Proof theory for locally finite many-valued logics: semi-projective logics

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Abstract

We extend the methodology in [5] to systematically construct analytic calculi for semi-projective logics – a large family of (propositional) locally finite many-valued logics. Our calculi, defined in the framework of sequents of relations, are proof search oriented and can be used to settle the computational complexity of the formalized logics. As a case study we derive sequent calculi of relations for Nilpotent Minimum logic and for Hajek's Basic Logic extended with the n-contraction axiom ($n \geq 1$). The introduced calculi are used to prove that the decidability problem in these logics is Co-NP complete.

Keywords: Many valued logic, analytic calculi, sequents of relations.

1. Introduction

Many-valued logics generalize classical logic by considering sets of truth values larger than the usual $\{0, 1\}$. The development of analytic calculi for these logics is not only an important theoretical task, useful to establish fundamental properties such as decidability, computational complexity or interpolation, but it is also the key to their applications. In analytic calculi proofs proceed indeed by stepwise decomposition of the formulas to be proved and this is a pre-condition for the automatization of proof search.

Analytic calculi should be developed within a suitable framework, ideally one easy to understand and flexible enough to handle a wide range of logics. Since its introduction by Gentzen, the sequent calculus has been the most popular. This framework is however not suitable to capture many-valued logics, which call for various generalizations of sequents. For instance, finite-valued logics were successfully formalized by many-placed (or labeled) sequent calculi, see e.g. the survey [6]. The resulting calculi are analytic, proof search oriented, and their construction –out of the truth tables of the connectives– is even computerized [7]. A useful framework to deal with infinite-valued logics is that of hypersequents [2], that are finite "disjunctions" of standard sequents. Analytic hypersequent calculi have been defined for several prominent many-valued logics, including the three logics formalizing Fuzzy Logic

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[17]: Gödel, Łukasiewicz and Product logic, or for Monoidal T-norm based logic MTL [14]; see [19] for an overview. Notwithstanding analyticity, hypersequent calculi are in general not suitable for proof search. For instance, termination is still an open problem for the calculus of MTL, which also does not help characterizing the computational complexity of the logic.

Hypersequents were generalized in [12] to finite disjunctions of "two sorts" of sequents, whose name suggests their intended meaning: $\leq$ and $<$ sequents. This allowed the definition of (uniform and) invertible rules for Gödel, Łukasiewicz and Product logic which also provided Co-NP decision procedures for these logics. The same framework was used in [20] to define analytic calculi for two many-valued logics characterized by finite ordinal sums of Łukasiewicz and Product t-norms, one of which is a conservative extension of Hajek's Basic fuzzy Logic BL [17]. Rules in these calculi are however tailored to the mentioned logics and have been discovered with considerable ingenuity. It is not known, for example, whether other many-valued logics can be captured in the same framework and, in the affirmative case, how to do it.

An important step towards the automated construction of analytic and proof search oriented calculi for many-valued logics is done in [5], with the introduction of *sequents of relations*, and of a methodology to construct such calculi for all *projective* logics. Intuitively a logic is projective if for each connective $\Box$, the value of $\Box(x_1, \ldots, x_n)$ is equal to a constant or to one of the $x_1, \ldots, x_n$. Prominent examples of projective logics are all finite-valued logics and Gödel logic.

In this paper we are interested in many-valued logics that have a locally finite variety as their equivalent algebraic semantics (*locally finite many-valued logics*). A variety is locally finite if every finitely generated algebra in it is finite. Locally finite many-valued logics are clearly 'tame' logics. For instance, all of them have the finite model property and the finite embeddability property. It follows that both provability and consequence relation are decidable. Despite of their good properties, various interesting locally finite many-valued logics lack an analytic calculus or nothing is known about their computational complexity. Sometimes the introduction of such calculus seems to be a difficult task, as in the case of the n-contractive BL-logics cnBL [8], extending BL with the n-contraction axiom ($n > 1$).

The aim of this paper is to introduce analytic calculi for locally finite many-valued logics with the following features: the calculi are defined in an algorithmic way, they are suitable for proof search and can be used to settle the computational complexity of the formalized logics. The emphasis is not to define such calculi for specific logics but to introduce *methodologies* to construct them in a uniform and systematic way.

A naive algorithm to define analytic calculi which are sound and complete for the whole family of locally finite many-valued logics is sketched in Section 2.1. The resulting calculi, which mirror the algebraic semantics of the formalized logics, are however far from being efficient and usable for actual proof search. To define analytic calculi having the desired features for a large class of locally finite many-valued logics we use sequents of relations, which are disjunctions of semantic predicates over formulas. By suitably generalizing the procedure in [5] we introduce a methodology to define such calculi for logics that are *semi-projective* or whose conservative extension is. Semi-projective logics properly contain projective logics. Examples of logics that are semi-projective but not projective are Nilpotent Minimum logic NM [14] and Gödel logic with an involutive negation [15, 16]. Logics having a proper conservative extension that is semi-projective include Weak Nilpotent Minimum logic [14] and cnBL, for $n > 1$. Our calculi, algorithmically defined starting from suitable semantic specifications of the considered logics, can be used to show that each formalized logic is Co-NP, provided that so is the validity of the calculus' axioms. As a case study we derive sequent calculi of relations for NM and for the logics cnBL$^+$ (with $n \geq 1$), which are conservative extensions of cnBL. These calculi provide Co-NP decision procedures for the formalized logics and first analytic calculi for cnBL.

2. Preliminaries

For all concepts of universal algebra we refer to [10] while for many-valued logics to [17].

The many-valued logics L we consider in this paper are algebraizable in the sense of Blok and Pigozzi [9] and have a locally finite variety $\mathcal{V}_L$ as their equivalent algebraic semantics. We further assume that $\mathcal{V}_L$ is a suitable variety of residuated lattices possibly with additional operations, and if * and $^+$ denote the deduction preserving interpretations of L-formulas into $\mathcal{V}_L$ equations and vice versa, then for every L-formula ϕ and $\mathcal{V}_L$ equation ε, ϕ^* is a single equation of the form $t(\phi) = s(\phi)$ and ε^+ is a single formula of L. For every formula ϕ and valuation v, $v(\phi)$ is a designated value iff ϕ^* is satisfied by v, that is $v(t(\phi)) = v(s(\phi))$. Hence, ϕ^* is valid in $\mathcal{V}_L$ iff $\mathrm{L} \models \phi$.

Henceforth we will often identify the formulas of a logic L with terms of its equivalent algebraic semantic $\mathcal{V}_L$. Moreover $A, B, \ldots$ will denote atomic propositions and $\phi, \psi, \ldots$ L-formulas.

2.1. A semantic calculus for locally finite many-valued logics

Given any logic L satisfying the above conditions, an analytic calculus for L – mirroring its algebraic semantics – can be easily defined as follows: Fix a natural number n and consider any L-formula ϕ containing n distinct atomic propositions. Let $\mathbf{F}_n$ be the free algebra of $\mathcal{V}_L$ in n generators. Since $\mathcal{V}_L$ is locally finite, $\mathbf{F}_n$ is finite, say $\mathbf{F}_n = t_1, \ldots, t_k$ constitutes the n-clone of the algebra. We may assume without loss of generality that $k \geq n$ and that $t_1, \ldots, t_n$ are the projections.

The object of our calculus are algebraic equations. Fix k new variables $y_1, \ldots, y_k$. For each m-ary connective $\Box$ and for each m-tuple of terms $t_{i_1}, \ldots, t_{i_m}$, there is an index $i(\Box, i_1, \ldots, i_m)$ such that $t_{i(\Box,i_1,\ldots,i_m)} = \Box(t_{i_1}, \ldots, t_{i_m})$, because $\Box(t_{i_1}, \ldots, t_{i_m})$ is an element of $\mathbf{F}_n$ and hence it is equal to one of the t_i. We then introduce the rule:

$$\frac{s = t}{s(y_{i(\Box,i_1,\ldots,i_m)}/\Box(y_{i_1}, \ldots, y_{i_m})) = t(y_{i(\Box,i_1,\ldots,i_m)}/\Box(y_{i_1}, \ldots, y_{i_m}))}$$

(note that each rule is unary, that is, it contains just one premise). The axioms of the proof system are identities, i.e., of the form $y_i = y_i$, for $i = 1, \ldots, k$.

A proof in this calculus is a labeled sequence ending in an axiom, and in which the label of each other node is derived by the label of its son using a calculus rule.

Let ϕ be any formula containing the atomic propositions $A_1, \dots, A_n$, and let ϕ' be obtained from ϕ by replacing A_i with y_i: $i = 1, \dots, n$. The search of a proof for ϕ proceeds according to the steps below:

Step 0 The last element $u_0 = w_0$ of the proof is defined to be $s(\phi') = t(\phi')$ where $s(x) = t(x)$ is the equation $\varepsilon(x)$ defining the designated elements.

The other elements of the proof are obtained by applying the rules of the calculus backwards, until we reach an equation of the form $y_h = y_j$. More precisely:

Step i Suppose that at step $i-1$ we have constructed the sequence of equations $u_{i-1} = w_{i-1}, u_{i-2} = w_{i-2}, \dots, u_0 = w_0$. If u_{i-1} and w_{i-1} are both variables, two cases can arise: 1. u_{i-1} and w_{i-1} are the same variable, then the construction of the proof ends with success (i.e., ϕ is provable), and 2. u_{i-1} and w_{i-1} are not the same variable then it ends with a failure (i.e., ϕ is not provable).

If at least one of u_{i-1} or w_{i-1} is not a variable, then at least one of them contains a subterm of the form $\Box(y_{i_1}, \dots, y_{i_m})$ where $\Box$ is an m-ary connective. Let $i(\Box, i_1, \dots, i_m)$ be such that $t_{i(\Box,i_1,\dots,i_m)} = \Box(t_{i_1}, \dots, t_{i_m})$. Thus u_i (resp., w_i) is obtained by replacing every occurrence of $\Box(y_{i_1}, \dots, y_{i_m})$ in u_{i-1} (resp., w_{i-1}) by $y_{i(\Box,i_1,\dots,i_m)}$.

At each step, at least one occurrence of a connective is eliminated and therefore the proof search ends after finitely many steps. Actually, the number of steps is bounded by the complexity of the equation $s(\phi') = t(\phi')$, and hence it is linear in the length of the input.

Example 2.1. *We prove $A_k \vee \neg A_k$ in the semantic calculus for classical logic. Here the terms $0, 1, A_k$ and $\neg A_k$ of the one generated Boolean algebra will be denoted by y_0, y_1, y_2 and y_3, respectively, $\phi = A_k \vee \neg A_k$ and $\phi' = y_2 \vee \neg y_2$. Designated elements are defined by the equation $\phi = 1$. Hence the final formula is $y_2 \vee \neg y_2 = 1$ and its proof is*

$$\dfrac{\dfrac{\dfrac{y_1 = y_1}{y_2 \vee y_3 = y_1}}{y_2 \vee \neg y_2 = y_1}}{y_2 \vee \neg y_2 = 1}$$

The semantic calculus is clearly not suitable for proof search. Already in the easy case of classical logic, the cardinality k of the free algebra on n generators is doubly exponential in n, and hence, given, for instance, a binary connective $\Box$, the index $i(\Box, i_1, i_2)$ may be much larger than i_1 and i_2. Also the number of rules explodes: we need a rule for each connective and for each pair i_1, i_2 with $i_1, i_2 \in [1, 2^{2^n}]$.

Assumption: For each logic L we will deal with, $\mathcal{V}_L$ is a suitable variety of residuated lattices possibly with additional operations, and ϕ^* is $\phi = 1$ and if ε is $\phi = \psi$, then ε^+ is $\phi \leftrightarrow \psi$.

In Section 3 we introduce an algorithm to automatically generate analytic calculi for a large class of such logics semantically characterized by locally finite varieties. Our procedure generalizes the method in [5], which is described in Section 2.2 below.

2.2. Sequents of Relations

Sequents of relations are a generalization of hypersequents introduced by Baaz and Fermüller in [5] (see also [4]). Hypersequents are multisets[2] of sequents understood as disjunctively connected at the external level, see e.g. [2]. A hypersequent has indeed the form

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n,$$

where each component $\Gamma_i \vdash \Delta_i$ is an ordinary sequent. If each $\Gamma_i \vdash \Delta_i$ is interpreted as the binary semantic predicate "$\bigwedge \Gamma_i$ implies $\bigvee \Delta_i$", a hypersequent can be seen as a finite disjunction of such binary semantic predicates. *Sequents of relations* generalize hypersequents to objects understood as a disjunction of *arbitrary* predicates belonging to a chosen semantic theory $\mathbf{T}$. These predicates can have any arity and various meaning. Examples of such predicates are "$\phi \leq \psi$", "$\phi < \psi$" or "$T_n(\phi)$" (meaning that the truth-value of ϕ is n).

In [5] it is shown how to use sequents of relations to automatedly define analytic calculi for a large family of many-valued logics — called *projective* — characterized by a special format of their semantics. Intuitively a logic L is projective if for each connective $\Box$, the value of $\Box(x_1, ..., x_n)$ is equal to a constant or to one of the $x_1, ..., x_n$. All finite-valued logics as well as (infinite-valued) Gödel logic are projective.

To describe projective logics we deal with semantic first-order theories whose intended range of discourse are sets of truth values. Additional conditions on the considered theories will be that they are function free and their set of universal formulas is decidable. To specify a projective logic associated to a theory we also need a notion of designated truth values (*designating predicate*). Any simple formula $Des(x)$ of $\mathbf{T}$ with exactly one free variable x may be chosen for this purpose, where a *simple formula* is any quantifier free formula of $\mathbf{T}$ built from atomic formulas using only conjunction and disjunction.

Example 2.2. *An example of such a semantic theory is the theory* $\mathbf{T}$ *of total orders with minimum* 0 *and maximum* 1*, based on the predicates* $<$ *and* $\leq$*. "*1*" is intended to be the only designated value. Therefore* $Des(x) := 1 \leq x$ *is the designating predicate. An axiomatization of this theory, whose universal formulas are well known to be decidable (see e.g.[3]), is*

$\forall x : \neg(x < x)$	(*Irrefl*$_<$)	$\forall x \forall y \forall z : (x < y \ \& \ y < z) \rightarrow x < z$	(*Trans*$_<$)
$\forall x : x \leq x$	(*Refl*$_\leq$)	$\forall x : 0 \leq x$	(*Min*$_\leq$)
$\forall x \forall y \forall z : (x \leq y \ \& \ y \leq z) \rightarrow x \leq z$	(*Trans*$_\leq$)	$\forall x : x \leq 1$	(*Max*$_\leq$)
$\forall x \forall y : x \leq y \vee y \leq x$	(*Lin*$_\leq$)	$\forall x \forall y : x < y \vee y \leq x$	(*Conn.*)
$\forall x \forall y : x < y \rightarrow \neg(y \leq x)$	(*Strict*)	$0 < 1$	(*Dist*)

Henceforth we write "$\mathbf{M}, \sigma \models \phi$" to denote that the formula ϕ is satisfied in a model $\mathbf{M}$ (of a semantic theory $\mathbf{T}$) under the valuation σ of elements of the domain of $\mathbf{M}$ to the free variables of ϕ. By "$\mathbf{T} \models \phi$" we mean that ϕ is valid in $\mathbf{T}$, i.e. ϕ is satisfied in *all* models of $\mathbf{T}$ for all valuations σ. Here we will consider theories $\mathbf{T}$ based on function free languages with finite signature. I.e., the atomic formulas of $\mathbf{T}$ are of the form $R(t_1, \ldots, t_k)$, where the t_i's are either variables or constants for truth values.

[2] If one prefers sequences over multisets as basic objects then an "external" permutation rule has to be added to the structural rules of the calculus.

Definition 2.1. *A logic L is* projective *if there is a classical, first-order theory* $\mathbf{T}$ *(the* semantic theory associated *to L) such that (1), (2) and (3) below hold:*

(1) $\mathbf{T}$ *has no function symbol, the constants of* $\mathbf{T}$ *coincide with the constants of L and the set Δ of universal formulas such that $\mathbf{T} \models \Delta$ is decidable.*

(2) For each n-ary connective $\Box$ of L there are simple *formulas $P^1_\Box(x_1, ..., x_n), ..., P^k_\Box(x_1, ..., x_n)$ and terms $t_1, ..., t_k$ of* $\mathbf{T}$ *which are either truth constants or in $\{x_1, ..., x_n\}$, such that:*

(2.i) For each model $\mathbf{M}$ *of* $\mathbf{T}$ *and valuation σ on* $\mathbf{M}$*, exactly one of the $P^i_\Box(x_1, ..., x_n)$ is satisfied in $\mathbf{M}$,σ.*

(2.ii) Let $\mathbf{M}^*$ be the model obtained by extending $\mathbf{M}$ *with the interpretation $\Box^{M^*}$ of any n-ary L-connective $\Box$ defined by $\Box^{M^*}(\sigma(x_1), ..., \sigma(x_n)) = \sigma(t_i)$ if $\mathbf{M}, \sigma \models P^i_\Box(x_1, ..., x_n)$. $\mathbf{M}^*$ is an algebraic model of L, i.e., for every valuation v (homomorphism from the algebra of L-formulas into $\mathbf{M}^*$) and for every theorem ϕ of L, $v(\phi)$ is a designated value.*

(3) Let $\psi^{M^*,\sigma}$ denote the truth value of ψ in the model $\mathbf{M}^*$ under σ. There is a simple formula $Des(x)$ of $\mathbf{T}$ *such that for each formula ψ of L, for each model* $\mathbf{M}$ *of* $\mathbf{T}$ *and for each valuation σ, one has: $\mathbf{M}, \sigma \models Des(\psi^{M^*,\sigma})$ iff $\psi^{M^*,\sigma}$ is a designated value of $\mathbf{M}^*$. Moreover $L = \{\phi \mid \mathbf{M} \models Des(\phi^{\mathbf{M}^*,\sigma})$ for all σ and all models* $\mathbf{M}$ *of* $\mathbf{T}\}$.

We express condition (2.ii) by the formula:

$$\Box(x_1, ..., x_n) = \begin{cases} t_1 & \text{if } P^1_\Box(x_1, ..., x_n) \\ \vdots & \vdots \\ t_m & \text{if } P^m_\Box(x_1, ..., x_n) \end{cases} \tag{1}$$

where each t_i is either a truth constant or in $\{x_1, ..., x_n\}$ and $P^i_\Box(x_1, ..., x_n)$ are simple formulas of the underlying semantic theory $\mathbf{T}$, whose free variables are among $\{x_1, \ldots, x_n\}$.

Remark 2.1. *When equality is not in the language of* $\mathbf{T}$*, we can define it as follows: let $x =^* y$ denote the conjunction of all formulas of the form*

$$\forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_i, x, x_{i+1}, \ldots, x_n) \Leftrightarrow P(x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_n))$$

where P ranges over all $(n+1)$-ary predicates of $\mathbf{T}$*, $n = 0, 1, \ldots,$ and $i = 1, \ldots, n$. This formula asserts that x and y behave in the same way with respect to any atomic formula. Since we assume that the language of* $\mathbf{T}$ *is finite $x =^* y$ is a formula, and we may assume it as a definition of equality. Moreover if we want $\mathbf{M}^*$ to be an algebraic model of L in the usual sense, we have to replace it by its quotient modulo the congruence $\theta = \{(a, b) : \mathbf{M}^* \models a =^* b\}$, cf. Definition 2.1.*

Notation: Henceforth in semantic theories we will denote classical conjunction, disjunction, implication and negation by $\sqcap$, $\sqcup$, $\Rightarrow$ and $\sim_c$, respectively ($\bigvee$ will stand for multiple disjunctions). $x \Leftrightarrow y$ is used as an abbreviation for $x \Rightarrow y \sqcap y \Rightarrow x$ and $x = y$ for $x \leq y \sqcap y \leq x$.

Example 2.3. *Gödel logic is projective. Given indeed the semantic theory in Example 2.2, its connectives can be expressed as (note that $\neg x := x \to 0$)*

$$x \to y = \begin{cases} 1 & \textit{if } x \leq y \\ y & \textit{if } y < x \end{cases} \qquad x \wedge y = \begin{cases} x & \textit{if } x \leq y \\ y & \textit{if } y < x \end{cases} \qquad x \vee y = \begin{cases} y & \textit{if } x \leq y \\ x & \textit{if } y < x \end{cases} \qquad \neg x = \begin{cases} 1 & \textit{if } x = 0 \\ 0 & \textit{otherwise} \end{cases}$$

As shown in [5] sequent calculi of relations for projective logics are defined as follows: Let L be any such logic and $\mathbf{T}$ its semantic theory. Let $R_1, \ldots, R_n$ be the predicate symbols of $\mathbf{T}$. An object of the calculus (*sequent of relations*) is then a finite multiset written in the form

$$R_{i_1}(\phi_1^1, \ldots, \phi_{r_1}^1) \mid \ldots \mid R_{i_k}(\phi_1^k, \ldots, \phi_{r_k}^k)$$

where for all $1 \leq j \leq k$: $i_j \in \{1, \ldots, n\}$, r_ℓ is the arity of R_{i_ℓ} and all ϕ_j^i are formulas of L. Each $R_{i_j}(\phi_1^1, \ldots, \phi_{r_j}^1)$ is called R_{i_j}-*component* of the sequent of relations.

Remark 2.2 ([5])**.** *Strictly speaking, the relational symbols R_{i_j} just* correspond *to the symbols of the language of* $\mathbf{T}$*, since the terms of the theory* $\mathbf{T}$ *are not formulas but variables and constants for truth values.*

A sequent calculus of relations, as usual, consists of axioms and rules. The latter are divided into structural and logical rules. Logical rules specify the behavior of connectives with respect to the relations $R_1, \ldots, R_n$ while the structural rules capture the intended interpretation of "|" as disjunction. Given a projective logic L based on a semantic theory $\mathbf{T}$, its sequent calculus of relations is defined as follows:

Axioms For each $\mathbf{T} \models \forall \bar{x} \bigvee_{1 \leq j \leq n} B_j$ where the B_j's are atomic formulas and $\bar{x}$ are the free variables in $\bigvee_{1 \leq j \leq n} B_j$. Let θ be any substitution of formulas for the variables $\bar{x}$. Then

$$B_1\theta \mid \ldots \mid B_n\theta$$

is an axiom. Notice that by Condition (1) in Definition 2.1 the set of axioms is recursive.

Structural rules Are external weakening and external contraction

$$\frac{H}{R \mid H}\ (\mathrm{EW}) \qquad\qquad \frac{R \mid R \mid H}{R \mid H}\ (\mathrm{EC})$$

where R is an arbitrary relation on formulas and H a possible empty *side sequent* (*of relations*).

Logical rules Let $\Box$ be any n-ary connective of L with the truth function (1) above. For each predicate symbol R of arity r and each position p, where $1 \leq p \leq r$, we have a rule $(\Box\colon R\colon p)$ for introducing $\Box$ at position p into an R-component of a sequent of relations. $(\Box\colon R\colon p)$ is obtained starting from the formula

$$\alpha_{(\Box\colon R\colon p)} = \bigvee_{1 \leq \ell \leq m} P_\Box^\ell(x_1, \ldots, x_n) \sqcap R(z_1, \ldots, z_r)\{t_\ell / z_p\}.$$

Take any conjunction of disjunctions of atomic formulas $\bigwedge_{1\leq j\leq s}\bigvee_{1\leq k\leq u_j} B_{j,k}$ that is equivalent in $\mathbf{T}$ to $\alpha_{(\Box:\, R:\, p)}$. Then we have the rule $(\Box\colon R\colon p)$

$$\frac{B_{1,1}\theta \mid ... \mid B_{1,u_1}\theta \mid H \quad \ldots \quad B_{s,1}\theta \mid ... \mid B_{s,u_s}\theta \mid H}{R(z_1,\ldots,z_r)\{\Box(x_1,\ldots,x_n)/z_p\}\theta \mid H}$$

where θ substitutes formulas for the variables $\{x_1,\ldots,x_n\}\cup\{z_1,\ldots,z_r\}-\{z_p\}$, and H is the side sequent of the rule.

3. Semi-projective logics

We extend the method in [5] to define analytic calculi for logics that are semi-projective or whose conservative extension is. Semi-projective logics properly contain projective logics. Examples of logics that are semi-projective but not projective are Nilpotent Minimum logic and Gödel logic with an involutive negation. Logics having proper extensions that are semi-projective include Weak Nilpotent Minimum logic and n-contractive BL-logics.

Intuitively, while connectives $\Box(x_1,\ldots,x_n)$ of projective logics do not make any calculation, that is under each interpretation σ their value is a constant or one of the actual values of the x_i, connectives of semi-projective logics can make a "limited amount" of calculations and evaluate to $\sigma(f(x_i))$, where f is a special unary connective of the logic. To capture semi-projective logics the idea is to handle applications of such unary connectives as they were atomic formulas and ”relax” the subformula property in logical rules accordingly. Examples of these unary connectives are the involutive negation ($\neg\phi = 1-\phi$), and, in the case of n-contractive BL logics, the operator $S^1(\phi)$, which represents the coatom of the component[3] ϕ belongs to, for all $\phi \neq 1$.

In contrast with the semantic theories associated to projective logics, those for semi-projective logics might contain unary function symbols.

Definition 3.1. *A logic L is* semi-projective *if there is a classical, first-order theory* $\mathbf{T}$ *(the* semantic theory associated *to L) such that the conditions (1)-(3) below hold:*

(1) $\mathbf{T}$ *contains unary function symbols $f_1,\ldots f_t$ ($t\leq 0$) corresponding to the homonym connectives of L, the constants of* $\mathbf{T}$ *coincide with the constants of L and the set of universal formulas which are valid in* $\mathbf{T}$ *is decidable.*

(2) For each n-ary connective $\Box$ of L and each f_p ($p\in\{1,\ldots,t\}$), there are basic terms s_i, t_j, s'_i, t'_j *of* $\mathbf{T}$*, which are truth constants, variables in $\{x_1,\ldots x_n\}$ or of the form $f_q(x_i)$ ($q\in\{1,\ldots,t\}$), and* simple formulas $P^1_{\Box}(s_1,\ldots,s_n),\ldots,P^m_{\Box}(s_1,\ldots,s_n)$, $P^1_{f_{p(\Box)}}(s_1,\ldots,s_n)$, $\ldots$, $P^m_{f_{p(\Box)}}(s'_1,\ldots,s'_n)$*, such that:*

(2.i) For each model $\mathbf{M}$ *of* $\mathbf{T}$ *and each valuation σ on* $\mathbf{M}$*, exactly one of the $P^i_{\Box}(s_1,\ldots,s_n)$ is satisfied in* $\mathbf{M}$*,σ and exactly one of the $P^j_{f_{p(\Box)}}(s'_1,\ldots,s'_n)$ is satisfied in* $\mathbf{M}$*,σ.*

[3] Such a component is a finite MV-algebra, and the coatom is the greatest element of the algebra which is strictly less than 1, see e.g. [13].

(2.ii) Let $\mathbf{M}^*$ be the model obtained by extending $\mathbf{M}$ with the interpretation $\Box^{M^*}$ and $f_p^{M^*}$ of the connectives of L defined by

(a) for each $p, q = 1, \dots, t$, $f_p^{M^*}(\sigma(x)) = \sigma(f_p(x))$ and $f_p^M(f_q(\sigma(x))) = \sigma(f_h(x))$, for some h in $\{1, \dots, t\}$

(b) for each n-ary connective $\Box$ ($n \geq 1$, $\Box \neq f_p$), $\Box^{M^*}(\sigma(x_1), \dots, \sigma(x_n)) = \sigma(t_i)$ if $\mathbf{M}, \sigma \models P^i_\Box(s_1, \dots, s_n)$. Moreover $f_p^{M,\sigma}(\Box(x_1, \dots, x_n)) = \sigma(t'_i)$ if $\mathbf{M}, \sigma \models P^i_{f_p(\Box)}(s'_1, \dots, s'_n)$.

$\mathbf{M}^*$ is an algebraic model of L.

(3) Let $\psi^{M^*,\sigma}$ denote the truth value of ψ in the model $\mathbf{M}^*$ under σ. There is a simple formula $Des(x)$ of $\mathbf{T}$ such that for each L-formula ψ, for each model $\mathbf{M}$ of $\mathbf{T}$ and for each valuation σ, one has: $\mathbf{M}, \sigma \models Des(\psi^{M^*,\sigma})$ iff $\psi^{M^*,\sigma}$ is a designated value of $\mathbf{M}^*$. Moreover $L = \{\phi \mid \mathbf{M} \models Des(\phi^{\mathbf{M}^*,\sigma})$ for all σ and all models $\mathbf{M}$ of $\mathbf{T}\}$.

We express condition (2.ii.(b)) by the formulas:

$$\Box(x_1, \dots, x_n) = \begin{cases} t_1 & \text{if } P^1_\Box(s_1, \dots, s_n) \\ \vdots & \vdots \\ t_m & \text{if } P^m_\Box(s_1, \dots, s_n) \end{cases} \tag{2}$$

$$f_p(\Box(x_1, \dots, x_n)) = \begin{cases} t'_1 & \text{if } P^1_{f_p(\Box)}(s'_1, \dots, s'_n) \\ \vdots & \vdots \\ t'_m & \text{if } P^m_{f_p(\Box)}(s'_1, \dots, s'_n) \end{cases} \tag{3}$$

where each t_i, t'_i, s_j, s'_j is either a truth constant or in $\{x_1, \dots, x_n, f_{p_1}(x_i), \dots, f_{p_n}(x_i)\}$, with $p, p_k \in \{1, \dots, t\}$ and $P^i_\Box, P^i_{f_p(\Box)}$ are simple formulas of the underlying semantic theory $\mathbf{T}$ whose free variables are among $\{x_1, \dots, x_n\}$.

Remark 3.1. *If equality is not in the language of $\mathbf{T}$ we can define it as $=^*$ similar to the case of projective logics (see Remark 2.1). For semi-projective logics the formula $x =^* y$ will stand for the conjunction of all formulas of the form*

$$\forall x_1 \dots \forall x_n (P(x_1, \dots, x_i, x, x_{i+1}, \dots, x_n) \Leftrightarrow P(x_1, \dots, x_i, y, x_{i+1}, \dots, x_n))$$

$$\forall x_1 \dots \forall x_n (P(x_1, \dots, x_i, f(x), x_{i+1}, \dots, x_n) \Leftrightarrow P(x_1, \dots, x_i, f(y), x_{i+1}, \dots, x_n))$$

where P ranges over all $(n+1)$-ary predicates of $\mathbf{T}$, f over all functions, $n = 0, 1, \dots$ and $i = 1, \dots, n$. Moreover, we assume that $\mathbf{T}$ proves all identities $f_p(f_q(x)) =^* f_h(x)$, where f_h satisfies Definition 3.1(2.ii).

Example 3.1. *Gödel logic extended with an involutive negation [15, 16] is semi-projective. Given indeed the semantic theory of Ex. 2.2 enriched with a function $\sim$ which is an order reversing involution, the negation of its connectives can be expressed as: $\sim(\sim x) = x$ and*

$$\sim(x \to y) = \begin{cases} 0 & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases} \qquad \sim(x \wedge y) = \begin{cases} \sim x & \text{if } x \leq y \\ \sim y & \text{if } y < x \end{cases} \qquad \sim(x \vee y) = \begin{cases} \sim y & \text{if } x \leq y \\ \sim x & \text{if } y < x \end{cases}$$

Proposition 3.1. *Each semi-projective logic L with finitely many connectives and constants is locally finite.*

Proof. Let $\mathcal{V}_L$ be the variety equivalent to L. It suffices to prove that for every n the free algebra of $\mathcal{V}_L$ on n generators (i.e.$\mathbf{F}_n$) is finite. Let $x_1, \ldots, x_n$ be the generators of $\mathbf{F}_n$. Without loss of generality we may assume that $x_1, \ldots, x_n$ are distinct variables. Let Δ be the set of basic terms which are either variables among $x_1, \ldots, x_n$ or constants or of the form $f(x_i)$, where f is a unary function corresponding to a special connective. Next let Θ be the set of all simple formulas of $\mathbf{T}$ of the form $P^i_\Box(t_1, \ldots, t_n)$ or $P^i_{f_p(\Box)}(t_1, \ldots, t_n)$ occurring in (2) and (3), where $t_1, \ldots, t_n \in \Delta$. Let Υ be the set of all maximally consistent (with $\mathbf{T}$) subsets of Θ. Note that Δ, Θ and Υ are all finite. We show that for every term t and for every $S \in \Upsilon$ there is a term $t_S \in \Delta$ such that for every model $\mathbf{M}, \sigma$ of $\mathbf{T} \cup S$ one has $\mathbf{M}, \sigma \models t = t_S$. The proof is by induction on t: If $t \in \Delta$, we take $t_S = t$ for every $S \in \Upsilon$. Assume $t = \Box(u_1, \ldots, u_k)$. By the induction hypothesis, every model $\mathbf{M}, \sigma$ of $\mathbf{T} \cup S$ satisfies $t = \Box((u_1)_S, \ldots, (u_k)_S)$, where $(u_i)_S \in \Delta$ for $i = 1, \ldots, k$. Finally, by (2) the connective $\Box$ splits into cases according to the conditions $P^1_\Box((u_1)_S, \ldots, (u_k)_S), \ldots, P^m_\Box((u_1)_S, \ldots, (u_k)_S)$. Now by the maximality of S, exactly one of the above conditions is consistent with S, and hence every model $\mathbf{M}, \sigma$ of $\mathbf{T} \cup S$, satisfies $t = \Box((u_1)_S, \ldots, (u_k)_S) = t_i$ for some i, where $t_i \in \Delta$ is provided by (2). The induction step for the case $t = f_p(\Box(u_1, \ldots, u_k))$ is similar, using condition (3) instead of (2).

Next, let Λ be the set of all sequences $(t_S : S \in \Upsilon)$. Then Λ is in turn finite, being a subset of Δ^Υ. We define the following equivalence on Λ: $(t_S : S \in \Upsilon) \equiv (u_S : S \in \Upsilon)$ iff for every $S \in \Upsilon$ and for every model $\mathbf{M}_S, \sigma$ of $\mathbf{T} \cup S$ one has $\mathbf{M}_S, \sigma \models u_S = t_S$. Let $[(t_S : S \in \Upsilon)]_\equiv$ denote the equivalence class of $(t_S : S \in \Upsilon)$ modulo $\equiv$. It is not difficult to see that the map $\Phi : t \mapsto [(t_S : S \in \Upsilon)]_\equiv$ is a bijection from F_n onto the quotient set $\Lambda/\equiv$. (To show that Φ is a function we use the fact that the algebra $\mathbf{M}^*$ is an an element of $\mathcal{V}_L$, while for proving that Φ is a bijection the fact that L is complete w.r.t. the class of all $\mathbf{M}^*$ such that $\mathbf{M}$ is a model of $\mathbf{T}$; in other words that $\mathcal{V}_L$ is the variety generated by the class of all $\mathbf{M}^*$ s.t. $\mathbf{M}$ is a model of $\mathbf{T}$). Being $\Lambda/\equiv$ finite, it follows that $\mathbf{F}_n$ is finite. $\Box$

Given a semi-projective logic L, a sequent calculus of relations $\mathcal{R}L$ for L is defined similarly as in Section 2.2. The only difference is the handling of the special connectives f_i. Indeed, instead of defining logical rules determining the behavior of those connectives w.r.t. the relations in the calculus, $\mathcal{R}L$ will contain rules for $f_i\Box$, for each connective $\Box$ of L, while formulas $f_i(A)$, where A is atomic, are not further decomposed.

Axioms and Structural rules Are exactly as in the case of projective logics, noticing that axioms might contain the connectives f_i of L corresponding to the functions of $\mathbf{T}$.

Logical rules If L contains a unary connective f_p corresponding to a function of $\mathbf{T}$, then for any n-ary connective $\Box$, for each predicate symbol R of arity r and each position p', where $1 \leq p' \leq r$, we have a rule ($f_p\Box$: R: p') introducing $f_p\Box(x_1, \ldots x_n)$ at position p' into an R-component of a sequent of relation. We distinguish two cases:

- If $\Box = f_q$, then we have the rule (cf. Definition 3.1(2.ii))

$$\frac{R(z_1,\ldots,z_r)\{f_h(x)/z_{p'}\}\theta \mid H}{R(z_1,\ldots,z_r)\{f_pf_q(x)/z_{p'}\}\theta \mid H}\ (f_pf_q\colon R\colon p')$$

  where θ substitutes formulas of L for the variables $\{x, z_1, \ldots, z_r\} - \{z_{p'}\}$, and H is the side sequent of relations of the rule.

- Otherwise, if $\Box$ satisfies the formula (2) above, to define $(f_p\Box\colon R\colon p')$ we start from the **T**-formula $\alpha_{(f_p\Box\colon R\colon p')}$:

$$\bigvee_{1\le\ell\le m} P^{\ell}_{f_p(\Box)}(s'_1,\ldots,s'_n)\sqcap R(z_1,\ldots,z_r)\{t'_\ell/z_{p'}\}$$

  (cf. formula (3)). Take any formula equivalent in **T** to $\alpha_{(f\Box\colon R\colon p')}$ of the form $\bigwedge_{1\le j\le s}\bigvee_{1\le k\le u_j} B_{j,k}$, where $B_{j,k}$ are atomic formulas of **T** (recall that $P^{l}_{f_p(\Box)}(s'_1,\ldots,s'_n)$ are simple formulas that is built from atomic formulas using conjunction and disjunction only). Then we have the $(f_p\Box\colon R\colon p')$ rule

$$\frac{B_{1,1}\theta \mid ... \mid B_{1,u_1}\theta \mid H \quad \ldots \quad B_{s,1}\theta \mid ... \mid B_{s,u_s}\theta \mid H}{R(z_1,\ldots,z_r)\{f_p\Box(x_1,\ldots,x_n)/z_{p'}\}\theta \mid H}$$

  where θ and H are similarly as above.

Moreover, for each n-ary connective $\Box$ of L satisfying formula (2) we have a rule $(\Box\colon R\colon p')$ introducing $\Box(x_1,\ldots x_n)$ at position p' into an R-component of a sequent of relations. The definition of $(\Box\colon R\colon p')$ is analogous to that of $(f_p\Box\colon R\colon p')$, starting here from the formula

$$\alpha_{(\Box\colon R\colon p')} = \bigvee_{1\le\ell\le m} P^{\ell}_{\Box}(x_1,\ldots,x_n)\sqcap R(z_1,\ldots,z_r)\{t_\ell/z_{p'}\}.$$

Remark 3.2. *Logical rules satisfy the subformula property up to the special connectives f_i.*

Soundness, completeness, decidability and computational complexity

A sequent of relations $\mathcal{S}$ is called *provable* in $\mathcal{R}L$ ($\vdash_{\mathcal{R}L}\mathcal{S}$, in symbols) if there is an upward tree of sequents of relations rooted in $\mathcal{S}$, such that every leaf is an axiom and every other sequent of relations is obtained from the ones standing immediately above it by application of one of the rules of $\mathcal{R}L$.

For the following statements let **T** be any semantic theory and L the semi-projective logic determined by **T**. Let $\mathcal{R}L$ be its corresponding sequent calculus of relations defined as described above.

It is easy to see that the rules of $\mathcal{R}L$ preserve soundness when read both top down and bottom up, i.e. they are sound and invertible. Indeed, for any sequent of relations

$$\mathcal{S} = R_1(\phi_{1,1},\ldots,\phi_{1,r_1}) \mid \ldots \mid R_n(\phi_{n,1},\ldots,\phi_{n,r_n})$$

we write $\mathbf{M}, \sigma \models \mathcal{S}$ to denote $\mathbf{M}, \sigma \models \beta^{\sigma}_{\mathcal{S}}$ where

$$\beta^{\sigma}_{\mathcal{S}} = \forall \overline{x} \bigvee_{1 \leq i \leq n} R_i(\phi^{\mathbf{M}^*,\sigma}_{i,1}, \ldots, \phi^{\mathbf{M}^*,\sigma}_{i,r_i})$$

(recall that for each formula ϕ of L, for each model $\mathbf{M}$ of $\mathbf{T}$ and for each valuation σ, there is a basic term t of $\mathbf{T}$ such that $\phi^{\mathbf{M}^*,\sigma}$ is equal to $\sigma(t)$).

Proposition 3.2. *Let* $\dfrac{S_1 \ldots S_k}{S}$ *be any rule of* $\mathcal{R}L$. *Then for each model* $\mathbf{M}$ *of* $\mathbf{T}$ *and each valuation* σ *on* $\mathbf{M}$*,*

$$\mathbf{M}, \sigma \models \mathcal{S} \quad \textit{iff} \quad \mathbf{M}, \sigma \models \mathcal{S}_i \ \textit{ for all } \ i = 1, \ldots k$$

Since the designating predicate $Des(x)$ is a simple formula, $Des(x)$ is equivalent to a formula $Des(x)'$ of form $\bigwedge_{1 \leq i \leq p} \bigvee_{1 \leq j \leq q_i} A_{i,j}$ where the $A_{i,j}$ are atomic formulas with at most one free variable x. Let ϕ be a formula of L, by

$$\mathcal{D}_1\{x/\phi\}, \ldots, \mathcal{D}_p\{x/\phi\}$$

we denote the sequence of sequents that correspond to the conjuncts of $Des(x)'$ if x is replaced by ϕ.

As usual, we define the *length* of a derivation as the number of inferences in a maximal branch of the derivation.

Theorem 3.1 (Soundness)**.** *If* $\mathcal{D}_1\{x/\phi\}, \ldots, \mathcal{D}_p\{x/\phi\}$ *are provable in* $\mathcal{R}L$ *then* ϕ *is valid in* L.

Proof. We show that for each sequent of relations $\mathcal{S}$, if $\vdash_{\mathcal{R}L} \mathcal{S}$ then $\mathbf{M}, \sigma \models \mathcal{S}$ for all σ and M of $\mathbf{T}$. Hence it follows that ϕ is valid in L. The proof is by induction on the length of the derivation of $\mathcal{S}$ in $\mathcal{R}L$. Base case: $\mathcal{S}$ is an axiom. Then $\mathbf{M}, \sigma \models \mathcal{S}$ for all σ and model $\mathbf{M}$ of $\mathbf{T}$. The inductive case immediately follows by Proposition 3.2. □

Definition 3.2. *A* quasi-atomic sequent *is a sequent of relations* $R_1(\phi_{1,1}, \ldots, \phi_{1,r_1}) \mid \ldots \mid R_n(\phi_{n,1}, \ldots, \phi_{n,r_n})$ *where each* $\phi_{i,j}$ *is either an atomic formula of* L *or of the form* $f_j(A)$ *where* A *is atomic.*

Theorem 3.2 (Completeness)**.** *If* ϕ *is valid in* L *then* $\mathcal{D}_1\{x/\phi\}, \ldots, \mathcal{D}_p\{x/\phi\}$ *are provable in* $\mathcal{R}L$.

Proof. By definition of L, $\mathbf{M} \models Des(\phi^{\mathbf{M}^*,\sigma})$, for all σ and $\mathbf{M}$ of $\mathbf{T}$ and therefore $\mathbf{M} \models \mathcal{D}_i\{x/\phi^{\mathbf{M}^*,\sigma}\}$ for each $i = 1, \ldots, p$. We show that $\mathcal{D}_i\{x/\phi\}$ is provable in $\mathcal{R}L$ for all $i = 1, \ldots, p$. This is done by stepwise decomposing ϕ by successively applying the logical rules of $\mathcal{R}L$. This way we eventually end up in quasi-atomic sequents. By Proposition 3.2 (invertibility of rules) each such sequent of relations is valid in $\mathbf{T}$ and hence it is an axiom or it derives an axiom by using weakening. We therefore get a derivation of $\mathcal{D}_i\{x/\phi\}$ in $\mathcal{R}L$ for each $i = 1, \ldots, p$. □

Notice that the contraction rule is not needed to prove the completeness of the calculus (i.e. contraction is an *admissible* rule).

Since the construction of the reduction trees is effective and the axioms are decidable by condition (1) in Definition 3.1 we obtain:

Corollary 3.1. *All semi-projective logics are decidable.*

The *size* of a sequent of relations S is the number of symbols occurring in formulas of S.

Proposition 3.3. *If the problem of determining the validity in* $\mathbf{T}$ *of quasi-atomic sequents is Co-NP then L is Co-NP.*

Despite having invertible rules and Co-NP decidable quasi-atomic sequents, we do not have yet Co-NP calculi for semi-projective logics since the rules applied upwards may increase the size of sequents of relations exponentially[4]. This problem is overcome by considering new sequent of relations rules which operate *simultaneously* on *all* occurrences of a compound formula in a sequent of relations (*generalized rules*).

Proof. Generalized rules are constructed as follows: let S be a sequent of relations containing (multiple occurrences of) $\psi = \Box(\phi_1, \ldots, \phi_n)$. We denote by S_i the sequent of relations obtained by replacing all occurrences of ψ in S with γ_i, where $\gamma_i = t_i(\phi_1, \ldots, \phi_n)$, that is a constant or a formula in $\{\phi_1, \ldots, \phi_n, f_{p_i}(\phi_j), \ldots\}$, according to condition (2). The condition $\bigvee_{i=1}^{m}(P^i_\Box(s_1(\phi_1, \ldots, \phi_n), \ldots, s_n(\phi_1, \ldots, \phi_n)) \sqcap S_i)$ (where the simple formulas $P^i_\Box$ are as in (2)) can be written as a conjunction of formulas $\bigvee_{i=1}^{p-1}(P^i_\Box(s_1(\phi_1, \ldots, \phi_n), \ldots, s_n(\phi_1, \ldots, \phi_n)) \sqcup \bigvee_{i=p+1}^{m}(P^i_\Box(s_1(\phi_1, \ldots, \phi_n), \ldots, s_n(\phi_1, \ldots, \phi_n)) \sqcup S_p$, for $p = 1, \ldots m$, each of which can be represented by a set of sequents of relations, which we denote by $D_{p,1} \mid S_p, \ldots, D_{p,k_p} \mid S_p$ ($k_p = 1$, if no $P^i_\Box$ contains a conjunction). It is easy to see that $\mathbf{M}, \sigma \models S$ if and only if $\mathbf{M}, \sigma \models D_{i,j} \mid S_i$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, i_j$.

The case $\psi = f_p\Box(\phi_1, \ldots, \phi_n)$ is analogous.

Let ϕ be any formula of L and $\mathcal{D}_1(x/\phi), \ldots, \mathcal{D}_p(x/\phi)$ be the sequence of sequents equivalent to $Des(\phi)$. We stepwise decompose each $\mathcal{D}_i(x/\phi)$ by successively applying the generalized rules above. This way we eventually end up in quasi-atomic sequents which are valid in $\mathbf{T}$ if and only if the formula ϕ is valid in L. We call the obtained trees *gen-reduction trees* and show that each of their branches has size polynomial in the size of ϕ.

Indeed, first notice that the length of each gen-reduction tree is linear in the size of ϕ as each application of a generalized rule simultaneously replace all formulas $\Box(\phi_1, \ldots, \phi_n)$ (or $f_p\Box(\phi_1, \ldots, \phi_n)$) by subformulas ϕ_i or formulas $f(\phi_i)$, with $i = 1, \ldots, n$, for some unary connective f. Hence once a formula is reduced, it does not appear anymore in the branch (and only its subformulas, possibly prefixed by a unary connective, are left).

Next notice that the number of $D_{i,j}$ and the number of components in each $D_{i,j}$ only depends on the connective $\Box$ (it is a constant). Moreover, the size $|S_i|$ of each S_i does not

[4]Each application of a rule reducing e.g. a formula $\Box(\Phi_1, \ldots \Phi_n)$ might produce *several* occurrences of the sub-formulas $\Phi_1, \ldots, \Phi_n$ and also duplicate other formulas in the sequent of relations. If each occurrence of these formula is handled separately, we can end up in a sequent of relations having exponential size.

exceed the size of S, and the size of each $D_{i,j}$ is linearly bounded in the size of ϕ (each $D_{i,j}$ consists of a fixed number of predicates only containing formulas of L which are simpler that ϕ). Thus the size of each son $D_{i,j} \mid S_i$ of S is bounded by $|S| + hn|\phi|$ for some constant h, n standing for the maximum number of predicates in $D_{i,j}$ and the maximum arity of predicates in $\mathbf{T}$. Hence the total size of a branch of a gen-reduction tree is bounded by $K|\phi|^2$ for some constant K.

Now a non-deterministic polynomial time algorithm to check the unprovability of a formula ϕ is the following: guess $i \in \{1, \ldots, p\}$, and guess a branch of the reduction tree of $\mathcal{D}_i(\phi/x)$. The guess may be done in polynomial time, because the height of the branch is linear in $|\phi|$. Checking whether the leaf is valid in $\mathbf{T}$ is in co-NP. $\square$

4. Examples

As case study, we show below how to define analytic sequent calculi of relations for nilpotent minimum logic NM, weak nilpotent minimum logic WNM and cnBL$^+$ ($n \geq 1$), which are n-contractive BL-logics extended by n unary connectives. The latter calculi are also analytic calculi for cnBL.

4.1. Nilpotent Minimum

Nilpotent minimum logic NM was introduced by Godo and Esteva in [14] as the logic of the nilpotent minimum t-norms. A semantic theory $\mathbf{T}_{\leq,<}$ for NM is the theory of total orders with minimum 0 and maximum 1 and with an order reversing involution $\sim$. $\mathbf{T}_{\leq,<}$ has a unary function symbol $\sim$, two predicate symbols $\leq$ and $<$, and two constants, 0 and 1 (see Example 3.1).

We state below the truth functions for the connectives of NM: disjunction ($\vee$), conjunctions ($\&$ and $\wedge$), implication ($\rightarrow$), and negation ($\sim$) in such a way that it gets clear that these connectives are semi-projective with respect to $\mathbf{T}_{\leq,<}$.

$$x\&y = \begin{cases} 0 & \text{if } x \leq \sim y \\ x & \text{if } \sim y < x \sqcap x \leq y \\ y & \text{if } \sim y < x \sqcap y < x \end{cases} \qquad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \sqcap \sim x < y \\ \sim x & \text{if } y < x \sqcap y \leq \sim x \end{cases}$$

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}$$

Moreover, $\sim (\sim x), \sim (x \wedge y)$ and $\sim (x \vee y)$ are as in Example 2.3, while

$$\sim (x\&y) = \begin{cases} 1 & \text{if } x \leq \sim y \\ \sim x & \text{if } \sim y < x \sqcap x \leq y \\ \sim y & \text{if } \sim y < x \sqcap y < x \end{cases} \qquad \sim (x \rightarrow y) = \begin{cases} 0 & \text{if } x \leq y \\ x & \text{if } y < x \sqcap y \leq \sim x \\ \sim y & \text{if } y < x \sqcap \sim x \leq y \end{cases}$$

Notice that the negation $\sim$ is a special connective corresponding to the homonym function in $\mathbf{T}_{\leq,<}$. We show how to derive the logical rules of the sequent calculus of relations for NM. For example: $\alpha_{(\&:\leq:l)} = ((x \leq \sim y) \sqcap (0 \leq z)) \sqcup ((x \leq y) \sqcap (\sim y < x) \sqcap (x \leq z)) \sqcup ((y < x) \sqcap (\sim y < x) \sqcap (y \leq z))$ is equivalent to $(x \leq \sim y) \sqcup (x \leq z) \sqcup (y \leq z)$, while $\alpha_{(\&:<:l)}$ is equivalent to $0 < z \sqcap (x \leq\sim y) \sqcup (x < z) \sqcup (y < z))$. Hence we have the rules (below l and r will abbreviate the left and the right side of the binary predicates, respectively):

$$\frac{H \mid \phi \leq\sim \psi \mid \phi \leq \gamma \mid \psi \leq \gamma}{H \mid \phi\&\psi \leq \gamma}\ (\& :\leq: l) \quad \frac{H \mid 0 < \gamma \quad H \mid \phi \leq\sim \psi \mid \phi < \gamma \mid \psi < \gamma}{H \mid \phi\&\psi < \gamma}\ (\& :<: l)$$

The right rules $(\& :\leq: r)$ and $(\& :<: r)$ for & are respectively

$$\frac{H \mid \gamma \leq 0 \mid \gamma \leq \phi \quad H \mid \gamma \leq 0 \mid \gamma \leq \psi \quad H \mid \gamma \leq 0 \mid\sim \psi < \phi}{H \mid \gamma \leq \phi\&\psi} \quad \frac{H \mid\sim \psi < \phi \quad H \mid \gamma < \phi \quad H \mid \gamma < \psi}{H \mid \gamma < \phi\&\psi}$$

$\alpha_{(\rightarrow:\leq:l)}$ is equivalent to $(1 \leq z \sqcup y < x) \sqcap y \leq z \sqcap \sim x \leq z$ while $\alpha_{(\rightarrow:<:r)}$ to $z < 1 \sqcap (x \leq y \sqcup z <\sim x \sqcup z < y)$. Hence:

$$\frac{H \mid 1 \leq \gamma \mid \psi < \phi \quad H \mid \psi \leq \gamma \quad H \mid\sim \phi \leq \gamma}{H \mid \phi \rightarrow \psi \leq \gamma}\ (\rightarrow:\leq: l) \quad \frac{H \mid \psi < \phi \quad H \mid \psi < \gamma \qquad \Lambda \mid\ \sim \phi < \gamma}{H \mid \phi \rightarrow \psi < \gamma}\ (\rightarrow:<: l)$$

The remaining rules for $\rightarrow$ are:

$$\frac{H \mid \phi \leq \psi \mid \gamma \leq\sim \phi \mid \gamma \leq \psi}{H \mid \gamma \leq \phi \rightarrow \psi}\ (\rightarrow:\leq: r) \quad \frac{H \mid \gamma < 1 \quad H \mid \phi \leq \psi \mid \gamma <\sim \phi \mid \gamma < \psi}{H \mid \gamma < \phi \rightarrow \psi}\ (\rightarrow:<: r)$$

The rules for $\vee$ and $\wedge$ are immediate; for instance

$$\frac{H \mid \phi \leq \gamma \quad H \mid \psi \leq \gamma}{H \mid \phi \vee \psi \leq \gamma}\ (\vee :\leq: l) \qquad \frac{H \mid \phi < \gamma \quad H \mid \psi < \gamma}{H \mid \phi \vee \psi < \gamma}\ (\vee :<: l)$$

$$\frac{H \mid \gamma \leq \phi \mid \gamma \leq \psi}{H \mid \gamma \leq \phi \vee \psi}\ (\vee :\leq: r) \qquad \frac{H \mid \gamma < \phi \mid \gamma < \psi}{H \mid \gamma < \phi \vee \psi}\ (\vee :<: r)$$

Finally, we need rules for negated compound formulas.

$$\frac{H \mid \phi \leq \psi}{H \mid\sim\sim \phi \leq \psi}\ (\sim\sim:\leq: l) \quad \frac{H \mid \psi \leq \phi}{H \mid \psi \leq\sim\sim \phi}\ (\sim\sim:\leq: r) \quad \frac{H \mid \phi < \psi}{H \mid\sim\sim \phi < \psi}\ (\sim\sim:<: l) \quad \frac{H \mid \psi < \phi}{H \mid \psi <\sim\sim \phi}\ (\sim\sim:<: r)$$

The rules for $\sim (\phi\wedge\psi)$ and $\sim (\phi\vee\psi)$ are easy and left to the reader. Those for $\sim (\phi \rightarrow \psi)$ and $\sim (\phi\&\psi)$ are (in the right rules for $\sim\rightarrow$ we will omit the side sequent of relation H):

$$\frac{H \mid \phi \leq \psi \mid \phi \leq \gamma \mid \sim \psi \leq \gamma}{H \mid\sim (\phi \rightarrow \psi) \leq \gamma}\ (\sim\rightarrow:\leq: l) \quad \frac{H \mid 0 < \gamma \quad H \mid \phi \leq \psi \mid \phi < \gamma \mid \psi < \gamma}{H \mid\sim (\phi \rightarrow \psi) < \gamma}\ (\sim\rightarrow:\leq: l)$$

$$\frac{\gamma \leq 0 \mid \gamma \leq \phi \quad \gamma \leq 0 \mid \gamma \leq\sim \psi \quad \gamma \leq 0 \mid \psi < \phi}{\gamma \leq\sim (\phi \rightarrow \psi)}\ (\sim\rightarrow:\leq: r) \quad \frac{\psi < \phi \quad \gamma < \phi \quad \gamma <\sim \psi}{\gamma <\sim (\phi \rightarrow \psi)}\ (\sim\rightarrow:<: r)$$

$$\frac{H \mid 1 \leq \gamma \mid\sim \psi < \phi \quad H \mid\sim \psi \leq \gamma \quad H \mid\ \sim \phi \leq \gamma}{H \mid\ \sim (\phi\&\psi) \leq \gamma}\ (\sim \& :\leq: l) \quad \frac{H \mid \phi \leq\sim \psi \mid \gamma \leq\sim \phi \mid \gamma \leq\sim \psi}{H \mid \gamma \leq\sim (\phi\&\psi)}\ (\sim \& :\leq: r)$$

$$\frac{H \mid \sim \psi < \phi\ H \mid \sim \psi < \gamma\ H \mid \sim \phi\ < \gamma}{H \mid \sim (\phi\&\psi) < \gamma}\ (\sim \& :<: l) \quad \frac{H \mid \gamma < 1 \quad H \mid \phi \leq\sim \psi \mid \gamma <\sim \phi \mid \gamma <\sim \psi}{H \mid \gamma <\sim (\phi\&\psi)}\ (\sim \& :<: r)$$

Example 4.1. *A proof of the formula* $(A \to B) \vee (B \to A)$ *in the calculus for NM is as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{A \leq B \mid B \leq A}{A \leq B \mid 1 \leq \sim A \mid 1 \leq B \mid B \leq A \mid 1 \leq \sim B \mid 1 \leq A}\,(\mathrm{EW})}{A \leq B \mid 1 \leq \sim A \mid 1 \leq B \mid 1 \leq B \to A}\,(\to:\leq:\mathrm{r})}{1 \leq A \to B \mid 1 \leq B \to A}\,(\to:\leq:\mathrm{r})}{1 \leq (A \to B) \vee (B \to A)}\,(\vee:\leq:\mathrm{r})$$

By $x^{\sim}$ we will denote $\sim x$ if x is not a negation of any other variable and x, if x is $\sim y$. (Similarly for $\sim \phi$).

Proposition 4.1. *The above calculus provides a Co-NP decision procedure for the validity problem in NM.*

Proof. By Proposition 3.3 it is enough to show that the validity in $\mathbf{T}_{\leq,<}$ of quasi-atomic sequents can be checked in polynomial time. Indeed, given any such sequent of relations H, H is valid in $\mathbf{T}_{\leq,<}$ iff its negation H^* is not satisfiable. To define H^* we first replace in H each atomic formula of NM with a variable symbol in such a way that equal formulas are replaced by the same variable. Now replace every relation in H of the form $x \leq y$ by $\{y < x, x^{\sim} < y^{\sim}\}$ and every relation $x < y$ by $\{y \leq x, x^{\sim} \leq y^{\sim}\}$. Let H^* be the set of formulas of $\mathbf{T}_{\leq,<}$ obtained in this way. H^* is not satisfiable in $\mathbf{T}_{\leq,<}$ iff H^* contains $1 < 0$, $1 \leq x_1 \leq \cdots \leq x_{n-1} < x_n$ or $x_1 \leq \cdots \leq x_n < 0$, for some x_i, or there is a cycle $x_1 \prec_1 x_2 \prec_2 \cdots \prec_n x_n \prec_{n+1} x_1$ where both $x_i \prec_i x_{i+1}$ and $x_n \prec_{n+1} x_1 \in H^*$, each $\prec_i$ is either $\leq$ or $<$ and at least one is $<$. Clearly this can be verified in polynomial time in the size of H. $\square$

This provides an alternative proof, e.g. w.r.t. [1], that the validity problem for NM is Co-NP complete (for the completeness it is enough to notice that interpretations in classical logic are particular interpretations in NM where each formula can take only value 0 or 1).

The proof of the proposition above also leads to the following explicit description of the calculus axioms.

Axioms are sequents of the form $S_1 | \ldots | S_n$ such that for some atomic formulas $A_1, \ldots, A_n$ and relations $\lhd_1, \ldots, \lhd_n$ (1) $\lhd_i$ is either $\leq$ or $<$ and (2) for $i < n$, S_i is either $A_i \lhd_i A_{i+1}$ or $A_{i+1}^{\sim} \lhd_i A_i^{\sim}$, and one of the following holds

- $S_1 | \ldots | S_n$ is a cycle, that is for at least one i, $\lhd_i$ is $\leq$, and S_n is either $A_n \lhd_n A_1$ or $A_1^{\sim} \lhd_n A_n^{\sim}$
- for at least one i, $\lhd_i$ is $\leq$ and either $S_1 = 0 \lhd_1 A_2$ $(\sim A_2 \lhd_1 1)$, or $S_n = A_n \lhd_n 1$ $(0 \lhd_1 \sim A_n)$
- S_1 is $0 \lhd_1 A_2$ and S_n is $A_n \lhd_n 1$ (the case $n = 1$ is $0 < 1$).

Remark 4.1. *Weak nilpotent minimum logic WNM is NM without the involutivity of negation [14]. That is, the negation of WNM is the same as in Gödel logic (cf. Example 2.3). Though not semi-projective, WNM becomes so when extended with a new connective c corresponding to the homonym function in its semantic theory. The semantic theory for WNM is that for NM without the order reversing involution and with the additional function c which satisfies* $x \leq c(x)$, $c(x) = \neg\neg(x)$, $c(\neg x) = \neg x$, *and in which the negation* $\neg$ *is such that* $\neg 1 = 0$ *and* $x \leq y \Rightarrow \neg y \leq \neg x$. *The truth functions for* $x \to y$, $x \star y$ *and* $\neg\star$ *where* $\star \in \{\&, \wedge, \vee\}$ *are as for NM. The remaining ones are:*

$$\neg(x \to y) = \begin{cases} 0 & \textit{if } x \leq y \\ \neg y & \textit{if } y < x \sqcap \neg x \leq y \\ c(x) & \textit{if } y < x \sqcap y < \neg x \end{cases} \qquad c(x\&y) = \begin{cases} 0 & \textit{if } x \leq \neg y \\ c(y) & \textit{if } \neg y < x \sqcap y < x \\ c(x) & \textit{if } \neg y < x \sqcap x \leq y \end{cases}$$

$$c(x \to y) = \begin{cases} 1 & \textit{if } x \leq y \\ \neg x & \textit{if } y < x \sqcap y \leq \neg x \\ c(y) & \textit{if } y < x \sqcap \neg x \leq y \end{cases} \qquad c(x \vee y) = \begin{cases} c(x) & \textit{if } y \leq x \\ c(y) & \textit{if } x < y \end{cases}$$

$$c(x \wedge y) = \begin{cases} c(y) & \textit{if } y \leq x \\ c(x) & \textit{if } x < y \end{cases} \qquad c(\neg x) = \neg(c(x)) = \neg x \quad c(c(x)) = c(x) \quad \neg(\neg x) = c(x)$$

A sequent calculus of relations for WNM can be easily derived using our methodology.

4.2. n-contractive BL$^{(+)}$-logics

Hajek's Basic Logic BL [17] is the logic of all continuous t-norms and their residua[5]. Its extension with the n-contraction axiom schema ($n \geq 1$)

$$\phi^n \to \phi^{n+1} \quad \text{where } \phi^k \text{ stands for } \phi \,\&\ldots\&\, \phi,\ k \text{ times}$$

was introduced in [8] (see also [11]), and called cnBL. In the particular case $n = 1$, cnBL coincides with Gödel logic. The logics cnBL "approximate" BL, being BL the intersection of all cnBL. The algebraic semantics of cnBL consists of those BL-algebras that are subdirect products of BL-chains that are ordinal sums of MV-chains with at most $n+1$ elements. Thus provability of a formula in cnBL is equivalent to its validity in all ordinal sums of MV-chains with at most $n+1$ elements.

Though the logics cnBL are not semi-projective, they become so when extended with unary connectives $S^1, \ldots, S^n$, where $S^1(x)$ denotes 1 if $x = 1$ and the coatom y of the component which x belongs to if $x < 1$ (y is a coatom if there is no z such that $y < z < 1$). $S^h(x)$ is $(S^1(x))^h$. The resulting logics are introduced in this section and called cnBL$^+$. Being cnBL$^+$ semi-projective, sequent calculi of relations can be defined using our methodology. These calculi also serve as analytic calculi for cnBL.

[5] A continuous t-norm is a continuous, commutative, associative, in both arguments monotonically increasing function $* : [0,1]^2 \to [0,1]$ such that $1 * x = x$ for all $x \in [0,1]$. The residuum of $*$ is a function $\to^*$: $[0,1]^2 \to [0,1]$ where $x \to^* y = \max\{z \mid x * z \leq y\}$.

Below we recall the definition of ordinal sum in the special case where all summands (components) are MV-chains with at most $n+1$ elements. We refer e.g. to [17, 8, 13] for more details. We assume that I is a totally ordered set with minimum i_0 and that for all $i \in I$, $\mathbf{A}_i$ is a non-trivial MV-chain with at most $n+1$ elements and that if $i \neq j$, then $A_i \cap A_j = \{1\}$. Then the ordinal sum $\bigoplus_{i\in I} \mathbf{A}_i$ is the algebra defined as follows:

The domain of $\bigoplus_{i\in I} \mathbf{A}_i$ is the union of all A_i. The top of $\bigoplus_{i\in I} \mathbf{A}_i$ is 1 (common to all summands). The bottom of $\bigoplus_{i\in I} \mathbf{A}_i$ is the bottom of $\mathbf{A}_{i_0}$. Implication and conjunction are:

$$x \to y = \begin{cases} x \to^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i,\, y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\},\, y \in A_j) \end{cases} \quad x \cdot y = \begin{cases} x \cdot^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\, y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\},\, x \in A_j) \end{cases}$$

Join and meet are defined in terms of $\cdot$ and $\to$ as $x \wedge y = x \cdot (x \to y)$ and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. When defining the ordinal sum $\bigoplus_{i\in I} \mathbf{A}_i$ we will tacitly assume that whenever the condition $A_i \cap A_j = \{1\}$ is not satisfied for all $i, j \in I$ with $i \neq j$, we will replace the $\mathbf{A}_i$ (called *components*) by isomorphic copies satisfying such condition.

Definition 4.1. *The language of cnBL$^+$ ($n \geq 1$) is that of cnBL extended with the unary connectives $S^1, \dots S^n$. A Hilbert axiomatization of cnBL$^+$ consists of (below $\phi \leftrightarrow \psi$ stands for $\phi \to \psi \wedge \psi \to \phi$):*

(1) The axioms of BL extended with the n-contraction axiom $\phi^n \to \phi^{n+1}$

(2) $\phi \to S^1(\phi)$

(3) $S^1(\phi)^n \to \phi^n$

(4) $(S^1(\phi) \to \psi) \to ((\psi \to S^1(\phi)) \vee ((\psi \to \phi) \to \phi)$

(5) $S^h(\phi) \leftrightarrow (S^1(\phi))^h$, for $h = 2, \dots, n$

The rules are modus ponens and the congruence rule for S^1:

$$\frac{\phi \leftrightarrow \psi}{S^1(\phi) \leftrightarrow S^1(\psi)}$$

The logic cnBL$^+$ is algebraizable and its equivalent algebraic semantics consists of the variety of cnBL$^+$-algebras, that is, the variety generated by the ordinal sum of MV-chains with cardinality $\leq n+1$, with the connective S^1 interpreted as follows: if $\phi = 1$, then $S^1(\phi) = 1$, otherwise $S^1(\phi)$ is the coatom of the component which ϕ belongs to.

Lemma 4.1.

(a) The axioms of cnBL$^+$ are valid in all cnBL$^+$-algebras.

(b) Any algebra in the signature of cnBL$^+$-algebras which satisfies all the axioms of cnBL$^+$ is a cnBL$^+$ -algebra.

(c) cnBL$^+$ is a conservative extension of cnBL

Proof. Throughout the proof we fix an arbitrary cnBL-chain $\mathbf{A}$ and a valuation σ, and we identify a formula with its truth value in $\mathbf{A}, \sigma$.

(a) The claim is trivial with the exception of axiom (4). If $\psi \leq S^1(\phi)$, then $\psi \to S^1(\phi) = 1$ and (4) holds. If $\psi > S^1(\phi)$, then either $\psi = 1$, and then $(\psi \to \phi) \to \phi = 1$, or ψ belongs to a component above the component of ϕ, and again $(\psi \to \phi) \to \phi = 1$, and (4) holds.

(b) Note that the n-contraction axiom holds in a BL-chain iff the chain is an ordinal sum of MV-algebras with $\leq n+1$ elements. We now prove that the axioms (1)-(4) force $S^1(\phi)$ to be interpreted as the coatom of the component which ϕ belongs to. First of all, axiom (2) implies that $S^1(\phi) = 1$ when $\phi = 1$, and that $\phi \leq S^1(\phi)$. Axioms (2) and (3) imply that ϕ and $S^1(\phi)$ are in the same component. Finally, (3) and (4) imply that if ϕ is not 1, then $S^1(\phi)$ is the coatom of the component ϕ belongs to. Indeed, we have already seen that ϕ and $S^1(\phi)$ are in the same component and $S^1(\phi) = 1$ iff $\phi = 1$. Hence if $\phi < 1$ then $S^1(\phi)$ cannot be greater than the coatom S^1 of the component ϕ belongs to. Now suppose that $S^1(\phi)$ is smaller than S^1. We interpret ψ as S^1. Then $S^1(\phi) \to \psi = 1$, but $\psi \to S^1(\phi) < 1$, as $S^1(\phi) < \psi$, and $(\psi \to \phi) \to \phi = \max\{\psi, \phi\} = \psi < 1$, contradicting axiom (4).

(c) Let ϕ be a formula in the language of cnBL. If ϕ is not a theorem of cnBL, then there is a cnBL-chain $\mathbf{A}$ that invalidates ϕ. For every $x \in \mathbf{A}$ and for $h = 1, ..., n$, define $S^h(x) = 1$ if $x = 1$, and $S^h(x) = c(x)^h$, where $c(x)$ denotes the coatom of the component which x belongs to, otherwise. By part (b), the resulting expansion $\mathbf{A}^+$ of $\mathbf{A}$ is a cnBL$^+$-algebra which invalidates ϕ. □

The semantic theory $\mathbf{T}^*$ *for cnBL*$^+$

$\mathbf{T}^*$ consists of a total preorder $(\ll_=)$ which determines an equivalence relation $\equiv$ on the components and a distribution of the equivalence classes w.r.t. $\equiv$ in the classes $C_{h,k}$ expressing the order of elements in each component.

The language of $\mathbf{T}^*$ contains the binary predicate symbol $\ll_=$, the unary predicates $C_{h,k}$, $1 \leq h \leq k \leq n$ and the unary function symbols S^h, $1 \leq h \leq n$ together with the constants $0, 1$. Moreover, the negations of $\ll_=$ and $C_{h,k}(x)$, that is $\ll$ and $C^*_{h,k}$ respectively, are also in $\mathbf{T}^*$. The intuitive meaning of these symbols is:

$S^1(x)$ denotes the coatom of the component which x belongs to, and $S^h(x)$ denotes $(S^1(x))^h$

$x \ll_= y$ means that either $y = 1$ or x and y are different from 1 and either they are in the same component or x is in a component below the component of y.

$x \ll y$ means that either $x < 1$ and $y = 1$ or x and y are not in the same component and x is in a component below the component of y. (Note that $x \ll y$ is equivalent to the negation of $y \ll_= x$).

$C_{h,k}(x)$ means that $x < 1$ belongs to a component with $k+1$ elements and $x = S^h(x)$.

$C^*_{h,k}(x)$ means that either $x = 1$ or x belongs to a component not having $k+1$ elements or x belongs to a component with $k+1$ elements and $x \neq S^h(x)$.

The designating predicate is $Des(x) := 1 \ll_= x$.

Notation: We will use $x \equiv y$ (meaning that x and y are in the same component and that either they are both equal to 1 or they are both less than 1) as an abbreviation for $(x \ll_= y) \sqcap (y \ll_= x)$, and $x \leq y$ (with the usual meaning) as an abbreviation for $(1 \ll_= y) \sqcup (x \ll y) \sqcup ((x \equiv y) \sqcap (\bigsqcup_{1 \leq i \leq j \leq k \leq n}(C_{j,k}(x) \sqcap C_{i,k}(y)))$. Notice that the order $\leq$ is uniquely determined by $\ll_=$ and by the relations $C_{i,k}$.

$\mathbf{T}^*$ is axiomatized as follows:

$$\forall x \forall y (x \ll y \Leftrightarrow \sim_c (y \ll_= x)) \quad \forall x (C^*_{h,k}(x) \Leftrightarrow \sim_c C_{h,k}(x)),\ 1 \leq h \leq k \leq n$$

Meaning: $\ll$ *is the complement of the inverse of* $\ll_=$ *and* $C^*_{h,k}$ *is the complement of* $C_{h,k}$*,*

$$\begin{array}{lll} \forall x(x \ll_= x) & 0 \ll 1 & \forall x \forall y \forall z ((x \ll_= y \sqcap y \ll_= z) \Rightarrow (x \ll_= z)) \\ \forall x \forall y (x \ll_= y \sqcup y \ll_= x) & & \forall x (0 \ll_= x \sqcap x \ll_= 1) \end{array}$$

Meaning: $\ll_=$ *is a linear preorder with minimum* 0 *and maximum* 1*,*

$$\forall x \forall y ((x \equiv y) \rightarrow (1 \ll_= x \sqcup (\bigvee_{s=1}^{n} \bigvee_{t,r=1}^{s} (C_{t,s}(x) \sqcap C_{r,s}(y))))) \qquad \bigvee_{i=1}^{n} C_{i,i}(0)$$

$\forall x (C_{j,i}(x) \Rightarrow (\sim_c C_{h,k}(x) \sqcap \sim_c (1 \ll_= x)),\ 1 \leq j \leq i \leq n, 1 \leq h \leq k \leq n$ and $j \neq h$ or $i \neq k$

Meaning: Each equivalence class with respect to $\equiv$ *which does not contain* 1 *is partitioned into classes* $C_{1,i}, \ldots, C_{i,i}$ *for some* i *with* $1 \leq i \leq n$*. This* i *is uniquely determined by the equivalence class (hence, if* $i \neq j$ *or* $h \neq k$*, then the classes* $C_{i,h}$ *and* $C_{j,k}$ *are disjoint). Moreover, the equivalence class of* 1 *is disjoint from all classes* $C_{h,k}$*, and* 0 *is the smallest element of its class, with respect to the relation* $\leq$*,*

$$\forall x (S^h(x) \equiv x) \quad \forall x (1 \ll_= x \sqcup \bigvee_{i=h}^{n} C_{h,i}(S^h(x)) \sqcup \bigvee_{k=1}^{h} C_{k,k}(S^h(x)) \text{ for } h = 1, \ldots, n$$

Meaning: $S^h(x)$ *is equivalent to* x*, and either* x *(and hence,* $S^h(x)$*) is equivalent to* 1*, or the component* x *belongs to has at least* $h+1$ *elements (including* 1*) and* $S^h(x)$ *occupies the* $h + 1^{st}$ *position, with respect to* $\leq$*, in that component (including* 1*, which occupies the first position), or the component* x *belongs to has less than* $h+1$ *elements and* $S^h(x)$ *occupies the last position in that component.*

Proposition 4.2. *The set of theorems of* $\mathbf{T}^*$ *which are universal formulas is decidable*

Proof. Follows by the finite model property: If a universal formula in m variables is not valid then it fails in a cnBL chain of at most $(n+1) \cdot (m+1)$ elements. □

An intuitive explanation of how the connectives S^h, with $h = 1, ..., n$, make cnBL$^+$ semi-projective is as follows: In cnBL $x \& y$ is one of x or y when either one of them is 1 or they belong to different components; when x and y are in the same component and are different from 1, $x\&y$ cannot be expressed in terms of unary connectives of cnBL (hence cnBL is not semi-projective). Using S^h we can instead express that x is equal to some power i of the coatom (i.e., $x = S^i(x)$), y is equal to some power j of the coatom (i.e., $y = S^j(x)$), and then $x\&y = S^{min\{k,i+j\}}(x)$, where k is the cardinality of the class they belong to. Similar considerations hold for the implication.

Lemma 4.2. *cnBL$^+$ is semiprojective and* $\mathbf{T}^*$ *is its semantic theory.*

Proof. Given a totally ordered set with minimum and maximum and a partition as indicated above, we can uniquely obtain a cnBL$^+$-algebra stipulating that:

- if x and y are not in the same equivalence class or one of them is equivalent to 1, then $x\&y = \min\{x, y\}$.
- if x, y are in the same class of cardinality, say k, then there are uniquely determined natural numbers $i, j \leq k$ such that $C_{i,k}(x)$ and $C_{j,k}(y)$. Then $x\&y = S^{\min\{k,i+j\}}(x) = S^{\min\{k,i+j\}}(y)$.
- If $x \leq y$, then $x \to y = 1$ and if $y \ll x$ then $x \to y = y$.
- If $y < x$ and x and y are in the same equivalence class and the cardinality of the class is k, then there are uniquely determined $i < j \leq k$ such that $C_{i,k}(x)$ and $C_{j,k}(y)$. Then $x \to y = S^{j-i}(x) = S^{j-i}(y)$.
- $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.
- $S^i(S^j(x)) = S^i(x)$; $S^i(x \vee y) = \max\{S^i(x), S^i(y)\}$; $S^i(x \wedge y) = \min\{S^i(x), S^i(y)\}$; $S^i(x\&y) = S^i(x)$ if $x \ll_= y$ and $S^i(x\&y) = S^i(y)$ otherwise; $S^i(x \to y) = 1$ if $x \leq y$ and $S^i(x \to y) = S^i(y)$ otherwise.

These conditions ensure that each cnBL$^+$ is semi-projective. To make it explicit we state below the truth functions of its connectives. For simplicity, we use (for $1 \leq h \leq m \leq n$) the abbreviations

$$\&^h_m(x, y) \quad \text{for} \quad (\mathrm{x \equiv y}) \sqcap (\bigsqcup_{\mathrm{i+j=h;i,j\geq 1}} (\mathrm{C_{i,m}(x) \sqcap C_{j,m}(y)}))$$

meaning intuitively that x and y are in the same component, the component has $m + 1$ elements and $x\&y = (S^1)^h$, where S^1 is the coatom of that component (recall that if $i+j \leq m$, then $S^i\&S^j = S^{i+j}$) and

$$\to^h_m (x, y) \quad \text{for} \quad (\mathrm{x \equiv y}) \sqcap (\mathrm{y < x}) \sqcap (\bigsqcup_{\mathrm{i-j=h;i\leq m,j\geq 1}} (\mathrm{C_{i,m}(x) \sqcap C_{j,m}(y)}))$$

meaning that x and y are in the same component, the component has $m + 1$ elements and $x \to y = (S^1)^h$, where S^1 is the coatom of that component.

The truth functions for the lattice connectives $\wedge$ and $\vee$ are exactly as in the case of NM. The remaining ones are:

$$x\&y = \begin{cases} x & \text{if} \quad x \ll y \text{ or } 1 \ll_= y \\ y & \text{if} \quad y \ll x \text{ or } 1 \ll_= x \\ S^h(x) & \text{if} \quad \&^h_m(x,y) \end{cases} \qquad x \to y = \begin{cases} 1 & \text{if} \quad x \leq y \\ y & \text{if} \quad y \ll x \\ S^h(x) & \text{if} \quad \to^h_m (x,y) \end{cases}$$

For all $h, k = 1, ..., n$:

$$S^h(x\&y) = \begin{cases} S^h(y) & \text{if} \quad y \ll x \\ S^h(x) & \text{if} \quad x \ll_= y \end{cases} \quad S^h(x \to y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ S^h(y) & \text{if} \quad y < x \end{cases} \quad S^h(S^k(x)) = S^h(x)$$

$$S^h(x \vee y) = \begin{cases} S^h(x) & \text{if } y \leq x \\ S^h(y) & \text{if } x < y \end{cases} \quad S^h(x \wedge y) = \begin{cases} S^h(y) & \text{if } y \leq x \\ S^h(x) & \text{if } x < y \end{cases}$$

To prove that $\mathbf{T}^*$ is the semantic theory of cnBL$^+$ we have to show, with reference to Definition 3.1, that cnBL$^+$ is sound and complete with respect to the class of all models $\mathbf{M}^*$ such that $\mathbf{M}$ is a model of $\mathbf{T}^*$. Now from a model $\mathbf{M}$ of $\mathbf{T}^*$ we obtain a model of cnBL$^+$ using the above definitions of $\&$, $\to$, $\vee$, $\wedge$ and S^h, $h = 1, ..., n$. Conversely, recall that cnBL$^+$ is complete with respect to the class of all cnBL$^+$-chains. Moreover from any cnBL$^+$-chain $\mathbf{A}$, we obtain a model $\mathbf{A}^-$ of $\mathbf{T}^*$ as follows:

(1) The universe of $\mathbf{A}^-$ is the universe of $\mathbf{A}$, and 0 and 1 are as in $\mathbf{A}$.

(2) $x \ll y$ holds if either $x < 1$ and $y = 1$ or $x < y$ and x, y are not in the same component.

(3) $x \ll_= y$ holds iff either $x \ll y$ or $x = y = 1$ or $x, y < 1$ and x, y are in the same component.

(4) $C_{h,k}(x)$ holds iff (a) $x < 1$, (b) x belongs to a component with $k+1$ elements, and (c) $x = S^h(x)$, that is, $x = c(x)^h$, where $c(x)$ is the coatom of the component which x belongs to.

It is readily seen (just using the definition of ordinal sum, the definition of $\mathbf{T}^*$ and the definition of connectives by cases in the semantic theory $\mathbf{T}^*$) that $\mathbf{A}^-$ is a model of $\mathbf{T}^*$ and that the algebraic structure of $(\mathbf{A}^-)^*$ is isomorphic to $\mathbf{A}$. $\square$

The calculus for cnBL$^+$

The logical rules of the sequent calculi of relations for cnBL$^{(+)}$ ($n \geq 1$) can be easily derived from the above truth functions using our methodology. We show as an example the rules for the connective $\&$. We start with the $\ll_=$ and the $\ll$-rules (below $\lhd$ stands for $\ll$ or $\ll_=$ and we omit the context H in all the rules). Let $K(i, j, h, k, \phi, \psi, \gamma)$ denote the sequent of relation $1 \ll_= \phi | 1 \ll_= \psi | \phi \ll \psi | \psi \ll \phi | C^*_{i,k}(\phi) | C^*_{j,k}(\psi)$. Then the rule $(\& : \lhd : l)$ is

$$\frac{\phi \ll_= \psi | \psi \lhd \gamma \quad \psi \ll_= \phi | \phi \lhd \gamma \quad \phi \ll 1 | \psi \lhd \gamma \quad \psi \ll 1 | \phi \lhd \gamma \quad K(i, j, k, \phi, \psi, \gamma) | S^h(\phi) \lhd \gamma}{\phi \& \psi \lhd \gamma}$$

(for all i, j, h, k such that $1 \leq i, j \leq k \leq n$ and $h = \min\{i + j, k\}$), and the rule $(\& : \lhd : r)$ is

$$\frac{\phi \ll_= \psi | \gamma \lhd \psi \quad \psi \ll_= \phi | \gamma \lhd \phi \quad \phi \ll 1 | \gamma \lhd \psi \quad \psi \ll 1 | \gamma \lhd \phi \quad K(i, j, k, \phi, \psi, \gamma) | \gamma \lhd S^h(\phi)}{\gamma \lhd \phi \& \psi}$$

(for all i, j, h, k such that $1 \leq i, j \leq k \leq n$ and $h = \min\{i + j, k\}$). The $(\& : C_{r,s})$ rules are:

$$\frac{\phi \ll_= \psi | C_{r,s}(\psi) \quad \psi \ll_= \phi | C_{r,s}(\phi) \quad \phi \ll 1 | C_{r,s}(\psi) \quad \psi \ll 1 | C_{r,s}(\phi) \quad K(i, j, k, \phi, \psi, \gamma) | C_{r,s}(S^h(\phi))}{C_{r,s}(\phi \& \psi)}$$

(for all i, j, h, k such that $1 \leq i, j \leq k$ and $h = \max\{i + j, k\}$). Finally, the $C^*_{r,s}$ rules are:

$$\frac{\phi \ll 1 | C^*_{r,s}(\psi) \quad \psi \ll 1 | C^*_{r,s}(\phi) \quad \phi \ll_= \psi | C^*_{r,s}(\psi) \quad \psi \ll_= \phi | C^*_{r,s}(\phi) \quad K(i, j, k, \phi, \psi, \gamma) | C^*_{r,s}(S^h(\phi))}{C^*_{r,s}}$$

(for all i, j, h, k such that $1 \leq i, j \leq k$ and $h = \max\{i + j, k\}$)

Remark 4.2. *The above rules, obtained by applying the definition of semi-projective connectives, can be simplified. For instance, it is not hard to check that the rules* $(\& : \lhd : l)$ *and* $(\& : \lhd : r)$ *are equivalent to the more elegant and easier to understand rules*

$$\frac{H \mid \phi \lhd \gamma | \psi \lhd \gamma}{H \mid \phi \& \psi \lhd \gamma}\ (\& : \lhd : l) \qquad\qquad \frac{H \mid \gamma \lhd \phi \quad H \mid \gamma \lhd \psi}{H \mid \gamma \lhd \phi \& \psi}\ (\& : \lhd : r)$$

Proposition 4.3. *Our calculi provide Co-NP decision procedures for the validity problem in each logic cnBL$^+$.*

Proof. We show that the validity in $\mathbf{T}^*$ of quasi-atomic sequents can be checked in polynomial time. The claim then follows by Proposition 3.3. By Lemma 4.2 a quasi-atomic sequent H is valid in $\mathbf{T}^*$ iff its negation H^* is not satisfiable in any cnBL$^+$-chain. Such a negation consists of the conjunction of: (a) all $\phi \ll \psi$ such that $\psi \ll_= \phi$ is a component of the sequent of relations H; (b) all $\phi \ll_= \psi$ such that $\psi \ll \phi$ is in H; (c) all $C_{i,k}(\phi)$ such that $C^*_{i,k}(\phi)$ is in H, and (d) all $C_{i,k}(\phi)$ such that $C^*_{i,k}(\phi)$ is in H. We want to check the satisfiability of H^*, first checking the satisfiability of the binary relations, and then of the unary ones. It is easy to see that the addition of the following relations does not change the satisfiability status of H^*: $0 \ll_{(=)} 1$, $0 \ll_= A$, $A \ll_= 1$, and $S^h(A) \ll_= S^k(A)$, for each atomic formula A in H and $h, k \leq n$ ($S^0(A) = A$). Hence, we assume that all these formulas are in H^*. (These conditions lead to the axioms (Ax0) and (Ax11) in the Appendix).

Set $A \ll^+_= B$ if either $A = B$ or there is a sequence $A_1 = A, \ldots, A_n = B$ such that for $i = 1, \ldots, n-1$ either $A_i \ll A_{i+1}$ or $A_i \ll_= A_{i+1}$ is a conjunct in H^*, and $A \ll^+ B$ iff in addition $A_i \ll A_{i+1}$ is a conjunct in H^* for some i. Moreover, set $A \equiv^+ B$ iff $A \ll^+_= B$ and $B \ll^+_= A$. It is clear that $\ll^+_=$ can be extended to a total preorder if and only if it is consistent, that is, there are no atoms A and B such that $A \ll^+_= B$ and $B \ll^+ A$ (Ax1). If $\ll^+_=$ is not consistent, then H^* is unsatisfiable, and hence H is an axiom. If $\ll^+_=$ is consistent, then let $a_1, ..., a_m$ be the equivalence classes with respect to $\equiv^+$. For each class a_i, let $C(a_i)$ be the set of formulas in H^* of the form $C_{h,k}(S^j(A))$ or $C^*_{h,k}(S^j(A))$ with $A \in a_i$, and set $a_i \ll^+_{(=)} a_j$ iff $a_i \neq a_j$ and $A \ll^+_{(=)} B$ for some $A \in a_i$ and $B \in a_j$. Below we introduce necessary conditions for the satisfiability of each $C(a_i)$ and show that these conditions are also sufficient. We distinguish two cases:

- $C(a_i)$ contains a positive formula, say $C_{h,k}(\phi)$.

It is easy to see that H^* is unsatisfiable if one of the conditions (1)-(5) below is violated:
(1) $C(a_i)$ cannot contain also its negation $C^*_{h,k}(\phi)$ (Ax2) and it is not the case that $\phi = S^i(B)$, where $h \neq i$ and either $i < k$ or $h < k$ (or both) (Ax3).
(2) a_i is not the equivalence class of 1 (Ax4).
(3) $C(a_i)$ cannot contain any formula of the form $C_{h',k'}(\phi)$ with $h' \neq h$ or $k' \neq k$ (Ax5).
(4) For all $B \in a_i$, $C(a_i)$ cannot contain any formula of the form $C_{h,k'}(B)$ with $k' \neq k$ (because any valuation satisfying H^* maps ϕ and B into the same component, which has cardinality $k+1$) (Ax6), or of the form $C^*_{i,k}(S^i(B))$ (Ax9), and $C(a_i)$ cannot contain all formulas of the form $C^*_{j,k}(B)$, with $j = 1, ..., k$ (Ax7).
(5) If $0 \in a_i$, then $C_{h,k}(\phi)$ is not $C_{h,k}(0)$ with $h < k$, and $C(a_i)$ cannot contain all $C^*_{h,h}(0)$, for $h = 1, \ldots, n$ (Ax8);

- $C(a_i)$ only contains negative formulas. Then:

(6) If a_i is not the equivalence class of 1 then for all $A \in a_i$, $C(a_i)$ cannot contain for a $h \leq n$ all $C^*_{i,i}(S^h(A))$ and $C^*_{h,k}(S^h(A))$ with $i \leq h$ and $h+1 \leq k \leq n$ (Ax9).
(7) One of the following cases has to occur (Ax10):
(7a) It is consistent to identify a_i with the equivalence class of 1. This formally means that: (i) for all $A \in a_i$ and for all B, we do not have $A \ll^+ B$, (ii) for all $k = 1, \ldots, m$ if $a_i \ll^+_= a_k$ then $C(a_k)$ cannot contain any positive formula $C_{h,l}(\phi)$. In this case we may extend $\equiv^+$ to a larger equivalence whose equivalence classes are $a_i, \ldots, a_{i-1}$ and the class containing all formulas in any class a_j such that $a_i \ll_= a_j$ are equivalent to 1. In this way, $C(a_i)$ is satisfied, as well as all $C(a_j)$ with $a_i \ll^+_= a_j$ (it suffices to interpret into 1 every variable in a_i and in a_j).
(7b) If (7a) is not satisfied, then there must be a fixed $k_i \leq n$ such that: (a) for all $A \in a_i$ there is an $i(A) \leq k_i$ such that $C^*_{i(A),k_i}(A) \notin C(a_i)$; (b) if in addition $A = S^h(B)$ for some B and for some $h \leq k_i$, then $C^*_{h,k_i}(A) \notin C(a_i)$, and (c) if in addition $A = S^h(B)$ for some B and for some $h > k_i$, then $C^*_{k_i,k_i}(A) \notin C(a_i)$ (Ax9).

Now suppose that $\ll^+_=$ is consistent and that none of conditions (1)-(7) is violated. Then, a model of H^* can be constructed as follows:

(a) Identify some equivalence classes according to (7a).

(b) Let $a'_1, \ldots, a'_h$ and $\ll'_=$ be the equivalence classes and their partial order after the identifications in (a). Extend $\ll_=$ to a total order so that no more equivalence classes are identified. Let $b_1 \ll' b_2 \ll' \cdots \ll' b_h$ be the equivalence classes in increasing order, where b_h is the equivalence class of 1 and b_1 is the equivalence class of 0. Take the ordinal sum $\mathbf{L} = \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_{h-1}$, and the valuation v defined as follows:

(b1) If $C(b_i)$ contains a positive formula $C_{h,k_i}(A)$, then let $\mathbf{L}_i$ be the MV-chain with k_i+1 elements, let c_i be its coatom, and let for $B \in b_i$ (B atomic) $v(B) = c_i^{h'}$, where h' is such that $C^*_{h',k_i}(B) \notin C(a_i)$ (this h' exists by (4)). Moreover if $\phi = S^h(B) \in b_i$, set $v(\phi) = c_i^h$. The defined interpretation is a model for $C(b_i)$ by (1)-(5).

(b2) If $1 \notin b_i$ and $C(b_i)$ does not contain any positive formula, then let $\mathbf{L}_i$ be the MV-chain with k_i elements where k_i is such that for all $B \in b_i$ there is an $h(B)$ such that

$C^*_{h(B),k_i}(B) \notin C(b_i)$ (k_i exists by (7b)). Let c_i be the coatom of $\mathbf{L}_i$, and set for all $B \in b_i$, $v(B) = c_i^{h(B)}$.

It is readily seen that H^* is satisfied in this way. □

The proof of the proposition above also leads to the syntactic description of the axioms of the sequent calculus of relations for cnBL$^+$ listed in the Appendix.

Remark 4.3. *By Lemma 4.1.(c) the calculus for cnBL$^+$ is also a calculus for cnBL.*

5. First-order logics?

Sequent calculi of relations work for propositional logics. A natural question is whether they can be extended to properly deal with quantifiers. The answer is negative, as shown by the example below. Consider a semantic theory based on the two binary relations "$\leq$" and "$<$". The natural rules for the universal quantifier are ($\lhd \in \{\leq, <\}$)

$$\frac{H \mid \phi(t) \lhd \psi}{H \mid \forall x \phi(x) \lhd \psi}\ (\forall : \lhd : l) \qquad \frac{H \mid \psi \lhd \phi(e)}{H \mid \psi \lhd \forall x \phi(x)}\ (\forall : \lhd : r)$$

where e is an eigenvariable, i.e., it does not appear in H and ψ. However the $(\forall :<: r)$ rule is not sound for most of many-valued logics with infinite truth values, as the value of a formula $\forall x \phi(x)$ under an interpretation σ is the infimum of the values of $\phi(a)$, for all elements a of the universe, and therefore we might have $\sigma(\psi) < \sigma(\phi(a))$ for all a while $\sigma(\psi) = \sigma(\forall x \phi(x))$. (Dually for the rule $(\exists :<: l)$ and supremum).

Notice that all quantifier rules are sound for so called *witnessed many-valued logics* [18], i.e. admitting only models in which the truth value of each universally quantified formula is the minimum of truth values of its instances (and dually for existential quantification and maximum). Our methodology could therefore be used to introduce sequent calculi of relations, for instance, for *witnessed* semi-projective logics with semantic theories based on the relations "$\leq$" and "$<$". These include witnessed Gödel logic (with or without the involutive negation) and witnessed NM. To prove the completeness of the resulting calculi with respect to the formalized logics we need a cut rule to simulate modus ponens, e.g. the rule below (cf. [4])

$$\frac{H \mid \phi \leq \psi \quad H' \mid \psi < \phi}{H \mid H'}\ (cut)$$

However (cut) is in general not admissible in first-order sequent calculi of relations. For instance, it is easy to see that the formula $\exists x(A(x) \to \forall y A(y))$ is not provable in the calculus for NM described in Section 4.1 augmented with the above quantifier rules (and the natural rules for $\sim\ \forall$ and $\sim\ \exists$). This formula, which is valid in all witnessed NM interpretations, has instead a derivation in the calculus extended with (cut). A proof of $1 \leq \exists x(A(x) \to \forall y A(y))$ is indeed constructed by cutting the (provable) sequents of relations $1 \leq \forall y A(y) \to \forall y A(y)$ and $\forall y A(y) \to \forall y A(y) < 1 \mid 1 \leq \exists x(A(x) \to \forall y A(y))$. A derivation of the latter sequent is the following ($\ldots$ abbreviates a component left unchanged in a rule application, and $(*)$ stands for an application of the rule $(\to:\leq: r)$ followed[6] by (EW)):

[6]Looking at the proof bottom up.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{1 \leq \sim A(a) \mid\ \sim A(a) < 1}{1 \leq A(a) \to \forall y A(y) \mid \ldots}\,{\scriptstyle (*)}}{\ldots \mid\ \sim A(a) < 1}\,{\scriptstyle (\exists:\leq:\mathrm{r})}}{\ldots \mid\ \sim \forall y A(y) < 1}\,{\scriptstyle (\sim\forall:<:\mathrm{l})}
\quad
\dfrac{\dfrac{1 \leq \forall y A(y) \mid \forall y A(y) < 1}{1 \leq A(a) \to \forall y A(y) \mid \ldots}\,{\scriptstyle (*)}}{\ldots \mid \forall y A(y) < 1}\,{\scriptstyle (\exists:\leq:\mathrm{r})}
\quad
\dfrac{\dfrac{\dfrac{A(a) \leq \forall y A(y) \mid \forall y A(y) < A(a)}{1 \leq A(a) \to \forall y A(y) \mid \ldots}\,{\scriptstyle (*)}}{\ldots \mid \forall y A(y) < A(a)}\,{\scriptstyle (\exists:\leq:\mathrm{r})}}{\ldots \mid \forall y A(y) < \forall y A(y)}\,{\scriptstyle (\forall:<:\mathrm{r})}
}{1 \leq \exists x (A(x) \to \forall y A(y)) \mid \forall y A(y) \to \forall y A(y) < 1}\,{\scriptstyle (\to:<:\mathrm{l})}
$$

Hence the addition of the natural quantifier rules to propositional sequent calculi of relations leads to calculi that are *not analytic*. The reason being that cut-elimination proofs in sequent calculi of relations strongly rely on the invertibility of rules (see e.g. the proof for the calculus for Gödel logic in [4]). This does not work anymore in presence of quantifier rules.

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Appendix

Let $\phi \lhd_1 \psi_1|\ldots|\psi_{n-1} \lhd_n \psi$ where for $i = 1, ..., n$, $\lhd_i$ is either $\ll$ or $\ll_=$. Below we write $\phi \ll^\star \psi$ if for at least one i, $\lhd_i$ is $\ll$; we write $\phi \ll^\star\!= \psi$ otherwise. The axioms of the sequent calculus of relations for cnBL$^+$ are the following.

Axioms are all quasi-atomic sequents containing (below $\|_{i=1}^n S_i$ abbreviates $S_1|\ldots|S_n$):

(Ax0) $0 \ll^\star_= \phi$, $\phi \ll^\star_= 1$, $0 \ll^\star 1$, $0 \ll^\star\!= 1$,

(Ax1) a cycle $\phi \ll^\star_= \phi$

(Ax2) $C_{h,k}(\phi)|C^*_{h,k}(\phi)$, for some $1 \leq h \leq k \leq n$

(Ax3) any component $C^*_{h,k}(S^p(\phi))$ where $h \neq p$ and either $h \neq k$ or $h = k < p$

(Ax4) $\phi \ll^\star 1|C^*_{h,k}(\phi)$, and $C^*_{h,k}(1)$, for $1 \leq h \leq k \leq n$

(Ax5) $C^*_{h,k}(\phi)|C^*_{h',k'}(\phi)$, when either $h \neq h'$ or $k' \neq k$

(Ax6) $\phi \ll^\star \psi|\psi \ll^\star \phi|C^*_{h,k}(\phi)|C^*_{h,k'}(\psi)$, with $h \leq k \leq n$, $h' \leq k' \leq n$ and $k \neq k'$

(Ax7) $\phi \ll^\star \psi|\psi \ll^\star \phi|C^*_{h,k}(\phi)\|_{i=1}^k C_{i,k}(\psi)$

(Ax8) $C^*_{h,k}(0)$ for any $h < k$, and $\|_{i=1}^n C_{i,i}(0)$

(Ax9) $\phi \ll^\star \psi|\psi \ll^\star \phi|C^*_{h,k}(\phi)|C_{i,k}(S^i(\psi))$ and $1 \ll^\star_= \phi\|_{i=1}^h C_{i,i}(S^h(\phi))\|_{i=h+1}^n C_{h,i}(S^h(\phi))$

Let $k \leq n$, $\mathrm{P} = \{U_1, \ldots, U_k\}$ be a partition of $\{1, \ldots, n\}$ into k nonempty pairwise disjoint sets, and let $\Sigma(U_i) = \|_{j' \in U_i}^{j \leq j'} C_{j,j'}(\phi_i)$. For every P

(Ax10) $\|_{i,j=1}^{k,(i \neq j)} \phi_i \ll^\star \phi_j \|_{i=1}^k 1 \ll^\star_= \phi_i|\Sigma(U_1)|\ldots|\Sigma(U_k)$ is an axiom (in the particular case $k = 1$ this is $1 \ll^\star_= \phi \mid C_{1,1}(\phi) \mid C_{1,2}(\phi) \mid \ldots \mid C_{n,n}(\phi)$). Moreover, for $1 \leq i \leq j \leq n$, $\|_{i,j=1}^{k,(i \neq j)} \phi_i \ll^\star \phi_j \|_{i=1}^k \psi \ll^\star \phi_i|C^*_{i,j}(\psi)\|_{r=1}^k \Sigma(U_i)$ is an axiom

(Ax11) all quasi-atomic sequents obtained from any of the previous axioms by replacing ϕ in any $\ll$ or $\ll_=$ component by $S^h(\phi)$, or $S^h(\phi)$ by either ϕ or $S^k(\phi)$, and by replacing in any component 1 by $S^h(1)$, or $S^h(1)$ by either 1 or $S^k(1)$. Moreover (Ax4), (Ax5) (in the case $k \neq k'$), (Ax6), (Ax7) and (Ax9) in which (some) ϕ are replaced by $S^h(\phi)$.