf{P}_{\mathcal{R}}$) are analogous to those for ($\mathsf{Mon}_{\mathcal{O}}$) and ($\mathsf{P}_{\mathcal{O}}$) respectively.

□

For proving the completeness of the calculus $\mathsf{G_{MD+}}$ with respect to the models defined in Def.3.4.13, we use the equivalent (Lem.3.4.8) hypersequent calculus $\mathsf{G^*_{MD+}}$ and define, for any hypersequent $\mathcal{G}$ which is not derivable in $\mathsf{G^*_{MD+}}$, a model which satisfies the negated translation of $\mathcal{G}$. Using those models we prove that if a hypersequent is not derivable, then there is a countermodel for it, therefore, by contraposition, if a hypersequent is valid in every model for $\mathsf{MD+}$, then it is derivable.

First, we define a countermodel $\mathfrak{M}_{\mathcal{G}} = (W^{\mathcal{G}}, \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}, \mathcal{N}^{\mathcal{G}}_{\mathcal{F}}, \mathcal{N}^{\mathcal{G}}_{\mathcal{R}}, \sigma^{\mathcal{G}})$ for a non-derivable hypersequent, starting from a failed proof search for it, using the procedure in Alg.2.

Definition 3.4.16 (countermodel from a saturated hypersequent $\mathcal{G}$) *The set $W^{\mathcal{G}}$ of all possible worlds of the model $\mathfrak{M}_{\mathcal{G}} = (W^{\mathcal{G}}, \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}, \mathcal{N}^{\mathcal{G}}_{\mathcal{F}}, \mathcal{N}^{\mathcal{G}}_{\mathcal{R}}, \sigma^{\mathcal{G}})$ is defined as the set of sequents in $\mathcal{G}$: $W^{\mathcal{G}} = \{[\Gamma \Rightarrow \Delta]_W \mid \Gamma \Rightarrow \Delta \in \mathcal{G}\}$; note that the form $[\Gamma \Rightarrow \Delta]_W$ is used in order to distinguish a sequent considered as a possible world from a sequent in the procedure.*

The valuation function $\sigma^{\mathcal{G}} : \mathsf{Var} \to \wp(W^{\mathcal{G}})$ associates to each p such that $p \in \mathsf{Var}$ the subset $\{[\Gamma \Rightarrow \Delta]_W \in W^{\mathcal{G}} \mid p \in \Gamma\}$ of $W^{\mathcal{G}}$ which includes all the sequents in $\mathcal{G}$ where p appears on the left-hand side.

Finally, the neighbourhood functions are defined as follows:

. $\mathcal{N}^{\mathcal{G}}_{\mathcal{O}}([\Gamma \Rightarrow \Delta]_W) = \{(X, Y) \in (\wp(W^{\mathcal{G}}) \times \wp(W^{\mathcal{G}})) \mid$ *there is a formula* $\mathcal{O}(\varphi/\psi) \in \Gamma$ *such that* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\} \subseteq X$ *and* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Sigma\} \subseteq$

Y *and* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Pi\} \subseteq Y^c\}$;

. $\mathcal{N}^{\mathcal{G}}_{\mathcal{F}}([\Gamma \Rightarrow \Delta]_W) = \{(X,Y) \in (\wp(W^{\mathcal{G}}) \times \wp(W^{\mathcal{G}})) \mid$ *there is a formula* $\mathcal{F}(\varphi/\psi) \in \Gamma$ *such that* $X \subseteq \{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\}$ *and* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Sigma\} \subseteq Y$ *and* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Pi\} \subseteq Y^c\}$;

. $\mathcal{N}^{\mathcal{G}}_{\mathcal{R}}([\Gamma \Rightarrow \Delta]_W) = \{(X,Y) \in (\wp(W^{\mathcal{G}}) \times \wp(W^{\mathcal{G}})) \mid$ *there is a formula* $\mathcal{R}(\varphi/\psi) \in \Gamma$ *such that* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\} \subseteq X$ *and* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Sigma\} \subseteq Y$ *and* $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Pi\} \subseteq Y^c\}$.

Before proceeding with the completeness proof, we need to show that the model in Def.3.4.16, built from a saturated hypersequent, represents a model for MD+. This amounts to prove that all the properties in Def.3.4.13 are satisfied by $\mathfrak{M}_{\mathcal{G}}$.

Lemma 3.4.17 *The model* $\mathfrak{M}_{\mathcal{G}}$ *in Def.3.4.16 is a model for* MD+.

Proof. The claim is proved by showing that the model $\mathfrak{M}_{\mathcal{G}}$ in Def.3.4.16 satisfies all the properties in Def.3.4.13

- (Property (1) of Def.3.4.13, corresponding to $\mathsf{Mon}_{\mathcal{O}}$)

  $(X,Z) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}([\Gamma \Rightarrow \Delta]_W)$ means that there is a formula $\mathcal{O}(\varphi/\psi) \in \Gamma$ such that $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\} \subseteq X$, $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Sigma\} \subseteq Z$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Pi\} \subseteq Z^c$. If there is a set of worlds Y such that $X \subseteq Y$, then the same formula $\mathcal{O}(\varphi/\psi) \in \Gamma$ is such that $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\} \subseteq Y$; hence also $(Y,Z) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}([\Gamma \Rightarrow \Delta]_W)$.

  With due adaptation, analogous arguments can be used for proving the properties $(3), (6)$ of Def.3.4.13, corresponding to $\mathsf{Mon}_{\mathcal{F}}$ and $\mathsf{Mon}_{\mathcal{R}}$, respectively.

- (Properties (2) -corresponding to $\mathsf{D}_{\mathcal{O}}$, (4) corresponding to $\mathsf{D}_{\mathcal{F}}$- and (5) -corr. to $\mathsf{D}_{\mathcal{O}\mathcal{F}}$- of Def.3.4.13)

  As those properties are very similar, we only give the proof that the last one is satisfied by a model $\mathfrak{M}_{\mathcal{G}}$ in Def.3.4.16: the proofs for the other two properties can be easily obtained by slightly modifying the one below.

  Let us assume $(X,Z) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}([\Gamma \Rightarrow \Delta]_W)$ and $(Y,V) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{F}}([\Gamma \Rightarrow \Delta]_W)$: we will show that $Z \neq V$ or $X \nsubseteq Y$.

  Since $(X,Z) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}([\Gamma \Rightarrow \Delta]_W)$ and $(Y,V) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{F}}([\Gamma \Rightarrow \Delta]_W)$, there is $\mathcal{O}(\varphi/\psi) \in \Gamma$ such that $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\} \subseteq X$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Sigma\} \subseteq Z$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Pi\} \subseteq Z^c$ and there is $\mathcal{F}(\theta/\chi) \in \Gamma$ such that $Y \subseteq \{[\Sigma \Rightarrow \Pi]_W \in$

$W^{\mathcal{G}} \mid \theta \in \Sigma\}$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \chi \in \Sigma\} \subseteq V$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \chi \in \Pi\} \subseteq V^c$. Then, by saturation conditions, there is a sequent $\Gamma' \Rightarrow \Delta' \in \mathcal{G}$ such that (a) $\psi \in \Gamma'$ and $\chi \in \Delta'$, or (b) $\chi \in \Gamma'$ and $\psi \in \Delta'$, or (c) $\varphi \in \Gamma'$ and $\theta \in \Delta'$.

In case (a) $[\Gamma' \Rightarrow \Delta']_W \in Z \bigcap V^c$ and in case (b) $[\Gamma' \Rightarrow \Delta']_W \in V \bigcap Z^c$, therefore, in both cases, $Z \neq V$. Case (c) is such that $[\Gamma' \Rightarrow \Delta']_W \in X \bigcap Y^c$, hence $X \nsubseteq Y$.

- (Property (7) of Def.3.4.13, corresponding to $\mathsf{P}_{\mathcal{R}}$)

  $(X, Y) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{R}}([\Gamma \Rightarrow \Delta]_W)$ means there is a formula $\mathcal{R}(\varphi/\psi) \in \Gamma$ such that $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \varphi \in \Sigma\} \subseteq X$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Sigma\} \subseteq Y$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \psi \in \Pi\} \subseteq Y^c$.

  Hence, by saturation conditions, there is a sequent $\Gamma' \Rightarrow \Delta' \in \mathcal{G}$ such that $\varphi \in \Gamma'$, which means X has a non empty subset, therefore $X \neq \varnothing$.

□

Now we can show that whenever a formula φ is on the left-hand side of a sequent $\Gamma \Rightarrow \Delta$ in $\mathcal{G}$, this sequent belongs to the truth set of φ, and when a formula is on the right-hand side of the sequent, it does belong to the complement of its truth set. Hence, by definition of satisfaction, $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \varphi$ if $\varphi \in \Gamma$, and $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \neg\varphi$ if $\varphi \in \Delta$

Lemma 3.4.18 (truth lemma) *Given a saturated hypersequent $\mathcal{G}$ and a sequent $\Gamma \Rightarrow \Delta \in \mathcal{G}$, (1) if $\varphi \in \Gamma$, then $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \varphi$, and (2) if $\psi \in \Delta$, then $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \neg\psi$.*

Proof. The claims (1) and (2) are proved together by induction on the complexities of φ and ψ.

The base cases where $\varphi = p$ or $\psi = p$ with $p \in \mathsf{Var}$ are trivial, and the same can be said for the cases of $\varphi = \bot$ or $\psi = \bot$.

If $\varphi = \theta \to \chi$, then, by saturation conditions, $\chi \in \Gamma$ or $\theta \in \Delta$. Hence, by induction hypotheses $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \nVdash \theta$ or $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \chi$ which means $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \varphi$.

If $\psi = \theta \to \chi$, then, by saturation conditions, $\theta \in \Gamma$ and $\chi \in \Delta$. By induction hypotheses this means $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \theta$ and $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \nVdash \chi$, therefore $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \neg\psi$.

If φ is a modal formula $\mathcal{O}(\theta/\chi)$, then, by I.H. and the definition of the neighbourhood function, $(\llbracket\theta\rrbracket, \llbracket\chi\rrbracket) \in \mathcal{N}^{\mathcal{G}}_{\mathcal{O}}([\Gamma \Rightarrow \Delta]_W)$, and, by Def.3.4.16, $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \mathcal{O}(\theta/\chi)$. The same reasoning applies if $\varphi = \mathcal{F}(\theta/\chi)$ or $\varphi = \mathcal{R}(\theta/\chi)$.

If ψ is a modal formula $\mathcal{O}(\theta/\chi)$, we need to prove that for no formula $\mathcal{O}(\xi/\zeta) \in \Gamma$ we have $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \xi \in \Sigma\} \subseteq \llbracket\theta\rrbracket$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \zeta \in \Sigma\} \subseteq \llbracket\chi\rrbracket$ and $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \zeta \in \Pi\} \subseteq \llbracket\chi\rrbracket^c$.

If there is no $\mathcal{O}(\xi/\zeta)$ in Γ, then the proof is trivial.

Otherwise, if there is $\mathcal{O}(\xi/\zeta) \in \Gamma$, by saturation conditions there is a sequent $\Gamma' \Rightarrow \Delta'$ in $\mathcal{G}$ such that one of the following conditions holds: (a) $\xi \in \Gamma'$ and $\theta \in \Delta'$, (b) $\zeta \in \Gamma'$ and $\chi \in \Delta'$, or (c) $\chi \in \Gamma'$ and $\zeta \in \Delta'$.

In case (a), by induction hypotheses, $\mathfrak{M}_{\mathcal{G}}, [\Gamma \Rightarrow \Delta]_W \Vdash \chi$ and $\mathfrak{M}_{\mathcal{G}}, [\Gamma' \Rightarrow \Delta']_W \Vdash \neg\theta$, hence, by definition of truth sets and satisfaction (Def.3.1.15) $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \xi \in \Sigma\} \nsubseteq [\![\theta]\!]$

By the same kind of argument, the case (b) is such that $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \zeta \in \Sigma\} \nsubseteq [\![\chi]\!]$, and in case (c) $\{[\Sigma \Rightarrow \Pi]_W \in W^{\mathcal{G}} \mid \zeta \in \Pi\} \nsubseteq [\![\chi]\!]^c$.

With due adaptations (the operators $\mathcal{O}$ and $\mathcal{R}$ are upward monotone in their first arguments, while $\mathcal{F}$ is downward monotone), the proofs for $\psi = \mathcal{F}(\theta/\chi)$ and $\psi = \mathcal{R}(\theta/\chi)$ are similar to the previous one. □

Given the previous lemmas, it is finally possible to prove that the calculus $\mathsf{G}_{\mathsf{MD+}}$ is complete with respect to the intended semantics.

Theorem 3.4.19 (Completeness) *For every sequent $\Gamma \Rightarrow \Delta$, if $\bigwedge\Gamma \rightarrow \bigvee\Delta$ is valid in every model for* MD+*, then $\vdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$.*

Proof. The claim is proved by contraposition.

If $\nvdash_{\mathsf{G}_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$, then $\nvdash_{\mathsf{G}^*_{\mathsf{MD+}}} \Gamma \Rightarrow \Delta$ (Lem.3.4.8), and by Lem. 3.4.11, the procedure in Alg. 2 terminates and rejects the input $\Gamma \Rightarrow \Delta$. Hence, by Lem. 3.4.18, $\mathfrak{M}_{\mathcal{G}}, [\Gamma' \Rightarrow \Delta']_W \Vdash \bigwedge\Gamma \wedge \neg \bigvee\Delta$ (where $\Gamma' \Rightarrow \Delta'$ represents the saturated version of $\Gamma \Rightarrow \Delta$, such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$): this means $\bigwedge\Gamma \rightarrow \bigvee\Delta$ is not valid in every model for MD+. □

3.4.3 An application

We use the introduced semantics to provide a formal analysis, extending that in [29], of the controversy around the Śyena sacrifice.

Recall the deontic statements involved in the controversy (see Section 2.4):

(A) *"One should not perform violence on any living being"*

(B) *"Someone who desires to kill an enemy should sacrifice with the Śyena"*

The following additional statements express the fact that sacrificing with Śyena implies killing an enemy and consequently also harming a living being:

(c) *"performing the Śyena entails killing an enemy"*

(d) *"killing an enemy entails harming a living being"*

Many explanations of the reasons why the sentences (A) and (B) are not contradictory have been proposed by Mīmāṃsā scholars. We consider below two different ways of formalizing such deontic statements (A) and (B), which correspond to the interpretations given by the Mīmāṃsā authors Kumārila and Prabhākara. We prove that in both cases the two commands and the assumptions (c) and (d) —which give informations about "facts" by connecting Śyena with the harming of living beings— do not give rise to contradictions, although (A) and (B) seem apparently conflicting. To do this, we use the semantics of our logic. We indeed construct models that match two solutions consistent with the thoughts of the two authors (in the case of Prabhākara, exactly the solution proposed by this author), making such solutions formally meaningful. Note that the solution of Prabhākara has been analysed in [29] in the context of the logic bMDL (and hence formalising (A) as a negative obligation).

Kumārila's and Prabhākara's perspectives differ essentially for the interpretation of the statement (B), enjoining people who desire to kill their enemies to perform the Śyena sacrifice. As it depends on a particular desire of obtaining the specific result mentioned in the statement (killing an enemy), the Śyena constitutes a *kāmya-karman* (elective) sacrifice. Hence, from Kumārila's perspective, the deontic content of the command is weaker than that of an injunction to perform a fixed or occasional sacrifice (or of a prohibition). This means that even the person who wants to kill an enemy is not properly compelled to perform the Śyena and the command should be formalized as a recommendation. On the other hand, for Prabhākara, elective sacrifices do not have a deontic content weaker than the others, they just have an *adhikāra* (eligibility condition) which is defined by a characteristic desire, different from the general desire for happiness or heaven of fixed and occasional sacrifices. Therefore, the deontic statement (B) should be formalized as an obligation. Hence, writing (B_K) for the formalization of statement (B) according to Kumārila's interpretation and (B_P) for its formalization according to Prabhākara's interpretation, the statements above concerning the Śyena sacrifice can be formalized as:

(A) $\mathcal{F}(\mathtt{harm}/\top)$

(B_K) $\mathcal{R}(\mathtt{\acute{S}y/des\_kill})$

(B_P) $\mathcal{O}(\mathtt{\acute{S}y/des\_kill})$

(c) $\mathtt{\acute{S}y} \rightarrow \mathtt{kill}$

(d) $\mathtt{kill} \rightarrow \mathtt{harm}$

Using Alg.2 and Lem.3.4.8, we can prove that none of the sequents
$\mathcal{F}(\mathtt{harm}/\top), \mathcal{R}(\mathtt{\acute{S}y/des\_kill}), (\mathtt{\acute{S}y} \rightarrow \mathtt{kill}), (\mathtt{kill} \rightarrow \mathtt{harm}) \Rightarrow \bot$ and
$\mathcal{F}(\mathtt{harm}/\top), \mathcal{O}(\mathtt{\acute{S}y/des\_kill}), (\mathtt{\acute{S}y} \rightarrow \mathtt{kill}), (\mathtt{kill} \rightarrow \mathtt{harm}) \Rightarrow \bot$ is derivable in $\mathsf{G}_{\mathsf{MD+}}$. The semantics is then used to "explain why" it is the case." Indeed, by using the construction in

Def.3.4.16 we can build a countermodel for (the saturated hypersequent corresponding to) each of those sequents. By Lem.3.4.18 the countermodel for the first sequent is such that the formula $\mathcal{F}(\texttt{harm}/\top) \wedge \mathcal{R}(\texttt{Śy}/\texttt{des\_kill}) \wedge (\texttt{Śy} \rightarrow \texttt{kill}) \wedge (\texttt{kill} \rightarrow \texttt{harm})$ (corresponding to Kumārila's interpretation) is satisfied and the countermodel for the second sequent satisfies $\mathcal{F}(\texttt{harm}/\top) \wedge \mathcal{O}(\texttt{Śy}/\texttt{des\_kill}) \wedge (\texttt{Śy} \rightarrow \texttt{kill}) \wedge (\texttt{kill} \rightarrow \texttt{harm})$ (corresponding to Prabhākara's interpretation).

To make the two solutions clearer and more explicative, we define simpler models (with respect to those defined by using the proof search algorithm) $\mathfrak{M}_K = (W_{\acute{S}y}, \mathcal{N}^K_{\mathcal{O}}, \mathcal{N}^K_{\mathcal{F}}, \mathcal{N}^K_{\mathcal{R}}, \sigma_{\acute{S}y})$ and $\mathfrak{M}_P = (W_{\acute{S}y}, \mathcal{N}^P_{\mathcal{O}}, \mathcal{N}^P_{\mathcal{F}}, \mathcal{N}^P_{\mathcal{R}}, \sigma_{\acute{S}y})$. Such models allow us to consider all the states that are consistent with the assumptions (c) and (d) and that are relevant for understanding the deontic statements (A) and (B). The domain $W_{\acute{S}y}$ is in both cases $\{w_i \mid 1 \leq i \leq 8\}$ such that $[\![\texttt{harm}]\!] = \sigma_{\acute{S}y}(\texttt{harm}) = \{w_2, w_3, w_4, w_6, w_7, w_8\}$, $[\![\texttt{kill}]\!] = \sigma_{\acute{S}y}(\texttt{kill}) = \{w_2, w_3, w_6, w_7\}$, $[\![\texttt{Śy}]\!] = \sigma_{\acute{S}y}(\texttt{Śy}) = \{w_2, w_6\}$, and $[\![\texttt{des\_kill}]\!] = \sigma_{\acute{S}y}(\texttt{des\_kill}) = \{w_5, w_6, w_7, w_8\}$.

The following figure represents the states $w_1, \cdots, w_8$ (denoted by circles) in the domain $W_{\acute{S}y}$, according to the valuation $\sigma_{\acute{S}y}$ of the two models $\mathfrak{M}_K$ and $\mathfrak{M}_P$. Note that a member (X, Y) of a neighbourhood is a pair of sets of states and each w_i (with $1 \leq i \leq 8$) in the domain $W_{\acute{S}y}$ has the same neighbourhoods. Hence, we represent the neighbourhood $\mathcal{N}^K_{\mathcal{R}}(w_i) = \mathcal{N}^P_{\mathcal{O}}(w_i)$ (which expresses recommendation in $\mathfrak{M}_K$ and obligation in $\mathfrak{M}_P$) by drawing arrows from each state in the second set Y of the pair to each one in the first set X of the pair. However, using the same representation for the neighbourhood $\mathcal{N}^K_{\mathcal{F}}(w_i) = \mathcal{N}^P_{\mathcal{F}}(w_i)$ (which expresses prohibition in both models), any state in the truth set $[\![\texttt{harm}]\!]$ would be reached by an arrow from each state in the domain, included itself. To simplify the figure, we represent any member (X, Y) of the prohibition-neighbourhood, by colouring grey the circles of all the states in the first set X of the pair, while the second set Y of the pair includes all the states in the domain. Intuitively this means that the states coloured in grey $(w_2, w_3, w_4, w_6, w_7, w_8)$ represent those where the command (A) has been violated. Indeed, if a pair of sets of states (X, Y) is in the prohibition-neighbourhood of a state w_i, it means that in w_i it is true that the states in X are states of violation "from the point of view of" the states in Y. As the deontic statement (A) is true at any state of the domain, in each state w_i it is true that $w_2, w_3, w_4, w_6, w_7, w_8$ are states of violation from the perspective of any other state. The statement (B), also true at any state of the domain, has more restrictive conditions. Hence in each state w_i it is true that w_4, w_8 (where the Śyena sacrifice is performed) are the deontically recommended (or the only deontically acceptable) states "from the point of view of" w_5, w_6, w_7, w_8 (where there is the desire to kill an enemy). Therefore, the states from which an arrow departs and which are not reached by any arrow (w_5, w_6, w_7) represent the ones that are not compliant with the command (B).

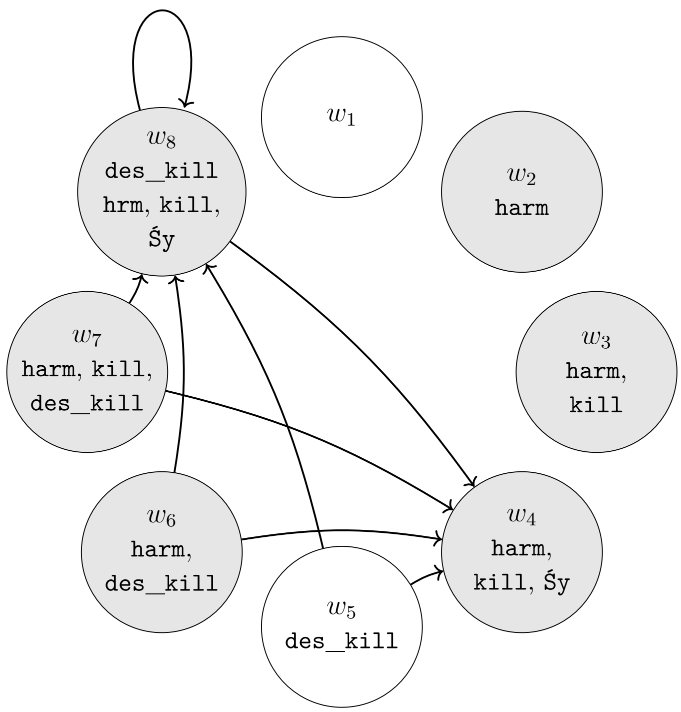


Figure 3.10: Models for the Śyena controversy

Kumārila's perspective Let us consider fist the model consistent with Kumārila's point of view. The neighbourhoods of a state w_i (with $1 \leq i \leq 8$) in the domain $W_{\acute{S}y}$ are defined as follows: $\mathcal{N}^K_{\mathcal{F}}(w_i) = \{(X,Y) \in (\wp(W_{\acute{S}y}) \times \wp(W_{\acute{S}y})) \mid X \subseteq \{w_2, w_3, w_4, w_6, w_7, w_8\}, Y = W_{\acute{S}y}\}$, $\mathcal{N}^K_{\mathcal{O}}(w_i) = \varnothing$ and $\mathcal{N}^K_{\mathcal{R}}(w_i) = \{(V,Z)) \in (\wp(W_{\acute{S}y}) \times \wp(W_{\acute{S}y})) \mid \{w_2, w_6\} \subseteq V, Z = \{w_5, w_6, w_7, w_8\}\}$.

We can observe that $\mathfrak{M}_K$ is a model for MD+, as it satisfies all the conditions in Def.3.4.13. In fact, as Def.3.4.13 does not contain any condition that relates the neighbourhood function for prohibitions with the one for recommendations, $\mathfrak{M}_K$ would have been a correct model for MD+ even if the prohibition (A) was explicitly stated to be enforced under the condition of desiring to kill an enemy, namely if (A) was formalized as $\mathcal{F}(\texttt{harm}/\texttt{des\_kill})$ and $\mathcal{N}^K_{\mathcal{F}}(w_i) = \{(X,Y) \in (\wp(W_{\acute{S}y}) \times \wp(W_{\acute{S}y})) \mid X \subseteq \{w_2, w_3, w_4, w_6, w_7, w_8\}, Y = \{w_5, w_6, w_7, w_8\}\}$ for any $w_i \in W_{\acute{S}y}$.

As the commands (A) and (B$_K$) are assumed to hold in any possible situation and therefore the neighbourhoods are the same for each state in the domain, we can prove that for any state w_i (with $1 \leq i \leq 8$) in the domain $W_{\acute{S}y}$ of $\mathfrak{M}_K$, we have $\mathfrak{M}_K, w_i \Vdash \mathcal{F}(\mathtt{harm}/\top) \wedge \mathcal{R}(\mathtt{\acute{S}y}/\mathtt{des\_kill}) \wedge (\mathtt{\acute{S}y} \rightarrow \mathtt{kill}) \wedge (\mathtt{kill} \rightarrow \mathtt{harm})$. Indeed, according to Def.3.4.14, $w_i \in [\![\mathcal{F}(\mathtt{harm}/\top)]\!]$ as $([\![\mathtt{harm}]\!], W_{\acute{S}y}) \in \mathcal{N}^K_{\mathcal{F}}(w_i)$ and $w_i \in [\![\mathcal{R}(\mathtt{\acute{S}y}/\mathtt{des\_kill})]\!]$ as $([\![\mathtt{\acute{S}y}]\!], [\![\mathtt{des\_kill}]\!]) \in \mathcal{N}^K_{\mathcal{R}}(w_i)$. Moreover, for simplicity, we did not consider states which are not consistent with the assumptions (c) and (d). Hence, $w_i \in [\![\mathtt{\acute{S}y} \rightarrow \mathtt{kill}]\!]$, as $w_1, w_3, w_4, w_5, w_7, w_8 \notin [\![\mathtt{\acute{S}y}]\!]$ and $w_2, w_3, w_6, w_7, w_8 \in [\![\mathtt{kill}]\!]$, i.e. for any w_i in $W_{\acute{S}y}$, $w_i \in [\![\mathtt{\acute{S}y}]\!]^c \cup [\![\mathtt{kill}]\!]$. For the same reason, any w_i in $W_{\acute{S}y}$ is such that $w_i \in [\![\mathtt{kill}]\!]^c \cup [\![\mathtt{harm}]\!]$ and therefore $w_i \in [\![\mathtt{kill} \rightarrow \mathtt{harm}]\!]$. This means that for any w_i in $W_{\acute{S}y}$, $w_i \in [\![\mathcal{F}(\mathtt{harm}/\top)]\!] \cap [\![\mathcal{R}(\mathtt{\acute{S}y}/\mathtt{des\_kill})]\!] \cap [\![(\mathtt{\acute{S}y} \rightarrow \mathtt{kill})]\!] \cap [\![(\mathtt{kill} \rightarrow \mathtt{harm})]\!]$, i.e. $\mathfrak{M}_K, w_i \Vdash \mathcal{F}(\mathtt{harm}/\top) \wedge \mathcal{R}(\mathtt{\acute{S}y}/\mathtt{des\_kill}) \wedge (\mathtt{\acute{S}y} \rightarrow \mathtt{kill}) \wedge (\mathtt{kill} \rightarrow \mathtt{harm})$.

Considering the characteristics of elective sacrifices from the perspective of Kumārila (recommendations), it is interesting to note that the states w_1, w_5 belonging to the complement of $[\![\mathtt{harm}]\!]$ —i.e. where the prohibition is not violated— are equally preferable to the others. Indeed, the desire to kill an enemy is different from the decision to do that and it does not brings with it any proper duty to perform the Śyena. The command (B_K) just states that the correct way to kill an enemy, if one desires to do so, is the Śyena sacrifice. This only means that the state w_7, where both `kill` and `des_kill` are true but `Śy` is not, is worse than w_8, verifying $\mathtt{kill}, \mathtt{des\_kill}$ and `Śy`.

Prabhākara's perspective Let us now examine the model $\mathfrak{M}_P = (W_{\acute{S}y}, \mathcal{N}^P_{\mathcal{O}}, \mathcal{N}^P_{\mathcal{F}}, \mathcal{N}^P_{\mathcal{R}}), \sigma_{\acute{S}y}$, consistent with Prabhākara's solution of the Śyena controversy. The version we present below differs from the one in [29] in that it interprets (A) as a prohibition. The neighbourhoods of a state $w_i \in W_{\acute{S}y}$ in $\mathfrak{M}_P$ are defined as follows: $\mathcal{N}^P_{\mathcal{F}}(w_i) = \{(X, Y) \in (\wp(W_{\acute{S}y}) \times \wp(W_{\acute{S}y})) \mid X \subseteq \{w_2, w_3, w_4, w_6, w_7, w_8\}, Y = W_{\acute{S}y}\}$, $\mathcal{N}^P_{\mathcal{O}}(w_i) = \{(T, U) \in (\wp(W_{\acute{S}y}) \times \wp(W_{\acute{S}y})) \mid \{w_2, w_6\} \subseteq T, U = \{w_5, w_6, w_7, w_8\}\}$ and $\mathcal{N}^P_{\mathcal{R}}(w_i) = \varnothing$.

Again we can observe that $\mathfrak{M}_P$ is a model for $\mathsf{MD+}$, as it satisfies all the conditions in Def.3.4.13. Unlike for $\mathfrak{M}_K$, in the case of $\mathfrak{M}_P$ we need also to compare the two neighbourhoods: by condition 5. of Def.3.4.13, it cannot be the case that the same neighbourhood of a state represents both prohibitions and obligations. This means we need to check that for any $(X, Y) \in \mathcal{N}^P_{\mathcal{F}}(w_i)$ and any $(T, U) \in \mathcal{N}^P_{\mathcal{O}}(w_i)$, either $X \nsubseteq T$ or $Y \neq U$. In this case we cannot verify the first condition $X \nsubseteq T$, as any X is included in $\{w_2, w_3, w_4, w_6, w_7, w_8\}$ and any T includes $\{w_2, w_6\}$; however, it is easy to verify that $Y \neq U$ is always valid as $W_{\acute{S}y} \neq \{w_5, w_6, w_7, w_8\}$. As in the case of Kumārila's point of view, the neighbourhoods are the same for each state in the domain, hence we can prove that for any $w_i \in W_{\acute{S}y}$

we have $\mathfrak{M}_P, w_i \Vdash \mathcal{F}(\texttt{harm}/\top) \wedge \mathcal{O}(\texttt{Śy}/\texttt{des\_kill}) \wedge (\texttt{Śy} \rightarrow \texttt{kill}) \wedge (\texttt{kill} \rightarrow \texttt{harm})$. The proof is analogous to the one for the formula corresponding to Kumārila's interpretation, as the domain, the evaluation function and the neighbourhood $\mathcal{N}^P_{\mathcal{F}}(w_i)$ for any $w_i \in W_{\acute{S}y}$ are defined in the same way as for $\mathfrak{M}_K$. Moreover, we can observe that for any $w_i \in W_{\acute{S}y}$, $(\{w_2, w_6\}, \{w_5, w_6, w_7, w_8\}) = ([\![\texttt{Śy}]\!], [\![\texttt{des\_kill}]\!]) \in \mathcal{N}^P_{\mathcal{O}}(w_i)$, hence $w_i \in [\![\mathcal{F}(\texttt{harm}/\top)]\!] \cap [\![\mathcal{O}(\texttt{Śy}/\texttt{des\_kill})]\!] \cap [\![(\texttt{Śy} \rightarrow \texttt{kill})]\!] \cap [\![(\texttt{kill} \rightarrow \texttt{harm})]\!]$ and therefore $\mathfrak{M}_P, w_i \Vdash \mathcal{F}(\texttt{harm}/\top) \wedge \mathcal{O}(\texttt{Śy}/\texttt{des\_kill}) \wedge (\texttt{Śy} \rightarrow \texttt{kill}) \wedge (\texttt{kill} \rightarrow \texttt{harm})$.

However, in this case the interpretation changes with respect to Kumārila's one. Indeed, now the states w_5, w_6, w_7, included in the truth set of `des_kill` and not in the one of `Śy`, represent situations where the obligation $\mathcal{O}(\texttt{Śy}/\texttt{des\_kill})$ is disrespected. Therefore, not only the state w_7 where the result of killing is obtained a non-Vedic way is worse than w_8, but also, from the point of view of the obligation, w_8 is preferable to w_5, where only `des_kill` is verified. This also means that the only "Vedic" state, where all the commands are complied with, is w_1: this is the only state in the intersection of w_1, w_5, the set of states where the prohibition is not violated, and w_1, w_2, w_3, w_4, w_8, the set of states where the obligation is not disrespected. Such analysis corresponds to a much stronger interpretation of the concept of "desire". Indeed, the desire for a specific result does not represent an element connecting the elective sacrifice to its result, but it identifies the addressee of a proper obligation. This means that the desire itself is not just a vague intention that can be ignored, but a decision already made, which is going to cause the wanted result. From Prabhākara's point of view we could probably add to the premisses the statement *"desiring to kill an enemy entails killing the enemy"* ($\texttt{des\_kill} \rightarrow \texttt{kill}$), making the states w_5, w_8 inconsistent. Hence, given the desire to kill an enemy, the situations where one performs the Śyena sacrifice represent the lesser evil. Furthermore, as there is at least one state where no command is transgressed, it is clear that agents are not forced to break a rule, if they are not already determined to do so. Thus, the model makes sense of Prabhākara's claim that *"the Vedas do not compel one to perform the malevolent sacrifice Śyena, they only say that it is obligatory"*, which some authors (see e.g. [118]) even considered meaningless.

Chapter 4

Defeasible Reasoning in Mīmāṃsā

The logic MD+ presented in the previous chapter is based on properties extracted from Mīmāṃsā texts and corresponding to some of the interpretative principles called *nyāya*s. Those *nyāya*s have been first formalized as Hilbert axioms and then translated into sequent rules, by using the method developed in [80].

However, not all the *nyāya*s can be converted into axioms; some of them indeed do not concern the interpretation of all the Vedic commands, but offer more general interpretative principles to resolve apparent contradictions in the Vedas (see also the Appendix).

Examples of such principles are the *bādha*s introduced in Section 2.4.1. In this chapter we will investigate a formal mechanism based on sequent calculus which allows us to mimic the reasoning with two of the major such principles: *Guṇapradhāna* (or *Sāmānya-viśeṣa*) and *vikalpa*. The former —known in contemporary formal logic as *specificity principle*— is used in Mīmāṃsā for dealing with two Vedic commands that cannot be both complied with at the same time and such that the conditions of one command are more specific than the conditions of the other one. According to the specificity/*Guṇapradhāna* principle, when the more specific conditions hold, the command with more general conditions is overruled by the other one. Reasoning with the specificity/*Guṇapradhāna* principle is tantamount to reason with norms that are subject to exceptions (i.e. they are *defeasible*). For example, a norm like *"one should not harm another person"* can be overruled by the exception *"one should harm another person to protect a child"*, but this does not mean the first norm loses its validity in general: it remains as a defeasible command.

In presence of two conflicting norms that can be applied exactly under the same conditions, Mīmāṃsā authors resort to the application of *vikalpa*, i.e. to consider the choice of which command to obey free. However, *vikalpa* is treated as the very last resort for avoiding the meaninglessness of commands and is applied only in cases of explicitly conflicting commands where any reinterpretation is impossible. E.g. the statements *"take the cup"* and *"do not take the cup"* in the same sacrifice, discussed in the XVII century text *Mīmāṃsānyāyaprakāśa* (see [38]).

The principles of specificity/*Guṇapradhāna* and *vikalpa* play a key role in Mīmāṃsā, as they allow to reason in the presence of possibly conflicting Vedic commands: they ensure that, for any given condition, it is possible to derive which commands in the Vedas are enforceable. This is particularly important from the perspective of Mīmāṃsā, as the set of Vedic commands is assumed to be consistent and it is considered to be the only source of knowledge for what concerns the duty. Hence, in any situation, the agents should derive unambiguously "what has to be done" from the commands in the sacred texts (plus their knowledge and beliefs about reality).

In this chapter the sequent calculus for $\mathsf{MD+}$ is extended to capture such mechanisms. In particular, the extended sequent calculus allows to reason using specificity/*Guṇapradhāna* in presence of *global factual assumptions* (a set of assumptions about "facts" and relations among them) and *deontic assumptions* (statements found in the Vedas expressing duties).

Most of the technical results in this chapter are taken from [32]. Based on that, the chapter is organized as follows. Section 4.1 contains a brief discussion on the reasons —rooted in Mīmāṃsā reasoning— behind our choices.

In order to introduce the essential idea behind our approach in a clear way, initially we will consider the restricted system presented in [31], corresponding to the logic MD with deontic assumptions consisting only of obligations. Hence, in Section 4.2 we will extend the calculus for the logic MD with "special" sequent rules that allow us to derive an obligation from a list of deontic assumptions when there is no more specific obligation that overrules it.

The rules in Section 4.2 constitute a special cases of the more general ones for reasoning with specificity/*Guṇapradhāna* in the multimodal system $\mathsf{MD+}$, introduced in section 4.3. The latter allows us to derive enforceable commands from a set of propositional global assumptions and a list of deontic assumption which can contain obligations, prohibitions, recommendations, and explicit exceptions (permissions) for all kinds of norm. The technical properties of the resulting system (cut-elimination and decidability) will be shown in Section 4.3.1.

Section 4.3.2 will present a way to adapt the resulting system for reasoning in presence of deontic assumptions coming from different sources. As discussed in Section 2.4.1, Mīmāṃsakas considered four sources of duty ordered in a hierarchical way. The new rules take this hierarchy into account, giving precedence, in case of conflict, to commands found in hierarchical higher sources.

Section 4.4 presents some applications of the formal system to Mīmāṃsā reasoning: in particular we will focus on the *vikalpa* principle according to which, in presence of a conflict between two deontic assumptions, any of the conflicting norm may be adopted as option.

As Mīmāṃsā authors looked at *vikalpa* as the last resort for avoiding the meaninglessness of any Vedic command, we will apply our sequent calculus for evaluating different interpretations of commands and choosing the one which gives rise to fewer applications of *vikalpa*. Our method has been implemented in the system at `http://subsell.logic.at/bprover/deonticProver/version1.1/`.

Finally, section 4.5 compares our system with some of the main approaches to defeasible deontic logics.

4.1 Motivations

Recall that our aim is to simulate Mīmāṃsā reasoning using formal methods. Before presenting the sequent system that uses specificity for conflict resolutions, we explain the choices we have done and how they are grounded on the Mīmāṃsā interpretation of the Vedic duties.

As already noticed in Section 3.2, the dyadic deontic operators of $\mathsf{MD+}$ can be read as special kinds of implicational formulas, where the second arguments (i.e. the conditions) represent the antecedents; as such, they pose a dilemma concerning the principle called *strengthening of the antecedents*. This principle states that an implicational formula with stronger conditions, e.g. $\psi \wedge \chi \rightarrow \varphi$, follows from a weaker one, $\psi \rightarrow \varphi$ (resp. a deontic formula $\mathcal{O}(\varphi/\psi \wedge \chi)$ follows from $\mathcal{O}(\varphi/\psi)$). This derivation not only seems very natural, but in general it is also useful for reasoning in presence of deontic assumptions, e.g a corpus of norms or a code of laws. Indeed, norms are rarely formulated in such a way that they cover all the possible cases, and a reasoner should identify the conditions such that norms are enforceable. E.g., given a norm like *"a private citizen should not physically harm another person"*, it is natural to conclude that *"a private citizen who is angry should not physically harm another person* (given the condition that the private citizen is angry)*"*. However, without additional mechanisms, dyadic deontic logics (e.g., [141, 27, 136, 82, 109]), as well as MD+, cannot reason on the conditions of dyadic deontic formulas. Indeed, adding an unrestricted downwards monotonicity rule for the second argument (i.e. adding the principle of *strengthening of the antecedents*) to $\mathsf{MD+}$ would allow the derivation of $\mathsf{op}(\varphi/\psi \wedge \chi)$ from

$\mathsf{op}(\varphi/\psi)$ for any deontic operator and any formulas φ, ψ, χ in the logic. For instance, if we add such a rule to $\mathsf{MD+}$, $\mathcal{O}(\varphi/\psi \wedge \chi)$ would follow from $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\neg\varphi/\psi \wedge \chi)$ would follow from $\mathcal{O}(\neg\varphi/\chi)$; hence, from $\mathcal{O}(\varphi/\psi \wedge \chi)$ and $\mathcal{O}(\neg\varphi/\psi \wedge \chi)$, by axiom $\mathsf{D}_{\mathcal{O}}$, a contradiction would follow.

Example 4.1.1. Let us consider the deontic assumptions (a) *"citizens must pay taxes"* ($\mathcal{O}(\texttt{pay}/\texttt{citizen})$), (b) *"citizens should not pay taxes if they have exemptions"* ($\mathcal{O}(\neg\texttt{pay}/\texttt{citizen} \wedge \texttt{exemption})$) and (c) *"citizens should not pay taxes if they are unemployed"* ($\mathcal{O}(\neg\texttt{pay}/\texttt{citizen} \wedge \texttt{unemployed})$). Clearly, we cannot use unrestricted monotonicity on the conditions of the dyadic deontic operator, as it would allow to derive from (a) the obligation (d) *"citizens must pay taxes if they have exemptions"* ($\mathcal{O}(\texttt{pay}/\texttt{citizen} \wedge \texttt{exemption})$), conflicting with (b).

In order to safely apply unrestricted monotonicity on the second argument of the deontic operator without allowing to derive conflicting obligations, we would need to reformulate each deontic assumption as explicitly including all the possible exceptions; e.g. the command (a) should be formalized as $\mathcal{O}(\texttt{pay}/\texttt{citizen} \wedge \neg\texttt{exemption} \wedge \neg\texttt{unemployed})$ *"citizens must pay taxes if they do not have exemptions and they are not unemployed ..."*.

However, this method is inefficient and often practically impossible, as it involves rephrasing each norm by taking into consideration all the related commands and all the states mentioned in their conditions. Moreover, in order to know which commands are enforced, one would need a complete description of the formulas that hold in a given circumstance. In the previous example, if we do not know whether the agents are unemployed or not and whether they have or not an exemption, we cannot apply any command. ■

Mīmāṃsā authors avoid conflicts generated by commands as the ones above, without resorting to inefficient methods and without cancelling one of the commands. They indeed regard the deontic statements with stronger conditions as exceptions to the more general ones, and consider all the deontic statements as *defeasible*. This means that Vedic norms are considered to hold in general and are never cancelled, but they are open to revision, as they can be "defeated" by exceptions or by stronger norms.

Hence, given conditions $\psi \wedge \chi \wedge \theta$ and two conflicting deontic assumptions $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\neg\varphi/\psi \wedge \chi)$, priority is given to the more specific command, whose conditions describe in a more detailed way circumstances that include $\psi \wedge \chi \wedge \theta$. This means $\mathcal{O}(\neg\varphi/\psi \wedge \chi \wedge \theta)$ is enforceable and $\mathcal{O}(\varphi/\psi \wedge \chi \wedge \theta)$ is not.

This way of reasoning, according to which deontic assumptions with stronger conditions (i.e. *more specific* deontic assumptions) override more general ones, is known in contemporary formal logic and Artificial Intelligence as *specificity principle* and it has been used in European

Law for centuries, expressed by the brocard *"Lex specialis derogat legi generali"*. Actually, it is the *Guṇapradhāna* (or *Sāmānya-viśeṣa*) *nyāya*, dating back not only to Kumārila's Tantravárttika (ca. 7th century CE), but even to the work of Śabara (ca. 3rd-5th c. CE). Since it allows to reason with norms that are subject to exceptions (*defeasible*), this principle excludes the monotonicity of the consequence relation $\vdash$, according to which $\Gamma \cup \{\mathcal{O}(\varphi/\psi)\} \cup \{\mathcal{O}(\neg\varphi/\psi \wedge \chi)\} \vdash \mathcal{O}(\varphi/\psi \wedge \chi \wedge \theta)$ follows from $\Gamma \cup \{\mathcal{O}(\varphi/\psi)\} \cup \vdash \mathcal{O}(\varphi/\psi \wedge \chi \wedge \theta)$.

The sequent rules we are going to introduce capture this kind of *non-monotonic* reasoning. They will prevent the conflicts by applying a restricted form of the aforementioned *strengthening of the antecedents* only to deontic assumptions (not to all the deontic formulas in the logic), using specificity/*Guṇapradhāna* to choose the enforceable command. Hence, in a sense, we leave the monotonicity of the consequence relation $\vdash$ for applying a (restricted) monotonicity on the conditions of the dyadic operators.

The property of monotonicity could indeed be understood in two connected but slightly different ways (see [95, 93, 17]), depending on which "level" it applies. On one hand, if the property of monotonicity concerns implicational formulas, as in the case of conditional logics and logics of counterfactuals (see Section 3.2), it corresponds to *strengthening of the antecedents* and goes under the name of *local* monotonicity. In that sense, we have observed that the operators of MD+ can be already considered (locally) non-monotonic.

On the other hand, the so called *global* monotonicity —which we simply refer to as "monotonicity"— concerns the consequence relation $\vdash$ and determines that a valid argument cannot be made invalid by adding new assumptions. As noticed above, adding rules capturing specificity to the system MD+ is tantamount to dropping this principle: an enforceable obligation $\mathcal{O}(\varphi/\psi \wedge \theta)$, which would follow from the deontic assumption $\mathcal{O}(\varphi/\psi)$ by monotoncity on the second argument of the operator, does not follow from the same assumption anymore if we add another deontic assumption $\mathcal{O}(\neg\varphi/\psi \wedge \theta)$.

Our approach involves the use of "special" sequent rules which allow to obtain all possible commands derivable by applying limited monotonicity on the conditions of (non-nested) deontic assumptions, "up to conflicting deontic statements", where the conflicts depend on the given set of facts. The addition of these rules to the system for MD+ allows us to better mimic Mīmāṃsā reasoning, as, in a sense, it introduces a restricted form of *local* monotonicity, renouncing to the *global* one. Indeed, as already observed in Section 3.2, the logic MD+ is so weak, that it does not give rise to inconsistencies, unless it is used for deriving commands from explicitly conflicting deontic assumptions that have incompatible contents and logically equivalent conditions, as in the case of $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\neg\varphi \wedge \theta/\psi)$. In contrast with this feature of the logic MD+, many examples of reasoning show that in fact Mīmāṃsā scholars considered the application of commands in conditions slightly different from the ones explicitly stated in the sacred texts; e.g. they discussed the enforceability of the prohibition *"a chariot maker* (i.e. a specific case of a member of the caste of *śūdra*s) *should not engage with the Veda"* based on the explicit Vedic command *"a śūdra should not engage with the Veda"*.

Basically, the idea is to formalize Mīmāṃsā reasoning applied for deciding which conditional norms can follow from the commands in the Vedas by adopting a cut-free sequent calculus. This allows the use of a limited form of monotonicity on the second argument of the deontic operators, for deriving only "undefeated" consequences of deontic assumptions. It is interesting to note that few aspects of Mīmāṃsā authors' reasoning, simulated by the sequent rules presented in this chapter, are shared by sequent-based mechanisms and cut-free Gentzen-style calculi developed in the context of research on non-monotonic reasoning in contemporary formal logic. Among the most important examples of such calculi, capturing a non-monotonic consequence relation, we have the calculi in [20] for a non-modal logic, in [60] for a deontic logic, in [121] for an argument-based system, and the systems developed in [35, 107] and [96]. The latter, in particular, apply the idea —also at the core of our system— of using a cut-free sequent calculus which prevents the derivation of overridden or invalidated conclusions, incompatible with other assumptions in sets called *control sets* or *defeater sets*. Moreover, similarly to our system, the sequent rules of the calculus in [20] make use of statements expressing that formulas are not derivable. As proved in [32], however, those *underivability statements* do not compromise the decidability of our system and do not affect its complexity, because of the postulation that the deontic assumptions constitute a closed set containing only non-nested formulas.

The design of our sequent system guarantees that, for any arbitrary condition, we can derive the enforceable (dyadic) commands from a given a set of Vedic norms and a set of facts. Moreover, it allows non-monotonic inferences only from deontic assumptions, while all the other derivations use the rules of the (globally) monotonic system MD+. This approach is inspired by the effort of Indian philosophers —in particular the Mīmāṃsā author Kumārila— to keep their arguments *"deductive* (hence not only analytical, but also monotonic) *as much as possible"* (see [127]). Indeed, the suspension of a command in exceptional circumstances indicated by a more specific one is read as an update of the readers' understanding of the original command, not as an update of the command itself (see Section 2.4.1). Hence the Indian philosophers' aim to keep their arguments as far as possible "deductive" does not contradict the very presence of principles like specificity/*Guṇapradhāna*.

4.2 Reasoning with global assumptions in MD

As anticipated, we start with the simplified logic MD, whose only modal operator is $\mathcal{O}(\cdot/\cdot)$.

Before presenting the sequent rules that capture the application of the Specificity/*Guṇapradhāna* principle in Mīmāṃsā and allow us to reason in presence of deontic assumptions and assumptions about the relations among facts, we introduce and clarify the key concepts which will be used through the chapter.

First, the notion of a condition being more specific than another one is interpreted –loosely following [54, p.281]– as the former implying the latter in the presence of (global) propositional assumptions. In other words, given a set $\mathfrak{F}$ of propositional formulas expressing *"facts"* about the world, we consider the proposition α *at least as specific* as the proposition β, if $\alpha \rightarrow \beta$ follows from $\mathfrak{F}$ in the logic MD.

As discussed in the previous section, a formal system suitable for representing Mīmāṃsā reasoning should derive a more specific obligation from a more general (unopposed) one only if the more general one is a deontic assumption, i.e. the (limited) monotonicity on the second argument of the deontic operator is applied only in the derivations from the set of assumptions. This means that the property of monotonicity on the second argument of the deontic operator does not hold for the derived statements in the logic, therefore $\mathcal{O}(\varphi/\psi \wedge \chi)$ cannot be obtained in the logic from the derived formula $\mathcal{O}(\varphi/\psi)$ (which is not a deontic assumption); hence also two formulas like $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\neg\varphi/\psi \wedge \chi)$, which are not deontic assumptions, cannot give rise to any conflict. This also implies that it is necessary to distinguish the deontic assumptions from the derived formulas: for the deontic operator $\mathcal{O}(\cdot/\cdot)$, we will write $\mathcal{O}_{\mathsf{pf}}(\cdot/\cdot)$ for deontic assumptions, i.e. *prima facie* commands[1]. The deontic assumptions constitute the formal representations of the explicit commands found in the *Veda*s (*Śrauta* norms).

Given the aforementioned interpretation of a condition being more specific than another one, the specificity principle can be interpreted as a constraint to the monotonicity of the prima facie deontic operators in their second arguments. For example, given the prima-facie obligation $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)\}$, i.e. a (translated, interpreted and formalized) explicit deontic statement in the *Veda*s, we can derive $\mathcal{O}(\varphi/\psi \wedge \chi \wedge \theta)$, while the same formula cannot be derived anymore from $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\varphi/\psi), \mathcal{O}_{\mathsf{pf}}(\neg\varphi/\psi \wedge \chi)\}$.

Let us further consider the following concrete example, which is a slightly simplified portion of a discussion found in Mīmāṃsā texts (ŚBh on PMS 6.1.12.44 —).

Example 4.2.1. Given the following Vedic deontic statements:

[1]In the following sections we will use a similar notation for the other operators of the logic MD+: for any deontic operator op, we will write $\mathsf{op}_{\mathsf{pf}}$ for deontic assumptions.

(a) *"A śūdra* (i.e., a member of the lowest cast, who carries out manual labour) *should not engage with the Vedas"* ($\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$);

(b) *"a chariot maker is a śūdra"* ($\mathtt{chmk} \to \mathtt{\acute{s}\bar{u}d}$);

(c) *"knowledge of the Vedas is necessary in order to perform sacrifices"* ($\mathtt{sacr} \to \mathtt{Ved}$).

By using monotonicity in the first argument (given by the axioms of the logic), we obtain

(a′) *"A śūdra should not perform any sacrifice"* ($\mathcal{O}(\neg\mathtt{sacr}/\mathtt{\acute{s}\bar{u}d})$).

Now, in order to show how it can generate contradictions, let us assume to have in the second argument unlimited downward monotonicity (see Section 3.1). By applying that rule, from (b) and (a′) we would obtain

(d) *"chariot makers should not perform any sacrifice"* ($\mathcal{O}(\neg\mathtt{sacr}/\mathtt{chmk})$).

However, (d) conflicts with another Vedic deontic statement:

(d′) *"chariot makers should perform a sacrifice"* ($\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$).

Assuming that this is the most accurate interpretation of the norms, then the most acceptable solution consists in suspending the efficacy of (a) for the case of chariot makers, i.e. ensuring that (d′) "blocks" the derivation of (d): this is tantamount to limiting monotonicity on the second argument of the operator.

Indeed, the $\mathsf{D}_{\mathcal{O}}$ axiom prevents conflicts among enforceable obligations (hence we could not have $\mathcal{O}(\neg\mathtt{sacr}/\mathtt{chmk})$ together with $\mathcal{O}(\mathtt{sacr}/\mathtt{chmk})$), and, because of the meaningfulness *nyāya* in Section 2.2, the *Veda*s are assumed not to contain any command which is straightforwardly inapplicable (hence we could not simply cancel the obligation (d′) $\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$).

This also means that the difference between prima facie deontic statements and derived ones is not only a technical requirement: the validity of deontic assumptions should be preserved as much as possible, while derived commands can be the result of a human error of interpretation and therefore can be cancelled.

In order to better understand the consequences of this characteristic, let us extend the example with another hypothetical condition, e.g. *being married* ($\mathtt{mar}$). Assuming that this condition is not mentioned in the texts, and adding to the system MD a limited form of monotonicity in the second argument of the deontic operator, from the premisses (a), (b), (c), (d′), it should be possible to derive the following obligations

(e) *"a married śūdra should not study the Vedas"* ($\mathcal{O}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar})$);

(f) *"a married śūdra who is a chariot maker should study the Vedas"* ($\mathcal{O}(\mathtt{Ved}/\mathtt{chmk} \wedge \mathtt{mar})$).

Let us consider the reasons why (f) should be derivable from (a), (b), (c), (d′) (similar reasons justify the derivability of (e)).

Because of (c), the action prescribed by (f) (*studying the Vedas*) is implied by the action prescribed by (d′), hence the obligation (f′) $\mathcal{O}(\mathtt{Ved}/\mathtt{chmk})$ would follow from (c) and (d′) by $\mathsf{Mon}_{\mathcal{O}}$. Moreover, the conditions of (f) (*being a married śūdra who is a chariot maker*) are more specific than the conditions of (d′), i.e. $\mathtt{chmk} \wedge \mathtt{mar} \rightarrow \mathtt{chmk}$ follows from the assumptions by using the rules of MD. Hence, the obligation (f) should follow from (d′) and (c) by $\mathsf{Mon}_{\mathcal{O}}$ and by monotonicity in the second argument of the deontic operator.

Considering a limited form of monotonicity in the second argument of the deontic operator, the derivation of (f) from (d′) (with the factual assumption (c)) should be possibly "blocked" by a conflicting deontic assumption, in the same way the prima facie command (d′) should "block" the derivation of (d) from (a) (with the factual assumptions (b) and (c)).

In this case, the derivation of (f) from (a), (b), (c), (d′) should not be "blocked", since there is no prima facie command which conflicts with (f) and such that its conditions are at most as specific as the ones of (f) and not less specific than any other prima facie command whose prescribed action implies the one of (f) (in this case only (d′)).

But, suppose that another Vedic statements is found, which explicitly supports one of the previously derivable ones, making it change its "status", e.g.

(e′) $\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar})$.

The conditions of the conflicting command (e′) are not less specific than the ones of (d′) (i.e. $\mathtt{chmk} \rightarrow \mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar}$ does not follow from the assumptions by using the rules of MD) and they are less specific than the ones of (f) (i.e. $\mathtt{chmk} \wedge \mathtt{mar} \rightarrow \mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar}$ follows from the assumptions by using the rules of MD). Hence, the obligation (f), together with (e), should be derivable from (a), (b), (c), (d′), but (f) should not be derivable anymore from (a), (b), (c), (d′), (e′), even if the obligation (e) changed only its "status" and not its content. ■

Remark 4.2.2 *The example above also represents a counterexample for the general rule of* Cautious Monotony*, one of the classical ones of non-monotonic logics [49]. According to this principle, the addition of a consequence to the set of its premisses does not reduce the set of conclusions derivable from that set of premisses. Writing* $\mid\!\sim$ *for a generic non-monotonic consequence relation and* φ, ψ, θ *for formulas, Cautious Monotony allows to infer* $\varphi, \psi \mid\!\sim \theta$ *from* $\varphi \mid\!\sim \psi$ *and* $\varphi \mid\!\sim \theta$*. Intuitively, as pointed out in [78], this principle ensures that if a derived statements is already assumed to hold, nothing changes in the system; i.e. the "status" of a statement which is known to be true by inference is not different from the "status" of a statement which is known to be true by assumption. The lack of this very intuitive rule is particularly interesting from the philosophical point of view. Indeed, it implies that the derived commands are normally not added or assimilated to the original norms in a given corpus of rules.*

Comparing Mīmāṃsā deontic reasoning with modern jurisprudence, we can see judicial decisions of courts as derived commands and the codes of laws as the sets of prima facie Vedic norms. In this sense, we could say that Mīmāṃsā reasoning is more similar to systems of civil law —which never consider judicial decisions as laws— than to the systems based on common law, where judicial decisions of courts are progressively included in the codes of laws.

What has been said highlights that, in order to decide whether a prescription is enforceable, we need to look at all the injunctions in the list $\mathfrak{L}_{\mathcal{O}}$ of deontic assumptions. The idea is to derive the enforceable obligation $\mathcal{O}(\alpha/\beta)$ if there is a prima facie injunction $\mathcal{O}_{\mathsf{pf}}(\gamma/\delta)$ in $\mathfrak{L}_{\mathcal{O}}$ such that α is implied by γ, β is at least as specific as δ, i.e. $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \beta \to \gamma$, and $\mathfrak{L}_{\mathcal{O}}$ does not contain any obligation which "blocks" the deduction of $\mathcal{O}(\alpha/\beta)$ from $\mathcal{O}_{\mathsf{pf}}(\gamma/\delta)$.

We say that the prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\gamma/\delta)$ is used as *base* in the derivation of $\mathcal{O}(\alpha/\beta)$ from the deontic assumptions.

The derivation of $\mathcal{O}(\alpha/\beta)$ from the base $\mathcal{O}_{\mathsf{pf}}(\alpha/\delta)$ is blocked by a more specific deontic assumption if $\mathfrak{L}_{\mathcal{O}}$ contains an obligation $\mathcal{O}_{\mathsf{pf}}(\epsilon/\zeta)$ such that the formulas α and ϵ are inconsistent (i.e., we can infer $\neg(\alpha \wedge \epsilon)$), β is at least as specific as ζ and ζ is at least as specific as δ.

The mechanism described above, however, is not yet suitable to derive only non conflicting commands. Indeed, it blocks the derivation of a command from a given deontic assumption (used as base) only if the latter conflicts with another more specific prima facie command; but it does not take into account the cases where another deontic assumption conflicts only with the command we want to derive and not with the given prima facie command used as base. Hence, for deriving $\mathcal{O}(\alpha/\beta)$, we need to add a condition stating that any other obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\lambda) \in \mathfrak{L}_{\mathcal{O}}$, such that $\mathfrak{F}$ entails $\beta \to \lambda$ and $\neg(\alpha \wedge \theta)$ in MD, is overruled by a more specific obligation $\mathcal{O}_{\mathsf{pf}}(\xi/\chi) \in \mathfrak{L}_{\mathcal{O}}$ supporting α and conflicting with θ.

Finally, since the operator $\mathcal{O}$ of MD is (upward) monotonic in its first argument, we would need to saturate the set of derivable commands under the rule $\mathsf{Mon}_{\mathcal{O}}$, i.e. apply the rule of monotonicity in the first argument for deriving all possible enforceable commands.

However, it is not clear whether we should first apply specificity for resolving conflicts among deontic assumptions and then saturate the resulting set of commands under the monotonicity rule (in line, e.g., with the suggested procedure for removing conflicts from specific logical structures in [83]) , or the rule should be applied to the initial commands, before removing conflicts. In order to understand what choosing between those two options entails, let us consider the following example.

Example 4.2.3. Let $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\varphi \wedge \psi/\theta), \mathcal{O}_{\mathsf{pf}}(\neg\varphi/\theta)\}$ be a list of deontic assumptions.

Since the two formulas $\varphi \wedge \psi$ and $\neg\varphi$ are incompatible, if we choose to first resolve conflicts in $\mathfrak{L}_{\mathcal{O}}$, the set of enforceable commands results in an empty list on which no saturation can be applied. Conversely, choosing to first saturate the original list $\mathfrak{L}_{\mathcal{O}}$ under monotonicity, we obtain many injunctions like e.g. $\mathcal{O}((\varphi \wedge \psi) \vee \neg\varphi/\theta)$, and then apply specificity for ruling out conflicts. In this case, for example, since $\mathcal{O}((\varphi \wedge \psi) \vee \neg\varphi/\theta)$ does not contradict $\mathcal{O}_{\mathsf{pf}}(\varphi \wedge \psi/\theta)$ or $\mathcal{O}_{\mathsf{pf}}(\neg\varphi/\theta)$, it is derivable as an enforceable command. This last approach certainly brings some counter-intuitive results, e.g. from $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\varphi/\theta), \mathcal{O}_{\mathsf{pf}}(\neg\varphi/\psi)\}$ we can derive $\mathcal{O}(\varphi \vee \neg\varphi/\theta \wedge \psi)$, that can be read as "under the conditions $\theta \wedge \psi$, it is obligatory to do something" with no indication of what to do. However, this choice has the advantage of blocking only the conflicting parts of a command —preserving as much as possible the power of deontic assumptions (Śrauta obligations)— and naturally lends itself to the application of *vikalpa*. Indeed, given two conflicting prima facie obligations $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ and $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$, and a condition under which in principle they both apply (i.e. a condition like $\psi \wedge \chi$, which is more specific than both ψ and χ), we can derive the command $\mathcal{O}(\varphi \vee \theta/\psi \wedge \chi)$ that prescribes to follow at least one of the original obligations.

As observed before, because of the *singleness nyāya* (*vākyabheda* in Sanskrit), it is not easy to find examples of commands of the form $\mathcal{O}_{\mathsf{pf}}(\varphi \wedge \psi/\theta)$ in Mīmāṃsā. However, such a formalization could be the interpretation of an obligation to perform a sacrifice together with a *viniyoga* prescription, i.e. an injunction that complements an action by specifying the instrument; e.g., the obligation *"one should sacrifice by using sour milk"* contains the prescription to perform a sacrifice (main action) and the instruction to use sour milk (*viniyoga* prescription). This obligation can be formalized as $\mathcal{O}_{\mathsf{pf}}(\texttt{sacr} \wedge \texttt{milk}/\top)$: in such cases, the discussions in Mīmāṃsā texts indicate that, even if the *viniyoga* command is contradicted or cannot be obeyed, the duty to perform the main action remains in place. Hence we should be able to derive $\mathcal{O}(\texttt{sacr}/\top)$ even in presence of $\mathcal{O}_{\mathsf{pf}}(\neg\texttt{milk}/\top)$. ■

Let us now give the formal definitions of the elements that will be employed in order to formalize the method sketched above. As a result, we will obtain a sequent calculus which derives obligations from a set of deontic assumptions, using specificity to solve the generated conflicts.

Definition 4.2.4 *Let the set $\mathfrak{F}$ of* (propositional) facts *be a finite set of atomic sequents (containing only propositional variables). The set $\mathfrak{F}$ is assumed to be* closed under contractions (i.e., if $(\Gamma, p, p \Rightarrow \Delta) \in \mathfrak{F}$ or $(\Gamma \Rightarrow p, p, \Delta) \in \mathfrak{F}$, then, respectively, $(\Gamma, p \Rightarrow \Delta) \in \mathfrak{F}$ or $(\Gamma \Rightarrow p, \Delta) \in \mathfrak{F}$) *and* closed under cuts (i.e., if $(\Gamma \Rightarrow \Delta, p) \in \mathfrak{F}$ and $(p, \Sigma \Rightarrow \Pi) \in \mathfrak{F}$, then $(\Gamma, \Sigma \Rightarrow \Delta, \Pi) \in \mathfrak{F}$). *Note that, since every propositional formula is equivalent to a formula* *in conjunctive normal form and sequents containing only propositional variables correspond to clauses of a formula in conjunctive normal form, using this definition we can stipulate arbitrary propositional formulas as facts.*

Let the list $\mathfrak{L}_{\mathcal{O}}$ *of* prima facie obligations *be a finite set* $\{\mathcal{O}_{\mathsf{pf}}(\varphi_1/\psi_1), \cdots, \mathcal{O}_{\mathsf{pf}}(\varphi_m/\psi_m)\}$ *of non-nested deontic formulas, i.e. such that for any* $1 \leq i \leq m$, φ_i, ψ_i *do not contain any deontic operator.*

In order for an obligation $\mathcal{O}(\varphi/\psi)$ to be derivable from a set $\mathfrak{F}$ of facts and a list $\mathfrak{L}_{\mathcal{O}}$ of prima facie obligations, the conditions (1) and (2) described below need to be satisfied.

Condition (1) requires that the derivable obligation $\mathcal{O}(\varphi/\psi)$ is implied (via upward monotonicity on the first argument and downward monotonicity on the second argument) by a less specific deontic assumption $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta) \in \mathfrak{L}_{\mathcal{O}}$ (referred to as *base*) and that there is no $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$ conflicting with the obligation we want to derive which is more general than that and more specific than the base. Implying $\mathcal{O}(\varphi/\psi)$ via upward monotonicity on the first argument and downward monotonicity on the second argument means that θ implies φ and ψ implies ζ, respectively; moreover the base should be such that there is no assumption "between" itself and the derivable formula which conflicts with the latter. However, there is no guarantee that an obligation can be derived only from one base, but rather we could derive the same command using as bases many different assumptions. Among other requirements, an obligation is derivable in the system if there is at least one deontic assumption which can be used as base for deriving such obligation.

Note that the clause *"there is no* $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$ *which is more specific than the base* $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta)$ *and more general than* $\mathcal{O}(\varphi/\psi)$*"* is equivalent to the universal statement *"for any* $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$, $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta)$ *is not more specific than* $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta)$ *or is not more general than* $\mathcal{O}(\varphi/\psi)$*"*. Hence, this first condition for the derivability of the obligation $\mathcal{O}(\varphi/\psi)$ can be formalized as follows:

(1) there is $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta) \in \mathfrak{L}_{\mathcal{O}}$ such that: $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \psi \Rightarrow \zeta$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \theta \Rightarrow \varphi$ and for all $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$ we have: $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \psi \Rightarrow \beta$ or $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \beta \Rightarrow \zeta$ or $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \alpha, \varphi \Rightarrow$
(where $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \Gamma \Rightarrow \Delta$ means that the sequent $\Gamma \Rightarrow \Delta$ is not derivable from the set of assumptions $\mathfrak{F}$ by using the rules of MD).

The condition above includes the choice mentioned in Ex.4.2.3. Indeed, it requires that any $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$ (is not more specific than $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta)$, or is not more general than $\mathcal{O}(\varphi/\psi)$ or) does not conflict with $\mathcal{O}(\varphi/\psi)$, instead of requiring that it does not conflict with $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta)$. This means that the conflicts are resolved by using specificity only after saturating the set of deontic assumptions under monotonicity.

Condition (2) expresses that, if $\mathcal{O}(\varphi/\psi)$ is derivable, then for any deontic assumption $\mathcal{O}_{\mathsf{pf}}(\eta/\tau)$ in the list $\mathfrak{L}_{\mathcal{O}}$ which conflicts with $\mathcal{O}(\varphi/\psi)$ and is less specific than that, there is at least another prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\chi/\xi)$ which overrules $\mathcal{O}_{\mathsf{pf}}(\eta/\tau)$. Such an obligation $\mathcal{O}_{\mathsf{pf}}(\chi/\xi)$ implies $\mathcal{O}(\varphi/\psi)$ (via upward monotonicity on the first argument and downward monotonicity on the second argument) and it is more specific than the conflicting assumption $\mathcal{O}_{\mathsf{pf}}(\eta/\tau)$. This second condition for deriving $\mathcal{O}(\varphi/\psi)$ is expressed by the following clause:

(2) for all $\mathcal{O}_{\mathsf{pf}}(\eta/\tau) \in \mathfrak{L}_{\mathcal{O}}$ we have: $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \psi \Rightarrow \tau$ or $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \eta, \varphi \Rightarrow$ or there is $\mathcal{O}_{\mathsf{pf}}(\chi/\xi) \in \mathfrak{L}_{\mathcal{O}}$ such that: $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \psi \Rightarrow \xi$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \xi \Rightarrow \tau$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \chi \Rightarrow \varphi$.

Example 4.2.5. Consider again the sentences in Ex.4.2.1:

(i) *"A śūdra* (i.e., a member of the lowest cast, who carries out manual labour) *should not engage with the Vedas"* ($\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$);

(ii) *"a chariot maker is a śūdra"* ($\mathtt{chmk} \rightarrow \mathtt{\acute{s}\bar{u}d}$);

(iii) *"knowledge of the Vedas is necessary in order to perform sacrifices"* ($\mathtt{sacr} \rightarrow \mathtt{Ved}$);

(iv) *"chariot makers should perform a sacrifice"* ($\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$).

Given a set of factual assumptions $\mathfrak{F} = \{\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}, \mathtt{sacr} \Rightarrow \mathtt{Ved}\}$ and a set of deontic assumptions $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d}), \mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})\}$, let us check if the obligation (v) *"chariot makers should study the Vedas"* ($\mathcal{O}(\mathtt{Ved}/\mathtt{chmk})$) can be derived from $\mathfrak{F}$ and $\mathfrak{L}_{\mathcal{O}}$, i.e. if it satisfies the clauses (1) and (2) above.

(1) The base $\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk}) \in \mathfrak{L}_{\mathcal{O}}$ is such that: $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \mathtt{chmk} \Rightarrow \mathtt{chmk}$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \mathtt{sacr} \Rightarrow \mathtt{Ved}$ and for $\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$ we have $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \mathtt{\acute{s}\bar{u}d} \Rightarrow \mathtt{chmk}$, and for $\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$ we have $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \mathtt{sacr}, \mathtt{Ved} \Rightarrow$.

(2) For $\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$ we have that there is $\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk}) \in \mathfrak{L}_{\mathcal{O}}$ such that: $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \mathtt{chmk} \Rightarrow \mathtt{chmk}$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \mathtt{sacr} \Rightarrow \mathtt{Ved}$. For $\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$ we have $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \nvdash \mathtt{sacr}, \mathtt{Ved} \Rightarrow$.

These two clauses confirm the intuition that (v) should be derivable from (i)–(iv) because:

• as required by condition (1), (v) is implied (via upward monotonicity on the first argument and downward monotonicity on the second argument) by (iv) and there is no other command which is more specific than (iv) and more general than (v);

• as required by condition (2), for any deontic assumption in the list $\mathfrak{L}_{\mathcal{O}}$ (i.e. (i) and (iv)), this does not conflict with (v), or it is not less specific than (v), or there is another prima facie obligation which overrules the conflicting one and implies (v) via upward monotonicity on the first argument. ■

Condition (2) hides another choice of interpretation: the overriding prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\chi/\xi)$ is not just inconsistent with $\mathcal{O}_{\mathsf{pf}}(\eta/\tau)$ and more specific than that, but it also supports the command $\mathcal{O}(\varphi/\psi)$ we are deriving. This choice is expressed by the condition $\mathfrak{F} \vdash \chi \Rightarrow \varphi$, instead of the weaker "condition of blocking the opponent" $\mathfrak{F} \vdash \chi, \eta \Rightarrow$.

Nonetheless, this choice is forced by the previous one of first saturating the set of deontic assumptions under monotonicity (on the first argument of the operator) before ruling out conflicts (see Ex. 4.2.3); in this case requiring only the weaker condition $\mathfrak{F} \vdash \chi, \eta \Rightarrow$ would indeed easily result in a contradiction, as the following example instantiates.

Example 4.2.6. If we change clause (2), not requiring that the deontic assumption $\mathcal{O}_{\mathsf{pf}}(\chi/\xi)$ which overrules the conflicting prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\eta/\tau)$ also implies $\mathcal{O}(\varphi/\psi)$, and requiring only the "condition of blocking the opponent" $\mathfrak{F} \vdash \chi, \eta \Rightarrow$, we would obtain a contradiction. To show that, let us consider an empty list $\mathfrak{F}$ of factual assumptions and the set $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\neg\varepsilon/\upsilon), \mathcal{O}_{\mathsf{pf}}(\varepsilon \wedge \gamma/\upsilon), \mathcal{O}_{\mathsf{pf}}(\neg\gamma \wedge \delta/\lambda), \mathcal{O}_{\mathsf{pf}}(\neg\delta/\lambda)\}$ of deontic assumptions.

Using $\mathcal{O}_{\mathsf{pf}}(\varepsilon \wedge \gamma/\upsilon)$ as base, we would derive $\mathcal{O}(\gamma/\upsilon \wedge \lambda)$. Indeed, the only prima facie command that conflicts with $\mathcal{O}(\gamma/\upsilon \wedge \lambda)$, is the obligation $\mathcal{O}_{\mathsf{pf}}(\neg\gamma \wedge \delta/\lambda)$, which is overruled by $\mathcal{O}_{\mathsf{pf}}(\neg\delta/\lambda)$; indeed, $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \upsilon \wedge \lambda \Rightarrow \lambda$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \lambda \Rightarrow \lambda$ and the "condition of blocking the opponent" holds, i.e. $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \neg\delta, \neg\gamma \wedge \delta \Rightarrow$ (while the stronger condition of supporting the derived obligation does not hold as it is not true that $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \neg\delta \Rightarrow \gamma$).

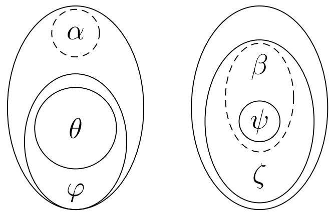
On the other hand, using $\mathcal{O}_{\mathsf{pf}}(\neg\gamma \wedge \delta/\lambda)$ as base, we would derive $\mathcal{O}(\neg\gamma/\upsilon \wedge \lambda)$. Again, there is only one prima facie command that conflicts with $\mathcal{O}(\neg\gamma/\upsilon \wedge \lambda)$, i.e. the obligation $\mathcal{O}_{\mathsf{pf}}(\varepsilon \wedge \gamma/\upsilon)$. But the latter is overruled by $\mathcal{O}_{\mathsf{pf}}(\neg\varepsilon/\upsilon)$, since $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \upsilon \wedge \lambda \Rightarrow \upsilon$ and $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \upsilon \Rightarrow \upsilon$ and the "condition of blocking the opponent" holds, i.e. $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \neg\varepsilon, \varepsilon \wedge \gamma \Rightarrow$. Notice, again, that the stronger condition of supporting the derived obligation does not hold: it is not true that $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}} \vdash \neg\varepsilon \Rightarrow \neg\gamma$.

Hence, requiring only the "condition of blocking the opponent", we would end up with the contradictory situation of deriving both $\mathcal{O}(\gamma/\upsilon \wedge \lambda)$ and $\mathcal{O}(\neg\gamma/\upsilon \wedge \lambda)$.

The deeper reasons for this behaviour lie in the interpretation of the overruling: given a list $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\varepsilon \wedge \delta\ /\upsilon), \mathcal{O}_{\mathsf{pf}}(\neg\varepsilon/\upsilon \wedge \gamma)\}$, the more specific obligation $\mathcal{O}_{\mathsf{pf}}(\neg\varepsilon/\upsilon \wedge \gamma)$ does not suspend the whole injunction $\mathcal{O}_{\mathsf{pf}}(\varepsilon \wedge \delta\ /\upsilon)$, but it only constitutes an exception to the conflicting part of it. This means that e.g. $\mathcal{O}(\delta/\upsilon \wedge \gamma)$ is derivable from $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\varepsilon \wedge \delta\ /\upsilon), \mathcal{O}_{\mathsf{pf}}(\neg\varepsilon/\upsilon \wedge \gamma)\}$. ■

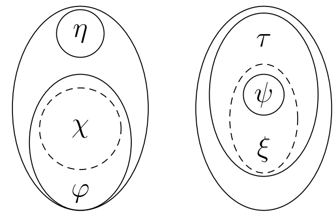
A graphical view of the two conditions above is in Fig. 4.1 and Fig. 4.2 ([31]): areas can be taken as formulas with containment representing entailment, i.e., more specific formulas are contained in less specific ones.

The conditions (1) and (2) above are translated into the sequent rules called *global assumption rules* (ga-rules $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}, \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$) in Fig.4.4. The name is due to the fact that



$\exists\, \mathcal{O}_{\mathsf{pf}}(\theta/\zeta) \in \mathfrak{L}_{\mathcal{O}}$ s.t. $\nexists\, \mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$

Figure 4.1: Condition (1). $\mathcal{O}(\varphi/\psi)$ is derivable if there is a deontic assumption $\mathcal{O}_{\mathsf{pf}}(\theta/\zeta)$ such that θ is more specific than φ (i.e. the area of θ is contained in the one of φ), ψ is more specific than ζ (i.e. ψ is contained in ζ), and for any $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta) \in \mathfrak{L}_{\mathcal{O}}$, ψ is not more specific (contained in) β, or β is not more specific (contained in) ζ, or α is not incompatible with φ (i.e. α is not contained in the complement of φ). Note that the dashed lines in this figure represent what should not be the case in order for $\mathcal{O}(\varphi/\psi)$ to be derivable.



$\forall\, \mathcal{O}_{\mathsf{pf}}(\eta/\tau) \in \mathfrak{L}_{\mathcal{O}}\ \exists\, \mathcal{O}_{\mathsf{pf}}(\chi/\xi) \in \mathfrak{L}_{\mathcal{O}}$

Figure 4.2: Condition (2). $\mathcal{O}(\varphi/\psi)$ is derivable if, for any $\mathcal{O}_{\mathsf{pf}}(\eta/\tau) \in \mathfrak{L}_{\mathcal{O}}$ such that η is incompatible with φ (i.e. η is contained in the complement of φ) and ψ is more specific than τ (i.e. ψ is contained in τ), there is another assumption $\mathcal{O}_{\mathsf{pf}}(\chi/\xi)$ such that χ implies (is contained in) φ and ξ is more specific than τ (i.e. ξ is contained in τ), but more general than ψ (i.e. ψ is contained in ξ). In this case the dashed lines indicate the conditions that should be verified in case there is a conflicting command $\mathcal{O}_{\mathsf{pf}}(\eta/\tau)$ with the mentioned characteristics.

those rules allow to reason from deontic and factual assumptions (in $\mathfrak{L}_{\mathcal{O}}$ and in $\mathfrak{F}$, respectively); such assumptions are called "global" because they are assumed to hold in every possible state of affairs (i.e. world in a model), not just in a particular one. As usual the L stands for left introduction rule and the R for right introduction rule; the superscript $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ indicates the deontic assumption used as base for the derivation. The rules $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ introduce a modal formula of the form $\mathcal{O}(\varphi/\psi)$ —on the left- and and on the right-hand side of a sequent, respectively— starting with the deontic assumption $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ used as base, and applying monotonicity on the second argument of the deontic operator "up to conflicting prima facie obligations". To write the rules in a concise way, following [32], we adopt the following notation:

Notation 4.2.7 *If $\mathcal{P}$ is a set of premisses, and $S = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$ is a set of sets of premisses, we write*

$$\frac{\mathcal{P} \cup [S]}{C}$$

for the set of rules

$$\left\{ \frac{\mathcal{P} \cup \mathcal{S}_1}{C}, \dots, \frac{\mathcal{P} \cup \mathcal{S}_n}{C} \right\}$$

Example 4.2.8. To understand how the notation works, consider the following rules, which allow to derive the sequent $\Gamma \Rightarrow \Delta$ from the premiss $\Theta \Rightarrow \Lambda$ and at least one of the premisses in the set $\{\Sigma_1 \Rightarrow \Pi_1,\ \Sigma_2 \Rightarrow \Pi_2,\ \Sigma_3 \Rightarrow \Pi_3\}$:

$$\frac{\Theta \Rightarrow \Lambda \quad \Sigma_1 \Rightarrow \Pi_1}{\Gamma \Rightarrow \Delta}\ r1 \qquad \frac{\Theta \Rightarrow \Lambda \quad \Sigma_2 \Rightarrow \Pi_2}{\Gamma \Rightarrow \Delta}\ r2 \qquad \frac{\Theta \Rightarrow \Lambda \quad \Sigma_3 \Rightarrow \Pi_3}{\Gamma \Rightarrow \Delta}\ r3$$

Using the notation defined above, we will write

$$\frac{\{\Theta \Rightarrow \Lambda\} \cup \begin{bmatrix} \Sigma_1 \Rightarrow \Pi_1 \\ \Sigma_2 \Rightarrow \Pi_2 \\ \Sigma_3 \Rightarrow \Pi_3 \end{bmatrix}}{\Gamma \Rightarrow \Delta}$$

Here we use the set notation for the premiss $\Theta \Rightarrow \Lambda$ to indicate the set containing the three copies of this premiss used in $r1$, $r2$, and $r3$. ■

For what concerns the statements in clauses (1) and (2) of the form "*for all* the deontic assumptions $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi) \in \mathfrak{L}_{\mathcal{O}}$ the sequent $\Gamma \Rightarrow \Delta$ is *not* derivable from the factual assumptions", they are called, for simplicity, *underivability* statements. From now on, we will write $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD}^{ga}\,cut}} \Gamma \Rightarrow \Delta$, indicating that $\Gamma \Rightarrow \Delta$ cannot be derived (Def.4.2.13) from the set $\mathfrak{F}$ of facts (on the basis of the set of deontic assumptions $\mathfrak{L}_{\mathcal{O}}$) by using the sequent system for the logic MD (in Fig.4.2), extended with the $\mathsf{ga}_{\mathfrak{F},\mathfrak{L}_{\mathcal{O}}}$ rules in Fig.4.4 and cut.

$$\frac{}{p \Rightarrow p}\ \mathsf{init} \qquad \frac{}{\bot \Rightarrow}\ \bot_L \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi)}\ \mathsf{Mon}_{\mathcal{O}} \qquad \frac{\varphi, \theta \Rightarrow \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow}\ \mathsf{D}_{\mathcal{O}}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{W}_L \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{W}_R \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{Con}_R$$

Figure 4.3: The calculus $\mathsf{G_{MD}}$

$$
\frac{\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} \\
\cup \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \eta, \varphi \Rightarrow \ \}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}} \right\} \\
\cup \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \eta, \varphi \Rightarrow \ \}\} \\ \cup \{\{\psi \Rightarrow \xi\} \cup \{\xi \Rightarrow \zeta\} \cup \{\tau \Rightarrow \varphi\} \mid \mathcal{O}_{\mathsf{pf}}(\tau/\xi) \in \mathfrak{L}_{\mathcal{O}}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}} \right\}
\end{array}}{\Rightarrow \mathcal{O}(\varphi/\psi)} \ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

$$
\frac{\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta, \varphi \Rightarrow \ \} \\
\cup \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \eta \Rightarrow \varphi)\}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}} \right\} \\
\cup \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \eta \Rightarrow \varphi\}\} \\ \cup \{\{\psi \Rightarrow \xi\} \cup \{\xi \Rightarrow \zeta\} \cup \{\tau, \varphi \Rightarrow \ \} \mid \mathcal{O}_{\mathsf{pf}}(\tau/\xi) \in \mathfrak{L}_{\mathcal{O}}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}} \right\}
\end{array}}{\mathcal{O}(\varphi/\psi) \Rightarrow} \ \mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

(where $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \Gamma \Rightarrow \Delta$).

Figure 4.4: The global assumption rules $\mathsf{ga}_{\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}}$ with $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}_{\mathcal{O}}$ as *base*. The rules allow to derive the formula $\mathcal{O}(\varphi/\psi)$ on the right and on the left hand side of a sequent, starting from the deontic assumption $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}_{\mathcal{O}}$, used as base.

Definition 4.2.9 *To clarify the rules' notation and to make it easier to refer to parts of them, we identify different blocks of premisses (here we use as example premisses of the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$, the terminology for the rule $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ is analogous).*

- *the first two premisses $\psi \Rightarrow \chi$ and $\theta \Rightarrow \varphi$ constitute the* standard block, *which has the role to guarantee that, in the absence of obstacles, the prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ is suitable for deriving the enforceable obligation $\mathcal{O}(\varphi/\psi)$.*
- *The premisses*

$$
\left\{ \left[\begin{array}{l} \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \psi \Rightarrow \zeta\}\big\} \\ \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \zeta \Rightarrow \chi\}\big\} \\ \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \eta, \varphi \Rightarrow \ \}\big\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}} \right\}
$$

  constitute the not-excepted block, *expressing that there are no obligations in $\mathfrak{L}_{\mathcal{O}}$ which are more specific and conflicting with respect to $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$. This set of premisses states that for every $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}}$, $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$ is not applicable (the obligation we want to* *derive is not more specific than $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$), or it is not more specific than the prescription used as base, or it does not conflict with the obligation we are deriving.*

- *The premisses*

$$\left\{\left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \eta,\varphi \Rightarrow\}\} \\ \{\{\psi \Rightarrow \xi\} \cup \{\xi \Rightarrow \zeta\} \cup \{\tau \Rightarrow \varphi\} \mid \mathcal{O}_{\mathsf{pf}}(\tau/\xi) \in \mathfrak{L}_{\mathcal{O}}\} \end{array}\right] \mid \mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}}\right\}$$

  constitute the no-active-conflict block*, which guarantees that there are no prima-facie obligations conflicting with the one we are deriving, such that they are not overruled by other more specific obligations in $\mathfrak{L}_{\mathcal{O}}$ that support $\mathcal{O}(\varphi/\psi)$. This set of premisses states that for every $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta) \in \mathfrak{L}_{\mathcal{O}}$, $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$ is not applicable, or it does not conflict with the obligation we are deriving, or there is another command $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$ which overrules $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$ and supports the obligation we are deriving.*
  This block is further divided in two (sub-)blocks:
  - *The first two underivability statements constitute the* no-conflict block*, which ensures that the formula $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$ does not conflict with the conclusion $\mathcal{O}(\varphi/\psi)$ or is not applicable ($\mathcal{O}(\varphi/\psi)$ does not apply in more specific situations than $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$).*
  - *The last three premisses constitute the* override block*, expressing that the deontic assumption $\mathcal{O}_{\mathsf{pf}}(\eta/\zeta)$ (possibly conflicting with $\mathcal{O}(\varphi/\psi)$) is overruled by a more specific prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$ that supports $\mathcal{O}(\varphi/\psi)$.*

Now, let us consider a simple example which is meant to give a general picture of how the "simplified" global assumption rules for obligations can be used for reasoning in Mīmāṃsā.

Example 4.2.10. Recall the statements in example Ex.4.2.1:

- *"A śūdra should not engage with the Veda"* ($\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$);
- *"a chariot maker is a śūdra"* ($\mathtt{chmk} \to \mathtt{\acute{s}\bar{u}d}$);
- *"knowledge of the Vedas is necessary in order to perform sacrifices"* ($\mathtt{sacr} \to \mathtt{Ved}$).
- *"chariot makers should perform a sacrifice"* ($\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$).

Their formalizations give the set of deontic assumptions $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk}), \mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})\}$ and the set of factual assumptions $\mathfrak{F} = \{\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d},\ \mathtt{sacr} \Rightarrow \mathtt{Ved}\}$.

Let us now consider the instance $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})}$ of the first scheme in Fig.4.4, where we use $\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$ as the base obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$.

$$
\frac{\begin{array}{l}
\{\psi \Rightarrow \mathtt{\acute{s}\bar{u}d}\} \quad \cup \quad \{\neg\mathtt{Ved} \Rightarrow \varphi\} \\[1ex]
\cup \left[\begin{array}{l}
\big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \mathtt{chmk}\}\big\} \\
\cup \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\big\} \\
\cup \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{sacr}, \varphi \Rightarrow\ \}\big\}
\end{array}\right] \\
\cup \left[\begin{array}{l}
\big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\big\} \\
\cup \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{\acute{s}\bar{u}d} \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\big\} \\
\cup \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \neg\mathtt{Ved}, \varphi \Rightarrow\ \}\big\}
\end{array}\right] \\[1ex]
\cup \left[\begin{array}{l}
\big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \mathtt{chmk}\}\big\} \\
\cup \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{sacr}, \varphi \Rightarrow\ \}\big\} \\
\cup \{\{\psi \Rightarrow \mathtt{chmk}\}\} \cup \{\{\mathtt{chmk} \Rightarrow \mathtt{chmk}\}\} \cup \{\{\mathtt{sacr} \Rightarrow \varphi\}\} \\
\cup \{\{\psi \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\} \cup \{\{\mathtt{\acute{s}\bar{u}d} \Rightarrow \mathtt{chmk}\}\} \cup \{\{\neg\mathtt{Ved} \Rightarrow \varphi\}\}
\end{array}\right] \\
\cup \left[\begin{array}{l}
\big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\big\} \\
\cup \big\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \neg\mathtt{Ved}, \varphi \Rightarrow\ \}\big\} \\
\cup \{\{\psi \Rightarrow \mathtt{chmk}\}\} \cup \{\{\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\} \cup \{\{\mathtt{sacr} \Rightarrow \varphi\}\} \\
\cup \{\{\psi \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\} \cup \{\{\mathtt{\acute{s}\bar{u}d} \Rightarrow \mathtt{\acute{s}\bar{u}d}\}\} \cup \{\{\neg\mathtt{Ved} \Rightarrow \varphi\}\}
\end{array}\right]
\end{array}}{\Rightarrow \mathcal{O}(\varphi/\psi)}\ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})}
$$

This rule can be used e.g. for deriving the obligation $\mathcal{O}(\neg\mathtt{sacr}/\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk})$:

$$
\frac{\begin{array}{l}
\{\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}\} \quad \cup \quad \{\neg\mathtt{Ved} \Rightarrow \neg\mathtt{sacr}\} \\[1ex]
\cup \big\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{chmk}\big\} \\
\cup \big\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \neg\mathtt{Ved}, \neg\mathtt{sacr} \Rightarrow\ \big\} \\[1ex]
\cup \big\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{chmk}\big\} \\
\cup \big\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \neg\mathtt{Ved}, \neg\mathtt{sacr} \Rightarrow\ \big\}
\end{array}}{\Rightarrow \mathcal{O}(\neg\mathtt{sacr}/\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk})}\ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})}
$$

However, since we now have a single application of the rule, we can also abandon the set notation for indicating the singletons of the premisses and right the rule application in the standard way:

$$
\frac{\begin{array}{l}
\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d} \qquad \neg\mathtt{Ved} \Rightarrow \neg\mathtt{sacr} \\
(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{chmk} \qquad (\text{for } \mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})) \\
(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \neg\mathtt{Ved}, \neg\mathtt{sacr} \Rightarrow \qquad (\text{for } \mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})) \\[1ex]
(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{chmk} \qquad (\text{for } \mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})) \\
(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\,\mathsf{cut}} \neg\mathtt{Ved}, \neg\mathtt{sacr} \Rightarrow \qquad (\text{for } \mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d}))
\end{array}}{\Rightarrow \mathcal{O}(\neg\mathtt{sacr}/\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk})}\ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})}
$$

(the indication of the deontic assumption under consideration next to the premisses in the *non-excepted* and *no-active-conflict* block are just meant to clarify the correspondence with the general form of the rule).

The notation above will be used in some of the following examples of rule application, when the roles of premisses in the rule are clear. ■

Note that, as in the whole system with prohibitions recommendations and permissions, the rules capturing specificity consider only prescriptions which directly block or support the obligation $\mathcal{O}(\varphi/\psi)$ in the conclusion. Hence, for the applicability of the rules, it does not make any difference if the command $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$ in the *override block* —which supports the obligation in the conclusion and defeats a less specific conflicting one— is, in turn, overruled by an obligation $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta)$ with a content α independent from φ. In order to consider all the possible chains of blocking (or supporting) prescriptions, we would need to add clauses for checking that a command $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$ in the *override block* is not blocked by a conflicting one $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta)$, at least as specific as $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$, or that $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$ is implied by another command $\mathcal{O}_{\mathsf{pf}}(\gamma/\delta)$ at least as specific as the conflicting one $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta)$. The same would need to be checked for such a supporting command $\mathcal{O}_{\mathsf{pf}}(\gamma/\delta)$ and so on: therefore, this chain would not be necessarily finite and apparently the system would not allow for cut-elimination.

Intuitively, this approach is based on the facts that, following Mīmāṃsā reasoning, we need to maximise the power of each deontic assumption in deriving commands, without admitting conflicts among such derived commands. The reasons lie in Mīmāṃsā scholars' effort in reducing the impact of conflicts. If we admitted that a command $\mathcal{O}_{\mathsf{pf}}(\tau/\xi)$ in the *override block* could be made unusable by a conflicting injunction $\mathcal{O}_{\mathsf{pf}}(\alpha/\beta)$, independent from the act enjoined by the obligation in the conclusion, then the number of derivable commands would be drastically reduced and, in a sense, this would give more importance to conflicts with respects to supports.

We present now the notion of derivation (valid *proto-derivation*), in presence of the global assumption rules $\mathsf{ga}_{\mathfrak{F},\mathfrak{L}_{\mathcal{O}}}$, i.e. the two sets of rules $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$.

Remark 4.2.11 *As will be shown in the analysis of the whole system (see Lem.4.3.11), the rules $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ for introducing a deontic formula on the left hand side of a sequent are needed for cut-free completeness, as they result from absorbing the $\mathsf{D}_{\mathcal{O}}$ axiom into the rules $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$.*

We write $\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}$ for the sequent calculus for the logic MD (in Fig.3.5) extended with the $\mathsf{ga}_{\mathfrak{F},\mathfrak{L}_{\mathcal{O}}}$ rules in Fig.4.4.

Definition 4.2.12 ([32]) *Let $\mathfrak{L}_{\mathcal{O}}$ and $\mathfrak{F}$ be as in Def.4.2.4. The rules in figure Fig.4.4 allow us to derive obligation formulas from $\mathfrak{L}_{\mathcal{O}}$ and $\mathfrak{F}$ by applying a limited form of monotonicity on the second argument of the deontic operator.*

A proto-derivation *from assumptions* $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$*, with conclusion* $\Gamma \Rightarrow \Delta$*, in the system* $\mathsf{G_{MD}}$ *extended with the rules* $\mathsf{ga}_{\mathfrak{F},\mathfrak{L}_{\mathcal{O}}}$*, is a finite labelled tree such that each internal node is labelled with a sequent and the label of every internal node is obtained from the labels of its children using the rules of* $\mathsf{G_{MD}}$ *or* $\mathsf{ga}_{\mathfrak{F},\mathfrak{L}_{\mathcal{O}}}$*, each leaf is labelled with an initial sequent, a sequent in* $\mathfrak{F}$*, or an* underivability statement $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \Sigma \Rightarrow \Pi$.

The notion of a proto-derivation in the system $\mathsf{G_{MD^{ga}}cut}$ *is defined analogously, but also permitting applications of the* cut rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{cut}\ .$$

The height *of a proto-derivation is the maximal length of a branch in the underlying tree plus one.*

For instance, the application of the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ in ex.4.2.10 represents a proto-derivation with conclusion $\Rightarrow \mathcal{O}(\neg\texttt{sacr}/\texttt{ś\=ud} \wedge \neg\texttt{chmk})$.

Now we define what it means for a proto-derivation (in Def.4.2.12) to be valid.

Definition 4.2.13 ([32]) *A proto-derivation from* $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$ *in the calculus* $\mathsf{G_{MD^{ga}}}$ *is* valid *if for each of the underivability statements* $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G_{MD^{ga}}cut}} \Sigma \Rightarrow \Pi$*, occurring as one of the leafs of that derivation, there is no valid proto-derivation of* $\Sigma \Rightarrow \Pi$ *in* $\mathsf{G_{MD^{ga}}cut}$ *from* $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$*; note that underivability statements are always evaluated in the system with the cut rule.*

If such a valid proto-derivation of a sequent $\Gamma \Rightarrow \Delta$ *does exist, we write* $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \vdash_{\mathsf{G_{MD^{ga}}}} \Gamma \Rightarrow \Delta$ *(or* $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \vdash_{\mathsf{G_{MD^{ga}}cut}} \Gamma \Rightarrow \Delta$ *if we are considering the system with the cut rule).*

To clarify the previous definition, we show below some simple examples of valid proto-derivations.

Example 4.2.14.

(a) Let us consider the following derivation:

$$\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\,\mathsf{init} \quad \dfrac{}{\bot \Rightarrow}\,\bot_L}{p, p \to \bot \Rightarrow}\,\to_L \quad \dfrac{}{q \Rightarrow q}\,\mathsf{init} \quad \dfrac{}{q \Rightarrow q}\,\mathsf{init}}{\mathcal{O}(p/q), \mathcal{O}(p \to \bot/q) \Rightarrow}\,\mathsf{D}_{\mathcal{O}}$$

It represents a proto-derivation, as each internal node is labelled with a sequent obtained from the sequents labelling the children of this node by using the rules of $\mathsf{G_{MD}}$ and each leaf is labelled with an initial sequent. Moreover, since the derivation does not contain any underivability statement, it is vacuously true that for each of the underivability statements occurring as a leaf there is no valid proto-derivation in $\mathsf{G_{MD^{ga}}cut}$.
Hence it represents a valid proto-derivation with conclusion $\mathcal{O}(p/q), \mathcal{O}(p \to \bot/q) \Rightarrow$.

(b) Let us consider the following derivation from the empty set $\mathfrak{F}$ of factual assumptions and the list of deontic assumptions $\mathfrak{L}_{\mathcal{O}}$ which contains only the obligation *"under any circumstance one must comply with the Vedic norms"* ($\mathcal{O}_{\mathsf{pf}}(\texttt{comply}/\top)$):

$$
\dfrac{\dfrac{\dfrac{\begin{array}{l}\{\texttt{śūd} \Rightarrow \top\} \qquad \{\texttt{comply} \Rightarrow \texttt{comply} \vee \texttt{work}\} \\ \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{comply}, \texttt{comply} \vee \texttt{work} \Rightarrow\} \\ \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{comply}, \texttt{comply} \vee \texttt{work} \Rightarrow\}\end{array}}{\Rightarrow \mathcal{O}(\texttt{comply} \vee \texttt{work/śūd})}\ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\texttt{comply}/\top)}}{p \to \bot \Rightarrow \mathcal{O}(\texttt{comply} \vee \texttt{work/śūd})}\ \mathsf{W}_L}{\Rightarrow (p \to \bot) \to \mathcal{O}(\texttt{comply} \vee \texttt{work/śūd})}\ \to_R
$$

According to the standard definitions of connectives, this represents a valid proto-derivation from$\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}$ with conclusion $\Rightarrow p \vee \mathcal{O}(\texttt{comply} \vee \texttt{work/śūd})$. Indeed it is a proto-derivation and for the underivability statement $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{comply}, \texttt{comply} \vee \texttt{work} \Rightarrow$ occurring as a leaf of that derivation, there cannot be a valid proto-derivation of $\texttt{comply}, \texttt{comply} \vee \texttt{work} \Rightarrow$ in $\mathsf{G_{MD^{ga}}cut}$ from $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$.

■

Remark 4.2.15 *Nested deontic formulas are not allowed as deontic assumptions, but it is possible to use the system* $\mathsf{G_{MD^{ga}}}$*for deriving such formulas, unlike, e.g., the known systems of Input/Output logic [88]. . Formulas of the form* $\mathcal{O}(\varphi/\mathcal{O}(\theta/\chi))$ *capture commands depending on other injunctions, like* "under the conditions of being obliged to pay taxes if one is self-employed, one should fill out the standardised form"*. However, the interpretation of nested deontic statements is not entirely clear, in particular in the first argument; for this reason, many deontic logics do not consider this kind of statements at all.*

Remark 4.2.16 *As the previous definition uses the notion of a valid proto-derivation for characterizing a valid proto-derivation, the definition might look circular. We will prove later, for the full system with the operators* $\mathcal{F}$ *and* $\mathcal{R}$ *and the permissions, that it is well-defined (Cor.4.3.17). Together with the decidability of the system, this result will follow from the* *redundancy of the cut rule (Thm.4.3.12). The (omitted) proof of cut-elimination (which can be found in [31]) for the restricted system* $\mathsf{MD}^{\mathsf{ga}}$ *represents a special case of Thm.4.3.12.*

Before moving to the full system for MD+, let us consider some example of how the calculus for MD extended with the global assumption rules can be used to mimic Mīmāṃsā reasoning.

Example 4.2.17. Let us consider again the statements in Ex.4.2.1.

We have a set of factual assumptions $\mathfrak{F} = \{\mathtt{chmk} \to \mathtt{\acute{s}\bar{u}d},\ \ \mathtt{sacr} \to \mathtt{Ved}\}$ and a list of deontic assumptions $\mathfrak{L}_{\mathcal{O}} = \{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})\ \ \mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})\}$.

We can show that the statement (i) *"a married śūdra should not study the Vedas"* ($\mathcal{O}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar})$) can be derived from $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}$ by using the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})}$:

$$\dfrac{\begin{array}{l}\{\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar} \Rightarrow \mathtt{\acute{s}\bar{u}d}\} \ \cup\ \{\neg\mathtt{Ved} \Rightarrow \neg\mathtt{Ved}\} \\ \\ \cup\ \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \neg\mathtt{Ved}, \neg\mathtt{Ved} \Rightarrow\ \} \\ \cup\ \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar} \Rightarrow \mathtt{chmk}\} \\ \\ \cup\ \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \neg\mathtt{Ved}, \neg\mathtt{Ved} \Rightarrow\ \} \\ \cup\ \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar} \Rightarrow \mathtt{chmk}\}\end{array}}{\Rightarrow \mathcal{O}(\neg\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar})}$$

Indeed the premisses in the standard block are derivable, recalling that the rule $\wedge_L$, as the other usual sequent rules for $\neg, \vee, \wedge$, is derivable using the rules in Fig.4.2 and the standard definitions of connectives in terms of $\bot, \to$ (see Def.3.1.1):

$$\dfrac{\dfrac{\dfrac{}{\mathtt{\acute{s}\bar{u}d} \Rightarrow \mathtt{\acute{s}\bar{u}d}}\,\mathsf{init}}{\mathtt{\acute{s}\bar{u}d}, \mathtt{mar} \Rightarrow \mathtt{\acute{s}\bar{u}d}}\,\mathsf{W}_L}{\{\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar} \Rightarrow \mathtt{\acute{s}\bar{u}d}\}}\,\wedge_L \qquad \dfrac{}{\{\neg\mathtt{Ved} \Rightarrow \neg\mathtt{Ved}\}}\,\mathsf{init}$$

Moreover, there is no valid proto-derivation in $\mathsf{G}_{\mathsf{MD}^{\mathsf{ga}}}\mathsf{cut}$ from $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$ for the two underivability statements: if $\neg\mathtt{Ved}, \neg\mathtt{Ved} \Rightarrow$ was derivable, $\mathfrak{F}$ would contain the contradiction $\neg\mathtt{Ved} \Rightarrow \mathtt{Ved}$, and, if $\mathtt{\acute{s}\bar{u}d} \wedge \mathtt{mar} \Rightarrow \mathtt{chmk}$ was derivable, $\mathfrak{F}$ would contain $\mathtt{\acute{s}\bar{u}d} \Rightarrow \mathtt{chmk}$ or $\mathtt{mar} \Rightarrow \mathtt{chmk}$, that are not actually in the set of factual assumptions.

From $\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}$ we can also derive the sentence (ii) *"a married śūdra who is a chariot maker should study the Vedas"* ($\mathcal{O}(\mathtt{Ved}/\mathtt{chmk} \wedge \mathtt{mar})$), by using the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})}$:

$$\{\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{chmk}\} \cup \{\texttt{sacr} \Rightarrow \texttt{Ved}\}$$

$$\cup \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{śūd} \Rightarrow \texttt{chmk}\}$$
$$\cup \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{sacr}, \texttt{Ved} \Rightarrow \}$$

$$\frac{\begin{array}{l}\cup \{\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{chmk}\} \cup \{\texttt{chmk} \Rightarrow \texttt{śūd}\} \cup \{\texttt{sacr} \Rightarrow \texttt{Ved}\} \\ \cup \{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{sacr}, \texttt{Ved} \Rightarrow \}\end{array}}{\Rightarrow \mathcal{O}(\texttt{Ved}/\texttt{chmk} \wedge \texttt{mar})}$$

Indeed, the sequents in the standard block and in the override (sub-)block are derivable:

$$\dfrac{\dfrac{\dfrac{}{\texttt{chmk} \Rightarrow \texttt{chmk}}\ \mathsf{init}}{\texttt{chmk}, \texttt{mar} \Rightarrow \texttt{chmk}}\ \mathsf{W}_L}{\{\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{chmk}\}}\ \wedge_L \qquad \dfrac{}{\{\texttt{sacr} \Rightarrow \texttt{Ved}\}}\ \mathfrak{F} \qquad \dfrac{}{\{\texttt{chmk} \Rightarrow \texttt{śūd}\}}\ \mathfrak{F}$$

and there is no valid proto-derivation in $\mathsf{G_{MD^{ga}}cut}$ from $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$ for the two underivability statements.

Note that in the application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\texttt{sacr/chmk})}$ above the override (sub-)block (containing the three premisses $\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{chmk}$, $\texttt{chmk} \Rightarrow \texttt{śūd}$and $\texttt{sacr} \Rightarrow \texttt{Ved}$) plays an essential role as it ensure that, for any prima facie obligation ($\mathcal{O}_{\mathsf{pf}}(\neg\texttt{Ved}/\texttt{śūd})$) which conflicts with the one we are deriving ($\mathcal{O}(\texttt{Ved}/\texttt{chmk} \wedge \texttt{mar})$), there is another applicable deontic statement ($\mathcal{O}_{\mathsf{pf}}(\texttt{sacr}/\texttt{chmk})$) that overrules the conflicting one.

This is the reason why, if the obligation (i) was assumed to be a deontic assumption ($\mathcal{O}_{\mathsf{pf}}(\neg\texttt{Ved}/\texttt{śūd} \wedge \texttt{mar})$) instead of a derived prescription, (ii) would not be derivable anymore; the no-active-conflict block relative to the formula $\mathcal{O}_{\mathsf{pf}}(\neg\texttt{Ved}/\texttt{śūd} \wedge \texttt{mar})$ would indeed be the following:

$$\left[\begin{array}{l}\{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{śūd} \wedge \texttt{mar}\}\} \\ \cup \{\{(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}}) \nvdash \neg\texttt{Ved}, \texttt{Ved} \Rightarrow \}\} \\ \cup \{\{\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{śūd}\} \cup \{\texttt{śūd} \Rightarrow \texttt{śūd} \wedge \texttt{mar}\} \cup \{\neg\texttt{Ved} \Rightarrow \texttt{Ved}\} \mid \mathcal{O}_{\mathsf{pf}}(\neg\texttt{Ved}/\texttt{śūd}) \in \mathfrak{L}_{\mathcal{O}}\} \\ \cup \{\{\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{chmk}\} \cup \{\texttt{chmk} \Rightarrow \texttt{śūd} \wedge \texttt{mar}\} \cup \{\texttt{sacr} \Rightarrow \texttt{Ved}\} \mid \mathcal{O}_{\mathsf{pf}}(\texttt{sacr}/\texttt{chmk}) \in \mathfrak{L}_{\mathcal{O}}\} \\ \cup \{\{\texttt{chmk} \wedge \texttt{mar} \Rightarrow \texttt{śūd} \wedge \texttt{mar}\} \cup \{\texttt{śūd} \wedge \texttt{mar} \Rightarrow \texttt{śūd} \wedge \texttt{mar}\} \cup \{\neg\texttt{Ved} \Rightarrow \texttt{Ved}\} \mid \mathcal{O}_{\mathsf{pf}}(\neg\texttt{Ved}/\texttt{śūd} \wedge \texttt{mar}) \in \mathfrak{L}_{\mathcal{O}}\}\end{array}\right]$$

Here it is easy to observe that there are valid proto-derivations in $\mathsf{G_{MD^{ga}}cut}$ from $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$ for the two underivability statements and no one of the triples of premisses in the last three lines contains only derivable sequents, as the conflicting deontic assumption $\mathcal{O}_{\mathsf{pf}}(\neg\texttt{Ved}/\texttt{śūd} \wedge \texttt{mar})$ is actually not overridden.

This example highlights the fact that the already mentioned principle of *cautious monotonicity* (Rmk.4.2.2) does not hold for the calculus GMD extended with the global assumption rules in Fig.4.2. This means that the only obligations which need to be considered (and possibly overridden) when deriving prescriptions via limited monotonicity on the second argument of the deontic operator are the ones in the list $\mathfrak{L}_{\mathcal{O}}$: from Mīmāṃsāperspective, a command which depends on human reasoning cannot block the enforceability of a Vedic norm.

Moreover, this reflects the fact that Mīmāṃsā authors assume that the Vedas do not contain useless commands, i.e. also commands which were already derivable without the need of an explicit deontic statement. Hence, if something which was already inferable is explicitly stated in the sacred texts, it must convey a content which is slightly different, stronger, or broader with respect to the corresponding derivable command. ■

4.3 Reasoning with global assumptions in MD+

We now consider the logic MD+ with the deontic operators for obligations ($\mathcal{O}(\cdot/\cdot)$), prohibitions ($\mathcal{F}(\cdot/\cdot)$) and recommendations ($\mathcal{R}(\cdot/\cdot)$). We extend the sequent calculus for the system MD+ (Fig.3.4.1) with the global assumption rules. The resulting calculus allows commands to be derived from a set of prima facie norms and in the presence of propositional global assumptions, using specificity for conflict resolution. We call the resulting system $\mathsf{MD+}^{\mathsf{ga}}$.

The introduction of global assumption rules, incorporating specificity, for the full system MD+, with all kinds of deontic assumptions, follows the same principles as for the case with only obligations. In this case it is necessary to consider how the deontic concepts are characterized and their mutual interactions.

Recommendations Let us consider first the notion of weak obligation, or recommendation: this turns out to be the simplest case, as the operator $\mathcal{R}(./.)$ does not interact with the other deontic operators, there are no permissions for it, and it is not restricted by the D axiom (Section 3.4). This means deriving conflicting recommendations $\mathcal{R}(\varphi/\psi), \mathcal{R}(\neg\varphi/\psi)$ is possible, provided that $\mathcal{R}(\bot/\chi)$ is not derivable for any formula χ.

Indeed, the operator for recommendations is characterized by the axiom P $((\varphi \to \bot) \to \neg(\mathcal{R}(\varphi/\psi)))$, expressing that the *Veda*s never recommend anything which is self-contradictory, and hence we only need to consider one global assumption rule (Fig.4.5), which allows us to rule out the sole prima facie recommendations $\mathcal{R}(\varphi/\psi)$ such that $\mathfrak{F}, \mathfrak{L} \vdash \varphi \to \bot$.

Before introducing the new global assumption rules for obligations and prohibitions, let us recall the concept of **permission** (Section 2.3.3) in Mīmāṃsā reasoning. Since their contents are usually actions that the agents would be already inclined to do, permissions cannot be

$$\frac{\psi \Rightarrow \chi \qquad \theta \Rightarrow \varphi \qquad (\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \theta \Rightarrow}{\Rightarrow \mathcal{R}(\varphi/\psi)}\ \mathcal{R}_R^{\mathcal{R}_{\mathsf{pf}}(\theta/\chi)}$$

Figure 4.5: The global assumption rules for recommendations, based on the prima-facie recommendation $\mathcal{R}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ ([32])

read as proper commands and represent only explicit exceptions to other commands. For this reason, they do not correspond to operators in MD+.

However, precisely as exceptions, they need to be considered at the level of deontic assumptions. Unlike other prima facie commands, a permission does not overrule a norm by commanding an opposite action, in such a way that one cannot keep following the overridden command and be compliant with the set of norms. Instead, permissions identify states where a command does not apply and the agent has in principle complete freedom of choice with respect to the performance of the previously obligatory (or prohibited) action. Hence, there is no permission relative to a recommendation, as the choice to perform the actions they prescribe is always considered free and not following norms of this kind does not lead to any form of undesirable result. It is important to emphasize the fact that permissions represent only exceptions to other previously stated commands, hence they do not necessarily correspond to the intuition behind permissions in natural language; the interpretation of permissions as exceptions has been studied by many authors in the field of deontic logic, e.g. [19]. As it will be shown later, this can have the counterintuitive consequences of making an action both obligatory and permitted under certain circumstances. E.g. when a permission "blocks" a command which in turn was blocking another injunction, the permission reinstates and supports the latter injunction, making the enjoined action both permitted and obligatory (or prohibited). The problems of such situations, where more specific permissions restore the enforceability of more general commands, have been analysed e.g. in [119].

The new list $\mathfrak{L}$ of deontic assumptions for MD+ contains not only prima facie obligations $\mathcal{O}_{\mathsf{pf}}(\cdot/\cdot)$, but also prima facie prohibitions $\mathcal{F}_{\mathsf{pf}}(\cdot/\cdot)$, prima facie recommendations $\mathcal{R}_{\mathsf{pf}}(\cdot/\cdot)$, exceptions to obligations, called *obligation-permissions* $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(./.)$, and exceptions to prohibitions (*prohibition-permissions*) $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(./.)$.

Even if permissions represent exceptions targeted to the corresponding operators, the behaviour of each operator for permission is not the same as the matching deontic operator, i.e. both $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(./.)$ and $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(./.)$ are upward monotone like $\mathcal{O}(./.)$, while $\mathcal{F}(./.)$ is downward monotone. Indeed, any permission —be it an exception to an obligation or to a prohibition— allows some agents to perform an action α which was previously prohibited or such that its opposite was obligatory.

Hence, any act β which is necessary for performing the permitted action ($\alpha \rightarrow \beta$) need to be allowed too. On the other hand, an act γ that is just sufficient to cause the permitted action ($\gamma \rightarrow \alpha$), but can be avoided, intuitively is not included in the permission.

Upward monotonicity in the first argument is then associated with deontic statements that give (any kind of) "positive reinforcement" for an action, while downward monotonicity characterize commands (prohibitions) which express "negative reinforcement". The fact that all permissions are in the first group reflects their formulation in the sacred texts: looking only at its grammatical form, a permission can hardly be distinguished from a prescription (*vidhi*).

For a better understanding of what that means from the "operational" point of view, let us consider the following example.

Example 4.3.1. Suppose to have the following deontic statements:

(a) *"one should sacrifice within a given ritual ψ, using three specific materials for oblation and calling the four priests concerned"* ($\mathcal{O}_{\mathsf{pf}}(\texttt{3Oblations} \wedge \texttt{4Priests/ritual}\psi)$);

(b) *"if one is performing the ritual ψ in certain conditions χ, it is permitted not to use the three specific materials for oblation or not to call the four priests concerned "* ($\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\neg\texttt{3Oblations} \vee \neg\texttt{4Priests/ritual}\psi \wedge \texttt{condition}\chi)$).

Since the obligation-permissions are upward monotonic, from the permission (b) the exemptions from using the materials for oblation and calling the four priests concerned should not follow. What is actually permitted is to choose not to perform one of the two actions, which means that their conjunction should not be considered obligatory given the further condition χ; however it is not said that one specific action is not obligatory anymore.

In other words, (b) only should block the derivation of $\mathcal{O}(\texttt{3Oblations}\wedge\texttt{4Priests/ritual}\psi\wedge \texttt{condition}\chi)$ for the conjunction of the two actions under conditions χ, but the obligations $\mathcal{O}(\texttt{3Oblations/ritual}\psi \wedge \texttt{condition}\chi)$ and $\mathcal{O}(\texttt{4Priests/ritual}\psi \wedge \texttt{condition}\chi)$ following from (a) should still be derivable.

This also shows how it is problematic to add the *aggregation principle* (see Ch.3 Section 3.2) to the system $\mathsf{MD}+^{\mathsf{ga}}$. Indeed, according to this principle, in presence of the rule $\mathsf{Mon}_{\mathcal{O}}$, the derivability of the two obligations $\mathcal{O}(\texttt{3Oblations/ritual}\psi \wedge \texttt{condition}\chi)$ and $\mathcal{O}(\texttt{4Priests/ritual}\psi \wedge \texttt{condition}\chi)$ would be equivalent to deriving $\mathcal{O}(\texttt{3Oblations} \wedge \texttt{4Priests/ritual}\psi \wedge \texttt{condition}\chi)$.

On the other hand, a permission (b′) $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\neg\mathtt{3Oblations} \wedge \neg\mathtt{4Priests}/\mathtt{ritual}\psi \wedge \mathtt{condition}\chi)$, allowing to avoid both the previously obligatory actions under conditions χ, would block not only the obligation $\mathcal{O}(\mathtt{3Oblations} \wedge \mathtt{4Priests}/\mathtt{ritual}\psi \wedge \mathtt{condition}\chi)$, but also all the obligations prescribing conflicting actions, whose conditions are implied by the conjunction $\mathtt{ritual}\psi \wedge \mathtt{condition}\chi$. ■

The situation is different for $\mathcal{F}_{\mathsf{pf}}(\cdot/\cdot)$ and $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\cdot/\cdot)$, as a formula like $\mathcal{F}_{\mathsf{pf}}(\varphi \wedge \theta/\psi)$ would mean only that the performance of both actions corresponding to φ and θ together is forbidden; hence, permitting their disjunction does not block any derivation from $\mathcal{F}_{\mathsf{pf}}(\varphi \wedge \theta/\psi)$, while permitting the conjunction of φ and θ would not block the prohibition of doing φ, θ and ξ together.

To clarify the behaviour described above, consider the following simple example, not related to Mīmāṃsā texts.

Example 4.3.2.

(i) *"one must not open the bank vault and enter"* ($\mathcal{F}_{\mathsf{pf}}(\mathtt{open\_vault} \wedge \mathtt{enter\_vault}/\top)$);

(ii) *"it is permitted to open the bank vault and enter if one is the bank manager"* ($\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\mathtt{open\_vault} \wedge \mathtt{enter\_vault}/\mathtt{manager})$).

Intuitively, the permission (ii) should block the prohibition (i) for the case where the person who opens the bank vault and enters is the manager.

However, (ii) should not block the derivation of the prohibition (iii) *"it is forbidden to open the bank vault and enter and wear a suit, if one is the bank manager"* ($\mathcal{F}(\mathtt{open\_vault} \wedge \mathtt{enter\_vault} \wedge \mathtt{suit}/\mathtt{manager})$).

This choice is motivated by the previous observations on the nature of permissions as "positive reinforcement" for a course of action: a prohibited action like *opening the bank vault and entering and wearing a suit*, that implies the permitted action but is not necessary for it, should not be included in the permission. This consideration could be made more intuitive by substituting the new conjunct "wear a suit" with a different action: (iii′) *"it is forbidden to open the bank vault and enter and steal money, even if one is the bank manager"* ($\mathcal{F}(\mathtt{open\_vault} \wedge \mathtt{enter\_vault} \wedge \mathtt{steal}/\mathtt{manager})$). ■

According to the example above, we need to define the formal properties of permissions in such a way that the relation of $\mathcal{O}_{\mathsf{pf}}(\cdot/\cdot)$ with $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\cdot/\cdot)$ and the relation of $\mathcal{F}_{\mathsf{pf}}(\cdot/\cdot)$ with $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\cdot/\cdot)$ are asymmetric. This means that an obligation-permission $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\neg\varphi/\psi)$ should attack all the consequences of an obligation $\mathcal{O}_{\mathsf{pf}}(\varphi/\chi)$ under condition $\psi \wedge \chi$, making sure that the obligation cannot be used for deriving anything under those circumstances.

Conversely, permissions-prohibition should, in a sense, "propagate in the opposite direction" with respect to prohibitions: an exception $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\varphi/\psi)$ to the command $\mathcal{F}_{\mathsf{pf}}(\varphi/\chi)$ should block the derivation of $\mathcal{F}(\varphi/\psi \wedge \chi)$, but it should allow the prima facie prohibition to be used for deriving other weaker consequences.

Obligations and Prohibitions This case is more complex than the case of rules $\mathsf{ga}_{\mathfrak{F},\mathfrak{L}_{\mathcal{O}}}$ in the previous section. This is due to the fact that those two operators not only are characterized by the axioms $\mathsf{D}_{\mathcal{O}}$ and $\mathsf{D}_{\mathcal{F}}$, respectively, meaning that conflicts should be ruled out by specificity, but they have also explicit exceptions ($\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(./.)$ and $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(./.)$, respectively), and, above all, they can interact. The presence of the axiom $\mathsf{D}_{\mathcal{O}\mathcal{F}}$, indeed, states that an obligation and a prohibition that are in conflict with each other should not be derivable: this means that obligations and prohibitions overrule each other according to the specificity principle.

Given the previous considerations, the construction of the global assumption rules should adhere to the following constraints:

(a) More specific conflicting obligations (resp. prohibitions) overrule less specific obligations (resp. prohibitions);

(b) More specific conflicting obligations overrule less specific prohibitions and vice versa;

(c) Exceptions to obligations (resp. prohibitions), i.e. obligation-permissions $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(./.)$ (resp. prohibitions-permissions $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(./.)$), override less specific obligations (resp. prohibitions), but have no relevance for the other operators.

(a) is given by the axiom $\mathsf{D}_{\mathcal{O}}$ (resp. $\mathsf{D}_{\mathcal{F}}$), stating that the system should not be able to derive two conflicting obligations (resp. prohibitions).

(b) is due to the axiom $\mathsf{D}_{\mathcal{O}\mathcal{F}}$, according to which, as mentioned, $\mathcal{O}(\varphi/\psi)$ and $\mathcal{F}(\varphi/\psi)$ are not derivable together.

The condition (c) is given by the already mentioned interpretation of permissions: they are meant to be "targeted" at identifying exceptions to particular norms and not to apply to all kinds of commands.

Following those constraints and the principle already applied in the previous section, we define the structures of the right rules for obligations and prohibitions.

Definition 4.3.3 (Structure of the right global assumption rules) *Consider a list $\mathfrak{L}$ of non-nested deontic formulas that contains (prima-facie) obligations, (prima-*

facie) prohibitions, (prima-facie) recommendations, obligation-permissions and prohibition-permissions, and let $\mathfrak{F}$ be a set of propositional facts as in Def.4.2.4.
An obligation $\mathcal{O}(\varphi/\psi)$ *follows from $\mathfrak{F}$ on the basis of $\mathfrak{L}$ if and only if:*

(1) *there is a prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ (the "base") in $\mathfrak{L}$ such that:*
(1.a) *the content of the obligation $\mathcal{O}(\varphi/\psi)$ we want to derive is supported by the prima-facie obligation used as base $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ (i.e. $(\mathfrak{F},\mathfrak{L}) \vdash \theta \Rightarrow \varphi$) and the condition ψ is at least as specific as χ (i.e. $(\mathfrak{F},\mathfrak{L}) \vdash \psi \Rightarrow \chi$)*
and
(1.b) *the prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ is not overruled by a more specific conflicting prima-facie obligation and it is not overridden by an obligation-permission, i.e., for every $\mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$ or $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$:*
$(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta$ *or* $(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi$ *or* $(\mathfrak{F},\mathfrak{L}) \nvdash \tau,\varphi \Rightarrow$
and
the prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ is not overruled by a more specific conflicting prima-facie prohibition, i.e., for every $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$:
$(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta$ *or* $(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi$ *or* $(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau$

(2) *For each prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$*
(2.a) *$\mathcal{O}_{\mathsf{pf}}(\tau/\zeta)$ does not conflict with $\mathcal{O}(\varphi/\psi)$, that we are deriving (i.e. $(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta$ or $(\mathfrak{F},\mathfrak{L}) \nvdash \tau,\varphi \Rightarrow$)*
or
(2.b) *$\mathcal{O}_{\mathsf{pf}}(\tau/\zeta)$ is overruled by a more specific obligation, obligation-permission, or prohibition supporting $\mathcal{O}(\varphi/\psi)$, that we are deriving. This means that:*
– there is $\mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$ or there is $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$ such that:
$(\mathfrak{F},\mathfrak{L}) \vdash \psi \Rightarrow \eta$ *and* $(\mathfrak{F},\mathfrak{L}) \vdash \eta \Rightarrow \zeta$ *and* $(\mathfrak{F},\mathfrak{L}) \vdash \xi \Rightarrow \varphi$
or
– there is $\mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$ such that:
$(\mathfrak{F},\mathfrak{L}) \vdash \psi \Rightarrow \eta$ *and* $(\mathfrak{F},\mathfrak{L}) \vdash \eta \Rightarrow \zeta$ *and* $(\mathfrak{F},\mathfrak{L}) \vdash \Rightarrow \varphi,\xi$

(3) *For each prima facie prohibition $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$:*
(3.a) *$\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ does not conflict with $\mathcal{O}(\varphi/\psi)$ (i.e. $(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta$ or $(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau$)*
or
(3.b) *$\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ is overruled by a more specific obligation, prohibition, or prohibition-permission. This means that*
– there is $\mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$ such that:
$(\mathfrak{F},\mathfrak{L}) \vdash \psi \Rightarrow \eta$ *and* $(\mathfrak{F},\mathfrak{L}) \vdash \eta \Rightarrow \zeta$ *and* $(\mathfrak{F},\mathfrak{L}) \vdash \xi \Rightarrow \varphi$

or

– *there is* $\mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$ *such that:*

$(\mathfrak{F}, \mathfrak{L}) \vdash \psi \Rightarrow \eta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash \eta \Rightarrow \zeta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash \; \Rightarrow \varphi, \xi$

or

– *there is* $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$ *such that:*

$(\mathfrak{F}, \mathfrak{L}) \vdash \psi \Rightarrow \eta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash \eta \Rightarrow \zeta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash \xi \Rightarrow \varphi$

(4) *For each prima facie obligation-permission* $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)$*:*

(4.a) $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)$ *does not constitute an exception to* $\mathcal{O}(\varphi/\psi)$

(i.e. $(\mathfrak{F}, \mathfrak{L}) \nvdash \psi \Rightarrow \zeta$ *or* $(\mathfrak{F}, \mathfrak{L}) \nvdash \varphi, \tau \Rightarrow$*)*

or

(4.b) $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)$ *is overruled by a more specific obligation. This means that*

– *there is a prima facie obligation* $\mathcal{O}_{\mathsf{pf}}(\xi/\eta)$ *such that:*

$(\mathfrak{F}, \mathfrak{L}) \vdash \psi \Rightarrow \eta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash \eta \Rightarrow \zeta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash \xi \Rightarrow \varphi$

The structure of the right global assumption rule for prohibitions is defined analogously, paying attention to the definitions of conflicts, as the operator $\mathcal{F}(\cdot/\cdot)$ *is downward monotonic in its first argument.*

Remark 4.3.4 *Notice that the formalization of the notion of conflicting norms depends on the types of prima facie deontic operators involved; two commands are conflicting if their contents (the formulas they have as first arguments) are incompatible in the following sense:*

- *The contents of two obligations* $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ *and* $\mathcal{O}_{\mathsf{pf}}(\tau/\zeta)$ *(or of an obligation* $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ *and an obligation-permission* $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)$*) are incompatible if they cannot be true at the same time, i.e. we can derive* $\neg(\varphi \wedge \tau)$ *(or, equivalently, the sequent* $\varphi, \tau \Rightarrow$ *) from the global assumptions.*
- *The contents of two prohibitions* $\mathcal{F}_{\mathsf{pf}}(\varphi/\psi)$ *and* $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ *are incompatible if it is impossible to avoid them at the same time, i.e. it is impossible for them to be both false at the same time. Hence* $\mathcal{F}_{\mathsf{pf}}(\varphi/\psi)$ *and* $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ *are conflicting if the formula* $\varphi \vee \tau$ *(resp. the sequent* $\Rightarrow \varphi, \tau$*) follows from the global assumptions.*
- *An obligation* $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ *and a prohibition* $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ *conflict if avoiding the forbidden action means avoiding also the obligatory one, or, equivalently, if bringing about the obligatory state also brings about the forbidden one. Those conditions are expressed by the possibility of deriving from the global assumptions the formula* $\varphi \rightarrow \tau$*, or, equivalently, the sequent* $\varphi \Rightarrow \tau$*.*
- *A prohibition-permission* $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\varphi/\psi)$ *conflicts with a prohibition* $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ *if performing*

$$
\frac{\begin{array}{ll}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} & \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow \}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow \}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \xi, \varphi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}}{\Rightarrow \mathcal{O}(\varphi/\psi)} \ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ and $(\mathfrak{F},\mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \Gamma \Rightarrow \Delta$.

Figure 4.6: The global assumption rules for obligations in $\mathsf{MD+}$.

$$
\frac{
\begin{array}{ll}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\varphi \Rightarrow \theta\} & \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \Rightarrow \tau, \varphi\}\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \Rightarrow \tau, \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{\Rightarrow \mathcal{F}(\varphi/\psi)} \; \mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ and $(\mathfrak{F},\mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}\,\mathsf{cut}}} \Gamma \Rightarrow \Delta$.

Figure 4.7: The right global assumption rules for prohibitions in MD+.

the permitted action means not avoiding the forbidden one, i.e. the formula $\varphi \to \tau$ resp. the sequent $\varphi \Rightarrow \tau$ is derivable from global assumptions.

On the basis of the structures in Def.4.3.3 and using the notation of 4.2.7 in the previous section, we build the right global assumption rules for obligation and prohibitions (Fig.4.6 and 4.7, respectively).

Again, for the rules in Fig.4.6 and 4.7, we classify the premisses by using the *blocks* defined in the previous section (Def.4.2.9). In the case of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (the situation for $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ is analogous), we have the following *blocks*.

- The *standard block*, consisting of the first two premisses $\psi \Rightarrow \chi$ and $\theta \Rightarrow \varphi$ and corresponding to (1.a) in Def.4.3.3. The premisses in this block state that the prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ is suitable for deriving $\mathcal{O}(\varphi/\psi)$.
- The *not-excepted block*, corresponding to (1.b) in Def.4.3.3 and consisting of the sets of underivability statements

$$\left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \zeta \Rightarrow \chi\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \tau, \varphi \Rightarrow \}\} \end{bmatrix} \,\middle|\, \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}$$

$$\left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \zeta \Rightarrow \chi\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \varphi \Rightarrow \tau\}\} \end{bmatrix} \,\middle|\, \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}$$

  The premisses in this block ensure that there is no relevant deontic statement which conflicts with $\mathcal{O}(\varphi/\psi)$ and overrules the prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$ used as base. This block expresses that any prima facie command which can conflict with the one used as base (so if the base is an obligation, we do not consider permissions-prohibition) is not applicable ($(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \psi \Rightarrow \zeta$), or it is not more specific than the base ($(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \zeta \Rightarrow \chi$), or it does not conflict with the command we are deriving (the third underivability statement in each of the two groups above).
- The *no-active-conflict block*, consisting of the last three sets of premisses and corresponding to (2), (3), (4) in Def.4.3.3, guarantees that any deontic statement is not applicable (expressed by the first underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta$ of each set), or it does not conflict with the one we are deriving (expressed by second underivability statement of each set), or it is overruled by a more specific command (expressed, in each

set of premisses, by the group of three premisses for each possibly overruling command). Let us consider the part of the no-active-conflict block relative to any possible obligation (the ones relative to prohibitions and suitable permissions are similar):

$$\left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow \}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \xi, \varphi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}$$

- The two underivability statements constitute the *no-conflict* (sub-)block, corresponding to (2.a), (3.a), (4.a) in Def.4.3.3. They express that any deontic statement in $\mathfrak{L}$ is not applicable in the situation ψ, or it does not conflict with $\mathcal{O}(\varphi/\psi)$.
- The three sets of three premisses represent the *override* (sub-)block, corresponding to (2.b), (3.b), (4.b) in Def.4.3.3. Each one of them states that there is a command such that it is applicable in the situation ψ (expressed by the first of the three premisses), it is more specific than the conflicting one (expressed by the second premiss) and it supports the command we are deriving (expressed by the third premiss).

As for $\mathsf{MD}^{\mathsf{ga}}$, the left rules for obligations and prohibitions represent the result of absorbing cuts between the principal formulas of the two rules $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$, and the other rules of the calculus for $\mathsf{MD}{+}^{\mathsf{ga}}$ (i.e. the rules of $\mathsf{G}_{\mathsf{MD}+}$).

Specifically, $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (Fig.4.8), which allows us to derive an obligation on the left hand side of a sequent from a prima-facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$, is obtained by saturating the rule set under cuts between $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}}$.

$\mathcal{F}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (Fig.4.9), deriving a prohibition on the left hand side of a sequent from the same assumption $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$, is given by the saturation under cuts between $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}\mathcal{F}}$.

In the same way, a cut between $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{F}}$ gives the rule $\mathcal{F}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ (Fig.4.10), and a cut between $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}\mathcal{F}}$ yields the rule $\mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ (Fig.4.11).

As anticipated in the previous section, if the set $\mathfrak{L}$ of deontic assumptions does not contain prohibitions and permissions, $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ coincide with the simplified rules in Fig.4.4.

$$
\dfrac{
\begin{array}{ll}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta, \varphi \Rightarrow\} & \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \Rightarrow \varphi, \tau\}\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi, \xi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi, \xi \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \Rightarrow \varphi, \tau\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi, \xi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi, \xi \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi, \varphi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{
\mathcal{O}(\varphi/\psi) \Rightarrow
}
\; \mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ and $(\mathfrak{F},\mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}\,\mathsf{cut}}} \Gamma \Rightarrow \Delta$.

Figure 4.8: The left assumption rule for obligations in MD+.

$$
\dfrac{
\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} \\
\cup \quad \left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \end{bmatrix} \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \end{bmatrix} \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{bmatrix} \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{bmatrix} \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \begin{bmatrix} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{bmatrix} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{
\mathcal{F}(\varphi/\psi) \Rightarrow
} \; \mathcal{F}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ and $(\mathfrak{F},\mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}\,\mathsf{cut}}} \Gamma \Rightarrow \Delta$.

Figure 4.9: The new left assumption rule for prohibitions in $\mathsf{MD+}$.

$$
\frac{
\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\Rightarrow \varphi, \theta\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\
\cup \{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\
\cup \{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\}
\end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\
\cup \{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\
\cup \{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \xi, \varphi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\
\{\{(\mathfrak{F}, \mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{
\mathcal{F}(\varphi/\psi) \Rightarrow
} \ \mathcal{F}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ and $(\mathfrak{F}, \mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+}^{ga}\,cut}} \Gamma \Rightarrow \Delta$.

Figure 4.10: The left assumption rule for prohibitions in MD+.

$$
\dfrac{
\begin{array}{ll}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\varphi \Rightarrow \theta\} & \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup\, \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup\, \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \end{array} \right] \middle|\ \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup\, \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup\, \{\{(\mathfrak{F},\mathfrak{L}) \nvdash\ \Rightarrow \tau, \varphi\}\} \end{array} \right] \middle|\ \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\ \} \end{array} \middle|\ \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\varphi \Rightarrow \xi\} \end{array} \middle|\ \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\ \} \end{array} \middle|\ \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \middle|\ \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash\ \Rightarrow \tau, \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\} \end{array} \middle|\ \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\varphi \Rightarrow \xi\ \} \end{array} \middle|\ \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi, \varphi \Rightarrow\} \end{array} \middle|\ \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \middle|\ \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau \Rightarrow \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\varphi \Rightarrow \xi\} \end{array} \middle|\ \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \middle|\ \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{
\mathcal{O}(\varphi/\psi) \Rightarrow
}\ \mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ and $(\mathfrak{F}, \mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \Gamma \Rightarrow \Delta$.

Figure 4.11: The new left assumption rule for obligations in MD+.

Let us now present a simple example of how the global assumption rules introduced above can be applied for reasoning with deontic and factual assumptions in Mīmāṃsā: we use the rules for checking if a given command is derivable from the list of Vedic commands. This check can be also performed with the help of a Prolog implementation of the system, available at `http://subsell.logic.at/bprover/deonticProver/version1.1/`.

Example 4.3.5. Consider the new formulations of the deontic statements in Ex.4.2.1:

(i) *"It is forbidden for a śūdra to study the Vedas"* ($\mathcal{F}_{\mathsf{pf}}(\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d}) \in \mathfrak{L}$);

(ii) *"a chariot maker is a śūdra"* ($\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}$);

(iii) *"performing sacrifices implies studying the Vedas"* ($\mathtt{sacr} \Rightarrow \mathtt{Ved}$);

(iii) *"chariot makers should perform a sacrifice"* ($\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk}) \in \mathfrak{L}$).

Hence we have $\mathfrak{L} = \{\mathcal{F}_{\mathsf{pf}}(\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d}), \mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})\}$ and $\mathfrak{F} = \{\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}, \mathtt{sacr} \Rightarrow \mathtt{Ved}\}$.

Using the rule $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ in Fig.4.7 we can derive e.g. the prohibition $\mathcal{F}(\mathtt{sacr}/\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk})$, by using the deontic assumption $\mathcal{F}_{\mathsf{pf}}(\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$ as base.

The standard block of this rule application is then constitute by two premisses stating that the prohibition used as base supports the prohibition we are deriving ($\mathtt{sacr} \Rightarrow \mathtt{Ved}$) and that it is applicable ($\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}$).

Both those premisses can be derived from the assumptions by using $\mathsf{G}_{\mathsf{MD+}}$:

$$\dfrac{}{\mathtt{sacr} \Rightarrow \mathtt{Ved}}\ \mathfrak{F} \qquad \dfrac{\dfrac{}{\mathtt{\acute{s}\bar{u}d}, \neg\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}}\ \mathsf{init}}{\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{\acute{s}\bar{u}d}}\ \wedge_L$$

The not-excepted block of this rule application contains, for each deontic statement in $\mathfrak{L}$, an underivability statement expressing that the deontic statement is not applicable, or that it is not more specific than the base, or that it does not conflict with what we want to derive. In this case, we have:

$$\text{for } \mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})\text{:}\ \ \nvdash \mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk} \Rightarrow \mathtt{chmk} \quad \text{and} \quad \text{for } \mathcal{F}_{\mathsf{pf}}(\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})\text{:}\ \ \nvdash \mathtt{Ved} \Rightarrow \mathtt{sacr}$$

For the no-active-conflict block of this rule application, the premisses should show that any deontic statement in $\mathfrak{L}$ is not applicable, not conflicting, or it is overruled by a more specific command. However, the underivability statements used in the not-excepted block already show that the assumption $\mathcal{O}_{\mathsf{pf}}(\mathtt{sacr}/\mathtt{chmk})$ is not applicable and the assumption $SForb(\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})$ is not conflicting. Hence they can be simply repeated, without making use of the overridden block.

Then, the application of the rule $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\mathtt{Ved}/\mathtt{\acute{s}\bar{u}d})}$ for deriving $\mathcal{F}(\mathtt{sacr}/\mathtt{\acute{s}\bar{u}d} \wedge \neg\mathtt{chmk})$ from $\mathfrak{F}, \mathfrak{L}$

is the following:

$$\dfrac{\dfrac{\{\texttt{sacr} \Rightarrow \texttt{Ved}\} \qquad \{\texttt{ś\=ud}, \neg\texttt{chmk} \Rightarrow \texttt{ś\=ud}\}}{\begin{array}{c}\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \texttt{ś\=ud} \wedge \neg\texttt{chmk} \Rightarrow \texttt{chmk}\} \\ \{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \texttt{Ved} \Rightarrow \texttt{sacr}\}\end{array}}}{\begin{array}{c}\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \texttt{ś\=ud} \wedge \neg\texttt{chmk} \Rightarrow \texttt{chmk}\} \\ \{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \texttt{Ved} \Rightarrow \texttt{sacr}\end{array}} $$

$$\dfrac{}{\Rightarrow \mathcal{F}(\texttt{sacr}/\texttt{ś\=ud} \wedge \neg\texttt{chmk})}\ \mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\texttt{Ved}/\texttt{ś\=ud})}$$

■

The definition of proto-derivation from $(\mathfrak{F},\mathfrak{L})$ in $\mathsf{G}_{\mathsf{MD+^{ga}}}$ and the definition of valid proto-derivation represent generalizations of the corresponding ones in the previous section for the system $\mathsf{G}_{\mathsf{MD^{ga}}}$ from the assumptions $(\mathfrak{F},\mathfrak{L}_{\mathcal{O}})$.

As for the simplified case in the previous section (Def.4.2.12), we define the derivability in the new system $\mathsf{MD+^{ga}}$ as follows.

Definition 4.3.6 *Let $\mathfrak{L}$ be a list of deontic assumptions, containing prima facie obligations, prohibitions, recommendations, permissions-obligation and permissions-prohibition. Moreover, let $\mathfrak{F}$ be a set of sequents as in Def.4.2.4.*

Let us call $\mathsf{G}_{\mathsf{MD+^{ga}}}$ the system $\mathsf{G}_{\mathsf{MD+}}$ extended with cut *and the rules*

$$\mathsf{ga}_{\mathfrak{L}} := \left\{ \mathsf{op1}_s^{\mathsf{op2}(\theta/\chi)} \;\middle|\; \begin{array}{c}\mathsf{op1} \in \{\mathcal{O},\mathcal{F}\},\ \mathsf{op2} \in \{\mathcal{O}_{\mathsf{pf}},\mathcal{F}_{\mathsf{pf}}\}, \\ \mathsf{op2}(\theta/\chi) \in \mathfrak{L},\ s \in \{L,R\}\end{array} \right\}.$$

(in figures 4.6, 4.6, 4.8, 4.10, 4.11, 4.9). A proto-derivation *from assumptions $(\mathfrak{F},\mathfrak{L})$, with conclusion $\Gamma \Rightarrow \Delta$, in $\mathsf{G}_{\mathsf{MD+^{ga}}}$, is a finite labelled tree such that each internal node is labelled with a sequent and the label of every internal node is obtained from the labels of its children using the rules of $\mathsf{G}_{\mathsf{MD+^{ga}}}$ and each leaf is labelled with an initial sequent, a sequent in $\mathfrak{F}$, or an* underivability statement $(\mathfrak{F},\mathfrak{L}_{\mathcal{O}}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Sigma \Rightarrow \Pi$.

The notion of a proto-derivation in the system $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}$ *is defined analogously, but also allowing for the use of* cut.

Definition 4.3.7 *A proto-derivation from $(\mathfrak{F},\mathfrak{L})$ in the calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}$ is* valid *if for each of the underivability statements $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Sigma \Rightarrow \Pi$, occurring as one of the leafs of that derivation, there is no valid proto-derivation of $\Sigma \Rightarrow \Pi$ in $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}$ from $(\mathfrak{F},\mathfrak{L})$.*

If there is a valid proto-derivation in $\mathsf{G}_{\mathsf{MD+^{ga}}}$ of a sequent $\Gamma \Rightarrow \Delta$, we write $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}} \Gamma \Rightarrow \Delta$ (or $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma \Rightarrow \Delta$ for system with the cut rule).

Before moving to consider the formal properties of the system $\mathsf{MD+^{ga}}$, let us show some simple examples of valid proto-derivations in that system.

Example 4.3.8. Let $\mathfrak{F} = \varnothing$ and let $\mathfrak{L}$ contain only the statements from Ex.4.3.2:

- *"one must not open the bank vault and enter"* ($\mathcal{F}_{\mathsf{pf}}(\texttt{open} \wedge \texttt{enter}/\top)$);
- *"it is permitted to open the bank vault and enter if one is the bank manager"* ($\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\texttt{open} \wedge \texttt{enter}/\texttt{manager})$).

As already noticed, the permission does not block the derivation of the "narrower" prohibition $\mathcal{F}(\texttt{open}\wedge\texttt{enter}\wedge\texttt{suit}/\texttt{manager})$ (*"one must not open the bank vault and enter and wear a suit, (even) if one is the bank manager"*), which is then derivable using $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\texttt{open}\wedge\texttt{enter}/\top)}$:

$$\frac{\begin{array}{c}\{\texttt{manager} \Rightarrow \top\} \quad \cup \quad \{\texttt{open} \wedge \texttt{enter} \wedge \texttt{suit} \Rightarrow \texttt{open} \wedge \texttt{enter}\} \\[1ex] \cup\, \{(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}}\,\mathsf{cut}} \texttt{open} \wedge \texttt{enter} \Rightarrow \texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}\} \\ \cup\, \{(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}}\,\mathsf{cut}} \Rightarrow \texttt{open} \wedge \texttt{enter}, \texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}\} \\[1ex] \cup\, \{(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}}\,\mathsf{cut}} \Rightarrow \texttt{open} \wedge \texttt{enter}, \texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}\} \\ \cup\, \{(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}}\,\mathsf{cut}} \texttt{open} \wedge \texttt{enter} \Rightarrow \texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}\}\end{array}}{\Rightarrow \mathcal{F}(\texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}/\texttt{manager})}$$

The premisses in the standard block, stating that the prohibition $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\texttt{open}\wedge\texttt{enter}/\top)}$ used as base is applicable and supports the prohibition $\mathcal{F}(\texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}/\texttt{manager})$, are derivable in $\mathsf{G_{MD+^{ga}}}$:

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow}\,\bot_L}{\Rightarrow \top}\,\neg_R}{\texttt{manager} \Rightarrow \top}\,\mathsf{W}_L \qquad \dfrac{\dfrac{\dfrac{}{\texttt{open} \wedge \texttt{enter} \Rightarrow \texttt{open} \wedge \texttt{enter}}\,\mathsf{init}}{\texttt{open} \wedge \texttt{enter}, \texttt{suit} \Rightarrow \texttt{open} \wedge \texttt{enter}}\,\mathsf{W}_L}{\texttt{open} \wedge \texttt{enter} \wedge \texttt{suit} \Rightarrow \texttt{open} \wedge \texttt{enter}}\,\wedge_L$$

Moreover, the first underivability statement in the not-excepted block (equal to the last one in the no-conflict sub-block), stating that the permission $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\texttt{open} \wedge \texttt{enter}/\texttt{manager})$ does not conflict with $\mathcal{F}(\texttt{open}\wedge\texttt{enter}\wedge\texttt{suit}/\texttt{manager})$, is such that there is no derivation in $\mathsf{G_{MD+^{ga}}}\,\mathsf{cut}$ with conclusion $\texttt{open} \wedge \texttt{enter} \Rightarrow \texttt{open} \wedge \texttt{enter} \wedge \texttt{suit}$. Otherwise, by applying the derived rule $\wedge_R$ (bottom-up), we would need a sequent $\texttt{open} \wedge \texttt{enter} \Rightarrow \texttt{suit}$ to be contained in $\mathfrak{F}$, but this set is assumed to be empty.

The same reasoning can be applied to the second underivability statement in the not-excepted block (equal to the first one in the no-conflict sub-block), stating that the prima facie prohibition $\mathcal{F}_{\mathsf{pf}}(\mathtt{open}\wedge\mathtt{enter}/\mathtt{manager})$ does not conflict with $\mathcal{F}(\mathtt{open}\wedge\mathtt{enter}\wedge\mathtt{suit}/\mathtt{manager})$ (in fact it implies the latter).

If there was a derivation in $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathtt{cut}$ with conclusion $\Rightarrow \mathtt{open}\wedge\mathtt{enter}, \mathtt{open}\wedge\mathtt{enter}\wedge\mathtt{suit}$, the set $\mathfrak{F}$ would contain a sequent with no formulas on the left hand side and at least one of the conjuncts above on the right hand side.

Hence, it is possible to state that there is a valid proto-derivation of $\mathcal{F}(\mathtt{open}\wedge\mathtt{enter}\wedge\mathtt{suit}/\mathtt{manager})$ from $\mathfrak{F}, \mathfrak{L}$ in $\mathsf{G}_{\mathsf{MD+^{ga}}}$. ■

Example 4.3.9. With respect to the previous example, the situation changes when the prohibition we derive is does not have a more specific content (formula in the first argument), but a more specific condition (formula in the second argument).

Let us consider $\mathfrak{F} = \varnothing$ and $\mathfrak{L} = \{\mathcal{F}_{\mathsf{pf}}(\mathtt{open}\wedge\mathtt{enter}/\top), \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\mathtt{open}\wedge\mathtt{enter}/\mathtt{manager})\}$ as in the previous example. Now we show that the more specific prohibition $\mathcal{F}(\mathtt{open}\wedge\mathtt{enter}/\mathtt{suit}\wedge\mathtt{manager})$ (*"one must not open the bank vault and enter, (even) if one is the bank manager and wears a suit"*) cannot be derived in the calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}$ by applying the rule $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\mathtt{open}\wedge\mathtt{enter}/\top)}$ (i.e. using $\mathcal{F}_{\mathsf{pf}}(\mathtt{open}\wedge\mathtt{enter}/\top)$ as base). Since the set $\mathfrak{L}$ of deontic assumptions does not contain any other prohibition or obligation that could be used as base for deriving the prohibition $\mathcal{F}(\mathtt{open}\wedge\mathtt{enter}/\mathtt{suit}\wedge\mathtt{manager})$, this also means that the prohibition is not derivable in $\mathsf{G}_{\mathsf{MD+^{ga}}}$ from the sets of assumptions $\mathfrak{F}, \mathfrak{L}$.

Let us consider the premisses and the underivability statements in the rule application below. The fact that the premisses in the standard block are both derivable means that the command used as base, if not overruled by the permission, would be in principle suitable for deriving the more specific prohibition $\mathcal{F}(\mathtt{open}\wedge\mathtt{enter}/\mathtt{suit}\wedge\mathtt{manager})$.

For the same reason, since the one used as base is the only prohibition in $\mathfrak{L}$, it is easy to observe that the parts of both the not-excepted block and the no-active-conflict block which are relative to that prohibition are not problematic. Indeed, they contain the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathtt{cut}} \Rightarrow \mathtt{open}\wedge\mathtt{enter}, \mathtt{open}\wedge\mathtt{enter}$: since $\mathfrak{F}$ is empty and $\Rightarrow \mathtt{open}\wedge\mathtt{enter}, \mathtt{open}\wedge\mathtt{enter}$ does not express a logical truth, there are no derivations from $\mathfrak{F}, \mathfrak{L}$ in $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathtt{cut}$ with such conclusion.

On the other hand, the parts of the not-excepted block and of the no-active-conflict block which are relative to the permission are not valid. The content of each underivability statement in those parts is indeed a sequent which represents a logical truth and hence for which there is a derivation in $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathtt{cut}$.

Moreover, the set of deontic assumptions $\mathfrak{L}$ does not contain any other command which supports the prohibition $\mathcal{F}(\mathtt{open} \wedge \mathtt{enter}/\mathtt{manager} \wedge \mathtt{suit})$ and that is applicable and more specific than the permission. Looking at the override sub-block in the last set of premisses (the part of the no-active-conflict block relative to the permission), we can observe that the only other command in $\mathfrak{L}$ (the prohibition $\mathcal{F}_{\mathsf{pf}}(\mathtt{open} \wedge \mathtt{enter}/\top)$) is not specific enough to override $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\mathtt{open} \wedge \mathtt{enter}/\mathtt{manager})$ ($\top \Rightarrow \mathtt{manager}$ is not derivable).

$$
\begin{array}{c}
\begin{array}{l}
\{\mathsf{suit} \wedge \mathsf{manager} \Rightarrow \top\} \quad \cup \quad \{\mathsf{open} \wedge \mathsf{enter} \Rightarrow \mathsf{open} \wedge \mathsf{enter}\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{suit} \wedge \mathsf{manager} \Rightarrow \mathsf{manager}\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{manager} \Rightarrow \top\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{open} \wedge \mathsf{enter} \Rightarrow \mathsf{open} \wedge \mathsf{enter}\}\}
\end{array} \right] \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\mathsf{manager}) \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{suit} \wedge \mathsf{manager} \Rightarrow \top\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \top \Rightarrow \top\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \mathsf{open} \wedge \mathsf{enter}, \mathsf{open} \wedge \mathsf{enter}\}\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\top) \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{suit} \wedge \mathsf{manager} \Rightarrow \top\}\} \\
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \mathsf{open} \wedge \mathsf{enter}, \mathsf{open} \wedge \mathsf{enter}\}\} \\
\left\{ \begin{array}{l} \{\mathsf{suit} \wedge \mathsf{manager} \Rightarrow \top\} \cup \{\top \Rightarrow \top\} \\ \cup \{\mathsf{open} \wedge \mathsf{enter} \Rightarrow \mathsf{open} \wedge \mathsf{enter}\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\top) \right\} \\
\left\{ \begin{array}{l} \{\mathsf{suit} \wedge \mathsf{manager} \Rightarrow \mathsf{manager}\} \cup \{\mathsf{manager} \Rightarrow \top\} \\ \quad \cup \{\mathsf{open} \wedge \mathsf{enter}, \mathsf{open} \wedge \mathsf{enter} \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\mathsf{manager}) \right\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\top) \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{suit} \wedge \mathsf{manager} \Rightarrow \mathsf{manager}\}\} \\
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \mathsf{open} \wedge \mathsf{enter} \Rightarrow \mathsf{open} \wedge \mathsf{enter}\}\} \\
\left\{ \begin{array}{l} \{\mathsf{suit} \wedge \mathsf{manager} \Rightarrow \top\} \cup \{\top \Rightarrow \mathsf{manager}\} \\ \quad \cup \{\mathsf{open} \wedge \mathsf{enter} \Rightarrow \mathsf{open} \wedge \mathsf{enter}\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\top) \right\}
\end{array} \right] \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\mathsf{open} \wedge \mathsf{enter}/\mathsf{manager}) \right\}
\end{array} \\
\hline
\Rightarrow \mathcal{F}(\mathsf{open} \wedge \mathsf{enter}/\mathsf{suit} \wedge \mathsf{manager})
\end{array}
$$

This small example also suggests that, in general, an optimal criterion for the interpretation and formalization of deontic statements is still missing. For instance, *"wear a suit"* can be seen as an action and put in the first argument of the deontic operator (as in the previous example), or it can be interpreted as a pre-existing condition and written in the second argument of the operator (as in this example). Though the different interpretation seems to just slightly change the intuitive meaning of the sentence, it modifies its message in such a way that *"one must not open the bank vault and enter and wear a suit, (even) if one is the bank manager"* is derivable, while the prohibition *"one must not open the bank vault and enter, (even) if one is the bank manager and wears a suit"* is not. ■

4.3.1 Formal properties of the sequent system

The main result of this section will be the cut-elimination theorem for the calculus $\mathsf{G}_{\mathsf{MD+ga}}$ (Thm.4.3.12). This theorem has a number of important consequences, in particular the fact that the notion of a derivation that uses underivability statements (i.e. a valid proto-derivation) is well-defined, consistency of the logic, and its decidability.

Remark 4.3.10 *Note that the well-definedness of the notion of valid proto-derivation depends on the fact that the global assumption rules allow to construct the set of consequences of the list of deontic assumptions* iteratively*.*

This approach differs from, e.g., the one in [71] (see section 4.5), where the set of consequences of the deontic assumptions is built by a fixed-point construction.

We start showing that in the presence of the cut rule, the left global assumption rules are redundant, as they can always be replaced by applications of other rules and cut. Hence, the system $\mathsf{G}_{\mathsf{MD+ga}}\mathsf{cut}$ derives the same set of sequents, with or without the rules $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}, \mathcal{F}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}, \mathcal{F}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$.

Lemma 4.3.11 (Redundancy of the left rules [32]) *If there is a valid proto-derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}_{\mathsf{MD+ga}}\mathsf{cut}$ from $(\mathfrak{F}, \mathfrak{L})$, then there is a valid proto-derivation of $\Gamma \Rightarrow \Delta$ from $(\mathfrak{F}, \mathfrak{L})$ in the system without the left global assumption rules $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}, \mathcal{F}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}, \mathcal{F}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}, \mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ in Figs.4.8, 4.9, 4.10, 4.11, respectively.*

Proof. We show that it is possible to simulate an application of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ by using cut between the conclusion of an application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and the conclusion of an application of $\mathsf{D}_{\mathcal{O}}$.

Similarly, we can simulate $\mathcal{F}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ with cuts between the conclusions of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}\mathcal{F}}$, $\mathcal{F}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ with cuts between $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{F}}$, and $\mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ by using cut between $\mathcal{F}_R^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}\mathcal{F}}$.

Notice that the rules $\mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathcal{F}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$, which derive an obligation starting from a prima facie prohibition and a prohibition starting from a prima facie obligation, are both simulated by using cut between the right rule for the other operator and the interaction rule $\mathsf{D}_{\mathcal{O}\mathcal{F}}$.

We prove only the case of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$, as the others are analogous. The only non trivial part is the adaptation of the underivability statements to the right rules.

Consider the application of the rule $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ as in the following figure (4.8):

$$
\dfrac{
\begin{array}{ll}
\{\psi\Rightarrow\chi\} \quad \cup \quad \{\theta,\varphi\Rightarrow\} & \\
\cup & \left\{\left[\begin{array}{l}
\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \psi\Rightarrow\zeta\right\}\right\} \\
\cup\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \zeta\Rightarrow\chi\right\}\right\} \\
\cup\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \tau\Rightarrow\varphi\right\}\right\}
\end{array}\right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta)\in\mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)\in\mathfrak{L}\right\} \\
\cup & \left\{\left[\begin{array}{l}
\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \psi\Rightarrow\zeta\right\}\right\} \\
\cup\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \zeta\Rightarrow\chi\right\}\right\} \\
\cup\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Rightarrow\varphi,\tau\right\}\right\}
\end{array}\right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta)\in\mathfrak{L}\right\} \\
\cup & \left\{\left[\begin{array}{l}
\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \psi\Rightarrow\zeta\right\}\right\}, \\
\cup\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \tau\Rightarrow\varphi\right\}\right\} \\
\cup\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\varphi,\xi\Rightarrow\}\end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\} \\
\cup\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\varphi\Rightarrow\xi\}\end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\} \\
\cup\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\varphi,\xi\Rightarrow\}\end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\}
\end{array}\right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta)\in\mathfrak{L}\right\} \\
\cup & \left\{\left[\begin{array}{l}
\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \psi\Rightarrow\zeta\right\}\right\}, \\
\cup\left\{\left\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Rightarrow\varphi,\tau\right\}\right\} \\
\cup\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\varphi,\xi\Rightarrow\}\end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\} \\
\cup\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\varphi\Rightarrow\xi\}\end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\} \\
\cup\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\varphi,\xi\Rightarrow\}\end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\}
\end{array}\right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta)\in\mathfrak{L}\right\} \\
\cup & \left\{\left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \psi\Rightarrow\zeta\}\} \\
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \tau\Rightarrow\varphi\}\} \\
\left\{\begin{array}{c}\{\psi\Rightarrow\eta\}\cup\{\eta\Rightarrow\zeta\} \\ \cup\{\xi,\varphi\Rightarrow\}\end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta)\in\mathfrak{L}\right\}
\end{array}\right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)\in\mathfrak{L}\right\}
\end{array}
}{\mathcal{O}(\varphi/\psi)\Rightarrow}\ \mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

First we transform each premiss of the form $\Gamma, \varphi \Rightarrow \Delta$ into $\Gamma \Rightarrow \neg\varphi, \Delta$ and premisses of the form $\Sigma \Rightarrow \varphi, \Pi$ into $\Sigma, \neg\varphi \Rightarrow \Pi$ by using the derived rules for negation on the right and on the left, respectively.

Then, each underivability statement of the form $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma \Rightarrow \varphi, \Delta$ is tranformed into $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma, \varphi \to \bot \Rightarrow \Delta$. Indeed, reasoning by contraposition, if $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma, \varphi \to \bot \Rightarrow \Delta$, we would obtain $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma \Rightarrow \varphi, \Delta$, by using the following derivation:

$$\dfrac{\dfrac{\overline{\Gamma, \varphi \Rightarrow \varphi, \bot, \Delta}}{\Gamma \Rightarrow \varphi, \varphi \to \bot, \Delta}\,{\to_R} \qquad \Gamma, \varphi \to \bot \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\,\mathsf{cut}$$

In the same way underivability statements of the form $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Sigma, \varphi \Rightarrow \Pi$ are transformed into $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Sigma \Rightarrow \varphi \to \bot, \Pi$.

With the premisses thus obtained, we can apply the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ with conclusion $\Rightarrow \mathcal{O}(\varphi \to \bot/\psi)$ and cut it with the conclusion of the rule $\mathsf{D}_{\mathcal{O}}$ as follows:

$$\dfrac{\dfrac{\vdots}{\Rightarrow \mathcal{O}(\varphi \to \bot/\psi)}\,\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)} \qquad \dfrac{\dfrac{\overline{\bot \Rightarrow}\,{\bot_L} \quad \overline{\varphi \Rightarrow \varphi}\,\mathsf{init}}{\varphi \to \bot, \varphi \Rightarrow}\,{\to_L} \quad \overline{\psi \Rightarrow \psi}\,\mathsf{init}}{\mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi \to \bot/\psi) \Rightarrow}\,\mathsf{D}_{\mathcal{O}}}{\mathcal{O}(\varphi/\psi) \Rightarrow}$$

This proof is made possible by the fact that the premisses of the left rules are those of the right ones, but with the formula φ (the content in the first argument of the command we derive) on the other side of the sequents. $\square$

The left global assumption rules are proved to be superfluous in the system $\mathsf{MD+^{ga}}$ in presence of the cut rule. However, the cut rule prevents a proof system from being analytic and also from being used to perform automated reasoning.

We show below that the cut rule can be eliminated from the calculus, "up to the underivability statements": if there is a valid proto-derivation of a sequent $\Gamma \Rightarrow \Delta$ from $(\mathfrak{F}, \mathfrak{L})$ in the sequent calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}$ with cut, then there is a valid proto-derivation of a sequent $\Gamma \Rightarrow \Delta$ from $(\mathfrak{F}, \mathfrak{L})$ in the sequent calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}$. However, the existence of a valid proto-derivation in the cut-free sequent calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}$ means that (Def.4.3.7) for each of the underivability statements occurring as one of the leafs of that derivation, there is no valid proto-derivation of the claimed underivable sequent in the calculus $\mathsf{G}_{\mathsf{MD^{ga}}}\mathsf{cut}$ with cut.

Theorem 4.3.12 (Cut elimination [32]) *If* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma \Rightarrow \Delta$, *then* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}} \Gamma \Rightarrow \Delta$.

Proof. We actually prove the elimination of the *multicut rule* (where φ^n stands for n occurrences of the formula φ)

$$\frac{\Gamma \Rightarrow \Delta, \varphi^n \quad \varphi^m, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{mcut}$$

Those rules are equivalent in the system, as cut is a case of mcut with $n = m = 1$ and mcut is derivable in $\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}$ by using $\mathsf{Con}_L, \mathsf{Con}_R$ and cut. Hence, showing that mcut can be eliminated in the sequent system for $\mathsf{MD+^{ga}}$ is equivalent to proving the redundancy of cut.

The proof follows the pattern of [130, Sec. 4.1.9], adapted for taking into account the underivability statements: we show how to eliminate topmost applications of the rule from a proto-derivation, preserving validity.

The claim of the theorem is proved by double induction on the complexity of the cut formula φ (depending on the number of logical connectives or modal operators occurring in it) and the sum of the heights of the derivations of the two premisses of the application of mcut.

If $\varphi = p \in \mathsf{Var}$ (the complexity of the cut formula is equal to 0), φ cannot be the principal formula of a logical, modal, or global assumption rule. Hence, by induction on the depths of the derivations, we can permute mcut into the premisses of the last applied rules, until it is absorbed by an application of weakening, or reaches the leaves of the proto-derivation.

If cut is applied to the leaves of the proto-derivation, at least one of its premisses is an initial sequent -in which case the mcut is eliminated in the standard way-, or its premisses are elements of $\mathfrak{F}$. If that is the case, considering that the set $\mathfrak{F}$ is closed under contraction and cuts, we replace the multicut with the corresponding element of $\mathfrak{F}$.

If the complexity of the cut formula is greater than 0 and the cut formula does not contain deontic operators, we proceed in the standard way (see e.g. [130]): by induction on the depth of the proto-derivation, mcut is permuted into the premisses of the last applied rules, until the cut formula is in an initial sequent or it is principal in the last rules of the derivations of both premisses of the multicut.

If the cut formula is a deontic formula and none of the last applied rules is a global assumption rule, then the proof follows the one for $\mathsf{G}_{\mathsf{MD+}}$ (see [80] for the general transformations).

If at least one of the premisses of mcut is the conclusion of a global assumption rule, we can distinguish three general cases for each operator $\mathsf{op} \in \{\mathcal{O}, \mathcal{F}, \mathcal{R}\}$:

(i) mcut between the conclusion of an application of $\mathsf{op}_R^{\mathsf{op}'(\theta/\chi)}$ or $\mathsf{op}_L^{\mathsf{op}'(\theta/\chi)}$ and the conclusion of an application of a modal rule of $\mathsf{G_{MD+}}$ which is not a global assumption rule ($\mathsf{Mon}_{\mathcal{O}}$, $\mathsf{Mon}_{\mathcal{F}}$, $\mathsf{Mon}_{\mathcal{R}}$, $\mathsf{D}_{\mathcal{O}}$, $\mathsf{D}_{\mathcal{F}}$, $\mathsf{P}_{\mathcal{R}}$, or $\mathsf{D}_{\mathcal{OF}}$), where the multicut has a non-empty conclusion;

(ii) mcut between the conclusion of an application of $\mathsf{op}_R^{\mathsf{op}'(\theta/\chi)}$ or $\mathsf{op}_L^{\mathsf{op}''(\theta/\chi)}$ and the conclusion of an application of a modal rule of $\mathsf{G_{MD+}}$ (not a global assumption rule), where the multicut has an empty conclusion;

(iii) mcut between the conclusion of an application of $\mathsf{op}_R^{\mathsf{op}'(\theta/\chi)}$ and the conclusion of an application of $\mathsf{op}_L^{\mathsf{op}'(\varsigma/\upsilon)}$.

We consider all the different cases for the simplest case of $\mathsf{op} = \mathcal{R}$ and for $\mathsf{op} = \mathcal{O}$ (The ones for the operator $\mathcal{F}$ are analogous).

Case (i): Let us first consider the case where the premisses of the mcut are the conclusions of an application of $\mathcal{R}_R^{\mathcal{R}_{\mathsf{pf}}(\theta/\chi)}$ and an application of $\mathsf{Mon}_{\mathcal{R}}$:

$$\dfrac{\dfrac{\psi \Rightarrow \chi \quad \chi \Rightarrow \psi \quad \theta \Rightarrow \varphi \quad (\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \theta \Rightarrow}{\Rightarrow \mathcal{R}(\varphi/\psi)}\ \mathcal{R}_R^{\mathcal{R}_{\mathsf{pf}}(\theta/\chi)} \quad \dfrac{\varphi \Rightarrow \upsilon \quad \psi \Rightarrow \varsigma \quad \varsigma \Rightarrow \psi}{\mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\varsigma/\upsilon)}\ \mathsf{Mon}_{\mathcal{R}}}{\Rightarrow \mathcal{R}(\varsigma/\upsilon)}\ \mathsf{mcut}$$

Writing on the left the (first three) premisses of $\mathcal{R}_R^{\mathcal{R}_{\mathsf{pf}}(\theta/\chi)}$ and on the right the ones occurring in the application of $\mathsf{Mon}_{\mathcal{O}}$, by induction hypothesis on the complexity of the cut formula, we have:

$$\dfrac{\psi \Rightarrow \chi \quad \varsigma \Rightarrow \psi}{\varsigma \Rightarrow \chi}\ IH \qquad \dfrac{\chi \Rightarrow \psi \quad \psi \Rightarrow \varsigma}{\chi \Rightarrow \varsigma}\ IH \qquad \dfrac{\theta \Rightarrow \varphi \quad \varphi \Rightarrow \upsilon}{\theta \Rightarrow \upsilon}\ IH$$

Hence, we can use the premisses thus obtained and an application of the right global assumption rule for recommendations to derive the conclusion of the application of mcut above:

$$\dfrac{\varsigma \Rightarrow \chi \quad \chi \Rightarrow \varsigma \quad \theta \Rightarrow \upsilon \quad (\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \theta \Rightarrow}{\Rightarrow \mathcal{R}(\varsigma/\upsilon)}\ \mathcal{R}_R^{\mathcal{R}_{\mathsf{pf}}(\theta/\chi)}$$

Let us now analyse the case where the premisses of the mcut are the conclusions of an application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (as in Fig.4.6) and an application of $\mathsf{Mon}_{\mathcal{O}}$:

$$
\dfrac{
\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \tau, \varphi \Rightarrow \}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \varphi \Rightarrow \tau\}\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \tau, \varphi \Rightarrow \}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \xi, \varphi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \varphi \Rightarrow \tau\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \tau, \varphi \Rightarrow \}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{\Rightarrow \mathcal{O}(\varphi/\psi)} \ \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

$$
\dfrac{\Rightarrow \mathcal{O}(\varphi/\psi) \qquad \dfrac{\varphi \Rightarrow \varsigma \quad \psi \Rightarrow \upsilon \quad \upsilon \Rightarrow \psi}{\mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\varsigma/\upsilon)} \ \mathsf{Mon}_{\mathcal{O}}}{\Rightarrow \mathcal{O}(\varsigma/\upsilon)} \ \mathsf{mcut}
$$

Writing as left premisses the sequents possibly occurring in the application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (in particular the premisses in the standard block and the first and last premisses of each overridden sub-block) and as right premisses the ones occurring in the application of $\mathsf{Mon}_{\mathcal{O}}$, by induction hypothesis on the complexity of the cut formula, we have:

$$\dfrac{\psi \Rightarrow \chi \quad \upsilon \Rightarrow \psi}{\upsilon \Rightarrow \chi}\ IH \qquad \dfrac{\theta \Rightarrow \varphi \quad \varphi \Rightarrow \varsigma}{\theta \Rightarrow \varsigma}\ IH$$

$$\dfrac{\psi \Rightarrow \eta \quad \upsilon \Rightarrow \psi}{\upsilon \Rightarrow \eta}\ IH \qquad \dfrac{\xi \Rightarrow \varphi \quad \varphi \Rightarrow \varsigma}{\xi \Rightarrow \varsigma}\ IH$$

Moreover, from the instances of the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \zeta$ and the premiss $\psi \Rightarrow \upsilon$ of $\mathsf{Mon}_{\mathcal{O}}$, we obtain the new underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \upsilon \Rightarrow \zeta$, reasoning by contradiction: if $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}cut}}} \upsilon \Rightarrow \zeta$, then we could apply cut to this and $\psi \Rightarrow \upsilon$ to obtain $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \zeta$, contradicting the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \zeta$.

By the same reasoning, from the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \tau, \varphi \Rightarrow$ and the premiss $\varphi \Rightarrow \varsigma$ of $\mathsf{Mon}_{\mathcal{O}}$, we obtain the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \tau, \varsigma \Rightarrow$ and from the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \varphi \Rightarrow \tau$ and the premiss $\varphi \Rightarrow \varsigma$ of $\mathsf{Mon}_{\mathcal{O}}$, we obtain the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \varsigma \Rightarrow \tau$.

Using the premisses thus obtained, we can have an application of the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ with conclusion $\Rightarrow \mathcal{O}(\varsigma/\upsilon)$.

Let us now consider the case where the two last applied rules are $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}}$ (such that the principal formulas of $\mathsf{D}_{\mathcal{O}}$ are different and only one of them is the cut formula).

We have the following application of mcut:

$$\dfrac{\Rightarrow \mathcal{O}(\varphi/\psi) \qquad \dfrac{\varphi, \varsigma \Rightarrow \quad \psi \Rightarrow \upsilon \quad \upsilon \Rightarrow \psi}{\mathcal{O}(\varphi/\psi), \mathcal{O}(\varsigma/\upsilon) \Rightarrow}\ \mathsf{D}_{\mathcal{O}}}{\Rightarrow \mathcal{O}(\varsigma/\upsilon)}\ \mathsf{mcut}$$

where the first premiss $\Rightarrow \mathcal{O}(\varphi/\psi)$ results from an application (below) of the right global assumption rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$.

$$
\dfrac{
\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \zeta \Rightarrow \chi\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \tau, \varphi \Rightarrow\}\}
\end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \zeta \Rightarrow \chi\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \varphi \Rightarrow \tau\}\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\}, \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \tau, \varphi \Rightarrow\}\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \xi, \varphi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\}, \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \varphi \Rightarrow \tau\}\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\} \\
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \tau, \varphi \Rightarrow\}\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{
\Rightarrow \mathcal{O}(\varphi/\psi)
} \; \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

Again, using the premisses in the standard block and the first and last premisses of each overridden sub-block of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$, together with the premisses of $\mathsf{D}_{\mathcal{O}}$, by induction hypothesis on the complexity of the cut formula, we obtain:

$$
\dfrac{\psi \Rightarrow \chi \quad \upsilon \Rightarrow \psi}{\upsilon \Rightarrow \chi}\; IH \qquad \dfrac{\theta \Rightarrow \varphi \quad \varphi, \varsigma \Rightarrow}{\theta, \varsigma \Rightarrow}\; IH
$$

$$
\dfrac{\psi \Rightarrow \eta \quad \upsilon \Rightarrow \psi}{\upsilon \Rightarrow \eta}\; IH \qquad \dfrac{\xi \Rightarrow \varphi \quad \varphi \Rightarrow \varsigma}{\xi \Rightarrow \varsigma}\; IH
$$

Moreover, reasoning by contradiction as before, from the instances of the underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \psi \Rightarrow \zeta$ and the premiss $\psi \Rightarrow \upsilon$ of $\mathsf{D}_{\mathcal{O}}$, we obtain the new underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \upsilon \Rightarrow \zeta$. In the same way, from the underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \tau, \varphi \Rightarrow$ and the premiss $\varphi, \varsigma \Rightarrow$ of $\mathsf{D}_{\mathcal{O}}$, we obtain the underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \tau \Rightarrow \varsigma$: if there was a valid proto-derivation from $\mathfrak{F}, \mathfrak{L}$ in $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$ for the sequent $\tau \Rightarrow \varsigma$, we could apply cut between that and $\varphi, \varsigma \Rightarrow$, obtaining a valid proto-derivation of the sequent $\tau, \varphi \Rightarrow$ and contradicting the underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \tau, \varphi \Rightarrow$. By the same reasoning, from the underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \varphi \Rightarrow \tau$ and the premiss $\varphi, \varsigma \Rightarrow$ of $\mathsf{D}_{\mathcal{O}}$, we obtain the underivability statement $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \Rightarrow \varsigma, \tau$.

Using the premisses thus obtained, we can have an application of the rule $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ with conclusion $\Rightarrow \mathcal{O}(\varsigma/\upsilon)$:

$$
\dfrac{
\begin{array}{l}
\{\upsilon \Rightarrow \chi\} \quad \cup \quad \{\theta, \varsigma \Rightarrow\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \upsilon \Rightarrow \zeta\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \zeta \Rightarrow \chi\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \tau \Rightarrow \varsigma\}\}
\end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \upsilon \Rightarrow \zeta\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \zeta \Rightarrow \chi\}\} \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \Rightarrow \varsigma, \tau\}\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \upsilon \Rightarrow \zeta\}\}, \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \tau \Rightarrow \varsigma\}\} \\
\cup \left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varsigma, \xi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varsigma \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varsigma, \xi \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \upsilon \Rightarrow \zeta\}\}, \\
\cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \Rightarrow \varsigma, \tau\}\} \\
\cup \left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varsigma, \xi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varsigma \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\
\cup \left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varsigma, \xi \Rightarrow\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l}
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \upsilon \Rightarrow \zeta\}\} \\
\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \tau \Rightarrow \varsigma\}\} \\
\left\{ \begin{array}{c} \{\upsilon \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi, \varsigma \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\}
\end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{\mathcal{O}(\varsigma/\upsilon) \Rightarrow} \ \mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

Similarly to the case above, the case where the two last applied rules are $\mathsf{Mon}_{\mathcal{O}}$ and $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ is proved by using an application of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$. Reasoning in the same way, the eliminability of the multicut rule is proved when the premisses mcut are the conclusions of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{OF}}$: in this case we derive the premisses of $\mathcal{F}_L^{\mathcal{O}(\theta/\chi)}$ and apply that rule.

Finally, the case where the last applied rules are $\mathsf{Mon}_{\mathcal{O}}$ and $\mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$ also uses the same method, finishing with an application of $\mathcal{O}_L^{\mathcal{F}_{\mathsf{pf}}(\theta/\chi)}$.

Case (ii): Since in this case the conclusion of the multicut rule is empty, the last applied rules are necessarily $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ and $\mathsf{D}_{\mathcal{O}}$ (with both the principal formulas of the latter as cut formulas). We claim that this cannot occur. Otherwise the premisses of mcut would be the conclusions of the application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (as in Fig.4.6) and of $\mathsf{D}_{\mathcal{O}}$:

$$
\frac{\begin{array}{ll}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} & \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \tau, \varphi \Rightarrow\ \}\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \zeta \Rightarrow \chi\}\} \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \varphi \Rightarrow \tau\}\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \tau, \varphi \Rightarrow\ \}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \xi, \varphi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\}, \\ \cup \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \varphi \Rightarrow \tau\}\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\Rightarrow \varphi, \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup & \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \tau, \varphi \Rightarrow\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}}{\Rightarrow \mathcal{O}(\varphi/\psi)} \; \mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

But let us consider the case where $\tau = \theta$ and $\zeta = \chi$. In this case we have the following set of underivability statements in the not-excepted block relative to the prima facie obligation $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$: $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \zeta$, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \zeta \Rightarrow \chi$, and $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \tau, \varphi \Rightarrow$.

The first underivability statement ($(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \zeta$) does not hold for $\zeta = \chi$, as $\psi \Rightarrow \chi$ is the first premiss of the standard block in the application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$, hence it is assumed that a valid proto-derivation of $\psi \Rightarrow \chi$ in $\mathsf{G_{MD^{ga}}cut}$ from $(\mathfrak{F}, \mathfrak{L}_{\mathcal{O}})$ exists. The second underivability statement ($(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \zeta \Rightarrow \chi$) cannot hold for $\zeta = \chi$, as $\zeta \Rightarrow \chi$ then expresses a logical truth. For what concerns the third underivability statement ($(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \tau, \varphi \Rightarrow$), it cannot hold (for $\tau = \theta$), as the sequent $\tau, \varphi \Rightarrow$ can be derived from the first premiss of $\mathsf{D}_{\mathcal{O}}$ ($\varphi, \varphi \Rightarrow$), by using W_L and Con_L. As a consequence, the proto-derivation of $\Rightarrow \mathcal{O}(\varphi/\psi)$ (represented by the application of the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ above) is not valid. Indeed, there is at least an obligation in $\mathfrak{L}$ ($\mathcal{O}_{\mathsf{pf}}(\theta/\chi)$) such that all the three underivability statements in the *not-excepted block* are false (i.e. there are valid proto-derivations for the sequents that should be underivable).

Case (iii): In this case the multicut rule is applied between the conclusions of two global assumption rules. Hence, let us assume that the premisses of the application of mcut are the conclusion $\Rightarrow \mathcal{O}(\varphi/\psi)$ of an application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ (as in Fig.4.6) and the conclusion of the application of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\varsigma/\upsilon)}$ (below).

As in Case (ii), we claim that this situation cannot occur. Otherwise the *no-active conflict* block of premisses of the application of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$, would have one of the following (set of) premisses:

(a.1) $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \upsilon$;

(a.2) $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \varsigma, \varphi \Rightarrow$;

(b) $\psi \Rightarrow \eta, \quad \eta \Rightarrow \upsilon, \quad \xi \Rightarrow \varphi$ for some $\mathcal{O}_{\mathsf{pf}}(\xi/\eta)$ or $\mathcal{P}_{\mathsf{pf}}^{\mathcal{O}}(\xi/\eta)$ from $\mathfrak{L}$;

(c) $\psi \Rightarrow \eta, \quad \eta \Rightarrow \upsilon, \quad \Rightarrow \varphi, \xi$ for some $\mathcal{F}_{\mathsf{pf}}(\xi/\eta)$ from $\mathfrak{L}$.

But it is easy to observe that none of those cases can happen. The premiss in (a.1) gives a contradiction with the premiss $\psi \Rightarrow \upsilon$ of the application of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\varsigma/\upsilon)}$, and the one in (a.2) contradicts the premiss $\varsigma, \varphi \Rightarrow$ of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\varsigma/\upsilon)}$. The premisses in (b) result in contradiction as the *not-excepted* block of the application of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\varsigma/\upsilon)}$ contains one of the premisses: $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \eta$, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \eta \Rightarrow \upsilon$, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \xi \Rightarrow \varphi$. Finally, the premisses in (c) contradict again one of the following premisses in the *not-excepted* block of the application of $\mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\varsigma/\upsilon)}$: $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \psi \Rightarrow \eta$, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \eta \Rightarrow \upsilon$, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}cut}}} \Rightarrow \varphi, \xi$.

$$
\dfrac{
\begin{array}{l}
\{\psi \Rightarrow \upsilon\} \quad \cup \quad \{\varsigma, \varphi \Rightarrow\ \} \\
\cup \quad \left\{ \left[\begin{array}{l} \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \zeta\}\big\} \\ \cup \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \zeta \Rightarrow \upsilon\}\big\} \\ \cup \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \tau \Rightarrow \varphi\}\big\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \zeta\}\big\} \\ \cup \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \zeta \Rightarrow \upsilon\}\big\} \\ \cup \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \Rightarrow \varphi, \tau\}\big\} \end{array} \right] \mid \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \left[\begin{array}{l} \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \zeta\}\big\}, \\ \cup \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \tau \Rightarrow \varphi\}\big\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi, \xi \Rightarrow\ \} \end{array} \mid \begin{array}{c} \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \\ \text{or } \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \end{array} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \left\{ \left[\begin{array}{l} \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \zeta\}\big\}, \\ \cup \big\{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \Rightarrow \varphi, \tau\}\big\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi, \xi \Rightarrow\ \} \end{array} \mid \begin{array}{c} \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \\ \text{or } \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \end{array} \right\} \\ \cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\varphi \Rightarrow \xi\} \end{array} \mid \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{F}(\tau/\zeta) \in \mathfrak{L} \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \tau \Rightarrow \varphi\}\} \\ \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup \{\xi, \varphi \Rightarrow\} \end{array} \mid \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L} \right\} \end{array} \right] \mid \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \right\}
\end{array}
}{
\mathcal{O}(\varphi/\psi) \Rightarrow
}
\ \mathcal{O}_L^{\mathcal{O}_{\mathsf{pf}}(\varsigma/\upsilon)}
$$

□

Recall that we write $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}}$ for the cut-free system, i.e., the calculus $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}}$ with the cut rule removed.

From the cut elimination theorem the equivalence of the systems with and without the cut rule follows.

Corollary 4.3.13 *For every $\mathfrak{F}, \mathfrak{L}$ we have*

$$(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad (\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}} \Gamma \Rightarrow \Delta\,.$$

Proof. The right-to-left direction corresponds to the statement of Thm.4.3.12. The other direction is straightforward, as the rules in $\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}$ are also in $\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}\,\mathsf{cut}$ and the underivability statements range over the same system for valid proto-derivations. □

Furthermore, the cut elimination theorem makes it possible to show that the notion of valid proto-derivation (Def.4.2.13, Def.4.3.7) is well defined.

Before showing the well-definedness of valid proto-derivation in $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$, let us define the notion of *modal nesting depth* of a formula.

Definition 4.3.14 (Modal nesting depth) *The modal nesting depth $mnd(\varphi)$ of a formula φ is the deepest nesting of modal operators contained by the formula. It is defined as follows:*

- $mnd(p) = 0$ *with* $p \in \mathsf{Var}$*;*
- $mnd(\bot) = 0$*;*
- $mnd(\theta \to \chi) = max(mnd(\theta), mnd(\chi))$*;*
- $mnd(\mathsf{op}(\theta/\chi)) = (max(mnd(\theta), mnd(\chi))) + 1$ *for any* $\mathsf{op} \in \{\mathcal{O}, \mathcal{F}, \mathcal{R}\}$*.*

To prove the well-definedness of the notion of valid proto-derivation, let us consider the following alternative *stratified* definition of this notion.

Definition 4.3.15 ([32]) *A* proto-derivation of rank n *in* $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$ *from* $(\mathfrak{F}, \mathfrak{L})$ *with conclusion* $\Gamma \Rightarrow \Delta$ *is a proto-derivation in* $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$ *from* $(\mathfrak{F}, \mathfrak{L})$ *with conclusion* $\Gamma \Rightarrow \Delta$ *such that*

(1) every formula occurring in the proto-derivation has modal nesting depth at most n hence every formula occurring in an underivability statement in the proto-derivation has modal nesting depth at most $n - 1$.

*For a natural number n, a proto-derivation is n-*valid *if it is of rank n and, for every $k < n$, for none of the underivability statements occurring in the proto-derivation there is a k-valid proto-derivation in* $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$ *from* $(\mathfrak{F}, \mathfrak{L})$*.*

The global assumption rules we introduced are such that the modal nesting depth of formulas in the underivability statements is strictly lower than that of the formulas in the conclusion. Hence the n-validity of a proto-derivation only depends on k-validity of its underivability statements, for $k < n$.

This means that the previous definition of a valid proto-derivation is not circular, but inductive: by using the Cut Elimination Theorem, we can prove that this definition is equivalent to the unrestricted one in Def.4.2.13, Def.4.3.7.

Theorem 4.3.16 ([32]) *Given any sequent $\Gamma \Rightarrow \Delta$ with modal nesting depth at most equal to n: there is a valid proto-derivation for it in* $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$ *from* $(\mathfrak{F}, \mathfrak{L})$*, if and only if there is a n-valid proto-derivation for it in* $\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}$ *from* $(\mathfrak{F}, \mathfrak{L})$*.*

Proof. The claim of the theorem is proved by induction on n, so let us assume that it holds for every $k < n$.

For the right-to-left direction, suppose that there is a valid proto-derivation $\mathfrak{D}$ of the sequent $\Gamma \Rightarrow \Delta$ from $(\mathfrak{F}, \mathfrak{L})$ in $\mathsf{G_{MD+^{ga}}cut}$. Then, by Cut Elimination (Thm.4.3.12) there is a valid proto-derivation $\mathfrak{D}'$ from $(\mathfrak{F}, \mathfrak{L})$ in $\mathsf{G_{MD+^{ga}}}$ with the same conclusion.

Since the rules of the cut-free calculus $\mathsf{G_{MD+^{ga}}}$ (included the global assumption rules) do not increases the modal nesting depth from the conclusion to the premiss(es), the maximal modal nesting depth of formulas occurring in the proto-derivation $\mathfrak{D}'$ is equal to the modal nesting depth of its conclusion $\Gamma \Rightarrow \Delta$, i.e. n. Moreover, the modal nesting depth of formulas occurring in the underivability statements in global assumption rules is strictly smaller than that of the conclusion; hence, the modal nesting depths of formulas in the underivability statements occurring in $\mathfrak{D}'$ are at most equal to $n-1$. This means the rank of the proto-derivation $\mathfrak{D}'$ is n. The modal nesting depths of the formulas occurring in the underivability statements are then strictly smaller than n, therefore we can apply the induction hypothesis for concluding that, for any $k \leq n-1$, there is no k-valid proto-derivation for the formulas in the underivability statements occurring in $\mathfrak{D}'$. This means the derivation is n-valid and, since any derivation in $\mathsf{G_{MD+^{ga}}}$ is also a derivation in $\mathsf{G_{MD+^{ga}}cut}$, we proved that, if there is a valid proto-derivation in $\mathsf{G_{MD+^{ga}}cut}$ from $(\mathfrak{F}, \mathfrak{L})$ for the sequent $\Gamma \Rightarrow \Delta$ with modal nesting depth at most n, then there is a n-valid proto-derivation in $\mathsf{G_{MD+^{ga}}cut}$ from $(\mathfrak{F}, \mathfrak{L})$ with the same conclusion.

For the other direction, asssume we have a n-valid proto-derivation with conclusion $\Gamma \Rightarrow \Delta$. As the modal nesting depth of formulas occurring in the underivability statements in global assumption rules is strictly smaller than that of the conclusion, we can apply the induction hypothesis, by which we obtain that there are no valid proto-derivations for any of the underivability statements occurring as premiss of an application of a global assumption rule in the derivation. Hence, by definition of a valid proto derivation (Def.4.2.13, Def.4.3.7) we have a valid proto-derivation with conclusion $\Gamma \Rightarrow \Delta$. $\square$

From Thm.4.3.16, together with the fact that the notion of n-validity (Def.4.3.15) is well-defined, we obtain the following corollary.

Corollary 4.3.17 (Well-definedness) *The notion of a valid proto-derivation is well-defined.*

The second main consequence of the Cut Elimination Theorem is the consistency of the system $\mathsf{MD+^{ga}}$: using Thm.4.3.12 we can show that the global assumption rules are compatible with the logic $\mathsf{MD+}$, i.e. they do not produce formulas (self contradictory recommendations, conflicting obligations or prohibitions) that can give rise to inconsistencies in $\mathsf{MD+}$.

Theorem 4.3.18 (Consistency) *For any set $\mathfrak{L}$ of deontic assumptions and any set $\mathfrak{F}$ of global propositional assumptions not containing the empty sequent ot the sequent $\Rightarrow \bot$, the underivability statement $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}} \Rightarrow \bot$ holds. Specifically, this also means:*

(a) there are no φ, ψ such that $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\neg\varphi/\psi)$;

(b) there are no φ, ψ such that $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{F}(\varphi/\psi) \wedge \mathcal{F}(\neg\varphi/\psi)$;

(c) there are no φ, ψ such that $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{O}(\varphi/\psi) \wedge \mathcal{F}(\varphi/\psi)$;

(d) there is no ψ such that $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{R}(\bot/\psi)$.

Proof. Recall that the calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}$ admits cut-elimination and therefore it has the subformula property. Moreover, looking at rules of $\mathsf{G}_{\mathsf{MD+^{ga}}}$, we can observe that there are no rules (schemas) with an empty conclusion and that every formula occurring in one of the premisses, including the underivability statements, is a subformula of a formula occurring in the conclusion or in $\mathfrak{L}$; we call this property the *subformula property relative to* $\mathfrak{L}$. Hence, all the rules in $\mathsf{G}_{\mathsf{MD+^{ga}}}$ have at least the *subformula property relative to* $\mathfrak{L}$; since, by assumption, the empty sequent is not in $\mathfrak{F}$, the empty sequent cannot be derived in the system $\mathsf{G}_{\mathsf{MD+^{ga}}}$ and, since W_R is the only rule introducing $\bot$ on the right hand side of a sequent, we cannot derive $\Rightarrow \bot$.

The statements (a)-(d) follow by using the derivability of the involved formulas on the left hand side of a sequent together with the cut rule. If one of those statement was derivable, then by using the cut rule it would be possible to derive the empty sequent, against the observations above.

For instance, consider the statement (a) $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\neg\varphi/\psi)$. If (a) was derivable, then we could have the following derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\mathsf{init}}{\varphi, \neg\varphi \Rightarrow}\,\neg_L \quad \dfrac{}{\psi \Rightarrow \psi}\,\mathsf{init} \quad \dfrac{}{\psi \Rightarrow \psi}\,\mathsf{init}}{\mathcal{O}(\varphi/\psi), \mathcal{O}(\neg\varphi/\psi) \Rightarrow}\,\mathsf{D}_{\mathcal{O}}}{\mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\neg\varphi/\psi) \Rightarrow}\,\wedge_L \qquad \Rightarrow \mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\neg\varphi/\psi)}{\Rightarrow}\,\mathsf{cut}
$$

but since the system has the subformula property relative to $\mathfrak{L}$, this means the empty sequent should be contained in $\mathfrak{F}$, against the assumption. $\square$

Finally, the Cut Elimination Theorem allows us to prove that the logic $\mathsf{MD+^{ga}}$ is decidable.

Indeed it is possible to show that we can restrict the proof search to proto-derivations in the system without the cut rule; hence, using the subformula property relative to $\mathfrak{L}$ of the rules of the cut free calculus $\mathsf{G}_{\mathsf{MD+^{ga}}}$, we can obtain a decision procedure.

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\ \mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \to_R$$

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{O}} \qquad \frac{\varphi, \theta \Rightarrow \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{O}} \qquad \frac{\varphi \Rightarrow}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{O}}$$

$$\frac{\theta \Rightarrow \varphi \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \mathcal{F}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{F}} \qquad \frac{\Rightarrow \varphi, \psi \qquad \theta \Rightarrow \chi \quad \chi \Rightarrow \theta}{\Gamma, \mathcal{F}(\varphi/\theta), \mathcal{F}(\psi/\chi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{F}} \qquad \frac{\Rightarrow \varphi}{\Gamma, \mathcal{F}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{F}}$$

$$\frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \mathcal{R}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{R}} \qquad \frac{\varphi \Rightarrow}{\Gamma, \mathcal{R}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{P}_{\mathcal{R}} \qquad \frac{\varphi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Delta, \mathcal{O}(\varphi/\psi), \mathcal{F}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_{\mathcal{OF}}$$

Figure 4.12: The system $\mathsf{G3}_{\mathsf{MD+ga}}$ without the assumption rules ([32]).

In order to show the decidability of the system, we will again build an algorithm for the proof search procedure: to do so, in the next definition we introduce a slightly different version of the calculus $\mathsf{G}_{\mathsf{MD+ga}}$ where the structural rules of of weakening and contraction are absorbed into the rules of the calculus for $\mathsf{MD+}^{\mathsf{ga}}$ and therefore (W_L), (W_R), (Con_L), (Con_R) can be eliminated.

Definition 4.3.19 ($\mathsf{G3}_{\mathsf{MD+ga}}$ **[32]**) *The system $\mathsf{G3}_{\mathsf{MD+ga}}$ is obtained from $\mathsf{G}_{\mathsf{MD+ga}}$ by dropping the weakening and contraction rules, absorbing weakening into the conclusion of the logical rules, and including the rules $\mathsf{P}_{\mathcal{O}}$ and $\mathsf{P}_{\mathcal{F}}$ for absorbing the contractions of the principal formulas of $\mathsf{D}_{\mathcal{O}}$ and $\mathsf{D}_{\mathcal{F}}$, respectively. The system without the global assumption rules is presented in Fig.4.12.*

Moreover, the global assumption rules in $\mathsf{G3}_{\mathsf{MD+ga}}$ are obtained from the rules in Figs.4.6-4.11 by absorbing weakening into the conclusion (only!). We write e.g. $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)^*}$ for the global assumption rule of $\mathsf{G3}_{\mathsf{MD+ga}}$ that has the same premisses as the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}$ of $\mathsf{G}_{\mathsf{MD+ga}}$ from Fig.4.6, but the conclusion $\Gamma \Rightarrow \mathcal{O}(\varphi/\psi), \Delta$.

A proto-derivation *in $\mathsf{G3}_{\mathsf{MD+ga}}$ from $(\mathfrak{F}, \mathfrak{L})$ is defined in the same way as for $\mathsf{G}_{\mathsf{MD+ga}}$, but in $\mathsf{G3}_{\mathsf{MD+ga}}$ it is possible for the leaves of the proto-derivation to be labelled with sequents of the form $\Gamma, \Sigma \Rightarrow \Pi, \Delta$, where $\Sigma \Rightarrow \Pi \in \mathfrak{F}$ and $\Gamma \Rightarrow \Delta$ is an arbitrary sequent.*

Also the notion of valid proto-derivation *is similar to the one for $\mathsf{G}_{\mathsf{MD+ga}}$; note that, as for $\mathsf{G}_{\mathsf{MD+ga}}$, the underivability statements range over $\mathsf{G}_{\mathsf{MD+ga}}\mathsf{cut}$.*

In order to prove that the calculus $\mathsf{G3}_{\mathsf{MD+ga}}$ is equivalent to $\mathsf{G}_{\mathsf{MD+ga}}$, we first need to prove that the weakening and contraction rules are admissible in $\mathsf{G3}_{\mathsf{MD+ga}}$. This is possible by using the invertibility of propositional rules of $\mathsf{G3}_{\mathsf{MD+ga}}$, which guarantees that the order of their appearance between two applications of modal rules in a derivation does not influence the conclusion.

To prove the invertibility of propositional rules $\rightarrow_R$ and $\rightarrow_L$ of $\mathsf{G3}_{\mathsf{MD+^{ga}}}$, we first prove the following lemma.

Lemma 4.3.20 (Generalized initial sequents [32]) *The generalized initial sequent rule*

$$\frac{}{\Gamma, \varphi \Rightarrow \varphi, \Delta}\ \mathsf{init}$$

for any formula φ is admissible in $\mathsf{G3}_{\mathsf{MD+^{ga}}}$.

Proof. We prove the lemma by induction on the complexity of the formula φ.

We show only the case for $\varphi = \mathcal{O}(\theta/\chi)$:

$$\frac{\dfrac{}{\theta \Rightarrow \theta}\ IH \quad \dfrac{}{\chi \Rightarrow \chi}\ IH \quad \dfrac{}{\chi \Rightarrow \chi}\ IH}{\Gamma, \mathcal{O}(\theta/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta}\ \mathsf{Mon}_{\mathcal{O}}$$

□

Lemma 4.3.21 (Invertibility of the propositional rules [32]) *The propositional rules $\rightarrow_R$ and $\rightarrow_L$ of $\mathsf{G3}_{\mathsf{MD+^{ga}}}$ are* height-preserving invertible *in the system. This means that, if there is a valid proto-derivation of their conclusion in $\mathsf{G3}_{\mathsf{MD+^{ga}}}$ with height at most n from $(\mathfrak{F}, \mathfrak{L})$, then for each of the premisses there is a valid proto-derivation with height at most n from $(\mathfrak{F}, \mathfrak{L})$ as well.*

Proof. The lemma is easily proved by induction on the height of the valid proto-derivation. □

Lemma 4.3.22 (Admissibility of Weakening and Contraction [32]) *The weakening and the contraction rules (W_L), (W_R), (Con_L), (Con_R) are* height-preserving admissible *in $\mathsf{G3}_{\mathsf{MD+^{ga}}}$: for any of these rules, if for each premiss there is a valid proto-derivation of height at most n from $(\mathfrak{F}, \mathfrak{L})$, then there is a valid proto-derivation of the rule's conclusion, with at most the same height.*

Proof. The claim is proved by induction on the depth of the valid proto-derivation. If the last applied rule before an application of contraction is propositional and the contracted formula is principal in this application of ($\rightarrow_L$) or ($\rightarrow_R$), we use the height-preserving invertibility of these rules (Lem.4.3.21) for applying the induction hypothesis.

As the global assumption rules of $\mathsf{G3}_{\mathsf{MD+^{ga}}}$ are obtained from the ones of $\mathsf{G}_{\mathsf{MD+^{ga}}}$ by absorbing weakening into the conclusion and they have only one principal formula, the cases where the last applied rule before an application of (W_L), (W_R), (Con_L), or (Con_R) is a global assumption rule are trivial.

Also the cases for modal rules are trivial as weakening is absorbed into their conclusions and the applications of $\mathsf{D}_{\mathcal{O}}$ or $\mathsf{D}_{\mathcal{F}}$ followed by (Con_L) can be simulated by using $\mathsf{P}_{\mathcal{O}}$ and $\mathsf{P}_{\mathcal{F}}$, respectively. For instance:

$$\dfrac{\dfrac{\varphi, \varphi \Rightarrow \quad \psi \Rightarrow \psi \quad \psi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\,\mathsf{D}_{\mathcal{O}}}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\,\mathsf{Con}_L \quad \text{is tranformed into} \quad \dfrac{\dfrac{\varphi, \varphi \Rightarrow}{\varphi \Rightarrow}\,IH}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\,\mathsf{P}_{\mathcal{O}}$$

□

Using the previous lemma, it is possible to prove that the calculus $\mathsf{G3}_{\mathsf{MD+^{ga}}}$ is equivalent to $\mathsf{G}_{\mathsf{MD+^{ga}}}$.

Lemma 4.3.23 (Equivalence of $\mathsf{G3}_{\mathsf{MD+^{ga}}}$ and $\mathsf{G}_{\mathsf{MD+^{ga}}}$ [32]) *Given the finite set $\mathfrak{F}$ of factual assumptions and a finite set $\mathfrak{L}$ of non nested prima-facie deontic assumptions, for any sequent $\Gamma \Rightarrow \Delta$, $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}} \Gamma \Rightarrow \Delta$ if and only if $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+^{ga}}}} \Gamma \Rightarrow \Delta$.*

Proof. We can transform a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3}_{\mathsf{MD+^{ga}}}$ into a a derivation with the same conclusion in $\mathsf{G}_{\mathsf{MD+^{ga}}}$, by simply substituting the applications of the rules, in which weakening and contraction have been absorbed, with application of the corresponding rules of $\mathsf{G}_{\mathsf{MD+^{ga}}}$, followed by suitable applications of (W_L), (W_R), (Con_L), or (Con_r).

For the other direction, we remove the applications of weakening and contraction rules by using the previous lemma Lem.4.3.22.

As the underivability statement range over the same system, this suffices to prove the equivalence of the two calculi. □

Using the previous lemma, we can construct the proof search procedure (Alg.3) for the calculus $\mathsf{G3}_{\mathsf{MD+^{ga}}}$. If it is possible to prove that the proof search algorithm terminates and accepts an input $\Gamma \Rightarrow \Delta$ if and only if there is a proto-derivation in $\mathsf{G}_{\mathsf{MD+^{ga}}}$ with $\Gamma \Rightarrow \Delta$ as conclusion, then $\mathsf{MD+^{ga}}$ is proved to be decidable. This means that, for any set $\mathfrak{F}$ of sequents as in Def.4.2.4, any set $\mathfrak{L}$ of non nested deontic assumptions and any sequent $\Gamma \Rightarrow \Delta$, it is always possible to decide whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Gamma \Rightarrow \Delta$ or not.

The formulation of algorithm Alg.3 is based on alternating Turing machines (see [24]), i.e. Alg.3 is formulated as a nondeterministic procedure which can make existential guesses and universal choices. For existential guesses the procedure is successful if at least one of the corresponding runs is, for universal choices if all of them are.

Algorithm 3: Proof-search procedure for $\mathsf{G3}_{\mathsf{MD+ga}}$ ([32])

Input: a tuple $(\mathfrak{F}, \mathfrak{L})$ of finite sets of propositional facts and prima-facie deontic statements and a sequent $\Gamma \Rightarrow \Delta$

Output: Is $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Gamma \Rightarrow \Delta$?

if $\bot \in \Gamma$ *or* $\Gamma \cap \Delta \neq \varnothing$ **then**
  halt and accept;
if *there is* $\Sigma \Rightarrow \Pi \in \mathfrak{F}$ *with* $\Sigma \subseteq \Gamma$ *and* $\Pi \subseteq \Delta$ **then**
  halt and accept;
existentially guess a rule scheme R (propositional, modal or assumption) from $\mathsf{G3}_{\mathsf{MD+ga}}$ or $\mathsf{ga}^{*}_{\mathfrak{L}}$ and a matching (tuple of) principal formula(s) from $\Gamma \Rightarrow \Delta$;
else if *R is a propositional rule scheme* **then**
  universally choose one of its premisses $\Sigma \Rightarrow \Pi$;
  check recursively whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$, output the answer and halt;
else if *R is a modal rule scheme* **then**
  universally choose one of its premisses $\Sigma \Rightarrow \Pi$;
  check recursively whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$, output the answer and halt;
else
  `/* Then` R `is an assumption schema */`
  universally choose a block Bl_x of premisses;
  if *Bl_x is the standard block* **then**
    Universally choose a premiss $\Sigma \Rightarrow \Pi$ in Bl_x;
    Recursively check whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$, output the answer and halt;
  else if *Bl_x is the non-excepted block* **then**
    universally choose a formula from $\mathfrak{L}$ and existentially guess a premiss $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$ from the block of premisses for this formula;
    Recursively check whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$, flip the answer and halt;
  else
    `/* Then` Bl_x `is the no-active-conflict block */`
    Universally choose a formula from $\mathfrak{L}$ and existentially guess a block Bl_y of premisses for this formula;
    if *Bl_y is the conflict block* **then**
      existentially guess a premiss $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$ from Bl_y;
      Recursively check whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$, flip the answer and halt;
    else
      `/* Then` Bl_y `is the override block */`
      existentially guess a formula from $\mathfrak{L}$ and universally choose a premiss $\Sigma \Rightarrow \Pi$ from the corresponding set of premisses;
      Recursively check whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G3}_{\mathsf{MD+ga}}\,\mathsf{cut}} \Sigma \Rightarrow \Pi$, output the answer and halt;
halt and reject;

Lemma 4.3.24 (Termination [32]) *Alg.3 terminates.*

Proof. Let n be the *size* of the input, i.e., the sum of the numbers of symbols in $\mathfrak{F}, \mathfrak{L}$ and $\Gamma \Rightarrow \Delta$, and let the *complexity* of a sequent be the number of occurrences of propositional or modal connectives in it.

An application of a propositional rule removes a propositional connective by replacing a (propositional) formula with its subformulas, reducing the complexity of the sequents. Hence, the number of applications of propositional rules is bounded by the number of subformulas of formulas in the conclusion, in $\mathfrak{F}$ or in $\mathfrak{L}$; therefore, it is finally bounded by the size n of the input.

Moreover, because of the form of modal rules and global assumption rules, together with the fact that $\mathfrak{L}$ does not contain nested modal formulas, the premisses of applications of those rules have maximal modal nesting depth strictly lower than the their conclusions. Hence, the recursive calls in lines 11, 16, 19, 24 and 27 are on sequents with strictly lower maximal nesting depth of the modal operators.

Therefore, applications of modal rules and global assumption rules reduce the maximal modal nesting depth of a sequent, which is also bounded by the size n of the input. Since there are only finitely many different rule schemes and, for each one of the existential guesses on rules applications, the recursive calls either reduce the complexity or decrease the maximal modal nesting depth of the sequent, a run of the algorithm necessarily makes a finite number of recursive calls, after which it either accepts with lines 2 or 4 or rejects with line 28. Thus, the algorithm terminates after a finite number of steps. $\square$

Lemma 4.3.25 (Correctness [32]) *Algorithm 3 accepts an input* $(\mathfrak{F}, \mathfrak{L}), \Gamma \Rightarrow \Delta$ *if and only if* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+ga}}} \Gamma \Rightarrow \Delta$.

Proof. The claim is proved by induction on the maximal modal nesting depth of $\Gamma \Rightarrow \Delta$.

For the left-to-right direction, from an accepting run of the algorithm we can easily build a cut-free proto-derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3}_{\mathsf{MD+ga}}$ by applying (backwards) the rules corresponding to the existential choices of the algorithm. By induction hypothesis, there are no valid proto-derivations in $\mathsf{G}_{\mathsf{MD+ga}}\mathsf{cut}$ from $(\mathfrak{F}, \mathfrak{L})$ for the underivability statements occurring in the proto-derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3}_{\mathsf{MD+ga}}$, therefore, this proto-derivation is valid. From that, by Lem.4.3.23, we obtain a valid proto-derivation in $\mathsf{G}_{\mathsf{MD+ga}}$.

For the other direction, let us assume that there is a valid proto-derivation for $\Gamma \Rightarrow \Delta$ in $\mathsf{G}_{\mathsf{MD+ga}}$. From that, by Thm.4.3.12 and Lem.4.3.23 it follows that there is a valid proto-derivation for $\Gamma \Rightarrow \Delta$ in $\mathsf{G3}_{\mathsf{MD+ga}}$. By induction hypothesis, the algorithm rejects all the underivability statements occurring as premisses of a global assumption rule in this proto-derivation; hence the algorithm accepts the recursive calls for existential and universal choices corresponding to the rules of the proto-derivation and therefore accepts the input.

□

Theorem 4.3.26 (Decidability [32]) *Given a set $\mathfrak{F}$ of factual assumptions, a set $\mathfrak{L}$ of non nested deontic assumptions and a sequent $\Gamma \Rightarrow \Delta$, it is always possible to decide whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \Gamma \Rightarrow \Delta$.*

Proof. Given a set $\mathfrak{F}$ of factual assumptions, a set $\mathfrak{L}$ of non nested deontic assumptions and a sequent $\Gamma \Rightarrow \Delta$, the proof search procedure on this input terminates in a finite number of steps (Lem.4.3.24) and accepts the input if and only if $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}} \Gamma \Rightarrow \Delta$ (Lem.4.3.25).

Hence for any triple $\mathfrak{F}, \mathfrak{L}, \Gamma \Rightarrow \Delta$ it is possible to decide whether $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\mathsf{cut}} \Gamma \Rightarrow \Delta$ or not. □

4.3.2 Adding a Superiority Relation

We now introduce a natural extension of the presented system, which allows us to capture the preference for a command over another command, independently from their applicability conditions. This is in particular useful for representing the hierarchy of sources as a criterion of prioritization of commands (see Section 2.4.1).

The rules in this section represent special cases of the ones introduced in [33]. Under `http://subsell.logic.at/bprover/deonticProver/version2.0/` it is also available a Prolog implementation of the decision procedure for the calculi in [33] (including $\mathsf{G}_{\mathsf{MD+}}$) extended with global assumption rules that also take into account superiority relations among norms.

In general a *superiority relation* is a binary relation $\prec$ on the set of deontic assumptions: for $\mathsf{op1}, \mathsf{op2} \in \{\mathcal{O}_{\mathsf{pf}}, \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}, \mathcal{F}_{\mathsf{pf}}, \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}, \mathcal{R}_{\mathsf{pf}}\}$, $\mathsf{op1}(\varphi/\psi) \prec \mathsf{op2}(\theta/\chi)$ intuitively expresses that the command $\mathsf{op2}(\theta/\chi)$ has higher authority than $\mathsf{op1}(\varphi/\psi)$ and cannot be overruled by the latter, independently from the relation between their conditions χ and ψ. Superiority relations of this kind are used e.g. in Defeasible Deontic Logic (see Section 4.5).

Since in Mīmāṃsā a transitive superiority relation is defined not on single norms but on sources of duty, we will incorporate the name of the sources in deontic assumptions and base the superiority relation between commands on the hierarchy of sources.

Recalling the discussion in (see Section 2.4.1), we will write `Śru` for śruti, `Smṛ` for smṛti, `Sad` for sadācāra, and `Ātm` for ātmatuṣṭi, and define the transitive relation $<$ on the set of sources $\mathsf{S} = \{\texttt{Śru}, \texttt{Smṛ}, \texttt{Sad}, \texttt{Ātm}\}$ such that $\texttt{Ātm} < \texttt{Sad}$, $\texttt{Sad} < \texttt{Smṛ}$, $\texttt{Smṛ} < \texttt{Śru}$.

Hence, for $\mathsf{op} \in \{\mathcal{O}_{\mathsf{pf}}, \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}, \mathcal{F}_{\mathsf{pf}}, \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}, \mathcal{R}\}$ and $\mathsf{so} \in \{\texttt{Śru}, \texttt{Smṛ}, \texttt{Sad}, \texttt{Ātm}\}$, each deontic assumption will have the form $\mathsf{so} : \mathsf{op}(\varphi/\psi)$, meaning that the command $\mathsf{op}(\varphi/\psi)$ is found in the source so.

Moreover, for $\mathsf{op1}, \mathsf{op2} \in \{\mathcal{O}_{\mathsf{pf}}, \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}, \mathcal{F}_{\mathsf{pf}}, \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}, \mathcal{R}\}$ and $\mathsf{so}_1, \mathsf{so}_2 \in \{\texttt{Śru}, \texttt{Smṛ}, \texttt{Sad}, \texttt{Ātm}\}$, $\mathsf{so}_1 : \mathsf{op1}(\varphi/\psi) \prec \mathsf{so}_2 : \mathsf{op2}(\theta/\chi)$ if and only if $\mathsf{so}_1 < \mathsf{so}_2$.

From the transitivity of the relation $<$ among the finite set of sources and the fact that each command has a source, it follows that, for any couple $\mathsf{so}_1, \mathsf{so}_2$, we have $\mathsf{so}_1 \nless \mathsf{so}_2$ or $\mathsf{so}_2 \nless \mathsf{so}_2$ and, correspondingly, for any couple of commands $\mathsf{so}_1 : \mathsf{op1}(\varphi/\psi)$, $\mathsf{so}_2 : \mathsf{op2}(\theta/\chi)$, necessarily $\mathsf{so}_1 : \mathsf{op1}(\varphi/\psi) \nprec \mathsf{so}_2 : \mathsf{op2}(\theta/\chi)$ or $\mathsf{so}_2 : \mathsf{op2}(\theta/\chi) \nprec \mathsf{so}_1 : \mathsf{op1}(\varphi/\psi)$.

This guarantees that, when considering the set of prima facie norms in conflict with a given command, we can always rule out those prima facie norms which are found in an inferior source with respect to the one of the given command.

Therefore, the modified global assumption rules, which incorporate the idea that a deontic assumption from an inferior source cannot overrule an assumption from a superior one, will be in a sense simplified. Indeed, the set of possibly conflicting norms to check will be restricted to the ones which are found in a source that is not inferior to the one of the command used as base. The modified right rule for obligations is given in Fig.4.13 (the other global assumption rules are adapted in a similar way).

The new conditions only restrict the set of deontic assumptions that the rules need to check in order to guarantee that $\mathcal{O}(\varphi/\psi)$ is derivable from $\mathsf{so}_{\mathsf{n}} : \mathcal{O}_{\mathsf{pf}}(\theta/\chi)$, without changing the sets of premisses.

Once the deontic assumption to be used as base is chosen, the new conditions restrict the set of prima facie commands to be checked, only to those which are found in a source at least at the same hierarchical level as the one in which we found the base. Hence, for example, in the procedure Alg.3 the new conditions restrict the choices at lines 18, 21, 26.

This means that the technical results (Cut-Elimination, Decidability), proved for the system $\mathsf{MD}+^{\mathsf{ga}}$, also hold for the logic $\mathsf{MD}+$ extended with global assumption rules incorporating the hierarchy of sources.

The following example shows how the new rules can be used to analyse arguments discussed in Mīmāṃsā texts.

Example 4.3.27. Let us consider the following statements, extracted and adapted from the discussions in the Mīmāṃsāsūtra (PMS 1.3.3–4) and in the subsequent commentaries by Śabara and (later) by Kumārila.

$$
\dfrac{
\begin{array}{l}
\{\psi \Rightarrow \chi\} \quad \cup \quad \{\theta \Rightarrow \varphi\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow \}\} \end{array} \right] \mid \mathsf{so_x} : \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L} \text{ or } \mathsf{so_x} : \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}, (\mathsf{so_x} \nprec \mathsf{so_n}) \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \zeta \Rightarrow \chi\}\} \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \end{array} \right] \mid \mathsf{so_x} : \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}, (\mathsf{so_x} \nprec \mathsf{so_n}) \right\} \\
\cup \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow \}\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathsf{so_y} : \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \xi, \varphi\} \end{array} \mid \mathsf{so_y} : \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathsf{so_y} : \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\}
\end{array} \right] \mid \mathsf{so_z} : \mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}, (\mathsf{so_z} \nprec \mathsf{so_n}) \right\} \\
\cup \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\}, \\ \cup\{\{(\mathfrak{F},\mathfrak{L}) \nvdash \varphi \Rightarrow \tau\}\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathsf{so_y} : \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\Rightarrow \varphi, \xi\} \end{array} \mid \mathsf{so_y} : \mathcal{F}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\} \\
\cup \left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathsf{so_y} : \mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\}
\end{array} \right] \mid \mathsf{so_z} : \mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}, (\mathsf{so_z} \nprec \mathsf{so_n}) \right\} \\
\cup \quad \left\{ \left[\begin{array}{l} \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \psi \Rightarrow \zeta\}\} \\ \{\{(\mathfrak{F},\mathfrak{L}) \nvdash \tau, \varphi \Rightarrow\}\} \\
\left\{ \begin{array}{c} \{\psi \Rightarrow \eta\} \cup \{\eta \Rightarrow \zeta\} \\ \cup\{\xi \Rightarrow \varphi\} \end{array} \mid \mathsf{so_y} : \mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}, (\mathsf{so_y} \nprec \mathsf{so_z}) \right\}
\end{array} \right] \mid \mathsf{so_z} : \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}, (\mathsf{so_z} \nprec \mathsf{so_n}) \right\}
\end{array}
}{
\Rightarrow \mathcal{O}(\varphi/\psi)
} \; \mathcal{O}_R^{\mathsf{so_n}:\mathcal{O}_{\mathsf{pf}}(\theta/\chi)}
$$

where $\mathsf{so_n} : \mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$, $\mathsf{so_x}, \mathsf{so_z}, \mathsf{so_y}$ vary over the ordered set of sources $\{\texttt{Śru}, \texttt{Smṛ}, \texttt{Sad}, \texttt{Ātm}\}$ and $(\mathfrak{F},\mathfrak{L}) \nvdash \Gamma \Rightarrow \Delta$ stands for $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G_{MD+ga}\,cut}} \Gamma \Rightarrow \Delta$.

Figure 4.13: The right assumption rule for obligations incorporating hierarcy of sources

- *The smṛti texts say that the whole udumbara* (the Indian fig tree) *post should be covered with a cloth during a* (specific) *ritual* ($\texttt{Smṛ} : \mathcal{O}_{\mathsf{pf}}(\texttt{cover/rit}) \in \mathfrak{L}$);
- *the śruti text on the same ritual speaks of "touching the post"* ($\texttt{Śru} : \mathcal{O}_{\mathsf{pf}}(\texttt{touch/rit}) \in \mathfrak{L}$);
- *"touching the post" would be impossible if the whole post was covered by the cloth* ($\texttt{cover}, \texttt{touch} \Rightarrow\ \in \mathfrak{F}$).

Since the two commands share the same conditions (represented by the fact that the obligations have the same formula in their second arguments), here, at first sight, it seems that we need to apply *vikalpa* principle.

However, as already mentioned, commands in smṛti have value only as they are based on the *Veda*s and clarify the content of śruti; hence, a contradiction between śruti and smṛti, if recognized as a genuine conflict, completely invalidate the "problematic" statements in the latter. Therefore, as shown by the following rule application, we will not be able to derive $\mathcal{O}(\texttt{cover/rit})$, while $\mathcal{O}(\texttt{touch/rit})$ is derivable, as well as $\mathcal{O}(\neg\texttt{cover/rit})$:

$$\frac{}{\texttt{rit} \Rightarrow \texttt{rit}}\ \mathsf{init} \qquad \frac{}{\texttt{touch} \Rightarrow \neg\texttt{cover}}\ \mathfrak{F}$$

$$\begin{array}{l} \left\{(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}\,cut}}} \texttt{touch}, \neg\texttt{cover} \Rightarrow \right\} for\ \ \texttt{Śru} : \mathcal{O}_{\mathsf{pf}}(\texttt{touch/rit}) \in \mathfrak{L} \\ (\texttt{Smṛ} < \texttt{Śru})\ \ for\ \ \texttt{Smṛ} : \mathcal{O}_{\mathsf{pf}}(\texttt{cover/rit}) \in \mathfrak{L} \end{array}$$

$$\frac{\begin{array}{l} \left\{(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}\,cut}}} \texttt{touch}, \neg\texttt{cover} \Rightarrow \right\} for\ \ \texttt{Śru} : \mathcal{O}_{\mathsf{pf}}(\texttt{touch/rit}) \in \mathfrak{L} \\ (\texttt{Smṛ} < \texttt{Śru})\ \ for\ \ \texttt{Smṛ} : \mathcal{O}_{\mathsf{pf}}(\texttt{cover/rit}) \in \mathfrak{L} \end{array}}{\Rightarrow \mathcal{O}(\neg\texttt{cover/rit})}\ \mathcal{O}_R^{\texttt{Śru}:\mathcal{O}_{\mathsf{pf}}(\texttt{touch/rit})}$$

The underivability of the weaker deontic assumption $\mathcal{O}(\texttt{cover/rit})$ involved in the conflict does not correspond to an instance of *vikalpa*: Mīmāṃsā authors seem more willing to accept a different authority for smṛti and śruti texts than to accept the temporary invalidation (through vikalpa) of śrauta commands. Indeed, the origin of the invalid command in the smṛti is explained as the result of human manipulation: some priests probably started covering the whole post with a cloth, as they would have obtained the cloth at the end of the ritual. This means that it is admitted that a passage of smṛti texts may not be related to the *Veda*s and hence it may not be authoritative at all. ■

4.4 Applications to Mīmāṃsā reasoning

In this section we will use the introduced formal tools for an analysis of Mīmāṃsā reasoning. We will first briefly discuss how of the system $\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}}$ captures the *vikalpa* principle, showing what kind of formal behaviour of the global assumption rules corresponds to the application of this principle. Then, we will use the system for comparing different interpretations of the deontic statements found in the sacred texts, mimicking the work of Mīmāṃsā scholars. As different interpretations of a set of commands give rise to different formalizations and interactions between those commands, the idea is to use our system to compare the different interpretations according to the number of applications of *vikalpa* they require. Indeed, as already mentioned, this principle for Mīmāṃsā scholars represents the last resort and the number of its applications should be minimized.

4.4.1 Vikalpa

The *vikalpa* principle intuitively states that if a set of deontic assumptions contains two incompatible norms which are totally equivalent from the point of view of prioritization criteria (like e.g. the specificity principle or the hierarchy of sources), the addressee of the commands could choose the one to follow. This principle is known in deontic logic as *disjunctive response*, thanks to which from the two conflicting assumptions $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\theta/\psi)$ we are able to derive at least the obligation of the disjunction $\mathcal{O}(\varphi \vee \theta/\psi)$ (see [54]). It results particularly important in normative reasoning as, intuitively, it states that, when it is impossible to be compliant with all the applicable norms, the agent should not "be paralysed" but take action to be compliant at least with one of the conflicting commands. The absence of such a principle, indeed, would imply that in a circumstance where two conflicting norms should be applicable, none of the commands is enforceable and hence the agent is free to violate both the norms.

Example 4.4.1. Consider the following deontic and factual assumptions. The example resembles what is known in deontic logic literature as the *"drowning twins dilemma"* ([92]), where two obligations towards identical twins hold, but it is impossible to be compliant with both of them.

- "if Anna is drowning, one should save Anna" ($\mathcal{O}_{\mathsf{pf}}(\texttt{save\_A}/\texttt{drown\_A})$);
- "if Bob is drowning, one should save Bob" ($\mathcal{O}_{\mathsf{pf}}(\texttt{save\_B}/\texttt{drown\_B})$);
- "it is impossible to save both Anna and Bob" ($\Rightarrow \neg(\texttt{save\_a} \wedge \texttt{save\_b})$).

Looking at the condition $\texttt{drown\_A} \wedge \texttt{drown\_B}$, we cannot derive $\mathcal{O}(\texttt{save\_A}/\texttt{drown\_A} \wedge \texttt{drown\_B})$ or $\mathcal{O}(\texttt{save\_B}/\texttt{drown\_A} \wedge \texttt{drown\_B})$, as none of the two conditions $\texttt{drown\_A}$, $\texttt{drown\_b}$ is more specific than the other.

Hence, without a principle like *vikalpa*, no command would be enforceable under the condition $\texttt{drown\_A} \wedge \texttt{drown\_B}$ and the agent wuould be in principle free to avoid saving anyone. On the other hand, as the application of *vikalpa* follows from the global assumption rules, the disjunctive command $\mathcal{O}(\texttt{save\_A} \vee \texttt{save\_B}/\texttt{drown\_A} \wedge \texttt{drown\_B})$ ("if Bob is drowning and Anna is drowning, one should save Bob or save Anna") is derivable in $\mathsf{G}_{\mathsf{MD}+^{\mathrm{ga}}}$ from the assumptions above. ■

However, both from a practical and a theoretical point of view, it is a very weak solution, as it gives no indication on which norm should be preferred, while common sense tends to always order norms' applications in a hierarchical way. Very interesting examples in this sense are given by thought experiments in ethics: the long story and many variants of ethical dilemmas like the trolley problem (see e.g.[39, 129]) show the importance given to criteria for choosing between the violations of two norms when at least one is inevitable, even in situations where the conflicting norms seem to have equal importance. Similar problems arise in the modern discussion on autonomous vehicles, as it could be necessary to choose e.g. what to hit in cases where an accident is impossible to be avoided. Hence, a principle like *vikalpa*, stating that it is enough to be compliant with at least one of the conflicting commands, chosen without any criterium, seems to be considered generally unsatisfactory. As already observed (Section 2.4.1), the same perspective seems to be shared by Mīmāṃsā scholars: in case of incompatible norms, they prefer the prioritization of one command over the others (by using the *bādha*s) and restrict the use of *vikalpa* to the cases where no one of the criteria of prioritization is applicable.

The *vikalpa* principle is also related to the phenomenon of *floating conclusions* for skeptical semantics in non-monotonic reasoning [87], i.e. the case where a conclusion (the content of the disjunctive command) is derivable via two conflicting arguments (the conflicting prima facie norms), while the contents of the single conflicting commands are not. Also this principle seems to be controversial, as many authors discuss if such a conclusion should be derivable (see e.g. [53, 70]). An interesting observation in [68] suggests that the acceptability of floating conclusions is inversely proportional to the "cost" of an error, in case one of those conclusions is wrong.

As discussed below, *vikalpa* is particularly important for simulating Mīmāṃsā reasoning in a formal logic context, exactly because it is very rarely used and it is considered the very last resort in case of conflicting Vedic norms: a small number of application of this principle can be considered an indicator of an acceptable argument from Mīmāṃsā point of view.

To explain the correspondence between the use of *vikalpa* in Mīmāṃsā reasoning, and the behaviour of the global assumption rules, let us consider the following example.

Example 4.4.2. From the perspective of the sequent calculus $\mathsf{MD}+^{\mathsf{ga}}$, capturing the *vikalpa* principle means that, if e.g. $\mathfrak{L} = \{\mathcal{O}_{\mathsf{pf}}(\varphi/\psi), \mathcal{O}_{\mathsf{pf}}(\theta/\chi)\}$ and $\mathfrak{F} = \{\varphi, \theta \Rightarrow\}$, for the condition $\psi \wedge \chi$, the system derives neither $\mathcal{O}(\varphi/\psi \wedge \chi)$ nor $\mathcal{O}(\theta/\psi \wedge \chi)$. Indeed, in a situation where $\psi \wedge \chi$ holds, since none of the prima-facie obligation is more specific and prevails, both φ and θ should be considered optional and therefore not strictly obligatory.

On the other hand, the prescription of $\mathcal{O}(\varphi \vee \theta/\psi \wedge \chi)$, i.e. the disjunction of the two original contents, is derivable. It expresses that the addressee of the commands can only choose between φ and θ, hence it is obligatory to perform at least one of them. ■

Generalising this intuition, we can prove the following proposition, which ensures that the *vikalpa* principle can, in a sense, be derived in $\mathsf{MD}+^{\mathsf{ga}}$.

The proposition guarantees that, given a set $\mathfrak{L}'_{\mathcal{O}}$ of possibly conflicting obligations, which in turn do not conflict with any other command in the list $\mathfrak{L}_{\mathcal{O}}$ of deontic assumptions, it is always possible to derive an obligation that has in its first argument the disjunction of all the contents of prescriptions in $\mathfrak{L}'_{\mathcal{O}}$. This means that, even if some obligations are incompatible under certain circumstances, we can always conclude that under those circumstances it is obligatory to perform at least one of the actions that were originally prescribed.

Proposition 4.4.3 ([32]) *Let $\mathfrak{L}'_{\mathcal{O}} \subseteq \mathfrak{L}$ be a set of prima facie obligations $\{\mathcal{O}_{\mathsf{pf}}(\varphi_1/\psi_1), \ \ldots, \ \mathcal{O}_{\mathsf{pf}}(\varphi_n/\psi_n)\}$ such that:*
(a) $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \varphi_i \Rightarrow$ *for any $i \leq n$;*
(b) $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigvee_{i\leq n} \varphi_i, \theta \Rightarrow$ *for any $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \setminus \mathfrak{L}'_{\mathcal{O}}$ with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigwedge_{i\leq n} \psi_i \Rightarrow \chi$*;*
(c) $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigvee_{i\leq n} \varphi_i \Rightarrow \theta$ *for any $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \setminus \mathfrak{L}'_{\mathcal{O}}$ with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigwedge_{i\leq n} \psi_i \Rightarrow \chi$*;*
(d) $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigvee_{i\leq n} \varphi_i, \theta \Rightarrow$ *for any $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \setminus \mathfrak{L}'_{\mathcal{O}}$ with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigwedge_{i\leq n} \psi_i \Rightarrow \chi$*.*
Then, $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \Rightarrow \mathcal{O}(\bigvee_{i\leq n} \varphi_i / \bigwedge_{i\leq n} \psi_i)$*.*

Proof. We prove the claim by showing that it is possible to derive $\Rightarrow \mathcal{O}(\bigvee_{i\leq n} \varphi_i / \bigwedge_{i\leq n} \psi_i)$ from $(\mathfrak{F}, \mathfrak{L})$ by applying $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\varphi_1/\psi_1)}$.

Using the propositional rules we obtain $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \varphi_1 \Rightarrow \bigvee_{i\leq n} \varphi_i$ and $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \bigwedge_{i\leq n} \psi_i \Rightarrow \psi_1$.

Moreover, for every $j \leq n$ we have $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \varphi_j, \bigvee_{i\leq n} \varphi_i \Rightarrow$, because otherwise we would obtain $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \varphi_j, \varphi_j \Rightarrow$, and hence $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD}+^{\mathsf{ga}}\,\mathsf{cut}}} \varphi_j \Rightarrow$, contradicting *(a)*.

Furthermore, by assumption *(b)*, for every $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \smallsetminus \mathfrak{L}'_{\mathcal{O}}$, either $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \bigwedge_{i\leq n}\psi_i \Rightarrow \chi$ or $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \theta, \bigvee_{i\leq n}\varphi_i \Rightarrow$. The analogous statement holds for every prima-facie deontic statement of the form $\mathcal{F}_{\mathsf{pf}}(\theta/\chi)$ (by assumption *(c)*) or $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\theta/\chi)$(by assumption *(d)*). Therefore, all the premisses of $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\varphi_1/\psi_1)}$ are available: by applying this rule, $\Rightarrow \mathcal{O}(\bigvee_{i\leq n}\varphi_i/\bigwedge_{i\leq n}\psi_i)$ is derivable. □

Note that the proposition above only shows that the disjunction $\bigvee_{i\leq m}\varphi_i$ of formulas from the set $\mathfrak{L}'_{\mathcal{O}}$ is not blocked. As there are no assumptions on whether two formulas $\varphi_i, \varphi_j \in \mathfrak{L}'_{\mathcal{O}}$ are jointly possible or not, the statement of the proposition applies, in particular, also when the assumptions are not jointly possible, i.e. when we can derive sequents of the form $\varphi_i, \varphi_j \Rightarrow$ with $1 \leq i, j \leq n$. It is easy to show (see, e.g., Ex.4.4.2) that if $(\mathfrak{F},\mathfrak{L}) \vdash \varphi_i, \varphi_j \Rightarrow$, then neither $\mathcal{O}(\varphi_i/\psi_i \wedge \psi_j)$ nor $\mathcal{O}(\varphi_j/\psi_i \wedge \psi_j)$ are derivable.

This means that the system $\mathsf{MD+}^{\mathsf{ga}}$ satisfies the principle of disjunctive response (resp. *vikalpa*).

As in the case of obligations in the previous proposition it can be shown that the corresponding statement for prohibitions also holds.

Proposition 4.4.4 ([32]) *Let $X = \{\mathcal{F}_{\mathsf{pf}}(\varphi_1/\psi_1), \ldots, \mathcal{F}_{\mathsf{pf}}(\varphi_n/\psi_n)\} \subseteq \mathfrak{L}$ be a set such that:*
(a) $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \Rightarrow \varphi_i$ for any $i \leq n$;
(b) $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \theta \Rightarrow \bigwedge_{i\leq n}\varphi_i$ for any $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \smallsetminus X$ with $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \bigwedge_{i\leq n}\psi_i \Rightarrow \chi$;
(c) $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \Rightarrow \bigwedge_{i\leq n}\varphi_i, \theta$ for any $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \smallsetminus X$ with $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \bigwedge_{i\leq n}\psi_i \Rightarrow \chi$;
(d) $(\mathfrak{F},\mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \theta \Rightarrow \bigwedge_{i\leq n}\varphi_i$ for any $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L} \smallsetminus X$ with $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \bigwedge_{i\leq n}\psi_i \Rightarrow \chi$.
Then $(\mathfrak{F},\mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+}^{\mathsf{ga}}}\,\mathsf{cut}} \Rightarrow \mathcal{F}(\bigwedge_{i\leq n}\varphi_i/\bigwedge_{i\leq n}\psi_i)$.

Proof. Analogous to the one for the case of obligations in Prop.4.4.3. □

The previous propositions ensure that we can always derive a command which includes as options all the actions prescribed by norms that are applicable under its conditions, independently from the relations among those actions. This means that the *vikalpa* principle is compatible with the structure of global assumption rules. However, the derivability of a disjunctive command of the form $\Rightarrow \mathcal{O}(\bigvee_{i\leq n}\varphi_i/\bigwedge_{i\leq n}\psi_i)$ (or $\Rightarrow \mathcal{F}(\bigwedge_{i\leq n}\varphi_i/\bigwedge_{i\leq n}\psi_i)$) does not necessarily imply that performing (or avoiding to perform) any action φ_i is a truly valid option.

Example 4.4.5. Given two conflicting obligations $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ and $\mathcal{O}_{\mathsf{pf}}(\neg\varphi/\psi \wedge \chi)$, according to Prop.4.4.3 we can derive $\mathcal{O}(\varphi \vee \neg\varphi/\psi \wedge \chi)$, but, as $\mathcal{O}(\neg\varphi/\psi \wedge \chi)$ is also derivable, the disjunctive obligation does not mean the agent has free choice of what to do when $\psi \wedge \chi$ hold. ■

A proper application of *vikalpa* occurs when none of the prima facie conflicting commands is derivable under some circumstances, hence commands like $\Rightarrow \mathcal{O}(\bigvee_{i\leq n}\varphi_i / \bigwedge_{i\leq n}\psi_i)$ or $\Rightarrow \mathcal{F}(\bigwedge_{i\leq n}\varphi_i / \bigwedge_{i\leq n}\psi_i)$ truly express optionality.

4.4.2 Comparing interpretations

The previous propositions show that the system $\mathsf{MD}+^{\mathsf{ga}}$ enables the reasoner to apply the *vikalpa* principle to obtain disjunctive deontic statements from conflicting prima-facie commands. However, as already mentioned, in Mīmāṃsā the application of *vikalpa* is considered the very last resort.

Following the scale of preferences shown in (Section 2.4.1), it seems that the most preferable option for Mīmāṃsakas in case of conflict is the re-interpretation of Śrauta commands. Note indeed that the norms in the Vedas can be subject to different readings. This means that, in their interpretation of the texts, Mīmāṃsā authors tried to avoid the use of *vikalpa*. Hence, in principle, two different interpretations of a set of commands could be compared and "ranked" according to the number of applications of this principle, preferring the ones with less instances of it.

With this in mind, the introduced system $\mathsf{MD}+^{\mathsf{ga}}$ can be used to mimic the methods employed in Mīmāṃsā in an informal manner, to decide between different interpretations of the deontic statements found in the *Veda*s.

The abstract idea behind this application of our formal system consists in analyzing the derived consequences of the deontic assumptions.

Let us consider natural language text, e.g. a passage of Vedic texts; because of the difficulty of understanding the use of Mīmāṃsakas' language and because of the ambiguity inherent in natural language (Sanskrit, in particular), there will almost certainly be many possible interpretations (corresponding to as many formalizations). The proposed methodology consists in approaching the choice among different interpretations by considering the consequences of each one of them, inferred by using a common system of background reasoning, e.g., here, the logic $\mathsf{MD}+^{\mathsf{ga}}$.

Once a set of consequences has been obtained for each interpretation (resp. formalization) of the original text, the resulting sets can be compared with respect to criteria laid down by experts, or, in general, by the authority designated for this purpose.

As observed, Mīmāṃsā authors implicitly considered the minimization of the number of applications of *vikalpa* as a suitable criterion in order to choose among different interpretations of the Vedic commands. However, in principle, other criteria can be adopted, involving a check for the consistency of the set of derived consequences, or checking whether some specific statements, considered essential, are derivable.

The proposed approach to the choice among different interpretations in Mīmāṃsā is illustrated in Fig.4.14.

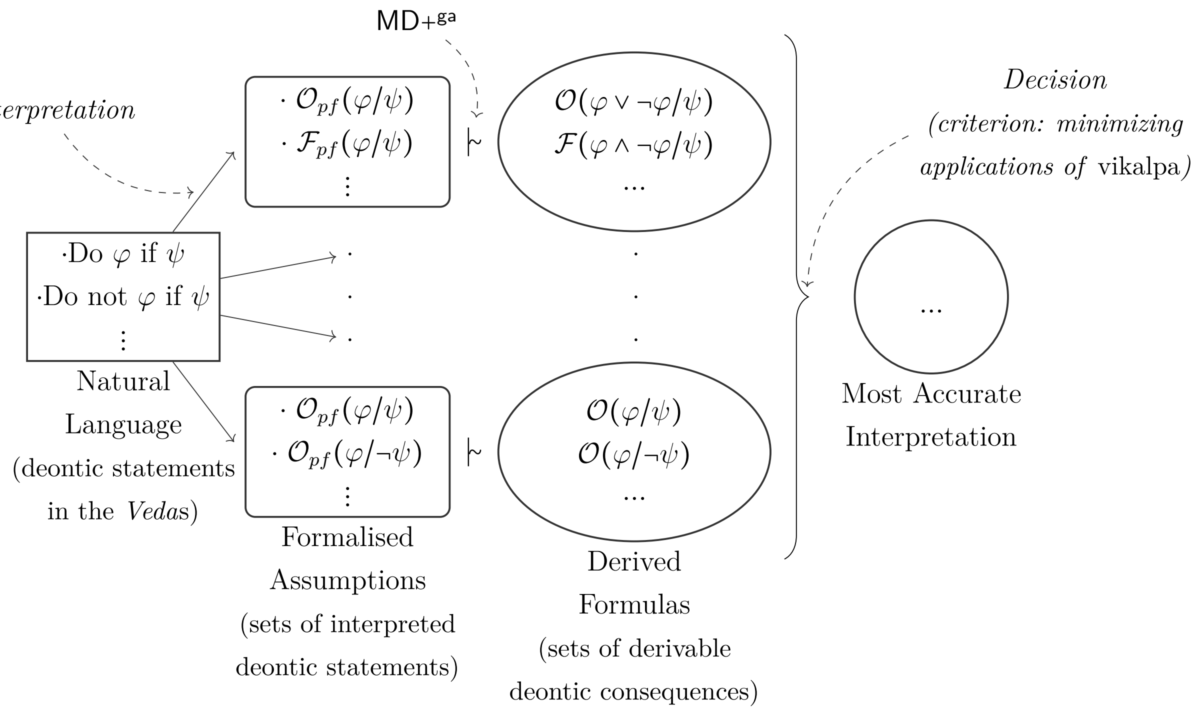


Figure 4.14: A method for choosing among different interpretations in Mīmāṃsā using the system MD+ga ([32])

Remark 4.4.6 *The outlined approach essentially consists in evaluating competing formalizations of a normative text by considering their different consequences under an assumed system of background reasoning (in our case the logic* MD+ga*) and comparing such consequences with respect to certain criteria (in our case* vikalpa*). The chosen criteria are clearly targeted to the system, but in general they involve a basic sanity check in the form of consistency, or checking whether certain desired conclusions are derivable (a similar idea is behind the* quality assurance procedures *in [84]). As well as the mechanism behind the global assumption rules, the introduced approach is flexible enough to be applied in different contexts.*

Specifically, it addresses the problems of validating the interpretation (or formalisation) of a normative text, and deciding between different interpretations (or formalisations) of a text of this kind. These subjects are common to many areas, including not only Philosophy and Hermeneutics, but also Legal Representation (see, e.g., [12, 84]).

However, the procedure in Fig.4.14 is targeted to Mīmāṃsā reasoning, not only because of the minimization of applications of vikalpa *as final criterion of choice, but also because of the logic* $\mathsf{MD+^{ga}}$ *as system of background reasoning.*

For what concerns the problem of formally representing the choice among different interpretations based on minimising the applications of *vikalpa*, it is first necessary to understand which specific behaviour of the system $\mathsf{MD+^{ga}}$ corresponds to an application of such principle. Indeed, as can be observed in Prop.4.4.3 and 4.4.4, an application of *vikalpa* cannot be simply identified with a derivable disjunctive command: due to the monotonicity of operators in their first arguments, it is not relevant whether the components of the disjunction are jointly possible or not.

At this point, it is important to recall that the rationale for avoiding *vikalpa* is to prevent the prima-facie deontic statements from becoming meaningless, i.e. becoming such that it can never be enforceable. Hence, looking at the set of derived deontic statements, we should not try to minimize the number of disjunctive derived statements, trying instead to maximize the number of commands corresponding to the deontic assumptions. As already mentioned, a deontic assumption $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ (or $\mathcal{F}_{\mathsf{pf}}(\varphi/\psi)$), conflicting with another deontic statement $\mathfrak{L}$, is involved in an application of the *vikalpa* principle only if the corresponding formula $\mathcal{O}(\varphi/\psi)$ (resp. $\mathcal{F}(\varphi/\psi)$) is not derivable from $(\mathfrak{L}, \mathfrak{F})$ using $\mathsf{MD+^{ga}}$.

For each deontic assumption $\mathsf{op}_{\mathsf{pf}}(\varphi/\psi)$ with $\mathsf{op} \in \{\mathcal{O}, \mathcal{F}\}$, the decision procedure in Alg.3 can be used for checking whether the sequent $\Rightarrow \mathsf{op}(\varphi/\psi)$, corresponding to that deontic assumptions, is derivable: therefore, Alg.3 enables us to identify the commands involved in an application of *vikalpa* and consequently to count them.

The following propositions formalize this intuition, showing that a case where a sequent corresponding to a deontic assumption is not derivable coincide with a case of a conflict that cannot be solved by using specificity, and hence to an application of *vikalpa*.

Proposition 4.4.7 ([32]) *For any $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi) \in \mathfrak{L}$, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}cut}} \Rightarrow \mathcal{O}(\varphi/\psi)$ holds if and only if at least one of the following conditions holds.*

(a) There is a $\mathcal{O}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$ with $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}cut}} \varphi, \tau \Rightarrow$ and $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}cut}} \Rightarrow \psi \leftrightarrow \zeta$;

(b) there is a $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$ with $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}cut}} \varphi, \tau \Rightarrow$ and $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}cut}} \Rightarrow \psi \leftrightarrow \zeta$;

(c) *there is a* $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta) \in \mathfrak{L}$ *with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \varphi \Rightarrow \tau$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \psi \leftrightarrow \zeta$.

Proof. Because of the form of the rules of $\mathsf{G_{MD+^{ga}}}$, $\Rightarrow \mathcal{O}(\varphi/\psi)$ can only be derived by using the rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\xi/\eta)}$ for some $\mathcal{O}_{\mathsf{pf}}(\xi/\eta) \in \mathfrak{L}$; but if one of the conditions (a)-(c) holds, then not all of the underivability statements in the not-excepted block of this rule hold. This proves the right to left direction of the proposition: if at least one of the conditions (a)-(c) holds, then $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \mathcal{O}(\varphi/\psi)$.

For the other direction, let us assume that $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \mathcal{O}(\varphi/\psi)$ holds.

Since the prima facie command $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ is in the set $\mathfrak{L}$ of deontic assumptions, $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \mathcal{O}(\varphi/\psi)$ also means that the sequent $\Rightarrow \mathcal{O}(\varphi/\psi)$ is not derivable by using the global assumption rule $\mathcal{O}_R^{\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)}$.

In this case, the first two premisses in the standard block are the initial sequents $\psi \Rightarrow \psi$ and $\varphi \Rightarrow \varphi$. Moreover, considering the premisses in the no-active-conflict block, it is clear that for any obligation $\mathcal{O}_{\mathsf{pf}}(\tau/\zeta)$, prohibition $\mathcal{F}(\tau/\zeta)$, or permission $\mathcal{P}^{\mathcal{O}}(\tau/\zeta)$ such that the premisses in the no-conflict block do not hold, there is a command in $\mathfrak{L}$ overruling the conflicting one, i.e. $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi)$ itself ($\psi \Rightarrow \psi$ and $\varphi \Rightarrow \varphi$ are initial sequents, and $\psi \Rightarrow \zeta$ is given by the fact that the premisses in the no-conflict block do not hold). Hence the sets of premisses in standard block and in the no-active-conflict block hold, and the invalid premisses should be in the not-excepted block. But this means that $\mathfrak{L}$ contains a deontic assumption $\mathcal{O}_{\mathsf{pf}}(\tau/\zeta)$, $\mathcal{F}_{\mathsf{pf}}(\tau/\zeta)$ or $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\tau/\zeta)$ satisfying the conditions (a)-(c). □

An analogous statement holds for prohibitions.

Proposition 4.4.8 ([32]) *Let* $\mathcal{F}_{\mathsf{pf}}(\varphi/\psi) \in \mathfrak{L}$. *Then* $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \mathcal{F}(\varphi/\psi)$ *holds if and only if at least one of the following holds.*

(a) *There is a* $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ *with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \varphi, \theta$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \psi \leftrightarrow \chi$;

(b) *there is a* $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ *with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \theta \Rightarrow \varphi$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \psi \leftrightarrow \chi$;

(c) *there is a* $\mathcal{O}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$ *with* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \theta \Rightarrow \varphi$ *and* $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G_{MD+^{ga}}\,cut}} \Rightarrow \psi \leftrightarrow \chi$.

Proof. The claim is proved in the same way as for the previous proposition Prop.4.4.7. □

Note that a permission can block the derivation of a command corresponding to a prima-facie norm, but, as permissions are meant to be only exceptions to other norms, it is not completely clear what *vikalpa* means when such deontic assumptions are involved.

Example 4.4.9. Let us consider a case of an obligation blocked by a permission-obligation (the case of a prohibition and a permission-prohibition is analogous).

- *"It is obligatory not to eat during a sacrifice"* ($\mathcal{O}_{\mathsf{pf}}(\neg\texttt{eat/sacr})$)
- *"it is permitted to eat a fruit during the same sacrifice"* $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\texttt{eat\_fruit/sacr})$
- *"eating a fruit implies eating"* $\texttt{eat\_fruit} \rightarrow \texttt{eat}$

Hence we have $\mathfrak{L} = \{\mathcal{O}_{\mathsf{pf}}(\neg\texttt{eat/sacr}), \mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\texttt{eat\_fruit/sacr})\}$ and $(\mathfrak{F}, \mathfrak{L}) \vdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}} \texttt{eat\_fruit}, \neg\texttt{eat} \Rightarrow$. According to Prop.4.4.8, $\mathcal{O}(\neg\texttt{eat/sacr}$ is not derivable in $\mathsf{G}_{\mathsf{MD+^{ga}}}$.

Because of the structure of global assumption rules, the obligation $\mathcal{O}(\neg\texttt{eat}\vee\texttt{eat\_fruit/sacr}$ is derivable, but in this case it is not clear if the two disjuncts $\neg\texttt{eat}$ and $\texttt{eat\_fruit}$ should be considered as options, since $\texttt{eat\_fruit}$ has never been obligatory.

Hence, similar cases are considered, according to Mīmāṃsā reasoning, mostly as "signals" indicating that the interpretations of deontic assumptions are wrong and the prima facie commands need to be formalized in a different way; e.g. the second statement can be reinterpreted as a proper obligation instead of a permission-obligation. ■

The propositions above show how a set $\mathfrak{L}$ of deontic assumptions —corresponding to a particular interpretation of the Vedic commands— can be evaluated, under a set $\mathfrak{F}$ of propositional assumptions about facts, with respect to the criterion of minimizing the number of applications of the *vikalpa* principle.

The procedure in Alg.3 enables us to check, for every formalized deontic assumption $\mathcal{O}_{\mathsf{pf}}(\varphi/\psi) \in \mathfrak{L}$ or $\mathcal{F}_{\mathsf{pf}}(\theta/\chi) \in \mathfrak{L}$, whether $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{O}(\varphi/\psi)$ (or $(\mathfrak{F}, \mathfrak{L}) \nvdash_{\mathsf{G}_{\mathsf{MD+^{ga}}}\mathsf{cut}} \Rightarrow \mathcal{F}(\theta/\chi)$, respectively), and identify exactly those prima-facie deontic formulas for which this holds. Those are, consequently, involved in applications of the *vikalpa* principle.

An analogous criterion can be used for identifying the presence of any deontic conflict or exception that is solved with an application of *vikalpa* or with an application of the specificity principle. The idea in this case is to check, for each obligation or prohibition $\mathsf{op}_{\mathsf{pf}}(\varphi/\psi)$ in the list of deontic assumptions, if the command is derivable under the most specific conditions, i.e. if $\mathsf{op}(\varphi/\bot)$ is derivable. If it is not, then necessarily there is a conflicting command $\mathsf{op}_{\mathsf{pf}}(\theta/\chi)$ which is at least as specific as $\mathsf{op}_{\mathsf{pf}}(\varphi/\psi)$ and any other prima facie command whose first argument implies φ.

Counting all the applications of *bādha*s in addition to *vikalpa* could be useful as, according to Mīmāṃsā scholars, an interpretation which does not give rise to conflicts is preferable to one which produces conflicts that can be solved by using specificity.

As the sketched method identifies the problematic assumptions and hence it also suggests which commands could be re-interpreted to lower the number of *vikalpa*'s instances, we can formalize those deontic statement in alternative ways, according to the principles in Section 2.3, and evaluate each set of formalized deontic assumptions for choosing the most accurate interpretation of the original set of norms.

The following examples show how $\mathsf{MD+^{ga}}$ can be used for evaluating and comparing different formalizations of the same set of Vedic deontic statements.

Example 4.4.10. Let us consider the following deontic statements, adapted in [117] from the discussion in ŚBh on PMS 10.8.1–4

(a) *"One should sacrifice with the mahāpitṛ-ritual in the manner of its archetypal sacrifice"*;

(b) *"In the mahāpitṛ-ritual, one should not choose the Hotṛ priest or the Ārṣeya priest"*.

However, the "manner" referred to in the first command consists in a conjunction of obligations, hence *(a)* should be read as:

(a′) *"One should sacrifice with the mahāpitṛ-ritual, using three specific materials for oblation and calling the Hotṛ priest, the Ārṣeya priest and the other two priests concerned"*

Hence we have the following formalizations:

(a′) $\mathcal{O}_{\mathsf{pf}}(\texttt{Three\_Oblations} \wedge \texttt{Hotṛ} \wedge \texttt{Ārṣeya} \wedge \texttt{Two\_Priests/Mahāpitṛ})$

(b) $\mathcal{F}_{\mathsf{pf}}(\texttt{Hotṛ} \vee \texttt{Ārṣeya/Mahāpitṛ})$

It is easy to observe that the corresponding enforceable commands $\mathcal{O}(\texttt{Three\_Oblations} \wedge \texttt{Hotṛ} \wedge \texttt{Ārṣeya} \wedge \texttt{Two\_Priests/Mahāpitṛ})$ and $\mathcal{F}(\texttt{Hotṛ} \vee \texttt{Ārṣeya/Mahāpitṛ})$ are not derivable in the logic using the global assumption rules: as their conditions are equivalent and the enjoined actions incompatible, they overrule each other. Hence an application of the *vikalpa* principle is detected.

As the design of the rules allows us to derive all the non-conflicting parts of the obligations, i.e. in this case $\mathcal{O}(\texttt{Three\_Oblations} \wedge \texttt{Two\_Priests/Mahāpitṛ})$, we can also see that the optionality concerns only the actions of inviting the Hotṛ priest and the Ārṣeya priest. This reflects the position of the opponent in the discussion in Mīmāṃsā texts (ŚBh on PMS 10.8.1–4), who propose to interpret (a′) as a set of commands and use *vikalpa* for the two problematic actions.

However, though the discussion in Mīmāṃsā texts identifies the (parts of the) deontic assumptions which would give rise to a conflict and includes the possibility of considering $\mathcal{O}(\texttt{Three\_Oblations} \wedge \texttt{Two\_Priests/Mahāpitṛ})$ enforceable and using *vikalpa* for the other two actions, the conclusive view (*siddhānta*) consists in a different solution. Mīmāṃsā authors give a hermeneutical explanation that not only allows to avoid *vikalpa* but essentially cancels the conflict. In Mīmāṃsā texts the statement (b) is indeed interpreted not as a proper command, but as a command-appendix (*vākyaśeṣa*) which clarifies the meaning of the general obligation (a). Thus, (a′) represents a wrong interpretation of (a) and (b) becomes a part of the previous prescription, now formalized as:

(a″) $\mathcal{O}_{\mathsf{pf}}(\texttt{Three\_Oblations} \wedge \neg(\texttt{Hotṛ} \vee \texttt{Ārṣeya}) \wedge \texttt{Two\_Priests/Mahāpitṛ})$.
Hence the conflict is cancelled. ■

Example 4.4.11 ([32])**.**

A similar example, that offers the opportunity to evaluate different interpretations of sentences, is loosely based on the discussion in Kumārila's Tantravārttika on 1.3.3–4.

Let us consider the following deontic statements:

(i) *"during a specific sacrifice it is forbidden to eat"*;

(ii) *"in the second part of this sacrifice it is also obligatory* (rewarded with good karman) *not to eat"*;

(iii) *"in the second part of this sacrifice it is also permitted to eat"*.

The commands (i) and (ii) are sufficiently clear and not conflicting with each other, therefore let us consider their interpretation fixed as:

(i) $\mathcal{F}_{\mathsf{pf}}(\texttt{eat/sacrifice})$

(ii) $\mathcal{O}_{\mathsf{pf}}(\neg\texttt{eat/sacrifice\_IIpart})$

For what concerns the third sentence, however, the situation is more complex. Even assuming that (iii) represents a permission, it is not clear if it should be read as an exception to the obligation (ii) (and formalized as $\mathcal{P}^{\mathcal{O}}_{\mathsf{pf}}(\texttt{eat/sacrifice\_IIpart})$), or to the prohibition (i) ($\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\texttt{eat/sacrifice\_IIpart})$).

Given the interpretation of the permission as an exception to the obligation (ii), we can derive the prohibition $\mathcal{F}(\texttt{eat/sacrifice})$ corresponding to (i) and, since the second part of a sacrifice represents a more specific condition than the whole sacrifice ($\texttt{sacrifice\_IIpart} \Rightarrow \texttt{sacrifice} \in \mathfrak{F}$) and the prohibition is not blocked by a more specific command, we can also derive $\mathcal{F}(\texttt{eat/sacrifice\_IIpart})$.

On the other hand, the obligation $\mathcal{O}(\neg\texttt{eat/sacrifice\_IIpart})$ corresponding to (ii) cannot be derived, i.e. the conflict between (ii) and (iii) can be dealt with only by using the *vikalpa* principle. This gives rise to a strange situation in which one is liable to a sanction if he eats at any moment of the sacrifice and, since no other obligation than $\mathcal{O}(\neg\texttt{eat} \vee \texttt{eat/sacrifice\_IIpart})$ is available for the second part of the sacrifice, it is said that in this circumstance one can be rewarded for doing something, but there is no way to know what.

The second possible interpretation of the permission (iii), as $\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\texttt{eat/sacrifice\_IIpart})$, gives rise to a suspension of the prohibition in (i) in the second part of the sacrifice, while it maintains the obligation (ii).

As it avoids applications of *vikalpa*, the latter is the preferred interpretation from the perspective of Mīmāṃsā authors.

Though, as mentioned in (Section 2.4), Mīmāṃsā authors, when possible, prefer a reinterpretation which cancels the conflict to the application of *bādha*s, here blocking the prohibition in the second part of the sacrifice seems the best outcome, as both the (defeasible) commands corresponding to (i) and (ii) are derivable. Not only avoids this interpretation the use of *vikalpa* but it also results in a more understandable situation, where the addressee of the commands is in general liable to a sanction if he eats during the sacrifice, except for the second part of it, but he will be rewarded if, even in this second part, he refrains from eating.

According to this interpretation, the actions of refraining form eating during the second part of the sacrifice could be seen as a *supererogatory* action, i.e. an act which is clearly recognized as good, but it is not strictly and explicitly required as a duty. However, this concept is not always easily applicable in Mīmāṃsā philosophy, as the only criterion for defining what is "good" is exactly being explicitly required as a duty in the *Veda*s. In the previous example, for instance, as it is not clear whether failing to perform an obligatory action results in some kind of undesirable consequence, it is also not clear whether the act of refraining form eating during the second part of the sacrifice can be considered "more than what is required" or not. On the other hand, if we considered "strictly required as a duty" only actions which, if not performed, give bad results and we assumed that failing to perform obligatory acts has no negative effects per se, we would risk to be forced to consider all the (strongly) obligatory actions as supererogatory. In view of the observations above, the actions that are more often recognized as supererogatory in Mīmāṃsā are the ones prescribed by those *kāmya-karman* sacrifices which are triggered by desires that are clearly altruistic or, at least, that do not imply, if satisfied, states where a prohibition or an obligation is not complied with.

In the example above, if (iii) is interpreted as an exemption to the prohibition, it seems that there are two different kinds of (equally binding) norms; for this reason it may recall the (modern) distinction between legal and moral obligations. ■

Example 4.4.12.

(i) *"law forbids Italian companies to sell weapons to Countries at war"*
($\mathcal{F}_{\mathsf{pf}}(\texttt{sell\_weapons/Countries\_war})$);

(ii) *"Italian companies should not sell weapons to Countries at war even in the absence of an official declaration of war"*
($\mathcal{O}_{\mathsf{pf}}(\neg\texttt{sell\_weapons/Countries\_war} \wedge \neg\texttt{declaration})$);

(iii) *"in the absence of an official declaration of war it is permitted for Italian companies to* *sell weapons to Countries at war"*

$\mathcal{P}^{\mathcal{F}}_{\mathsf{pf}}(\texttt{sell\_weapons/Countries\_war} \wedge \neg\texttt{declaration})$.

As in the previous example, the permission is better interpreted as an exception to the prohibition, as otherwise it would block the derivation of the enforceable command corresponding to the second deontic assumption, giving rise to a counterintuitive case of *vikalpa*.

Here the operators $\mathcal{O}_{\mathsf{pf}}(\cdot/\cdot)$ express moral duties, while $\mathcal{F}_{\mathsf{pf}}(\cdot/\cdot)$ represent legal prohibitions. Since $\mathcal{F}(\texttt{sell\_weapons/Countries\_war} \wedge \neg\texttt{declaration})$ cannot be derived, it is said that Italian companies selling weapons to Countries at war in the absence of an official declaration of war (e.g. in civil wars) are not legally punishable. However $\mathcal{O}(\neg\texttt{sell\_weapons/Countries\_war} \wedge \neg\texttt{declaration})$ can be derived, meaning that it is still (morally) obligatory to refrain from selling weapons to Countries at war, even if there is no official declaration of war. ■

4.4.2.1 Limits of formal methods

The problem of evaluating the interpretations of commands in Mīmāṃsā is a complex subject, that cannot be reduced to the simple operation of avoiding meaninglessness of commands and *vikalpa*. In line with the considerations in Section 2.4.1, we can order the methods used in Mīmāṃsā for resolving conflicts according to a decreasing scale of preferences, where the best option consists in interpreting the commands in order to avoid the application of *vikalpa* and *bādha*s and the pointlessness or purposelessness of the commands at stake.

Given the complexity of the matter, however, the (re)interpretation of commands (in order to minimize not only the applications of *vikalpa*, but also the number of deontic conflicts) cannot be dealt with only using formal logic. Indeed, some cases in which the choice is between two interpretations which generate the same number of conflicts can be handled only by referring to hermeneutic and linguistic considerations. In such cases formal logic can help only clarifying the structure of commands and identifying all the consequence of each interpretation.

The following example, discussed e.g. in Kumarila's Ślokavārttika on PMS 1.1.2 (*Codanāsūtra*), shows a case where the re-interpretation of commands cannot be guided by the criterion of minimizing the applications of *vikalpa*. This example has been analysed together with the Sanskritists and the experts in Mīmāṃsā involved in the already mentioned project *Reasoning Tools for Deontic Logic and Applications to Indian Sacred Texts*.

Example 4.4.13.

(1) *"One should never physically harm another living being"*;

(2) *"one must sacrifice an animal during a Vedic annual ritual"*.

We analyse below three different options for dealing with the conflict generated by (1) and (2).

I. The first option consists in acknowledging a conflict between the two statements and solving it by using the specificity/*Guṇapradhāna* principle, i.e. by suspending the effectiveness of the first command in the conditions indicated by the second one.

The two commands are indeed formalized in the most natural way as a prohibition and an obligation:

(I.1) $\mathcal{F}_{\mathsf{pf}}(\texttt{harm}/\top)$;

(I.2) $\mathcal{O}_{\mathsf{pf}}(\texttt{kill\_animal/Vedic\_sacrifice})$.

Moreover, it is assumed that killing an animal involves physically harming a living being ($\texttt{kill\_animal} \Rightarrow \texttt{harm} \in \mathfrak{F}$) and that the conditions of (I.2) are more specific than the ones of (I.1) ($\texttt{Vedic\_sacrifice} \Rightarrow \top$ is always true). Hence, using the global assumption rules, we can derive $\mathcal{F}(\texttt{harm}/\top)$, $\mathcal{O}(\texttt{kill\_animal/Vedic\_sacrifice})$ and $\neg\mathcal{F}(\texttt{kill\_animal/Vedic\_sacrifice})$.

As expected, considering the set $C_{\mathfrak{L}}$, containing every formula that appears as the condition of a command in $\mathfrak{L} = \{\mathcal{F}_{\mathsf{pf}}(\texttt{harm}/\top), \mathcal{O}_{\mathsf{pf}}(\texttt{kill\_animal/Vedic\_sacrifice})\}$, the prohibition in (I.1) cannot be extended to all the members of $C_{\mathfrak{L}}$ more specific than its own condition. This means that there is a conflict solved by using the specificity principle.

However, as noticed in Section 2.4.1, the use of *bādha*s, like the specificity/*Guṇapradhāna*, is acceptable but not optimal for Mīmāṃsā authors, especially when the blocked command is a prohibition: in general, it seems that Mīmāṃsakas used to choose a hermeneutical approach aimed at reducing the conflict to an erroneous interpretation.

Therefore, although not wrong, the option I is not the preferred one.

II. The conflict is cancelled by reinterpreting (one of) the commands.

Reading the first command as a negative obligation ($\mathcal{O}_{\mathsf{pf}}(\neg\texttt{harm}/\top)$) would give rise to an application of the specificity principle similar to the one in I. Moreover, by interpreting it as the exclusion $\mathcal{O}_{\mathsf{pf}}(\texttt{harm}/\bot)$ (see Section 2.3.2,) it would be made inapplicable, which would run against the *meaningfulness* nyāya; hence, we choose to rather reinterpret the second command.

In particular, considering that recommendations and obligations are both expressed by the Sanskrit word *vidhi*, it is possible to reinterpret the conflicting commands as a prohibition and a recommendation:

(II.1) $\mathcal{F}_{\mathsf{pf}}(\texttt{harm}/\top)$;

(II.2) $\mathcal{R}_{\mathsf{pf}}(\texttt{kill\_animal/Vedic\_sacrifice})$.

However, also this interpretation is not completely satisfactory. Indeed, though it does not give rise to conflicts ($\mathcal{F}(\texttt{harm}/\top)$, $\mathcal{F}(\texttt{harm/Vedic\_sacrifice})$, $\mathcal{R}(\texttt{kill\_animal/Vedic\_sacrifice})$ are derivable), this reading radically changes the meaning of the second statement and this is not always coherent with the characteristics of the command and with the other parts of the text. Moreover, as in the case of the previously mentioned *Śyena* sacrifice, since the recommendation (II.2) violates a prohibition, we would be forced to admit that the condition `Vedic_sacrifice` already represents a suboptimal state and, if it necessarily involves killing an animal, performing it will give rise to a sanction.

Hence, reinterpreting the commands could be a viable solution, although not optimal.

III. The third option —the one accepted by all Mīmāṃsā authors— consists in avoiding the occurrence of a conflict by reinterpreting the facts and the relations among them. This means assuming that the conditions of the prohibition are not more general than the one of the obligation, or that ritual killing of an animal does not entail the forbidden act.

Since, as mentioned earlier (see Section 2.3.1), each prohibition needs for the forbidden action to be previously known as obligatory or desirable, it should be supposed that what is forbidden is not the harming of living beings in itself, but the harming with the intention to do it for one's own advantage (`des_harm`).

Hence, while the second deontic statement is formalized as in case I., for the first one there are two possibilities:

(III.1′) $\mathcal{F}_{\mathsf{pf}}(\texttt{harm/des\_harm})$;

(III.1″) $\mathcal{F}_{\mathsf{pf}}(\texttt{harm} \wedge \texttt{des\_harm}/\top)$.

In both cases the conflict is cancelled, as the prohibition is not blocked by the obligation because the latter is not more specific ($\texttt{Vedic\_sacrifice} \Rightarrow \texttt{des\_harm}$ is not derivable given (III.1′)), or because the contents are not incompatible ($\texttt{Vedic\_sacrifice} \Rightarrow \texttt{des\_harm}$ is not derivable given (III.1′)). However, even if they could seem equivalent from an "operational" point of view, (III.1″) is preferred, as it has the advantage of leaving the prohibition very general and to state that what is prescribed in the sacred texts is not an action that in other conditions is strictly forbidden, but an act different from the prohibited one.

■

4.5 Related works

Prioritization principles like specificity have been widely used to capture defeasible reasoning in the context of non-monotonic logics[2] and they have been applied as mechanisms of conflict resolution in legal reasoning [3]. Many different systems and methods have been introduced in the literature for choosing between two conflicting defeasible rules, in such a way that the overridden one is not deleted, but it is not derivable anymore when the other applies. Though there are similarities between some of those systems and the one presented here (e.g. Horty's syntactic approach [69]), the characteristics of their formalisms and rules of inference make it impossible to use them "out of the box" for representing Mīmāṃsā deontic reasoning. Some of them result unsuitable for such a goal because of their base logic, different from MD+ (e.g. [122]), or because their mechanisms for applying prioritization principles are developed within general frameworks that do not enable us to express the difference between the "statuses" of explicit commands in the texts and of derived ones, resulting from human reasoning (e.g. the argumentation-based approach in [109], and Deontic Default logic in [71]). Some other important systems that deal with deontic conflict resolutions in the framework of non monotonic logic (e.g., Defeasible Deontic logic in [59] and Input/Output logic in[88]) use explicit priorities among rules, instead of general prioritization principles as in Mīmāṃsā.

In this section we present a brief outline of some of the best-known approaches tackling the lack of monotonicity in deontic logics: in particular we compare to our mechanism the approaches of Deontic Default logic, Defeasible Deontic logic and Input/Output logic. Not based on sequent calculi, these approaches represent more general frameworks that can be extended to handle different deontic operators. However, as pointed out below, their properties make it impossible to use one of those "as it is" to express the normative reasoning developed by Mīmāṃsā authors.

Deontic Default Logic is one of the first systems of non-monotonic deontic logic. Introduced by J. Horty ([67, 69, 71]), Deontic Default Logic is based on Reiter's *Default Logic* [114]. In Reiter's default logic new rules of inference, called *default rules*, are added to classical logic and the inference relation is modified in order to include and make sense of those rules. Roughly speaking, a default rule in Reiter's logic can be seen as a triple $(\varphi : \psi/\theta)$ such that θ should be concluded if φ holds and ψ is consistent with the conclusion θ; a *default theory* is a couple $\Delta = \langle \mathfrak{W}, \mathfrak{D} \rangle$, where $\mathfrak{W}$ is a set of propositional formulas and $\mathfrak{D}$ is a set of

[2]A survey of this complex field is beyond the scope of the present work; for an overview see [86].

[3]Good recent overviews and comparisons are given, e.g., in [13] and [22].

default rules. The *extensions* (sets of conclusions) of a default theory Δ are defined as the fixed points of the operator Γ_Δ, which employs the default theory for mapping any set of formulas $\mathcal{S}$ into the set of formulas $\Gamma_\Delta(\mathcal{S})$ such that:

1. $\mathfrak{W} \subseteq \Gamma_\Delta(\mathcal{S})$
2. $Cn(\Gamma_\Delta(\mathcal{S})) = \Gamma_\Delta(\mathcal{S})$ (where $Cn(\Gamma_\Delta(\mathcal{S}))$ is the closure of $\Gamma_\Delta(\mathcal{S})$ under classical consequence
3. for any $(\varphi : \psi/\theta) \in \mathfrak{D}$, if $\varphi \in \Gamma_\Delta(\mathcal{S})$ and $\neg\psi \notin \Gamma_\Delta(\mathcal{S})$, than $\theta \in \Gamma_\Delta(\mathcal{S})$.

The basic idea behind Horty's approach essentially consists in considering default theories of the form $\Delta_\mathcal{O} = \langle \mathfrak{W}_\mathcal{O}, \mathfrak{D}_\mathcal{O} \rangle$ where the *context* $\mathfrak{W}_\mathcal{O}$ corresponds to the set of what we called propositional facts, and $\mathfrak{D}_\mathcal{O}$ contains conditional obligations, thus corresponding to our list $\mathfrak{L}$ of prima facie injunctions. Intuitively an obligation $\mathcal{O}(\varphi/\psi)$ represents a (default) rule which commits a reasoner to bring about φ if ψ holds and, in addition, φ itself is consistent with the set of conclusions obtained by a reasoner. The formula $\mathcal{O}(\varphi/\psi)$ is derivable iff φ is contained in the extensions of the theory $\langle \mathfrak{W}_\mathcal{O} \bigcup \{\psi\}, \mathfrak{D}_\mathcal{O} \rangle$ where $\mathfrak{W}_\mathcal{O}$ has been extended with the antecedent of the deontic conditional.

With respect to our system, a construction like the one in [71], which uses extensions, has the advantage of leaving the choice between the "credulous" approach (an obligation is derivable if it belongs to at least one extension) and the "skeptical" one (an obligation is derivable if it belongs to all extensions). The approach of $\mathsf{MD}+^{\mathsf{ga}}$, on the other hand, is not credulous, as it does not allow to derive conflicting obligations like $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\neg\varphi/\psi)$, and it is not skeptical, because it does not consider all the possible chains of blocking (or supporting) commands. Our approach has been defined "generous" in [33], as it aims at deriving as many non-conflicting commands as possible, by using monotonicity on the second argument of the deontic operators. However, systems like the ones in [71] are also characterized by aspects which are not optimal for representing Mīmāṃsā reasoning. In particular, in such systems there is no distinction between the "statuses" of prima facie and derived obligations, as extensions depend both on deontic assumptions and on prescriptions which themselves are derivable from the first ones. Moreover, unlike our system, Horty's logics do not allow to derive nested obligations or to use them in the list of assumptions; in contrast, in the system presented in this chapter the only restriction on nested obligations is that modal formulas are not allowed as arguments of prima facie commands. As observed in Sec.4.3.1, this restriction in our system is necessary to guarantee that the modal nesting depth of formulas in the underivability statements is lower than that of the formulas in the conclusion, and hence to prove the decidability of the system.

While the logics in [71] make use of priorities among (deontic) defaults, the system introduced in [69] is in many respects much more similar to the one presented here. Indeed, there the notion of extension of an *"ought context"* (a default theory where the set of default rules is substituted by a set of conditional obligations as defined above) has a built-in mechanism capturing the specificity principle. The extension of an ought context in [69] is defined as the closure under classical consequence of the the set of formulas containing the context, plus all the formulas φ such that (a) there is $\mathcal{O}(\varphi/\psi)$ in the set of conditional obligations of the ought context, (b) the conjunction of the formulas in the extended context is more specific than ψ, (c) $\mathcal{O}(\varphi/\psi)$ is not overridden by a more specific command, and (d) the same extension does not contain $\neg\varphi$. Similarly to the system presented here, an injunction $\mathcal{O}(\varphi/\psi)$ in the set of conditional obligations is *overridden* if there is another obligation $\mathcal{O}(\chi/\theta)$, applicable in the context (i.e. θ is more specific than the extended set of factual assumptions), which is more specific than $\mathcal{O}(\varphi/\psi)$ (i.e. θ is more specific than ψ) and incompatible with $\mathcal{O}(\varphi/\psi)$ (i.e. φ and χ are inconsistent in the context). The most evident difference between such logic and the system presented here is that the specificity principle is applied before saturating the set of conditional obligations of the ought context under monotonicity (see Ex.4.2.3). Hence, for example, an ought statement overridden by another one cannot be reinstated by a more specific obligation that implies it; this also implies that the system does not satisfy the *vikalpa* principle (disjunctive response), required by Mīmāṃsā.

Defeasible Deontic Logic represents another important approach to normative reasoning rooted in non-monotonic logics. First introduced by G. Governatori and A. Rotolo in [59], it was thoroughly developed over the last 15 years, see e.g. [5, 57, 58]. Here the system assumed as a base is *Defeasible Logic* (see [100, 85, 4]), characterized by the use of strict non-defeasible rules, defeasible ones, undercutting rules called *defeaters*, and a binary relation over the set of rules (called *superiority relation*), that allows to solve conflicts between defeasible rules. As already mentioned, the presence of *defeaters*, which *undercut* rules (i.e. give exemptions to them in specific states) instead of *rebutting* them (i.e. commanding to perform different incompatible actions in specific states), represents a very important feature for a non monotonic logic. Indeed they allow us to express explicit exceptions to a general law, even when the latter is not overridden by another one with a conflicting content; for instance, consider the statements:

(a) "you ought not to eat during a sacrifice"

(b) "you ought to eat during a sacrifice if it lasts more than three days"

(c) “it is not forbidden to eat during a sacrifice if you are ill”.

The injunction (b) should be interpreted and formalized as a norm (in its turn defeasible, as liable to be invalidated by possible new assumptions), which overrules (i.e. *rebuts*) the conflicting more general defeasible norm (a). On the other hand, (c) does not prescribe an action in conflict with (a), it just specifies a state where the prohibition (a) is not enforced (i.e. (c) *undercuts* (a)), hence it is interpreted as a permission (in our logic) and as a *defeater* in Defeasible Logic.

In Defeasible Deontic Logic, commands are interpreted as defeasible rules of the form

$$r_x : A_1, \cdots, A_n \Longrightarrow_{\mathcal{O}} C$$

where r_x is the name of the rule, $A_1, \cdots, A_n$ are formulas of the language which constitute the antecedents of the rule r_x, and C is the conclusion; a superiority relation between rules (the norm r_y is weaker than r_y) is usually expressed as

$$r_y < r_z$$

For deriving $\mathcal{O}(C)$ using a rule r_x, it is then necessary to check that its antecedents hold and that all the possible rules $r_1, \cdots, r_m$ whose consequent conflicts with C are either not applicable (i.e., undercut by a defeater), or weaker that an applicable rule supporting C (i.e., rebutted by a stronger rule).

Defeasible Deontic Logic has proved to be a very flexible tool in many applications. E.g., it can be extended to allow for nested formulas both in the antecedents and consequents of the rules, and to define the superiority relation as a hierarchical order on the whole set of commands. However, not to increase the complexity of the system, it is normally assumed that the antecedent and consequent $A_1, \cdots, A_n, C$ of a rule r_x are modalised literals, hence they can contain deontic operators, but, unlike in our system, the full propositional language is not used. Furthermore, in our system the specificity principle is built in the rules for reasoning in presence of deontic and factual assumptions. In contrast, in Defeasible Deontic logic there is no fixed mechanism of prioritization, but explicit preferences between two rules are defined (similar to the superiority relations presented in Section 4.3.2). This means, on one hand, that it is not necessary to check all the norms in a set for determining whether a deontic formula is derivable or not from a given rule, but it is enough to check that it does not conflict with the norms that take priority over the given one. On the other hand, it also means that it is not necessary that more specific rules always overrule more general ones.

The different behaviour of the formal system for representing Mīmāṃsā deontic reasoning depends on the general tendency of Mīmāṃsā scholars towards the use of global methods and comprehensive rules of interpretation. In other words, it seems that Mīmāṃsā scholars prefer to use fixed mechanisms like *Guṇapradhāna* and *vikalpa*, which can be generally applied to any situation satisfying some minimal requirements, rather than explicitly indicating for any pair of conflicting deontic assumptions an ad hoc priority relation.

Input/Output Logics are relatively young systems of non-monotonic deontic logics. *Input/Output Logics*, have been introduced by D. Makinson and L. van der Torre in [88, 89, 90] and further developed in the last few years (see e.g. [105, 123]).

The basic idea behind those logics is to treat conditional norms not as formulas bearing truth values, but as mechanisms for transforming information that arrives as an input into new outputs; then, the role of the formal system is just to *"prepare information before it goes in as input, [...] unpack output as it emerges and, if needed, coordinate the two in certain ways"* ([91]).

A norm is represented as an ordered pair (φ, ψ), where φ and ψ are formulas of the propositional language such that φ, called *body* of the rule, constitutes a possible input, and ψ —the *head* of the rule— the corresponding output. For example, a deontic statement like "one should not eat during a sacrifice" would be translated into a norm where the *body* expresses the state of affairs corresponding to an ongoing sacrifice and the *head* expresses the state where the norm is complied with, hence where the agents do not eat.

Given a set G of such norms and a set of formulas Φ, $G(\Phi) = \{\psi | (\varphi, \psi) \in G \text{ for some } \varphi \in \Phi\}$ is defined as the set containing the formulas corresponding to the heads of those rules whose bodies are included in Φ. The set $out(G, \Phi)$ represents the set of formulas that correspond to the consequences resulting from the application of the rules in G to the inputs in Φ. Depending on the definition of the operator $out(G, \Phi)$, for any set of rules G and set of formulas Φ, four different systems of Input/Output logic are defined. Each of the following operators $out_n(G, \Phi)$ (with $n = 1, 2, 3, 4$) defines a different set of outputs resulting from the rules in G and the inputs in Φ and hence a system characterized by specific rules.

- *Simple-minded output*: $out_1(G, \Phi) = Cn(G(Cn(\Phi))$ (where $Cn(\Phi)$ is the closure of Φ under classical consequence);
- *Basic output*: $out_2(G, \Phi) = \bigcap\{Cn(G(\Theta)) | \Phi \subseteq \Theta \text{ and } \Theta \text{ is complete}\}$ (where Θ is *complete* if it is maximally consistent or contains all the formulas of the language);
- *Reusable simple-minded output*: $out_3(G, \Phi) = \bigcap\{Cn(G(B)) | \Phi \subseteq B = Cn(B) \supseteq G(B)\}$;

- *Reusable basic output*: $out_4(G, \Phi) = \bigcap\{Cn(G(\Theta)) | \Phi \subseteq \Theta \supseteq G(\Theta)$ and Θ is complete$\}$.

For each of those operations, it is also possible to consider the so called *throughput* version, where inputs are automatically also outputs: in this case the *throughput* versions of Basic output and Reusable basic output operations collapse into classical consequence.

In order to clarify the idea behind the approach of Input/Output logics and the necessity of different characterizations of the set of output, let us consider the following example, highlighting the limits of *simple-minded output* operation.

Example 4.5.1. Let the set of norms contain the two injunctions "men should not tell lies" and "women should not tell lies" ($G = \{(\mathtt{man}, \neg\mathtt{lie}), (\mathtt{woman}, \neg\mathtt{lie})\}$). Given the set of possible inputs $\Phi = \{\mathtt{man} \vee \mathtt{woman}\}$, as the set of all the bodies of the norms in G does not share elements with the set of consequences of the inputs in Φ, no rule can be applied. Hence we have $out_1(G, \Phi) = Cn(G(Cn(\Phi)) = G(Cn(\Phi)) = \varnothing$.

However, this result could be considered counterintuitive in the deontic context: since the output $\neg\mathtt{lie}$ can be obtained from each of the two disjuncts of the input, it should be obtained also from the disjunction. As it is observed below, this depends on the fact that the *simple-minded output* is not characterized by the rule of *Disjoining input* (OR).

Note that also the system $\mathsf{MD+}$ does not satisfy the property necessary for deriving e.g. $\mathcal{O}(\neg\mathtt{lie}/\mathtt{man} \vee \mathtt{woman})$ from $\mathcal{O}(\neg\mathtt{lie}/\mathtt{man})$ and $\mathcal{O}(\neg\mathtt{lie}/\mathtt{woman})$: the injunction $\mathcal{O}(\neg\mathtt{lie}/\mathtt{man} \vee \mathtt{woman})$ can only be derived from the two commands $\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{lie}/\mathtt{man})$ and $\mathcal{O}_{\mathsf{pf}}(\neg\mathtt{lie}/\mathtt{woman})$ as deontic assumptions, by using the Global Assumption Rules. ■

As proved in [88], the four operators $out_n(G, \Phi)$ (with $n = 1, 2, 3, 4$) in the previous list —corresponding to the four systems of *Simple-minded output*, *Basic output*, *Reusable simple-minded output* and *Reusable basic output*— can be also characterized by using four different subsets of the following derivation rules:

- Strengthening input (SI):

$$\frac{(\theta, \psi) \quad \theta \in Cn(\varphi)}{(\varphi, \psi)}$$

- Conjoining of output (AND):

$$\frac{(\theta, \psi) \quad (\theta, \varphi)}{(\theta, \psi \wedge \varphi)}$$

- Weakening output (WO):

$$\frac{(\theta, \psi) \quad \varphi \in Cn(\psi)}{(\theta, \varphi)}$$

- Disjoining input (OR):

$$\frac{(\theta,\psi) \quad (\varphi,\psi)}{(\theta\vee\varphi,\psi)}$$

- Cumulative transitivity (CT):

$$\frac{(\theta,\psi) \quad (\theta\wedge\psi,\varphi)}{(\theta,\varphi)}$$

Given a subset R of $\{SI, AND, WO, OR, CT\}$, a couple (φ,ψ) is said to be derivable from a set of norms G using the derivation rules from R if and only if (φ,ψ) is an element of the smallest set which contains the norms in G and which is closed under the rules in R.

Hence out_1 corresponds to derivability using $\{SI, AND, WO\}$, out_2 to $\{SI, AND, WO, OR\}$, out_3 to $\{SI, AND, WO, CT\}$, and out_4 to derivability using all the rules; moreover, the versions where inputs are allowed to be outputs can be obtained just by adding the following rule.

- Identity (ID):

$$\frac{}{(\varphi,\varphi)}$$

It can be easily observed that the listed rules mostly correspond to principles which have already proved to be unsuitable for representing Mīmāṃsā reasoning. For instance, (SI) represents the *strengthening of the antecedents*, translated as unlimited monotonicity in the second argument of deontic operators in MD+, (OR) and (CT) are weaker forms of monotonicity on the antecedents: in particular, (CT) constitutes the equivalent of *cautious monotonicity* (Ex.4.2.1) at the level of formulas, and the derivation rule (AND) is the already mentioned *aggregation principle* (Section 3.2). Moreover, the equivalent of nested obligations is admissible only in throughput versions of the logics, where, however, the undesirable (Section 3.2) (ID) rule holds.

The reasons why the approach of those logics is so different from the one of other systems of non monotonic deontic logics lies in their main purpose: Input/Output logics seem to be specifically designed for dealing with the problem of *Contrary-To-Duty* norms. For the same reason, those logics have the detachment principles as their core, allowing even a stronger version of deontic detachment, which in the language of MD+ can be expressed as follows:

$$\frac{\mathcal{O}(\varphi/\theta) \quad \mathcal{O}(\theta/\psi)}{\mathcal{O}(\varphi/\psi)}$$

Chapter 5

Conclusions

This thesis discussed a formal approach, based on deontic logic, to the analysis of the principles and structures of reasoning used by the authors belonging to the Mīmāṃsā school of philosophy. Key of our analysis were the *nyāya*s, the interpretative principles aimed at making sense of the prescriptive portion of the Vedas, elaborated by Mīmāṃsā scholars over more than 2000 years.

Core of the thesis are some prominent *bādha*s —the *nyāya*s developed with the specific objective of resolving conflicts among Vedic commands by giving priority to one command over the others. Our main focus has been on the *bādha* called *Guṇapradhāna* or *Sāmānya-viśeṣa*, which states that a command with more specific conditions overrules a more general one. Known in contemporary logic and Artificial Intelligence as *specificity principle*, *Guṇapradhāna* also corresponds to the principle of European Law expressed by the maxim *"lex specialis derogat legi generali"*. The formal mechanism we developed for capturing the specificity principle turned out to naturally express also the rule of reasoning called *vikalpa*. According to this principle, when two (or more) conflicting commands are such that none of them can take priority over the other(s), any of the conflicting norms may be adopted as option. Corresponding to the principle known in the field of deontic logic as *disjunctive response*, *vikalpa* is considered by Mīmāṃsā authors the last resort, as it forces the reasoner to consider the discarded commands as not applicable, at least until the next choice. For this reason Mīmāṃsā authors interpret the Vedic commands in such a way that the number of applications of *vikalpa* is minimized. We have developed a method —implemented in the computer program available at `http://subsell.logic.at/bprover/deonticProver/version1.2/`— allowing to count the applications of *vikalpa* for any set of Vedic norms interpreted and formalized. Hence, simulating the reasoning of Mīmāṃsā authors, the minimization of the number of *vikalpa* applications is used as a criterion for choosing among different interpretations (and consequently different formalizations) of the same deontic statements found in the Vedas.

The base formal system employed to analyse Mīmāṃsā reasoning is the logic MD+, which we have introduced by "extracting" its formal properties from Mīmāṃsā texts. MD+ is a non-normal dyadic deontic logic obtained by extending classical propositional logic with three deontic operators, meant to express the deontic concepts of obligation, prohibition and recommendation, as recognized in Mīmāṃsā texts. The logic is defined by using a Hilbert system, consisting of a set of axioms and a small number of inference rules, here only the rule of Modus Ponens. The axioms of this system, characterizing the properties of deontic operators, represent the formalizations of some deontic *nyāya*s found in Mīmāṃsā texts, translated, interpreted and abstracted with the help of Sanskitists and experts in Indian philosophy.

In order to model the reasoning of Mīmāṃsā authors in our formal framework, e.g. checking whether a desired conclusion follows from some assumptions, we adopted a proof-theoretic approach. Hence, the Hilbert system used for introducing MD+ is transformed into a cut-free sequent calculus, using the methods elaborated in [80]. The conclusion is proved to be sound and complete with respect to the corresponding neighbourhood-style semantics. Refining [29], the semantic approach has been used to analyse a widely debated example of apparently conflicting commands from the sacred texts from the point of view of two different authors.

The calculus for MD+ has been then extended with "special" sequent-style rules capturing specificity/*Guṇapradhāna*. Such rules allow us to derive a command applicable under some conditions from a set of Vedic norms, only if it is implied by a more general norm in the set and there are no more specific conflicting command enforceable under the same circumstances. We showed how to modify those rules for capturing another *bādha*, i.e. the prioritization based on a fixed hierarchy of the reliable sources of duty, spoken or written. Indeed, Mīmāṃsā authors recognized three other sources of duty besides the *śruti* (the Vedic texts), namely, in decreasing order of importance, *smṛti* (the "recollected texts", meant to clarify and explain some contents of the Vedas), *sadācāra* (the behaviour of righteous people, acquainted with the in the Vedas) and *ātmatuṣṭi* (the inner feeling of approval by people studied the Vedas). Since they are all based on the knowledge of the Vedas, they are all inferior to the sacred texts as sources for learning the duty; moreover, each one seems to found its authority in the one immediately above it in the hierarchy, hence less valuable sources never add up. This means that even many consistent commands from *smṛti*, *sadācāra* and *ātmatuṣṭi* cannot overrule one conflicting norm which is explicitly stated in the Vedas. The extended rules derive a norm applicable under some conditions, only if it is implied by a more general norm in the set and there are no conflicting commands that are more specific or found in a more important source, enforceable under the same circumstances.

Finally, the formal system has been used to simulate Mīmāṃsā reasoning on some concrete examples debated by the authors. As the results obtained with formal methods match many points of view of various Mīmāṃsā authors, the system can mimic the reasoning of Mīmāṃsā scholars, making clear and explicit some of the reasoning steps they applied.

Remarkably enough, the results presented in this thesis only scratch the surface of the research opportunities offered by formal approaches to the study of Mīmāṃsā reasoning. These approaches can indeed clarify and provide a better understanding of Mīmāṃsā reasoning, and offer new stimuli for the deontic logic community. We mention below some of the possible directions for future work, as a direct continuation of this thesis.

Future work

A first natural research direction still to be explored, is the refinement, modification and expansion of the logic MD+, to include important principles of deontic logic and, mainly, new *nyāya*s as they are translated, analysed and extracted from Mīmāṃsā texts. For instance, it would be interesting to adapt the global assumption rules in order to extend the logic with the already mentioned *aggregation principle*; though not yet explicitly found in Mīmāṃsā reasoning, this principle is very common in deontic logics (its importance is highlighted e.g. in [3]) and would allow comparisons with other systems.

From a more philosophical point of view, an important modification of the logic would consist in a mechanism for reasoning about the concept of expiation. Indeed, Sanskrit texts speak of the possibility of performing expiations (*prāyaścitta*) when a command has been disrespected. To reason about "reparatory commands", the logic should be also refined for allowing to formalize the different consequences of disobedience for each kind of commands; the concepts of *violation* and the consequent *sanction*, as they are conceived in Mīmāṃsā, have not yet been carefully investigated by the Sanskritists or formalized by the logicians. The issue of dealing with expiations and commands which can be applied only in states of violation is strongly connected with the problem of Contrary-To-Duty prescriptions. As already mentioned, in the system MD+ extended with the global assumption rules no strategies have been developed for dealing with such commands; hence, in principle, cases of Contrary-To-Duty obligations collapse on cases of commands overriding each others ([132, 133]). The reasons for such a behaviour of the system are due to the fact that Contrary-To-Duty (CTD) sacrifices are commonly interpreted as *kāmya-karman* (Section2.3.2), as they are performed with the specific goal of weakening or compensating for a violation. Hence they are interpreted and formalized as recommendations by Kumārila and therefore they cannot override obligations and prohibitions. An attempt to adapt the system discussed in this thesis to give an account of CTD injunctions is in [33]. The idea there is to keep track of the more general injunction for "more ideal" situations by representing commands with two different operators for "violations" and "sanctions". In this way it is possible to express the fact that a CTD command (specifically, an expiation) may override and cancel the sanction caused by the disrespect of a norm, but it does not cancel the violation of this norm. Such an approach to the formal representation of CTD norms may be suitable for capturing the interpretation of the Mīmāṃsā author Prabhākara.

A related direction for future works concerns the development of different logics for simulating and comparing the thoughts of different authors. We already sketched here a first distinction between Kumārila's and Prabhākara's interpretation of elective sacrifices, respectively as recommendations and as proper obligations conditioned by a specific desire. However, the topic is worthy of more thorough and extensive investigations, to determine the characteristics of each deontic concept from the perspective of the two authors. Furthermore, the formalization of the logic at the basis of Maṇḍana's analysis of Vedic norms is still an open problem. The latest Mīmāṃsā main author, indeed, seems to substitute the concept of "instrumentality" to the deontic content of commands: Vedic norms do not express duties but instructions for obtaining happiness (obligations), for avoiding a sanction (prohibitions) and for achieving a specific goal (what Kumārila indicates as weak obligations or recommendations). Hence, it appears that deontic logic is not a suitable framework for representing Maṇḍana's reasoning, which probably would be better captured by agency logic (see e.g., [131]), which accounts for the notion of instrumentality, or logics of causation (see. e.g., [18]).

A different point of reflection for future works is the development of a suitable semantic characterisation for the logic including the global assumption rules. Following examples like the one in Section 3.4.3, this would allow the analysis of controversies that make use of *bādha*s as mechanisms of conflict resolution.

With regard to the use of *bādha*s, the investigation in this thesis also raised an interesting research direction that seems to deserve a more thorough analysis: the formalization of other principles from Kumārila's list in the appendix. Whilst many of the procedural principles and of the principles concerning preferable interpretations of commands are connected with linguistic aspects of Sanskrit and practical considerations on the performance of sacrifices, some seem suitable for a formal representation. Among them, the *economy principle* —stating that a command that conflicts with the minimum number possible of other norms should be preferred to one which contradicts many norms— appears to lend itself to a formal analysis, especially given that already the system presented here allows to count the conflicts among Vedic norms. Another category of *bādha*s that seems to be well suited for an analysis in the context of formal logic is the one we called *no empty rule*. This essentially expresses the fact that each command should convey a duty and be enforceable at least under some circumstances. As our logic already prevents the derivation of commands which do not convey any duty or convey impossible duties, the aspect of the *no empty rule* principle that seems to be most interesting for logicians is the fact that each Vedic norm should be applicable. Hence, more than a principle for resolving conflicts, it represents a general hermeneutic rule which allows to prioritize an interpretation of one or more commands over another interpretation of the same Vedic deontic statements. From this point of view, it is not very different from the (meta-)rule we used for simulating Mīmāṃsā authors' choices among possible interpretations of a set of Vedic commands; indeed, what we count in this case are precisely the Vedic norms that cannot be enforced under any situation. However, in the system presented here such principle is just an external criterion of choice, while the formalization of the *no empty rule* as a formal mechanism like the sequent-style rules capturing specificity is still an open problem.

A different but very interesting possible future work would involve the analysis of Mīmāṃsā in other contexts of formal reasoning. In particular, as already observed about the controversies in Mīmāṃsā texts, the structure of argumentation is dialectical, namely, when discussing a specific topic, the structure of the text is that of a dialogue between upholders of different theories. Through the dialogue, indeed, the prevailing thesis is enriched with new explanations and confutations of the other views. Hence, an approach based on Argumentation Theory can be used for simulating such dialogues, allowing to analyse what kind of claims are used for objecting or defending a view.

A further subject of future research that has been raised by our analysis is the connection and comparison between Mīmāṃsā reasoning —also applied in *Dharmaśāstra*, i.e. Indian jurisprudence— and the tradition of European jurisprudence. Besides the immediate experiment of adapting the system presented in this thesis for analysing examples from *Dharmaśāstra* and from European Law, this research field is much broader. Indeed, it involves many aspects of the two juridical traditions, such as argumentative methods, interpretative principles and historical development. To narrow the investigation, a first step could consist in considering specific groups or schools that appear to share many traits in their methods and cultural background. In particular a good starting point could be a comparison between the Italian tradition of exegesis of the *Corpus Juris Civilis* in the low Middle Ages —focusing on the schools of *Glossatores* (11th–12th c.) and *Commentatores* (13th–14th c.)— and the Sanskrit commentators on the bodies of law in 6th–12th c. South Asia. Such traditions share some core principles, as the assumption that the authoritative texts (the *Corpus Juris Civilis* in Europe and the Vedas in India) are perfect, i.e. internally consistent and meaningful in all their parts. In connection with that, they seem to have a common tendency towards the systematization of the analysis of authoritative texts. This implies that both the groups assumed that the norms in the corpus of laws are universally valid, hence they should be understandable independently not only from the single interpretation of the agent subject to such norms, but also from the possible intentions attributed to the legislator(s), author(s) of the corpus of laws. For this reason they make use of general interpretative principles, developing them starting from concrete cases of dubious interpretation in the authoritative texts, or borrowing them from another discipline, i.e. Scholastic theology for Glossatores and Commentatores and Mīmāṃsā for South Asian commentators. From such philosophical and religious systems, the two groups of juridical schools also inherit their mentality, which values the authority of tradition over the innovation and exegesis over originality, giving great importance to the role of commentaries. Moreover, the European and South Asian traditions adopted the (dialectical) argumentative structures elaborated in the contexts of Scholastic theology and Mīmāṃsā, respectively. The structure observed in Mīmāṃsā remarkably resembles that of *quaestiones* of European Scholasticism, where the supporters of the winning view needs to justify their theses quoting passages from the texts recognized as authoritative (called indeed *auctoritates*) and replying to each objection, so that the final solution is clear and accepted by everyone. It is interesting to note that in both European and Indian juridical tradition there is the tendency to work with flexible principle and consider the assumptions liable to be revised if shown to be inconsistent with other premisses or norms. This seems to suggest that both the traditions are characterized by a way of reasoning that is inductive, as the principle often are extracted and abstracted from concrete examples in the authoritative texts, and defeasible, as the rules admit exceptions and revisions. An analysis of such aspects from the point of view of formal logic could contribute to the debate on defeasible logic in Indian philosophy, with particular attention to the possibility of representing the theories of inference developed by some philosophical schools of ancient India as examples of non-monotonic reasoning ([101, 127]).

Appendix

This section contains an annotated translation of a list of "blocking elements" (*bādha*s), used by Mīmāṃsā authors to prioritize Vedic deontic statements and solve possible conflicts among them. These elements first appear in a section (*balābala-adhikaraṇa*) of Kumārila's (sub-)commentary TV on (ŚBh on) PMS 3.3.14 and has been further elaborated upon in Someśvara Bhaṭṭa's *Nyāyasudhā*, one of the most important commentaries on Kumārila Bhaṭṭa's *Tantravārttika*.

The list here has been translated by the Sanskritists working on the project *Reasoning Tools for Deontic Logic and Applications to Indian Sacred Texts*. The classification, comments and examples below are the result of discussions and close cooperation with them and with the experts in Sanskrit and Indian philosophy who have collaborated on the project in the last three years, notably Lawrence J. McCrea (Cornell University), Andrew Ollett (University of Chicago), Shishir Saxena (Ahmedabad University) and mainly Sudipta Munsi (University of Cagliari).

The list contains 34 *bādha*s[1], i.e. *"invalidating elements"*, which can allow a command to be "temporarily overruled" by another one, under restricted circumstances that make them conflicting.

As already mentioned (see Section 2.4.1), the role of *bādha*s is not uncontroversial. Indeed, if the effectiveness of a Vedic norm can be suspended under certain circumstances, then the Vedic commands are not "fixed", but could need to be updated; this is clearly inconceivable, as the Vedas represent the only source of knowledge about the duty and no other authority can give the power of updating Vedic commands. Kumārila seems to explain the phenomenon by stating that the meaning of Vedic norms is always "fixed", but in some (rare) cases the single norm does not represent a complete description of a duty, which is instead given by the system of Vedic norms that concerns that duty, including the conflicting one. Once the

[1]Depending on the translation and categorization of the principles, some of them can be considered as two (meta-)rules with similar contents; in this case the list includes 36 *bādha*s.

readers know the whole system of norms concerning a specific duty, they can use *bādha*s to update their understanding of Vedic commands, by solving seeming conflicts.

For some of the principles in the list, we provide abstract examples which are meant to explain the application of *bādha*s in the context of Mīmāṃsā reasoning.

Moreover, we tentatively classified the following *bādha*s using the major categories (indicated in square brackets below the statement of the principle) of *principles concerning the instruments of knowledge*, *principles concerning the hierarchy of sources*, *principles concerning the nature of purposes of commands*, *principles concerning the sequence of ritual actions in a sacrifice*, *principles concerning the preferable interpretation of commands*, *principles meant to avoid the "emptiness"* (non-applicability) *of a command*, *principles concerning the "economy" of a choice of interpretation*, *different forms of the specificity principle*, *principles concerning the relation between principal action and auxiliaries in a sacrifice*, *principles concerning the relation between archetypes and ectypes*.

Among the principles in these categories, this thesis has formally analysed the specificity principle and the hierarchy of sources. Other *bādha*s which could be suitable for a formal representation are the one we called "no empty rule", and the economy principle (see Section 2.4.1), which will be subject of future work.

1. ***Pratyakṣa* (direct perception) defeats *anumāna* (inference), like in the case of mirage.**
   [*Concerning The Instruments Of Knowledge*]
2. **Each respective *pramāṇa* (instrument of knowledge) defeats the corresponding *pramāṇābhāsa* (seeming instrument of knowledge).**
   [*Concerning The Instruments Of Knowledge*]
3. ***Śruti* (the Vedas) defeats *smṛti* (traditional texts based on the Vedas).**
   [*Hierarchy Of Sources*]
4. **A *smṛti* contradicted (*vigīta*) [and composed by] *anāptas* (not experts) is defeated by one non contradicted [and composed] by *āptas* (experts).**
   [*Hierarchy Of Sources*]

   The *smṛti*s are believed to be based on the Vedas, but composed by human authors. They therefore contain material which is scattered in the Veda in a more easily accessible way and/or contain material which was originally present in Vedic branches now lost. Hence, unlike for the *śruti*, if there are conflicting deontic statements in those texts, it is possible to assume that the ones which are stronger (non contradicted) have been composed by more expert authors and should defeat the others.

5. **A *dṛṣṭārtha smṛti* (traditional text about something visible) is defeated by one which is *adṛṣṭārtha* (traditional text about something not visible).**
   [*Concerning The Nature Of Purposes*]
   This principle seems to apply only to deontic statements of the same kind (obligations, prohibitions or recommendations) and which are found in the same kind of source (*smṛti* texts). It concerns the physical effects of ritual actions: for instance a command *"you should bless* (not visible effect) *the rice"* overrules *"you should cook* (visible effect) *the rice"*.
6. **A *smṛti* based on *liṅga* (inferential sign, *probans*) or on *arthavāda* (commendatory statement) is defeated by one based on a *śruti* (direct mention in the Veda).**
   [*Hierarchy Of Sources*]
7. ***Ācāra* (custom) is defeated by *smṛti.***
   [*Hierarchy Of Sources*]
   For example, the custom of wearing the bride's dress on the 4th day from marriage is defeated by the *smṛti* text of the form of *"one should give the bride's dress to the knower of the Sūrya hymn"*.
8. ***Ācāra* of more expert people defeats the *ācāra* [of less expert people].**
   [*Hierarchy Of Sources*]
   For instance, the (linguistic) custom of the the *Mleccha*s —adept in the linguistic transactions of the elders (*vṛddhavyavahārābhiyuktānāṃ mlecchānām*)— of applying the word *"pīlu"* to elephants is defeated by the custom of the Āryas —who are considered to be more expert in the same field— of applying this word to a (certain) tree.
9. **Something doubtful is defeated by something certain (*saṃdigdha*).**
   [*Hierarchy Of Sources* – It could seem a general rule on knowledge, but its position suggests that its similarity with the fourth principle, hence it probably represents a case of *bādha* concerning the hierarchy of sources.]
   For example, in a specific ritual, the (doubtful/ambiguous) prescription to mix stones with anything, inferred from the expression *"stones mixed with"*, is invalidated through the (precise) one to mix them with butter, derived from the eulogy of clarified butter which is done in the ritual and has no purpose if it is not understood as an injunction to mix the stones with clarified butter.
10. **Something whose support is weaker is defeated by something whose support is stronger.**

[*Concerning The Instruments Of Knowledge*]

For example, the statement *"one should do the altar after doing the grass-broom"* (concerning the order in which single subsidiary ritual actions should be performed) is defeated by the *smṛti* text *"one should wash one's mouth in case one sneezes"*. This means that the sequence involved in the first sentence can be changed (interrupted) by the ritual in the second sentence. Washing the mouth is indeed a sacrifice (a main ritual action), while the sequence is a property of sacrifices: as ordered by the relation possessors-properties, a sacrifice is a content stronger than a sequence of subsidiary ritual actions. Hence the deontic statement concerning rituals have stronger support than the ones concerning sequences.

11. **The thing which comes at the end is defeated by the thing which was there at the beginning.**

    [*Concerning The Sequence Of Ritual Actions In A Sacrifice*]

    This rule regards only the opening and the closing statement within a larger text prescribing a complex ritual (for the other cases, it seems that the principle to be applied is the No. 22in this list).

    We can imagine that such a principle applies in cases where there is a sequence like e.g. (i) *"you should perform the ritual for Agni"*, (ii) *"you should perform the subsidiary ritual action for α"*, (iii) *"you should perform the subsidiary ritual action for β"*, (iv) *"you should perform the subsidiary ritual action for γ"*, (v) *"you should complete the ritual for Agni and Soma"*: the last statement (v) is invalidated by (i).

12. **Something which is for a purpose which is completely *adṛṣṭa* (unseen) is defeated by something whose "unseenness" is regulated *(by the sacred texts).***

    Fore example, consider two conflicting injunctions *"you should do α"* and *"you should do the opposite of α by means of β"*. The second command, specifying a method (β) for performing the prescribed ritual act, overrules the other one, which does not give indications on methods.

    [*Concerning The Nature Of Purposes*]

13. **The fact of being a direct/proximate contributor is defeated by the fact of being an indirect/distant contributor (*sāmavāyika*) (presumably the *sannipatyopakāraka*).**

    [*Concerning Interpretation*]

    Something is considered a proximate contributor if it is an essential element of a sacrifice, e.g. the sacrificial object or the deity of sacrifice, as without them a sacrifice cannot be completed and loses its identity. Hence, the interpretation of a deontic statement in a sacrifice as concerning such elements that "constitute" the sacrifice (proximate contributors) is defeated by the interpretation of the statement as concerning e.g. the substance one is using for the sacrifice (indirect/distant contributor). In other words, if the interpretation of a prescribed action within a sacrifice is not clear, it should probably be read as an indirectly-contributing auxiliary. For example, if the readers find the rule to pronounce a sacrificial spell, they should rather do it on the substance they are using for the sacrifice than on the sacrifice itself.

14. **An injunction which has several purposes is defeated by one which has a single purpose.**

    [*No Empty Rule*]

    For instance, a prescription like *"one should sacrifice with curds"* prescribes two things, namely the fact that one should sacrifice and the use of curds. By contrast *"one should offer the Agnihotra"* prescribes only one thing (the offering called *Agnihotra*, as *"Agnihotra"* is only an own name and does not add any information which would need to be followed). If such two prescriptions are conflicting, the one which prescribes two actions is overruled, otherwise no part of the one with only one purpose would ever be used (it would become an empty rule).

    As a simple abstract example (not found in the texts), let us consider the following statements: (i) *"one should perform the sacrifice α (which implies chanting the hymn β)"* (ii) *"one should perform the sacrifice α by chanting the hymn γ and use curds"*. The prescription (i) overrules the first part of prescription (ii). In this way both prescription (i) and the second part of prescription (ii) are applied and no norm is empty.

15. **The fact of a word having several meanings is defeated by the fact of having one only.**

    [*Concerning Interpretation* – It represents a linguistic (grammatical) principle of interpretation.]

16. **The invalidation of** (too) **many things is defeated by the invalidation of lesser number of things.**[2]

    [*Economy Principle*]

    In general, a rule (set of rules) which invalidates as few injunctions as possible is preferable to one which invalidates many injunctions.

[2]The translation here already represents a first step of interpretation; Kumārila indeed uses only the words *bahubādho alpabādhena*, literally: *"the suspension (or invalidation) of many things is suspended by the suspension of few things"*.

17. **Something having its origin in one of the** (four) Vedas is defeated by that thing (whose application is) **prescribed in another Veda.**

    [*Concerning The Relation Between Archetypes And Ectypes*]

    The prescriptions concerning the application of a sacrifice are in general more relevant than the "origin" of that sacrifice. Assuming, for instance, that all the sacrifices in one Veda (A) should be performed in a certain way (1) and all the sacrifices in another Veda (B) should be performed according to the method (2), if the archetype of a sacrifice α is in (A), but its application is in (B), then it should be performed using the method (2). For example, the hymns of the *Sāmaveda* should be sung in a loud voice, while the hymns of the *Yajurveda* should be chanted in a low tone. The *Vāravantī Sāma* hymn —which finds its origin in the *Sāmaveda*— is being prescribed during the installation of the sacrificial fire which is done by someone who has studied the *Yajurveda*. Since the use (application) of *Vāravantī Sāma* hymn is laid down in the *Yajurveda*, in accordance with the Yajurvedic system, it will be chanted in low tone, even though such a ritual hymn has its origin in the *Sāmaveda*.

18. **Something prescribed by another *śākhā* (Vedic recension) is defeated by what is prescribed in one's own *śākhā*.**

    [*Hierarchy Of Sources*]

    Each Vedic collection is preserved in one or more "recensions", i.e., redactions. Some of them might entail differences in commands, which are dealt with by this rule: the recension related to someone/someone's family/someone's group is in principle stronger for this person than the ones related to others.

19. **What is *nitya* (fixed) is defeated by what is *naimittika* (triggered by a specific occasion).**

    [*Specificity Principle*]

20. **Something** (a prohibition) **meant for a ritual purpose alone (*kratvartha*) is blocked by something desired by human beings (*puruṣārtha*)** (a prohibition regarding the person)**.**

    [*Concerning The Nature Of Purposes*]

21. **What has been learnt outside the context is defeated by what is read in the same context.**

    [*Concerning The Instruments Of Knowledge*]

    For example, the number of *Sāmidhenī* hymns (hymns for the purpose of kindling fire) is 15, but there is also a sentence stating *"one should utter 17* Sāmidhenī *hymns"*.

However, this does not occur within the (same) context, so the number 17 read outside the context will be blocked by the number 15 read within the context.

22. **In case of contradiction in the sequence, the preceding one is defeated by the subsequent one.**
    [*Concerning The Sequence Of Ritual Actions In A Sacrifice*]
23. What belongs to the *prakṛti* (archetype) is defeated by what belongs to the *vikṛti* (ectype).
    [*Specificity Principle*]
24. **What is based on the statement of a *prayoga* (performance) [in the *prakṛti*] is defeated by what is based on a *codaka* (rules of transfer between *prakṛti* and *vikṛti*).**
    [*Specificity Principle*]
25. **What is purposeless is defeated by what has a purpose.**
    [*No Empty Rule*]
26. **The sequence in the *Brāhmaṇa*s is defeated by the sequence in the *mantra*s.**
    [*Hierarchy Of Sources* – As already mentioned, the *Brāhmaṇa*s and the the *mantra*s are different parts of the Vedas.]
27. **What is based on the *devatās* (deity) is defeated by what is based on the *dravya* (substance).**
    [*Concerning The Relation Between Archetypes And Ectypes*]
    If there are two ectypes of the same archetype and they cannot be both performed, then the one which has the same substance as the archetype overrules the one with the same deities of the archetype.
28. **What is transmitted thereafter is defeated by what is transmitted before.**
    [*Concerning The Sequence Of Ritual Actions In A Sacrifice*]
    It seems that No.11 is a sub-case of it; however, No.11 is not redundant, as the opening statement and the closing one could be in the reverse order, or there could be ectype sacrifices for which the opening statement is derived and is not explicitly said.
29. **The little is defeated by the much.**
    [*Concerning Interpretation*]
    For instance, in case there are two alternative ritual actions, the one which should be performed for more days is in general preferable the one which should be performed for less days.
30. **A subordinate element is defeated by the principle one.**

[*Concerning The Relation Between Principal Action And Auxiliaries In A Sacrifice*]
The principle is relative to the different components of a sacrifice: the elements involved in the performance of the main ritual action is more important than the elements involved in the performance of the auxiliary acts of the same sacrifice.

31. **Something that has been enjoined in general is defeated by something enjoined in particular.**
    [*Specificity Principle*]
32. **Something which has been prescribed with a scope is defeated by something which [would] not have a scope (unless it defeated the other thing).**
    [*No Empty Rule*]
33. **An auxiliary is defeated by something principal.**
    [*Concerning The Relation Between Principal Action And Auxiliaries In A Sacrifice*]
    This principle is similar to No.30, opposing main act and auxiliaries.
34. **A feature of the auxiliary is defeated by a feature of the principal.**
    [*Concerning The Relation Between Principal Action And Auxiliaries In A Sacrifice* – Consequence of No.33.]
    For example, the performance of a principal ritual action —the *Soma* sacrifice— and some of its auxiliaries (e.g. the *Dīkṣaṇīya Iṣṭi* sacrifice) are prescribed for the same period, i.e. the new/full-moon day. However, it is not possible to perform the principal action and its auxiliaries on the same day. Hence the full/new-moon time period of the principal defeats the full/new-moon time period of the auxiliaries: the property (here the time-period) of the principal defeats the feature of the auxiliary.

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