Algebraic proof theory: hypersequents and hypercompletions ☆

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Abstract

We continue our program of establishing connections between proof-theoretic and order-algebraic properties in the setting of substructural logics and residuated lattices. Extending our previous work that connects a strong form of cut-admissibility in sequent calculi with closure under MacNeille completions of corresponding varieties, we now consider hypersequent calculi and more general completions; these capture logics/varieties that were not covered by the previous approach and that are characterized by Hilbert axioms (algebraic equations) residing in the level $\mathcal{P}_3$ of the substructural hierarchy. We provide algebraic foundations for substructural hypersequent calculi and an algorithm to transform $\mathcal{P}_3$ axioms/equations into equivalent structural hypersequent rules. Using residuated hyperframes we link strong analyticity in the resulting calculi with a new algebraic completion, which we call hyper-MacNeille.

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1. Introduction

The combination of syntactic and semantic methods in logic provides a double-edged sword for solving problems. Introduced in [14], the term algebraic proof theory refers to a research line aiming at connecting proof theory and universal order-algebra in a novel way that goes beyond merely combining results of the two fields by rather integrating their techniques. The techniques investigated in [14] are cut-admissibility (on the proof theoretic side) and order

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theoretic completions (on the algebraic side), by means of which the existence of *strongly analytic* sequent calculi (derivations from atomic assumptions contain only subformulas of the formulas to be proved) for substructural logics are linked to the closure of the corresponding varieties of residuated lattices under MacNeille completions. Existing work in the spirit of algebraic proof theory usually refers to sequent calculi; e.g., in the context of modal logics [30] characterizes analyticity via non-deterministic matrix semantics (see also [29] and references therein), while [8, 9] investigate the bounded proof property using one-step algebras.

In this paper we expand the realm of algebraic proof theory to *hypersequent calculi* – a natural generalization of sequent calculi – and explore their connections to more general order theoretic completions in the setting of substructural logics. The latter are logics weaker than classical logic that lack some of the axioms corresponding to the structural rules implicit in Gentzen's systems: exchange, weakening and contraction. Substructural logics encompass among many others, intuitionistic logic, as well as linear, many-valued and relevance logics. They are axiomatic extensions of *full Lambek calculus* **FL** and their algebraic semantics form varieties of pointed residuated lattices, also called *FL-algebras*.

Many, but not all, substructural logics possess analytic sequent calculi. Analyticity usually follows from the redundancy of the special rule *cut* (which corresponds to modus ponens in Hilbert systems, and to transitivity in algebra) and is a key in establishing many important properties of the formalized logics; these include decidability, the Herbrand theorem, interpolation, as well as various algebraic properties, see, e.g., the monograph [21].

In our previous studies [12, 14] we addressed the question:

- Which Hilbert axioms can be transformed into structural sequent rules that preserve strong analyticity when added to **FL**?

and showed that for a large class of axioms (i.e. those in the class $\mathcal{N}_2$ of the *substructural hierarchy*, a syntactic classification of axioms/equations introduced in [12, 13, 14]) this question can be reformulated as:

- Which algebraic equations over FL-algebras are preserved by MacNeille completions?

In [14] we introduced an algorithm for extracting structural sequent rules from axioms/equations belonging to the class $\mathcal{N}_2$ and showed that the calculus obtained by adding these rules to **FL** is strongly analytic if and only if the corresponding variety is preserved by MacNeille completions. These results were obtained by using *residuated frames* [20], a relational semantics resembling Kripke frames but applicable also to non-distributive settings. The results in [12, 14] also reveal that the expressive power of structural sequent rules is limited to $\mathcal{N}_2$ axioms/equations and that higher levels of the hierarchy call for calculi based on formalisms more expressive than sequents.

Various extensions of sequent calculi have been introduced during the last three decades in order to present analytic calculi for logics that seem to resist an analytic sequent calculus formalization. In an ideal classification of the various proof theoretic frameworks according to their expressive power, hypersequent calculi can be seen as the "next level" after the sequent calculus. A hypersequent consists of a multiset of Gentzen sequents separated by a new structural connective "|" intuitively understood in a disjunctive way. Many substructural logics that cannot be captured via analytic sequent calculi possess instead analytic hypersequent calculi; this is for instance the case of various fuzzy logics [32] due to the presence of prelinearity $(\alpha \to \beta) \vee (\beta \to \alpha)$ which is beyond the class $\mathcal{N}_2$ and is naturally captured by a hypersequent structural rule (Avron's communication rule [3]).

While the existing setting of algebraic proof theory deals with the class $\mathcal{N}_2$ of the substructural hierarchy, in [12, 13] we investigated the next level (the class $\mathcal{P}_3$) separately in the proof theoretic and in the algebraic setting. Indeed, [12] contains an algorithm for extracting structural rules in the hypersequent calculus from axioms/equations corresponding to a subclass of $\mathcal{P}_3$, all in a commutative setting, while [13] investigates closure under MacNeille completions applied to subdirectly irreducible algebras for varieties defined by axioms in a subclass of $\mathcal{P}_3$ equations.

This paper provides a comprehensive account of the connections between proof theory and algebra for the hypersequent calculus, the $\mathcal{P}_3$ level of the substructural hierarchy, and a new type of completion. The key tools for our investigation are *residuated hyperframes*, a new relational semantics which generalizes residuated frames. Residuated hyperframes support both a proof of strong analyticity in hypersequent calculi and a proof of the preservation of equations under a new type of algebraic completion, which we call *hyper-MacNeille*. Though more involved than the subdirect MacNeille completion used in [13], the hyper-MacNeille completion preserves more existing infinitary joins and meets; it is not always regular, namely it does not preserve all existing joins and meets, but it is regular for certain well-behaved algebras.

The paper is organized as follows. Section 2 contains the basic notions and a summary of the results in [14] (including the substructural hierarchy). Section 3 presents the hypersequent calculus for full Lambek calculus (and extensions), its algebraic foundations and the notion of equivalence between structural hypersequent rules/clauses and axioms/equations. Note that although hypersequent calculi have been successfully used to capture specific substructural logics, a precise definition of the meaning of the symbol "|" and of the equivalence between structural rules and axioms in the noncommutative case was still lacking. On the way of providing algebraic foundations of hypersequents, we describe the semantic interpretation of "|" which is not the lattice join unless the algebraic models are subdirectly irreducible or contain a version of prelinearity, see e.g. [32]. For general algebras, "|" actually corresponds to ∇, a form of disjunction also considered in the setting of abstract algebraic logic [17]; for substructural logics/residuated lattices ∇ consists of a combination of the usual disjunction and iterated conjugates [10], which account for the lack of commutativity, as in group theory.

Section 4 presents an algorithm for extracting equivalent structural rules/clauses out of $\mathcal{P}_3^\flat$ axioms/equations. This class, defined by refining the class $\mathcal{P}_3$ of the substructural hierarchy, includes many interesting axioms/equations such as prelinearity, weak excluded middle and weak nilpotent minimum. Under the additional syntactic condition of *acyclicity* or in presence of integrality (weakening), structural rules/clauses can be further transformed into well-behaved ones, called *analytic* structural rules/clauses.

Residuated hyperframes are introduced in Section 5, and used in Section 6 to show the following results.

1. If $\mathcal{R}$ is a set of analytic structural rules, then $\mathbf{HFL}(\mathcal{R})$, the hypersequent version of $\mathbf{FL}$ extended with $\mathcal{R}$, is strongly analytic.
2. If a set $\mathcal{E}$ of equations is equivalent to a set of analytic structural clauses, then the variety $\mathsf{FL}(\mathcal{E})$ of FL-algebras defined by $\mathcal{E}$ admits hyper-MacNeille completions.

Section 7 proves the converse direction of 1. and a partial converse direction of 2. (restricted to the commutative case), thus establishing a strong connection between acyclicity (a syntactic condition), strong analyticity (a proof theoretic property) and closure under hyper-MacNeille completions (an algebraic property). The main results are summarized in Theorem 7.3.

Section 8 concludes the paper by discussing the expressive power of structural hypersequent rules and the structure of the substructural hierarchy.

2. Preliminaries

We first recall some basic definitions in substructural logics (Section 2.1) and their algebraic semantics (Sections 2.2, 2.3). We then introduce the substructural hierarchy, a central concept in our previous works [12, 13, 14] and summarize the main results of [14] (Section 2.4).

2.1. Substructural logics and strong analyticity

Below we introduce *full Lambek calculus* $\mathbf{FL}$, the base logic that we consider, using a sequent calculus formalism.

The *formulas* of $\mathbf{FL}$ are built from propositional variables $p, q, r, \ldots$ and constants 1 (unit) and 0 (dual unit/negation constant) by using the binary connectives $\wedge$ (conjunction/meet), $\vee$ (disjunction/join), $\cdot$ (fusion/product/multiplication), $\backslash$ (left implication/division) and $/$ (right implication/division). We denote by Fm the set of all formulas. It is convenient to consider the following algebraic structure

$$\mathbf{Fm} := (Fm, \wedge, \vee, \cdot, \backslash, /, 1, 0),$$

called the *term algebra* of $\mathbf{FL}$. We will use $\neg\alpha$ and $\alpha \leftrightarrow \beta$ as abbreviations for $\alpha\backslash 0$ and $(\alpha\backslash\beta) \wedge (\beta\backslash\alpha)$. We also write $\alpha\beta$ for $\alpha \cdot \beta$. Since the constant 1 is the unit of the fusion operation, we naturally adopt the convention that $\alpha_1 \cdots \alpha_n$ denotes 1 when $n = 0$. Finally α^n denotes $\alpha \cdot \cdots \cdot \alpha$ (n times).

$$\frac{\Sigma \Rightarrow \alpha \quad \Gamma, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (cut) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (id) \qquad \frac{}{\Rightarrow 1}\ (1r)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \Pi}\ (\cdot l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, 1, \Delta \Rightarrow \Pi}\ (1l)$$

$$\frac{\Sigma \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \alpha\backslash\beta, \Delta \Rightarrow \Pi}\ (\backslash l) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\backslash r) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0l)$$

$$\frac{\Sigma \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta/\alpha, \Sigma, \Delta \Rightarrow \Pi}\ (/l) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (/r) \qquad \frac{}{0 \Rightarrow}\ (0r)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \Pi \quad \Gamma, \beta, \Delta \Rightarrow \Pi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee r_1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee r_2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \Pi}\ (\wedge l_1) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \Pi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \Pi}\ (\wedge l_2) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r)$$

Figure 1: Inference rules of **FL**

A *sequent* of **FL** is an expression of the form $\Gamma \Rightarrow \Pi$, where Γ stands for a (possibly empty) sequence of formulas and Π for a *stoup*, i.e., it is either a formula or empty. The inference rules in Figure 1 define the base logic **FL** (α, β are metavariables for formulas, Γ, Δ, Σ for formula sequences and Π for stoups). For a set of formulas $\mathcal{F} \cup \{\alpha\}$, we write $\mathcal{F} \vdash_{\mathbf{FL}} \alpha$ if the sequent $\Rightarrow \alpha$ is derivable from the sequents $\{\Rightarrow \beta : \beta \in \mathcal{F}\}$ by using the rules in Figure 1. We also write $\mathcal{F}_1 \vdash_{\mathbf{FL}} \mathcal{F}_2$ if $\mathcal{F}_1 \vdash_{\mathbf{FL}} \alpha$ holds for every $\alpha \in \mathcal{F}_2$.

A *substructural logic* **L** is a set of formulas closed under substitution and deduction with respect to $\vdash_{\mathbf{FL}}$ (i.e., $\mathbf{L} \vdash_{\mathbf{FL}} \alpha$ implies $\alpha \in \mathbf{L}$). We write $\Phi \vdash_{\mathbf{L}} \alpha$ if $\Phi \cup \mathbf{L} \vdash_{\mathbf{FL}} \alpha$ holds.

Given a set $\mathcal{E}$ of axioms, we write $\mathbf{L}(\mathcal{E})$ for the substructural logic axiomatized by $\mathcal{E}$, see, e.g., [22, 21]. Typical axioms added to **FL** are

$$(e)\ \alpha \cdot \beta\backslash\beta \cdot \alpha, \qquad (c)\ \alpha\backslash\alpha \cdot \alpha, \qquad (i)\ \alpha\backslash 1, \qquad (o)\ 0\backslash\alpha.$$

Axioms (i) and (o) are jointly denoted by (w). We use the standard notation for substructural logics defined by these axioms: $\mathbf{FL_e}$ for **FL** with (e), $\mathbf{FL_{ew}}$ for $\mathbf{FL_e}$ with (w) and **Int** for $\mathbf{FL_{ew}}$ with (c) (which is *intuitionistic logic*).

It is often the case that although not equal nor equivalent, $\alpha\backslash\beta$ and β/α are interchangeable in certain contexts. For instance, in every substructural logic **L**, we have $\vdash_{\mathbf{L}} \alpha\backslash\beta$ iff $\vdash_{\mathbf{L}} \beta/\alpha$; also, if **L** includes (e) (i.e. **L** is commutative) the two formulas are interchangeable in any context. In such cases, we often write $\alpha \to \beta$ rather than $\alpha\backslash\beta$ or β/α.

Every substructural logic can be obtained by adding suitable axioms to **FL**. However, the presence of additional axioms destroys a fundamental property of the sequent calculus **FL**, namely cut-admissibility (or cut-elimination in its algorithmic version). This property, which establishes the redundancy of the rule (*cut*), usually ensures that proofs only consist of formulas already contained in the statement to be proved (*subformula property*). For this reason it is preferable to add to **FL** rules that preserve cut-admissibility, rather than axioms. When the added rules do not mention any connective or constant (i.e., they are *structural rules*) the resulting system is modular and cut-admissibility can be verified by investigating only the new rules. For instance, the axioms (e), (c), (i) and (o) above can be replaced by the following structural rules, which preserve cut-admissibility when added to **FL** (see [14] for a general definition of structural rule):

$$\frac{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}\ (e) \qquad \frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (c) \qquad \frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (i) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (o)$$

Given a set $\mathcal{R}$ of structural rules, we write $\mathbf{FL}(\mathcal{R})$ for the sequent calculus obtained by adding the rules in $\mathcal{R}$ to **FL**. A fundamental question in proof theory is which axioms can be transformed into structural rules that preserve cut-admissibility. Actually, in a general setting, a preservation of a condition stronger than plain cut-admissibility is of interest. This is expressed by the following definition.

Definition 2.1. A set $\mathcal{S}$ of sequents is said to be *elementary* if each sequent in $\mathcal{S}$ consists of atomic formulas and $\mathcal{S}$ is closed under cuts: if S contains $\Sigma \Rightarrow p$ and $\Gamma, p, \Delta \Rightarrow \Pi$, it also contains $\Gamma, \Sigma, \Delta \Rightarrow \Pi$.

We say that $\mathbf{FL}(\mathcal{R})$ is *strongly analytic* if for any elementary set $\mathcal{S}$ of sequents and sequent Θ, if Θ is derivable from $\mathcal{S}$ in $\mathbf{FL}(\mathcal{R})$ then Θ has a cut-free derivation from $\mathcal{S}$ which has the *subformula property.*

Thus strong analyticity combines a stronger form of cut-admissibility in presence of (atomic) premises with the subformula property. The latter is mentioned explicitly, because in a very general setting one could define peculiar structural rules which permit cut-admissibility but do not preserve the subformula property. This is for instance the case of the following rule:

$$\frac{\Gamma, \alpha \Rightarrow \alpha}{\Gamma \Rightarrow}$$

Remark 2.2. We often include the constants $\top$ (true) and $\bot$ (false) in **FL**; the resulting logic is denoted by $\mathbf{FL}_\bot$. The results in our paper hold for both **FL** and $\mathbf{FL}_\bot$.

2.2. Algebraic semantics

The logic **FL** is algebraizable and its algebraic semantics is the variety of pointed residuated lattices, also known as FL-algebras.

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$,

$$a \cdot b \leq c \iff b \leq a \backslash c \iff a \leq c/b.$$

We refer to the last property as *residuation*. An *FL-algebra* is a residuated lattice $\mathbf{A}$ with a distinguished element $0 \in A$.

An equation (identity) is an expression of the form $t = u$, where t and u are terms/formulas; note that this includes expressions of the form $t \leq u$, which is a shorthand for $t = t \wedge u$. We use symbols and, or, $\Longrightarrow$ to denote the conjunction, disjunction and implication of first-order logic, respectively. By a *clause*, we mean an expression of the form $(0 \leq m < n)$:

$$t_1 \leq u_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \ \mathsf{or} \ \ldots \ \mathsf{or} \ t_n \leq u_n, \qquad (q)$$

where all variables are assumed to be universally quantified. The equations $t_1 \leq u_1, \ldots, t_m \leq u_m$ are called the *premises* and $t_{m+1} \leq u_{m+1}, \ldots, t_n \leq u_n$ the *conclusions*. The clause (q) is *disjunctive* if there are no premises (i.e., $m = 0$). It is a *quasiequation* if there is only one conclusion (i.e. $n = m + 1$).

Let $\mathbf{A}$ be an FL-algebra and f a *valuation* into $\mathbf{A}$, namely a homomorphism $f : \mathbf{Fm} \longrightarrow \mathbf{A}$. Then we say that f *satisfies* (q) and write $\mathbf{A}, f \models (q)$ if $f(t_i) \leq f(u_i)$ for all $1 \leq i \leq m$ implies $f(t_j) \leq f(u_j)$ for some $m + 1 \leq j \leq n$. We say that $\mathbf{A}$ *satisfies* (q) and write $\mathbf{A} \models (q)$ if every valuation into $\mathbf{A}$ satisfies (q). More generally, let K be a class of FL-algebras and $\mathcal{C} \cup \{(q)\}$ a set of clauses. We write $\mathcal{C} \models_{\mathsf{K}} (q)$ if the following holds: for every algebra $\mathbf{A} \in \mathsf{K}$ and every valuation f into $\mathbf{A}$, if f satisfies all clauses in $\mathcal{C}$, then f also satisfies (q).

We often identify a formula α with equation $1 \leq \alpha$. We say that an FL-algebra $\mathbf{A}$ *satisfies* a formula α and write $\mathbf{A} \models \alpha$ if $\mathbf{A} \models 1 \leq \alpha$. More generally, given a class K of FL-algebras and a set $\mathcal{F} \cup \{\alpha\}$ of formulas, we write $\mathcal{F} \models_{\mathsf{K}} \alpha$ whenever $\{1 \leq \beta : \beta \in \mathcal{F}\} \models_{\mathsf{K}} 1 \leq \alpha$.

We denote by FL the variety of FL-algebras; given a set $\mathcal{E}$ of axioms (equations), we denote by $\mathsf{FL}(\mathcal{E})$ the variety of FL-algebras that satisfy all axioms (equations) in $\mathcal{E}$. Below is a standard algebraization result, see, e.g. [21].

Theorem 2.3. *If $\mathbf{L}$ is a substructural logic, $\mathsf{L} := \mathsf{FL}(\mathbf{L})$ and $\mathcal{F} \cup \{\alpha\}$ is a set of formulas, then*

$$\mathcal{F} \vdash_{\mathbf{L}} \alpha \iff \mathcal{F} \models_{\mathsf{L}} \alpha.$$

Moreover, the map $\mathbf{L} \mapsto \mathsf{L}$ gives a dual order-isomorphism between the lattice of all substructural logics and that of all varieties of FL-algebras.

We conclude this subsection by introducing algebraic counterparts to structural rules, see [14].

Definition 2.4. An equation $t \leq u$ is said to be *structural* if t is a product of variables and u is either 0 or a variable. A clause is *structural* if it is composed of structural equations.

It is clear that every structural rule in the sequent calculus naturally corresponds to a structural quasiequation. For instance, the structural rule

$$\frac{\Gamma \Rightarrow \Pi \quad \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \ (min)$$

corresponds to

$$x \leq z \text{ and } y \leq z \Longrightarrow xy \leq z. \tag{min}$$

The correspondence will be later extended to structural rules in the hypersequent calculus and structural clauses.

2.3. Completions

A *completion* of an FL-algebra $\mathbf{A}$ is a pair $(\mathbf{B}, e)$ where $\mathbf{B}$ is a complete FL-algebra and $e : \mathbf{A} \longrightarrow \mathbf{B}$ is an embedding. A completion $(\mathbf{B}, e)$ is *regular* if e preserves all existing joins and meets in $\mathbf{A}$.

A *homomorphism* h between two completions $(\mathbf{B}_1, e_1)$ and $(\mathbf{B}_2, e_2)$ of $\mathbf{A}$ is a homomorphism $h : \mathbf{B}_1 \longrightarrow \mathbf{B}_2$ for which the following diagram commutes:

$$\begin{tikzcd}
\mathbf{B}_1 \arrow[rr, "h"] & & \mathbf{B}_2 \\
& \mathbf{A} \arrow[ul, "e_1"'] \arrow[ur, "e_2"] &
\end{tikzcd}$$

Two completions $(\mathbf{B}_1, e_1)$ and $(\mathbf{B}_2, e_2)$ are *isomorphic* if there is a bijective homomorphism between them. It is clear that if $(\mathbf{B}, e)$ is a completion of $\mathbf{A}$, there is an isomorphic one $(\mathbf{B}', e')$ such that $\mathbf{B}'$ is an *extension* of $\mathbf{A}$ (i.e., $\mathbf{A}$ is a subalgebra of $\mathbf{B}'$) and e' is the inclusion map. Hence we will often think of completions just as complete extensions.

Given a class K of FL-algebras, we say that K *admits completions* if every $\mathbf{A} \in \mathsf{K}$ has a completion (actually a complete extension, if K is closed under isomorphisms) $\mathbf{B}$ in K.

A completion $\mathbf{B}$ of $\mathbf{A}$ is said to be

- *join-dense*, if every element $x \in B$ is a join of elements from A:

$$x = \bigvee C, \qquad \text{for some } C \subseteq A.$$

- *meet-dense*, if every element $x \in B$ is a meet of elements from A:

$$x = \bigwedge C, \qquad \text{for some } C \subseteq A.$$

Let $\mathbf{A}$ be an FL-algebra. It is well known that its lattice reduct $(A, \wedge, \vee)$ admits a join-dense and meet-dense completion $(\overline{A}, \wedge, \vee)$ that is unique up to isomorphism, called the *MacNeille completion* [4, 36]. We may extend the concept to FL-algebras. While there are several choices when extending the non-lattice operations to $\overline{A}$, it is the following one that works (see [37] for a rationale).

Theorem 2.5. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ *be an FL-algebra and* $(\overline{A}, \wedge, \vee)$ *be the MacNeille completion of the lattice reduct* $(A, \wedge, \vee)$ *with* $A \subseteq \overline{A}$*. We extend the multiplication and divisions of* $\mathbf{A}$ *to* $\overline{A}$ *by:*

$$\begin{array}{rcl} x \cdot y & := & \bigvee\{a \cdot b : a \leq x,\ b \leq y,\ a, b \in A\}, \\ x \backslash y & := & \bigwedge\{a \backslash b : a \leq x,\ y \leq b,\ a, b \in A\}, \\ y / x & := & \bigwedge\{b / a : a \leq x,\ y \leq b,\ a, b \in A\}. \end{array}$$

Then $\overline{\mathbf{A}} := (\overline{A}, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ *is an FL-algebra that is a completion of* $\mathbf{A}$.

Such $\overline{\mathbf{A}}$ is always regular, and is called the *MacNeille completion* of $\mathbf{A}$. A concrete construction will be described in Section 5.1.

2.4. Substructural hierarchy

In [14] we addressed the question of which (sets of) axioms $\mathcal{E}$ are equivalent to structural sequent rules $\mathcal{R}$ (i.e., $\mathbf{L}(\mathcal{E}) = \{\alpha \in Fm : \vdash_{\mathbf{FL}(\mathcal{R})} \alpha\}$) such that $\mathbf{FL}(\mathcal{R})$ is strongly analytic. To provide a *systematic* answer to this question we introduced the substructural hierarchy [14] (and [12], in the commutative case), which suitably classifies axioms in the language of $\mathbf{FL}_\perp$ or, equivalently, equations over FL-algebras possibly extended with $\top$ and $\perp$.

The idea behind the substructural hierarchy $(\mathcal{P}_n, \mathcal{N}_n)$ is to track polarity alternations of connectives/operations. The classes $\mathcal{P}_n$ and $\mathcal{N}_n$ stand indeed for axioms/equations with leading positive and negative connectives, where (1, $\perp$, $\vee$, $\cdot$) are positive and (0, $\top$, /, $\backslash$, $\wedge$) are negative, see [1].

Definition 2.6. For each $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas (terms) are defined as follows:

(0) $\mathcal{P}_0 := \mathcal{N}_0 :=$ the set of variables.

(P1) $1, \perp$ and all formulas in $\mathcal{N}_n$ belong to $\mathcal{P}_{n+1}$.

(P2) If $\alpha, \beta \in \mathcal{P}_{n+1}$, then $\alpha \vee \beta, \alpha \cdot \beta \in \mathcal{P}_{n+1}$.

(N1) $0, \top$ and all formulas in $\mathcal{P}_n$ belong to $\mathcal{N}_{n+1}$.

(N2) If $\alpha, \beta \in \mathcal{N}_{n+1}$, then $\alpha \wedge \beta \in \mathcal{N}_{n+1}$.

(N3) If $\alpha \in \mathcal{P}_{n+1}$ and $\beta \in \mathcal{N}_{n+1}$, then $\alpha \backslash \beta, \beta / \alpha \in \mathcal{N}_{n+1}$.

Namely $\mathcal{P}_{n+1}$ is the set generated from $\mathcal{N}_n$ by means of finite (possibly empty) joins and products, and $\mathcal{N}_{n+1}$ is generated from $\mathcal{P}_n \cup \{0\}$ by means of finite (possibly empty) meets and divisions with denominators from $\mathcal{P}_{n+1}$.

By residuation, any equation ε can be written as $1 \leq t$. We say that ε belongs to $\mathcal{P}_n$ ($\mathcal{N}_n$, resp.) if t does.

As shown in [14], formulas in each class admit the following normal forms.

Lemma 2.7.

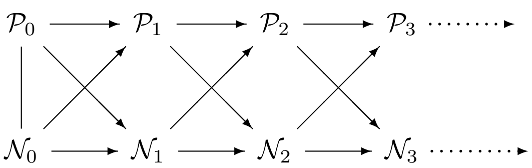

Figure 2: The Substructural Hierarchy

Class	Equation	Name
$\mathcal{N}_2$	$\alpha \to 1$	left weakening (integrality)
	$0 \to \alpha$	right weakening
	$\alpha \cdot \beta \to \beta \cdot \alpha$	exchange (commutativity)
	$\alpha \to \alpha \cdot \alpha$	contraction
	$\alpha \cdot \alpha \to \alpha$	expansion
	$\alpha^n \to \alpha^m$	knotted axioms $(n, m \geq 0)$
	$\neg(\alpha \wedge \neg\alpha)$	no-contradiction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$	weak nilpotent minimum
	$\alpha \cdot (\alpha \backslash 1) \leftrightarrow 1$	ℓ-group
	$\bigvee_{i=0}^{k}(\alpha_i \to \bigvee_{i \neq j} \alpha_j)$	Kripke models of width $\leq k$
	$\alpha_0 \vee (\alpha_0 \to \alpha_1) \vee \cdots \vee (\alpha_0 \wedge \cdots \wedge \alpha_{k-1} \to 0)$	Kripke models with size $\leq k$
$\mathcal{N}_3$	$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$	distributivity
	$(\alpha \backslash (\alpha \cdot \beta)) \backslash \beta$	cancellativity
	$(\alpha \wedge \beta) \to \alpha \cdot (\alpha \to \beta)$	divisibility
	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Łukasiewicz axiom

Figure 3: Some known axioms in substructural logics

(P) If $\alpha \in \mathcal{P}_{n+1}$, then α is equivalent to $\bot$ or $\beta_1 \vee \cdots \vee \beta_m$, where each β_i is a product of formulas in $\mathcal{N}_n$.

(N) If $\alpha \in \mathcal{N}_{n+1}$, then α is equivalent to $\top$ or $\bigwedge_{1 \leq i \leq m} \gamma_i \backslash \beta_i / \delta_i$, where each β_i is either 0 or a formula in $\mathcal{P}_n$, and each γ_i and δ_i are products of formulas in $\mathcal{N}_n$.

We have $\mathcal{P}_n \cup \mathcal{N}_n \subseteq \mathcal{P}_{n+1} \cap \mathcal{N}_{n+1}$ for every n. Hence the substructural hierarchy can be depicted as in Figure 2 (the arrows stand for inclusions among the classes).

Remark 2.8. A recent paper [24] shows that *any* formula is $\mathbf{FL_e}$-equivalent to a set of formulas in $\mathcal{N}_3$. Thus the hierarchy collapses to the level $\mathcal{N}_3$ in the commutative case.

Some examples of axioms classified into the hierarchy are in Figure 3.

In [14] we have investigated in depth the first classes of the hierarchy (up to $\mathcal{N}_2$). The main results can be summarized by 1-3 below:

1. Every axiom (resp. equation) in $\mathcal{N}_2$ can be transformed into an equivalent set of structural rules (resp. quasiequations).

2. Let $\mathcal{E}$ be a set of $\mathcal{N}_2$ axioms (equations). The following are equivalent.

- $\mathsf{FL}(\mathcal{E})$ admits MacNeille completions.
- $\mathsf{FL}(\mathcal{E})$ admits completions.
- $\mathcal{E}$ is equivalent to a set $\mathcal{R}$ of structural rules such that $\mathbf{FL}(\mathcal{R})$ is strongly analytic.

Remark 2.9.

- This indicates that MacNeille completions are the strongest completion method for $\mathcal{N}_2$ equations; whenever such an equation is preserved by some completions, it is necessarily preserved by MacNeille completions.
- This shows that strong analyticity of a sequent calculus and closure under completions of a variety of FL-algebras are essentially the same thing, as far as $\mathcal{N}_2$ axioms and equations are concerned. The common step for both lies in the transformation of axioms (equations) into structural rules (quasiequations) and the residuated frame construction (see Section 5.1).
- Not all $\mathcal{N}_2$ equations satisfy the above conditions. For example the $\mathcal{N}_2$ equation $x\backslash x \leq x/x$ is not preserved by any completions, and accordingly, the formula $(\alpha\backslash\alpha)\backslash(\alpha/\alpha)$ is not equivalent in $\mathbf{FL}$ to any set of structural rules enjoying strong analyticity.

3. In presence of the left weakening axiom $\alpha \to 1$ (integrality $x \leq 1$), all the statements in 2 hold.

Remark 2.10. This means that MacNeille completions work for all varieties of integral FL-algebras axiomatized by $\mathcal{N}_2$ equations, and we have strong analyticity for all integral substructural logics axiomatized by $\mathcal{N}_2$ formulas. The problem is *completely* settled for these varieties and logics.

The purpose of this paper is to systematically extend the above results to a class of equations and axioms wider than $\mathcal{N}_2$. We focus on $\mathcal{P}_3$ – the next level of the substructural hierarchy. To do that we employ the hypersequent calculus and a new completion method inspired by that.

3. Non-commutative hypersequent calculus

Introduced by Avron in [2], the hypersequent calculus arises by extending Gentzen sequent calculus with a meta-disjunction " | " in order to refer to many (a multiset of) sequents, instead of just one. As shown in [12] the hypersequent calculus captures axioms/equations in the class $\mathcal{P}_3$, in presence of weakening and exchange.

Although hypersequent calculi have been defined for many logics including fuzzy, modal and superintuitionistic logics (see, e.g., [3, 32, 28]), there is no general account for them in substructural logics, in particular in absence of exchange (non-commutative calculi). Even though various structural rules have been proposed as "equivalent" to logical axioms, none of the proposed definitions of equivalence between axioms and rules (e.g., [32, 12]) works in the general substructural logic setting.

In this section we present the hypersequent calculus formalism for general substructural logics and set up its algebraic foundations. Most of the results in this section are known (e.g. [19, 25, 10]) and serve as background for developing our correspondence between formulas/equations and structural hypersequent rules.

We first introduce the hypersequent calculus **HFL** for full Lambek calculus (Section 3.1), then address the problem of how to understand and interpret the meta-disjunction " | " properly. As we will see, this requires the notion of iterated conjugates (Section 3.2). After a short account on how to interpret hypersequents in FL-algebras (Section 3.3), we turn to the algebraic side, and make sense of the symbol " | " in terms of subdirect representations of algebras (Section 3.4). A general notion of equivalence between axioms and structural rules is finally introduced in Section 3.5.

3.1. The system **HFL** *for full Lambek calculus*

In addition to the meta-level implication ($\Rightarrow$) and fusion (,), present in the sequent calculus, the hypersequent calculus contains the meta-level disjunction (|). A *hypersequent* Ξ is indeed a multiset of sequents written as $\Theta_1 \mid \ldots \mid \Theta_n$, and each Θ_i is called a *component*.

Throughout this paper we will consider single-conclusion hypersequents, i.e., hypersequents whose components have at most one formula on the right-hand side of $\Rightarrow$. We will use the following syntactic *metavariables*:

$\alpha, \beta, \gamma, \ldots$	formulas
$\Gamma, \Delta, \Sigma, \ldots$	formula sequences
$\Pi, \Pi_1, \Pi_2, \ldots$	stoups
$\Theta, \Theta_1, \Theta_2, \ldots$	sequents
$\Xi, \Xi_1, \Xi_2, \ldots$	hypersequents

The calculus **HFL** consists of the following *inference rules*:

- the hypersequent version

$$\frac{\Xi \mid \Theta_1 \quad \cdots \quad \Xi \mid \Theta_m}{\Xi \mid \Theta}\ (r)$$

of each rule of **FL** (cf. Figure 1) of the form

$$\frac{\Theta_1 \quad \cdots \quad \Theta_m}{\Theta}\ (r), \quad \text{with } m \geq 0.$$

- the external structural rules of weakening and contraction[1]:

$$\frac{\Xi}{\Xi \mid \Theta}\ (EW) \qquad \frac{\Xi \mid \Theta \mid \Theta}{\Xi \mid \Theta}\ (EC)$$

As examples of the former rules, we have:

$$\frac{}{\Xi \mid \alpha \Rightarrow \alpha}\ (id) \qquad \frac{\Xi \mid \Sigma \Rightarrow \alpha \quad \Xi \mid \Gamma, \alpha, \Delta \Rightarrow \Pi}{\Xi \mid \Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (cut) \qquad \frac{\Xi \mid \alpha, \Gamma \Rightarrow \beta}{\Xi \mid \Gamma \Rightarrow \alpha \backslash \beta}\ (\backslash r)$$

Notions of rules, rule instances, derivations, strong analyticity and so on, defined for sequents and sequent calculi transfer unscathed to hypersequents and hypersequent calculi.

Let $\mathcal{H} \cup \{\Xi\}$ be a set of hypersequents. If Ξ is derivable from the premises in $\mathcal{H}$, we write $\mathcal{H} \vdash_{\mathbf{HFL}} \Xi$. As before, we also write $\mathcal{F} \vdash_{\mathbf{HFL}} \alpha$ if $\{\Rightarrow \beta : \beta \in \mathcal{F}\} \vdash_{\mathbf{HFL}} \Rightarrow \alpha$.

Taking hypersequent versions of sequent calculi alone is not enough to obtain calculi for new logics; indeed we have $\vdash_{\mathbf{HFL}} \alpha$ if and only if $\vdash_{\mathbf{FL}} \alpha$. The benefit of considering hypersequent calculi is that they support the addition of new structural rules that act on various sequents inside the hypersequents. It is this type of rules that increases the expressive power of the hypersequent calculus with respect to the sequent calculus.

Example 3.1. A typical example of a structural rule in the hypersequent calculus is Avron's communication rule [3] (see Figure 4 for its non-commutative counterpart):

$$\frac{\Xi \mid \Sigma_2, \Gamma \Rightarrow \Pi \quad \Xi \mid \Sigma_1, \Gamma' \Rightarrow \Pi'}{\Xi \mid \Sigma_1, \Gamma \Rightarrow \Pi \mid \Sigma_2, \Gamma' \Rightarrow \Pi'}\ (com),$$

by means of which we can prove the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\beta \Rightarrow \beta}\ (id) \quad \cfrac{}{\alpha \Rightarrow \alpha}\ (id)}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}\ (com)}{\Rightarrow \alpha \to \beta \mid \ \Rightarrow \beta \to \alpha}\ (\to r)}{\cfrac{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha) \mid \ \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (EC)}\ (\vee r)$$

By extending **HFL** with (com), (e), (c), and (w) (see Section 2.1), we obtain a hypersequent calculus for propositional Gödel logic that enjoys cut-elimination [3].

$$\frac{\Xi \mid \Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi}{\Xi \mid \Sigma_1 \Rightarrow \ \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi}\ (em) \qquad \frac{\Xi \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi \quad \Xi \mid \Gamma', \Sigma_1, \Delta' \Rightarrow \Pi'}{\Xi \mid \Gamma, \Sigma_1, \Delta \Rightarrow \Pi \mid \Gamma', \Sigma_2, \Delta' \Rightarrow \Pi'}\ (com)$$

$$\frac{\Xi \mid \Sigma_1, \Sigma_2 \Rightarrow}{\Xi \mid \Sigma_1 \Rightarrow \ \mid \Sigma_2 \Rightarrow}\ (lq) \qquad \frac{\{\Xi \mid \Gamma_i, \Sigma_j, \Delta_i \Rightarrow \Pi_i\}_{0 \leq i,j \leq k, i \neq j}}{\Xi \mid \Gamma_0, \Sigma_0, \Delta_0 \Rightarrow \Pi_0 \mid \ \ldots \ \mid \Gamma_k, \Sigma_k, \Delta_k \Rightarrow \Pi_k}\ (Bw_k)$$

$$\frac{\{\Xi \mid \Gamma_i, \Sigma_j, \Delta_i \Rightarrow \Pi_i\}_{0 \leq i \leq k-1;\, i+1 \leq j \leq k}}{\Xi \mid \Gamma_0, \Delta_0 \Rightarrow \Pi_0 \mid \ \ldots \ \mid \Gamma_{k-1}, \Sigma_{k-1}, \Delta_{k-1} \Rightarrow \Pi_{k-1} \mid \Sigma_k \Rightarrow}\ (Bc_k)$$

Figure 4: Some structural rules

More examples of structural rules can be found in Figure 4. The general format of a *structural rule* in the hypersequent calculus is:

$$\frac{\Xi \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Xi \mid \Upsilon_m \Rightarrow \Psi_m}{\Xi \mid \Upsilon_{m+1} \Rightarrow \Psi_{m+1} \mid \ \cdots \ \mid \Upsilon_n \Rightarrow \Psi_n}\ (r)$$

where for each $1 \leq i \leq n$,

- Υ_i is a (possibly empty) sequence that consists of metavariables for formula sequences and for formulas,
- Ψ_i is empty, a metavariable for stoups or a metavariable for formulas.

Given a set $\mathcal{R}$ of structural rules, we write $\mathbf{HFL}(\mathcal{R})$ for the calculus obtained by adding $\mathcal{R}$ to **HFL**. Now our central task in this paper can be formulated as follows:

- Given a set $\mathcal{E}$ of axioms, we would like to find an "equivalent" set $\mathcal{R}$ of structural rules that preserve strong analyticity. How and when is it possible?

To address this question, we first need to clarify what it means that $\mathcal{E}$ is "equivalent" to $\mathcal{R}$. The rest of this section is devoted to this issue.

3.2. Iterated conjugates

We start by discussing the meaning of the separator $|$. In hypersequent calculi containing all three basic sequent structural rules or the (com) rule in Fig.4, " $|$ " is usually interpreted as the logical connective $\vee$, see, e.g., [32]. This interpretation does not work however for weaker hypersequent calculi extending **HFL**, whose rule soundness requires the ∇ disjunction introduced below.

Recall that we have two distinct notions of entailment: $\vdash_{\mathbf{FL}} \alpha \Rightarrow \beta$ implies $\alpha \vdash_{\mathbf{FL}} \beta$, but the converse does not hold in general, due to the lack of the rules

[1]External exchange (EE) is implicit by considering hypersequents as multisets of sequents.

$(e), (c), (i)$. In other words the deduction theorem fails in its usual form, but is still valid in the weaker form of the following theorem. A way to compensate for this is to use conjugates [10].

A *conjugate* of a formula α is either $\lambda_\beta(\alpha) := (\beta\backslash\alpha\beta) \wedge 1$ or $\rho_\beta(\alpha) := (\beta\alpha/\beta) \wedge 1$ for some formula β, see e.g. [25]. Conjugates allow us to simulate (e) and (i), as follows:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \lambda_\beta(\alpha), \Delta \Rightarrow \Pi} \qquad \frac{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \rho_\beta(\alpha), \beta, \Delta \Rightarrow \Pi} \qquad \frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \lambda_\beta(\alpha), \Delta \Rightarrow \Pi}$$

We also have derivations from $\Rightarrow \alpha$ to $\Rightarrow \lambda_\beta(\alpha)$ and $\Rightarrow \rho_\beta(\alpha)$, namely:

$$\alpha \vdash_{\mathbf{FL}} \lambda_\beta(\alpha), \qquad \alpha \vdash_{\mathbf{FL}} \rho_\beta(\alpha). \tag{1}$$

Since a conjugate is needed each time (e) or (i) is simulated, the conjugate operator has to be iterated to simulate an arbitrary number of (e) and (i).

Definition 3.2. An *iterated conjugate* of α is a formula of the form $\mu_{\beta_1} \cdots \mu_{\beta_n}(\alpha)$, where $n \geq 0$ and each μ_{β_i} is either λ_{β_i} or ρ_{β_i}. The set of all iterated conjugates of α is denoted by $\flat\alpha$. More generally, if $\alpha_1, \ldots, \alpha_n$ are formulas of $\mathbf{FL}$, $\flat\alpha_1 \vee \cdots \vee \flat\alpha_n$ (resp. $\flat\alpha_1 \cdot \cdots \cdot \flat\alpha_n$) denotes the set of all formulas of the form $\alpha'_1 \vee \cdots \vee \alpha'_n$ (resp. $\alpha'_1 \cdot \cdots \cdot \alpha'_n$) where $\alpha'_i \in \flat\alpha_i$ for $1 \leq i \leq n$.

To simulate (c), we take a product of iterated conjugates. As a consequence, we have the so called parameterized local deduction theorem [22]:

Theorem 3.3. *Let $\mathbf{L}$ be a substructural logic and $\Phi \cup \{\alpha, \gamma\}$ a set of formulas:*

$$\Phi, \alpha \vdash_{\mathbf{L}} \gamma \iff \Phi \vdash_{\mathbf{L}} \alpha'_1 \cdots \alpha'_n \to \gamma$$

for some $n \geq 0$ and $\alpha'_1, \ldots, \alpha'_n \in \flat\alpha$.

When $\mathbf{L}$ is over $\mathbf{FL_e}$ or $\mathbf{FL_{ew}}$, the above theorem can be much simplified. Indeed, we have

$$\Phi, \alpha \vdash_{\mathbf{L}} \gamma \iff \Phi \vdash_{\mathbf{L}} (\alpha \wedge 1)^n \to \gamma \text{ for some } n \geq 0$$

if $\mathbf{FL_e} \subseteq \mathbf{L}$, and

$$\Phi, \alpha \vdash_{\mathbf{L}} \gamma \iff \Phi \vdash_{\mathbf{L}} \alpha^n \to \gamma \text{ for some } n \geq 0$$

if $\mathbf{FL_{ew}} \subseteq \mathbf{L}$. Hence we naturally adopt the following convention.

- In substructural logics over $\mathbf{FL_e}$, $\flat\alpha$ denotes the singleton set $\{\alpha \wedge 1\}$.
- In substructural logics over $\mathbf{FL_{ew}}$, $\flat\alpha$ just denotes $\{\alpha\}$.

As a consequence of Theorem 3.3, we obtain a meta-disjunction defined by means of the $\flat$ operation. We define

$$\alpha \nabla \beta := \flat\alpha \vee \flat\beta.$$

As seen below, this (strong) disjunction, which will be used to interpret the symbol "|", inherits the properties of hypersequents, one of which is *proof-by-cases*.

Corollary 3.4. *Let $\mathbf{L}$ be a substructural logic and $\Phi \cup \{\alpha, \beta, \gamma\}$ a set of formulas:*

$$\Phi, \alpha \vdash_{\mathbf{L}} \gamma \text{ and } \Phi, \beta \vdash_{\mathbf{L}} \gamma \iff \Phi, \alpha \nabla \beta \vdash_{\mathbf{L}} \gamma.$$

Proof. By (1), we have $\alpha \vdash_{\mathbf{L}} \alpha\nabla\beta$ and $\beta \vdash_{\mathbf{L}} \alpha\nabla\beta$, which imply the ($\Leftarrow$) direction.

($\Rightarrow$) Assume that $\Phi, \alpha \vdash_{\mathbf{L}} \gamma$ and $\Phi, \beta \vdash_{\mathbf{L}} \gamma$. By Theorem 3.3, we obtain $\Phi \vdash_{\mathbf{L}} \alpha'_1 \cdots \alpha'_m \to \gamma$ and $\Phi \vdash_{\mathbf{L}} \beta'_1 \cdots \beta'_n \to \gamma$ where each α'_i (resp. β'_j) belongs to $\flat\alpha$ (resp. $\flat\beta$). Let δ be the product of all $\alpha'_i \vee \beta'_j$ ($1 \leq i \leq m$, $1 \leq j \leq n$) lexicographically ordered with respect to (i, j). By distributivity of multiplication over join, δ is a join of products, each of which is the form $\prod \delta_{ij}$, lexicographically ordered as above, where $\delta_{ij} \in \{\alpha'_i, \beta'_j\}$; we will show that each such product is less or equal to $(\alpha'_1 \cdots \alpha'_m) \vee (\beta'_1 \cdots \beta'_n)$, using the fact that $\delta_{ij} \leq 1$. Indeed, if for each i there exists j_i with $\delta_{ij_i} = \alpha'_i$, then $\prod \delta_{ij} \leq \prod \delta_{ij_i} = \alpha'_1 \cdots \alpha'_m$. On the other hand, if there exists i_0 such that for all j we have $\delta_{i_0 j} = \beta'_j$, then $\prod \delta_{ij} \leq \prod \delta_{i_0 j} = \beta'_1 \cdots \beta'_n$. Therefore, δ implies $(\alpha'_1 \cdots \alpha'_m) \vee (\beta'_1 \cdots \beta'_n)$, so $\Phi \vdash_{\mathbf{L}} \delta \to \gamma$. Since δ is a product of members of $\flat\alpha \vee \flat\beta$, we conclude $\Phi, \alpha\nabla\beta \vdash_{\mathbf{L}} \gamma$. $\square$

See [17] for a general study of (parameterized) disjunctions and the proof-by-cases property. It is the ∇ disjunction that the symbol " | " of the hypersequent calculus is intended to denote. More precisely, we consider the following translation of a sequent Θ into a formula $\Theta^\flat$, and a hypersequent Ξ into a set $\Xi^\flat$ of formulas:

$$\begin{array}{rcl}
(\alpha_1, \ldots, \alpha_n \Rightarrow \beta)^\flat & := & \alpha_1 \cdots \alpha_n \backslash \beta, \\
(\alpha_1, \ldots, \alpha_n \Rightarrow \ \ )^\flat & := & \alpha_1 \cdots \alpha_n \backslash 0, \\
(\Theta_1 | \cdots | \Theta_n)^\flat & := & \Theta_1^\flat \nabla \cdots \nabla \Theta_n^\flat = \flat(\Theta_1^\flat) \vee \cdots \vee \flat(\Theta_n^\flat).
\end{array}$$

Finally we define $\mathcal{H}^\flat := \bigcup \{\Xi^\flat : \Xi \in \mathcal{H}\}$ for a set $\mathcal{H}$ of hypersequents.

Remark 3.5. $(\Theta_1 | \cdots | \Theta_n)^\flat$ amounts to $\{(\Theta_1^\flat \wedge 1) \vee \cdots \vee (\Theta_n^\flat \wedge 1)\}$ in the commutative case, and to $\{\Theta_1^\flat \vee \cdots \vee \Theta_n^\flat\}$ in the commutative and integral case.

A prominent feature of hypersequent calculi is that one can reason separately in each component, a principle that we call the *local reasoning principle*:

Lemma 3.6. *Let Ξ_0, Ξ_1, Ξ_2 be hypersequents and $\mathcal{R}$ a set of structural rules:*

$$\Xi_1 \vdash_{\mathbf{HFL}(\mathcal{R})} \Xi_2 \implies \Xi_0 \mid \Xi_1 \vdash_{\mathbf{HFL}(\mathcal{R})} \Xi_0 \mid \Xi_2.$$

Proof. We simply add the context Ξ_0 to all hypersequents in the derivation $\Xi_1 \vdash_{\mathbf{HFL}(\mathcal{E})} \Xi_2$. $\square$

Lemma 3.7. $\Xi \vdash_{\mathbf{HFL}} \Xi^\flat$ *holds for every hypersequent Ξ. Moreover if $\mathcal{R}$ is a set of structural rules, $\mathcal{H}$ is a set of hypersequents and Θ is a sequent, then*

$$\mathcal{H}^\flat \vdash_{\mathbf{L}(\mathcal{R})} \Theta^\flat \Longrightarrow \mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \Theta,$$

where $\mathbf{L}(\mathcal{R}) := \{\alpha \in Fm : \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha\}$ *(cf. Section 3.5).*

Proof. Suppose that $\Xi = \Theta_1 \mid \cdots \mid \Theta_n$. First note that $\Theta_i^\flat \vdash_{\mathbf{FL}} \flat(\Theta_i^\flat)$ holds by (1), so we have $\Theta_i \vdash_{\mathbf{HFL}(\mathcal{R})} \Rightarrow \flat(\Theta_i^\flat)$. Hence by Lemma 3.6 we obtain $\Theta_1 \mid \cdots \mid \Theta_n \vdash_{\mathbf{HFL}(\mathcal{R})} \Rightarrow \flat(\Theta_1^\flat) \mid \cdots \mid \Rightarrow \flat(\Theta_n^\flat)$, from which $\Xi \vdash_{\mathbf{HFL}} \Xi^\flat$ follows by using $(\vee r)$ and (EC).

The second claim easily follows from the first one, since $\mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \mathcal{H}^\flat$, $\mathcal{H}^\flat \vdash_{\mathbf{HFL}(\mathcal{R})} \Theta^\flat$ and $\Theta^\flat \vdash_{\mathbf{HFL}(\mathcal{R})} \Theta$. $\square$

Remark 3.8. In the above lemma, the conclusion Θ cannot be replaced by a hypersequent Ξ. Namely, $\Xi^\flat \vdash_{\mathbf{HFL}} \Xi$ does *not* hold in general (e.g. take Ξ to be $\Rightarrow \alpha \mid \Rightarrow \beta$), while it does, for instance, in the case of fuzzy logics (see [32]) where the hypersequent $\alpha \vee \beta \Rightarrow \alpha \mid \alpha \vee \beta \Rightarrow \beta$ is derivable and hence $\Rightarrow \alpha \mid \Rightarrow \beta$ follows from $\Rightarrow \alpha \vee \beta$. This intriguing fact is the main source of complications when developing hypersequent calculi for substructural logics.

3.3. From structural rules to clauses

Recall that we have identified a formula α of **FL** with the equation $1 \leq \alpha$ of FL-algebras. This extends to an identification of hypersequents with disjunctive clauses as follows:

$$\begin{array}{lll} \alpha_1, \cdots, \alpha_n \Rightarrow \beta & \text{with} & \alpha_1 \cdots \alpha_n \leq \beta, \\ \alpha_1, \cdots, \alpha_n \Rightarrow & \text{with} & \alpha_1 \cdots \alpha_n \leq 0, \\ \Theta_1 \mid \ldots \mid \Theta_n & \text{with} & \Theta_1 \text{ or } \cdots \text{ or } \Theta_n. \end{array}$$

This allows us to extend the semantic consequence relation $\models_{\mathsf{K}}$, where K is a class of FL-algebras, to a relation between hypersequents. Given a set $\mathcal{H} \cup \{\Xi\}$ of hypersequents, we write $\mathcal{H} \models_{\mathsf{K}} \Xi$ if every $\mathbf{A} \in \mathsf{K}$ and every valuation f into $\mathbf{A}$ which satisfies all hypersequents in $\mathcal{H}$ also satisfies Ξ under the above identification.

Example 3.9. $\models_{\mathsf{K}} \alpha, \beta \Rightarrow \beta \mid \alpha \Rightarrow$ means $\models_{\mathsf{K}} \alpha\beta \leq \beta$ or $\alpha \leq 0$.

Accordingly, we identify a structural rule with a *structural clause* as follows. Let

$$\frac{\Xi \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Xi \mid \Upsilon_m \Rightarrow \Psi_m}{\Xi \mid \Upsilon_1 \Rightarrow \Psi_{m+1} \mid \cdots \mid \Upsilon_n \Rightarrow \Psi_n}\ (r)$$

be a structural rule. First, we associate to each metavariable $(\alpha, \Gamma, \Pi, \Delta$ etc.) a variable $(x, y, z, w$, etc.) and we call this mapping $^\bullet$. Then each $\Upsilon_i \Rightarrow \Psi_i$ can be transformed into an equation $\Upsilon_i^\bullet \leq \Psi_i^\bullet$.

Example 3.10.

$$\begin{array}{lll} \Gamma, \alpha, \Gamma \Rightarrow \alpha & \mapsto & yxy \leq x, \\ \Gamma, \Delta, \Gamma \Rightarrow \Pi & \mapsto & ywy \leq z, \\ \Gamma \Rightarrow & \mapsto & y \leq 0. \end{array}$$

Now (r) is identified with the following structural clause:

$$\Upsilon_1^\bullet \leq \Psi_1^\bullet \text{ and } \cdots \text{ and } \Upsilon_m^\bullet \leq \Psi_m^\bullet \Longrightarrow \Upsilon_{m+1}^\bullet \leq \Psi_{m+1}^\bullet \text{ or } \cdots \text{ or } \Upsilon_n^\bullet \leq \Psi_n^\bullet. \quad (r^\bullet)$$

Note that the metavariable Ξ is dropped and the distinction between the different types of metavariables (i.e., those for formulas, formula sequences and stoups) is ignored. Hence there are various ways to read back a structural rule from a structural clause; a canonical way will be described in Section 4.4.

Given a set $\mathcal{R}$ of structural rules (clauses), we denote by $\mathsf{FL}(\mathcal{R})$ the class of FL-algebras which satisfy all rules (clauses) in $\mathcal{R}$.

Soundness with respect to the algebraic interpretation is obvious.

Lemma 3.11. *Let $\mathcal{H} \cup \{\Xi\}$ be a set of hypersequents and $\mathcal{R}$ a set of structural rules:*

$$\mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \Xi \quad \Longrightarrow \quad \mathcal{H} \models_{\mathsf{FL}(\mathcal{R})} \Xi.$$

3.4. Subdirect representation

The hypersequent calculus allows us to decompose $\Theta_1^\flat \nabla \cdots \nabla \Theta_n^\flat$ into a hypersequent $\Theta_1 \mid \cdots \mid \Theta_n$ and to work separately on each component using the local reasoning principle (Lemma 3.6). In algebra, the concept of subdirect representation supports a similar decomposition and local reasoning. In what follows we recall some elementary facts in universal algebra concerning subdirect representation. See, e.g., [11] for more information.

Given algebras $\mathbf{A}$ and $\{\mathbf{A}_i\}_{i \in I}$ of the same type, Recall that $\mathbf{A}$ is called a *subdirect product* of $\{\mathbf{A}_i\}_{i \in I}$ if there is an embedding $e : \mathbf{A} \longrightarrow \prod_{i \in I} \mathbf{A}_i$ which is surjective onto each coordinate, namely if $p_i : \prod_{i \in I} \mathbf{A}_i \longrightarrow \mathbf{A}_i$ is the projection map onto the ith coordinate, $e_i := p_i \circ e : \mathbf{A} \longrightarrow \mathbf{A}_i$ is surjective for every $i \in I$. Throughout this paper, we write $\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i$ to mean that $\mathbf{A}$ is a subdirect product of $\{\mathbf{A}_i\}_{i \in I}$, and $e_i : \mathbf{A} \longrightarrow \mathbf{A}_i$ for the canonical map $p_i \circ e$ above.

Subdirect products correspond to intersections of congruences.

Lemma 3.12. *Let $\mathbf{A}$ be an algebra and let $\{\theta_i\}_{i \in I}$ be congruences on $\mathbf{A}$. Then $\theta := \bigcap_{i \in I} \theta_i$ is also a congruence and we have*

$$\mathbf{A}/\theta \hookrightarrow \prod_{i \in I} \mathbf{A}/\theta_i.$$

Conversely, if θ is a congruence on $\mathbf{A}$ and algebras $\{\mathbf{A}_i\}_{i \in I}$ satisfy

$$\mathbf{A}/\theta \hookrightarrow \prod_{i \in I} \mathbf{A}_i,$$

then there are congruences $\{\theta_i\}_{i \in I}$ on $\mathbf{A}$ such that $\theta = \bigcap_{i \in I} \theta_i$ and $\mathbf{A}_i \cong \mathbf{A}/\theta_i$.

If we choose θ_i $(i \in I)$ so that $\bigcap_{i \in I} \theta_i = \triangle$ (the identity congruence), then we obtain $\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}/\theta_i$.

An algebra $\mathbf{A}$ is said to be *subdirectly irreducible* if it cannot be expressed as a subdirect product in a nontrivial way. More precisely, $\mathbf{A}$ is subdirectly irreducible if $\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i$ implies $e_i : \mathbf{A} \cong \mathbf{A}_i$ for some $i \in I$.

For instance, let a, b be two distinct elements in A, and consider the class of congruences θ such that $(a, b) \notin \theta$. By Zorn's lemma, there is a maximal one in the class, which we denote by $\tau_{a,b}$. Then $\mathbf{A}/\tau_{a,b}$ is a subdirectly irreducible algebra. To see this, suppose that $\mathbf{A}/\tau_{a,b}$ is a subdirect product of $\{\mathbf{A}_i\}_{i\in I}$. By Lemma 3.12, there are congruences $\{\theta_i\}_{i\in I}$ such that $\bigcap_{i\in I}\theta_i = \tau_{a,b}$ and $\mathbf{A}_i \cong \mathbf{A}/\theta_i$. We have $\tau_{a,b} \subseteq \theta_i$ for every $i \in I$. Because of $(a,b) \notin \tau_{a,b} = \bigcap_{i\in I}\theta_i$, there is a θ_i such that $(a,b) \notin \theta_i$. By maximality $\tau_{a,b} = \theta_i$. This proves that $\mathbf{A}/\tau_{a,b}$ is subdirectly irreducible.

We have thus established a fundamental fact in universal algebra that every algebra $\mathbf{A}$ admits a *subdirect representation*.

Theorem 3.13. *Every algebra* $\mathbf{A}$ *is a subdirect product of subdirectly irreducible algebras.*

Proof. The intersection of all $\tau_{a,b}$ with $(a,b) \in A^2$ and $a \neq b$ is $\triangle$. Hence $\mathbf{A} = \mathbf{A}/\triangle \hookrightarrow \prod_{(a,b)\in A^2, a\neq b} \mathbf{A}/\tau_{a,b}$ by Lemma 3.12. □

Notice that $\mathbf{A} \hookrightarrow \prod_{i\in I}\mathbf{A}_i$ means that $\mathbf{A}$ is (isomorphic to) a *subalgebra* of the *product* of $\{\mathbf{A}_i\}_{i\in I}$ and each $\mathbf{A}_i$ is a *homomorphic image* of $\mathbf{A}$. Since all these three operations preserve equations, an equation ε is satisfied in $\mathbf{A}$ iff it is satisfied in every $\mathbf{A}_i$ $(i \in I)$.

Given a class K of algebras, we denote by K_{SI} the class of its subdirectly irreducible members. By all the above, we conclude:

Corollary 3.14. *For every variety* V *and every set* $\mathcal{E} \cup \{\varepsilon\}$ *of equations,*

$$\mathcal{E} \models_{\mathsf{V}} \varepsilon \iff \mathcal{E} \models_{\mathsf{V}_{SI}} \varepsilon.$$

Proof. The $(\Rightarrow)$ direction is obvious. For the converse direction, observe that any $\mathbf{A} \in \mathsf{V}$ admits a subdirect representation $\mathbf{A} \hookrightarrow \prod_{i\in I}\mathbf{A}_i$ with each $\mathbf{A}_i \in \mathsf{V}_{SI}$. We have $\mathcal{E} \models_{\mathbf{A}} \varepsilon$ since even infinitary quasiequations, namely infinitary clauses with exactly one disjunct in the conclusion, are preserved under subalgebras and products. □

The following lemma proved in [19] shows that subdirect representation indeed provides us with a way to decompose a disjunction $\alpha\nabla\beta$ and to reason about it locally.

Lemma 3.15. *Let* $\mathbf{A}$ *be a subdirectly irreducible FL-algebra and* $a, b \in A$. *Then* $1 \leq a\nabla b$ *if and only if* $1 \leq a$ *or* $1 \leq b$.

Here $\flat a$ denotes the set of elements obtained by applying an iterated conjugate operator to a, and $a\nabla b := \flat a \vee \flat b = \{a' \vee b' : a' \in \flat a,\ b' \in \flat b\}$ as before. Expression $1 \leq a\nabla b$ means that $1 \leq c$ holds for every $c \in a\nabla b$.

Proof. $(\Leftarrow)$ Follows by noticing that $1 \leq a$ implies $1 \leq \lambda_c(a)$ and $1 \leq \rho_c(a)$ for every $c \in A$. $(\Rightarrow)$ Assume by contradiction that $1 \leq a\nabla b$ holds for $1 \not\leq a$ and $1 \not\leq b$. We then obtain two congruences θ_a and θ_b respectively generated by $(1, 1 \wedge a)$ and $(1, 1 \wedge b)$ in A^2. θ_a and θ_b are different from $\triangle$ since $1 \neq 1 \wedge a$ and $1 \neq 1 \wedge b$. It is enough to show that $\theta_a \cap \theta_b = \triangle$, since by Lemma 3.12 it implies:

$$\mathbf{A} = \mathbf{A}/\triangle \hookrightarrow \mathbf{A}/\theta_a \times \mathbf{A}/\theta_b,$$

contradicting the subdirect irreducibility of $\mathbf{A}$.

We may think of each element of A as a propositional variable and of formulas as built from the variables in A. The identity map $f(c) := c$ then gives rise to a canonical valuation into $\mathbf{A}$. The formula set $\mathcal{F}(\mathbf{A}) := \{\alpha \in Fm : \mathbf{A}, f \models 1 \leq \alpha\}$ completely describes the structure of $\mathbf{A}$ in the sense that $\mathcal{F}(\mathbf{A}) \vdash_{\mathbf{FL}} c \leftrightarrow d$ holds iff $c = d$. Now suppose that $(c, d) \in \theta_a \cap \theta_b$. We then obtain:

$$\mathcal{F}(\mathbf{A}), a \vdash_{\mathbf{FL}} c \leftrightarrow d \quad \textsf{and} \quad \mathcal{F}(\mathbf{A}), b \vdash_{\mathbf{FL}} c \leftrightarrow d$$

where a, b, c, d are considered to be formulas, since adding a to $\mathcal{F}(\mathbf{A})$ means that we consider the logical consequences of $1 \leq a$, namely $1 = 1 \wedge a$, in $\mathbf{A}$.

But then $\mathcal{F}(\mathbf{A}), a\nabla b \vdash_{\mathbf{FL}} c \leftrightarrow d$ follows by Corollary 3.4, and we obtain $\mathcal{F}(\mathbf{A}) \vdash_{\mathbf{FL}} c \leftrightarrow d$ because $a\nabla b \subseteq \mathcal{F}(\mathbf{A})$. Hence $c = d$ and we conclude that $\theta_a \cap \theta_b = \triangle$. □

3.5. Equivalence between axioms and structural rules

Any set $\mathcal{R}$ of structural rules defines a substructural logic by

$$\mathbf{L}(\mathcal{R}) := \{\alpha \in Fm : \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha\}$$

However, a little care is needed since it is not always the case that one can recover the derivability relation $\vdash_{\mathbf{HFL}(\mathcal{R})}$ from the logic $\mathbf{L}(\mathcal{R})$.

For instance, consider the hypersequent version of the rule discussed in [14], i.e.

$$\frac{\Xi \mid \quad \Rightarrow}{\Xi \mid \Gamma, \Gamma \Rightarrow}\ (abn).$$

Observe that $\mathbf{L}(abn) := \mathbf{L}(\{(abn)\}) = \mathbf{FL}$ (because the rule (abn) is unusable when proving a formula without any assumption), whereas

$$0 \vdash_{\mathbf{HFL}(abn)} pp \to 0, \qquad 0 \not\vdash_{\mathbf{FL}} pp \to 0$$

for any propositional variable p. Accordingly, it was pointed out in [14] that (the sequent version of) (abn) is not equivalent to any axiom. Since our central task is to find a set of structural rules equivalent to a given axiom, we will not consider such "abnormal" rules.

Definition 3.16. A set $\mathcal{R}$ of structural rules is *normal* if

$$\mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha \quad \Longleftrightarrow \quad \mathcal{F} \vdash_{\mathbf{L}(\mathcal{R})} \alpha$$

for every set $\mathcal{F} \cup \{\alpha\}$ of formulas.

Recall that $\mathcal{F} \vdash_{\mathbf{L}(\mathcal{R})} \alpha$ just means $\mathcal{F} \cup \mathbf{L}(\mathcal{R}) \vdash_{\mathbf{FL}} \alpha$. The $\Leftarrow$ direction always holds since $\vdash_{\mathbf{HFL}(\mathcal{R})} \alpha$ for every $\alpha \in \mathbf{L}(\mathcal{R})$. On the other hand, we have seen that $0 \vdash_{\mathbf{HFL}(abn)} pp \to 0$ but not $0 \vdash_{\mathbf{L}(abn)} pp \to 0$.

There is an alternative way to define normality.

Lemma 3.17. *A set $\mathcal{R}$ of structural rules is normal if and only if* $\mathsf{FL}(\mathcal{R})_{SI} = \mathsf{FL}(\mathbf{L}(\mathcal{R}))_{SI}$.

Proof. ($\Rightarrow$) We always have $\mathsf{FL}(\mathcal{R}) \subseteq \mathsf{FL}(\mathbf{L}(\mathcal{R}))$. Conversely, let $\mathbf{A} \in \mathsf{FL}(\mathbf{L}(\mathcal{R}))_{SI}$ and

$$\frac{\Xi \mid \Theta_1 \quad \cdots \quad \Xi \mid \Theta_m}{\Xi \mid \Theta_{m+1} \mid \cdots \mid \Theta_n}\ (r)$$

be (an instance of) a structural rule in $\mathcal{R}$. Let $\mathcal{F} := \{\Theta_1^\flat, \ldots, \Theta_m^\flat\}$ and $\mathcal{G} := \Theta_{m+1}^\flat \nabla \cdots \nabla \Theta_n^\flat$. We have $\mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R})} \mathcal{G}$ by Lemma 3.7. Hence by normality $\mathcal{F} \vdash_{\mathbf{L}(\mathcal{R})} \mathcal{G}$, so $\mathcal{F} \models_{\mathbf{A}} \mathcal{G}$ by Theorem 2.3. Since $\mathbf{A}$ is subdirectly irreducible, we have $\{\Theta_1, \ldots, \Theta_m\} \models_{\mathbf{A}} \Theta_{m+1} \mid \cdots \mid \Theta_n$, that is, $\mathbf{A}$ satisfies (r).

($\Leftarrow$) Assume $\mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha$ for a set $\mathcal{F} \cup \{\alpha\}$ of formulas. We then have $\mathcal{F} \models_{\mathsf{FL}(\mathcal{R})_{SI}} \alpha$ by Lemma 3.11, so $\mathcal{F} \models_{\mathsf{FL}(\mathbf{L}(\mathcal{R}))_{SI}} \alpha$. By Corollary 3.14 and Theorem 2.3, we conclude $\mathcal{F} \vdash_{\mathbf{L}(\mathcal{R})} \alpha$. $\square$

We are now ready to state the fundamental relationship among the main consequence relations $\vdash_{\mathbf{HFL}(\mathcal{R})}$, $\vdash_{\mathbf{L}(\mathcal{R})}$, $\models_{\mathsf{FL}(\mathcal{R})}$, $\models_{\mathsf{FL}(\mathcal{R})_{SI}}$ and $\models_{\mathsf{FL}(\mathbf{L}(\mathcal{R}))}$.

Theorem 3.18. *Let $\mathcal{R}$ be a normal set of structural rules, $\mathcal{H}$ a set of hypersequents and Θ a sequent. The following are equivalent:*

1. $\mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \Theta$.
2. $\mathcal{H} \models_{\mathsf{FL}(\mathcal{R})} \Theta$.
3. $\mathcal{H} \models_{\mathsf{FL}(\mathcal{R})_{SI}} \Theta$.
4. $\mathcal{H}^\flat \models_{\mathsf{FL}(\mathbf{L}(\mathcal{R}))} \Theta^\flat$.
5. $\mathcal{H}^\flat \vdash_{\mathbf{L}(\mathcal{R})} \Theta^\flat$.

Proof. ($1 \Rightarrow 2$) By Lemma 3.11.
($2 \Rightarrow 3$) Trivial.
($3 \Rightarrow 4$) By Lemma 3.17 we have $\mathcal{H} \models_{\mathsf{FL}(\mathbf{L}(\mathcal{R}))_{SI}} \Theta$, so $\mathcal{H}^\flat \models_{\mathsf{FL}(\mathbf{L}(\mathcal{R}))_{SI}} \Theta^\flat$ by Lemma 3.15, that implies $\mathcal{H}^\flat \models_{\mathsf{FL}(\mathbf{L}(\mathcal{R}))} \Theta^\flat$ by Corollary 3.14, noting that $\mathsf{FL}(\mathbf{L}(\mathcal{R}))$ is a variety and $\mathcal{H}^\flat \cup \{\Theta^\flat\}$ is a set of formulas.
($4 \Rightarrow 5$) By Theorem 2.3.
($5 \Rightarrow 1$) By Lemma 3.7. $\square$

There are various candidates for the definition of equivalence between axioms and structural rules. The previous theorem implies that some of them do coincide.

Corollary 3.19. *Let $\mathcal{R}$ be a set of structural rules and $\mathcal{E}$ a set of axioms. The following are equivalent:*

1. $\mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha \iff \mathcal{F} \vdash_{\mathbf{L}(\mathcal{E})} \alpha$ *for every set $\mathcal{F} \cup \{\alpha\}$ of formulas.*
2. $\mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \Theta \iff \mathcal{H}^\flat \vdash_{\mathbf{L}(\mathcal{E})} \Theta^\flat$ *for every set $\mathcal{H}$ of hypersequents and every sequent Θ.*
3. $\mathsf{FL}(\mathcal{R})_{SI} = \mathsf{FL}(\mathcal{E})_{SI}$ *and $\mathcal{R}$ is normal.*
4. $\mathbf{L}(\mathcal{R}) = \mathbf{L}(\mathcal{E})$ *and $\mathcal{R}$ is normal.*

Proof. ($1 \Rightarrow 4$) Obviously $\mathbf{L}(\mathcal{R}) = \mathbf{L}(\mathcal{E})$. Normality follows since $\mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha$ iff $\mathcal{F} \vdash_{\mathbf{L}(\mathcal{E})} \alpha$ iff $\mathcal{F} \vdash_{\mathbf{L}(\mathcal{R})} \alpha$.
($4 \Rightarrow 3$) By Lemma 3.17, noting that $\mathsf{FL}(\mathbf{L}(\mathcal{E})) = \mathsf{FL}(\mathcal{E})$.
($3 \Rightarrow 2$) By Theorem 3.18, $\mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \Theta$ iff $\mathcal{H} \models_{\mathsf{FL}(\mathcal{R})_{SI}} \Theta$ iff $\mathcal{H} \models_{\mathsf{FL}(\mathcal{E})_{SI}} \Theta$ iff $\mathcal{H}^\flat \models_{\mathsf{FL}(\mathcal{E})} \Theta^\flat$ iff $\mathcal{H}^\flat \vdash_{\mathbf{L}(\mathcal{E})} \Theta^\flat$.
($2 \Rightarrow 1$) Immediate. $\square$

On the basis of this corollary, it is reasonable to say that $\mathcal{E}$ and $\mathcal{R}$ are *equivalent* if either of the above conditions holds. This naturally induces an equivalence between two normal sets $\mathcal{R}, \mathcal{R}'$ of structural rules: $\mathcal{R}$ and $\mathcal{R}'$ are equivalent if $\mathsf{FL}(\mathcal{R})_{SI} = \mathsf{FL}(\mathcal{R}')_{SI}$.

4. From $\mathcal{P}_3^\flat$ axioms to structural rules

Having established the right notion of equivalence between axioms and structural hypersequent rules, we generalize the argument in [14] and show how to transform a large class of axioms into such rules.

In Section 4.1 we identify the class $\mathcal{P}_3^\flat$ of equations/axioms in the substructural hierarchy that can be dealt with by the techniques developed so far, and then define a procedure that transforms each $\mathcal{P}_3^\flat$ equation into a set of structural clauses (Section 4.2). The clauses we obtain are further transformed into what we call analytic clauses, under the additional assumption of acyclicity (Section 4.3). We finally turn to the proof-theoretic side and define a canonical translation of structural clauses into structural rules in the hypersequent calculus (Section 4.4). All together we obtain an algorithm that transforms any given acyclic $\mathcal{P}_3^\flat$ axiom into structural hypersequent rules that will be shown in Section 6 to preserve strong analyticity when added to **HFL**.

4.1. The Class $\mathcal{P}_3^\flat$

As shown in the previous section, it is not the internal lattice disjunction $t \vee u$ but the external one $t \nabla u$ (i.e., $\flat t \vee \flat u$) that can be dealt with by the hypersequent calculus or by the subdirect representation. For this reason we consider a slight modification of the class $\mathcal{P}_3$ of the substructural hierarchy (Definition 2.6). Informally, the new class, denoted by $\mathcal{P}_3^\flat$, is obtained by replacing the outermost $t \vee u$ with $t \nabla u$. Hereafter we treat $\flat t$ as if it were a single term (even though it actually denotes a set of terms).

Definition 4.1 ([13])**.** For each $n \geq 0$, we denote by $\mathcal{P}_{n+1}^\flat$ the set of terms generated from $\{\flat t : t \in \mathcal{N}_n\}$ by finite joins and products. More precisely:

- If $t \in \mathcal{N}_n$, then $\flat t \in \mathcal{P}_{n+1}^\flat$.
- $1, \bot \in \mathcal{P}_{n+1}^\flat$.
- If $t, u \in \mathcal{P}_{n+1}^\flat$, then $t \vee u, t \cdot u \in \mathcal{P}_{n+1}^\flat$.

We say that an equation $1 \leq t$ belongs to $\mathcal{P}_n^\flat$ if t does[2].

Remark 4.2. $\mathcal{P}_n^\flat \subseteq \mathcal{P}_n$ for $n \geq 3$ in the commutative case (recall that in this case we identify $\flat t$ with $t \wedge 1$ and $\mathcal{P}_n^\flat$ coincides with the class $\mathcal{P}'_n$ of [12]), and $\mathcal{P}_n^\flat = \mathcal{P}_n$ in the commutative and integral case.

4.2. From $\mathcal{P}_3^\flat$ equations to structural clauses

We show how to transform $\mathcal{P}_3^\flat$ equations into structural clauses. The procedure is an extension of the one in [14], which applies to $\mathcal{N}_2$ equations; see also [13] for $\mathcal{P}_3$ equations in the commutative and integral case[3].

For the purpose of this section, it is convenient to express a clause

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \ldots \text{ or } t_n \leq u_n \qquad (q)$$

as a pair $P \Longrightarrow C$ of sets of equations, where $P = \{t_1 \leq u_1, \ldots, t_m \leq u_m\}$ and $C = \{t_{m+1} \leq u_{m+1}, \ldots, t_n \leq u_n\}$. Thus $P \Longrightarrow C$ means that, under each particular valuation, if *all* of the equations in P hold, then *some* of the equations in C hold.

The following easy observation, often referred to as *Ackermann's lemma* [18], is indeed the key of the transformation in [12, 13, 14].

Lemma 4.3. *Every clause $P \Longrightarrow C \cup \{t \leq u\}$ is equivalent to each of the following:*

$$\begin{array}{rcl} P \cup \{x \leq t\} & \Longrightarrow & C \cup \{x \leq u\}, \\ P \cup \{u \leq x\} & \Longrightarrow & C \cup \{t \leq x\}, \end{array}$$

where x is a fresh variable not occurring in any of the equations in $P \cup C \cup \{t \leq u\}$.

Proof. Follows by the transitivity of $\leq$ and the instantiation of x with a suitable term (t or u). □

The next two lemmas pertain to the treatment of products in $\mathcal{P}_3^\flat$ equations.

Lemma 4.4. *Let $t(x)$ be a term in the language $\{\vee, \cdot, 1, \bot\}$ which contains at most one occurrence of x. Let $\mathbf{A}$ be an FL-algebra and f a valuation into $\mathbf{A}$ such that $f(w) \leq 1$ for every variable w. Then, for every pair of variables y, z,*

$$\mathbf{A}, f \models t(y) \cdot t(z) \leq t(y \cdot z) \leq t(y) \wedge t(z).$$

Proof. The second inequality is due to the monotonicity of $t(x)$. The first inequality is proved by induction on the structure of $t(x)$. The crucial case is when $t(x) = t_1(x) \vee t_2$, where we need to verify

$$\mathbf{A}, f \models (t_1(y) \vee t_2) \cdot (t_1(z) \vee t_2) \leq t_1(y \cdot z) \vee t_2.$$

[2] It follows from the lemmas below that the set of terms generated from $\{t \wedge 1 : t \in \mathcal{N}_n\}$ by finite products and ∇'s could have served as an alternative definition of $\mathcal{P}_{n+1}^\flat$, in the sense that we obtain the same equations $1 \leq t$, up to equivalence of sets of equations.

[3] The page `https://www.logic.at/tinc/webaxiomcalc/` contains an implementation of the procedure.

This follows from the induction hypothesis and the fact that $t_1(y)$, $t_1(z)$ and t_2 are all below 1 when interpreted by f. $\square$

Lemma 4.5. *Let $t(x)$ be a term that is generated from $\{\flat u : u \in \mathcal{N}_n\} \cup \{x\}$ by finite joins and products and that contains at most one occurrence of x. Let $\mathbf{A}$ be an FL-algebra, then*

$$\mathbf{A} \models 1 \leq t(\flat u \cdot \flat v) \iff \mathbf{A} \models 1 \leq t(\flat u) \text{ and } 1 \leq t(\flat v).$$

Proof. Let f be a valuation. Then $f(u') \leq 1$ for every iterated conjugate $u' \in \flat u$. Hence the claim follows from the previous lemma. $\square$

Our transformation procedure consists of four steps.

STEP 1

Let ε be a $\mathcal{P}_3^\flat$ equation. By Lemma 2.7 we can assume that it has the form $1 \leq \bigvee \prod \flat s_{ij}$ with $s_{ij} \in \mathcal{N}_2$; here $\bigvee$ denotes a finite join and $\prod$ a finite product.

By repeatedly applying Lemma 4.5, we may remove all products. As a result, we obtain a set of equations of the form $1 \leq \flat t_1 \vee \cdots \vee \flat t_n$ with each $t_i \in \mathcal{N}_2$. We replace each such equation with a disjunctive clause

$$1 \leq t_1 \text{ or } \cdots \text{ or } 1 \leq t_n.$$

It is equivalent to $1 \leq \flat t_1 \vee \cdots \vee \flat t_n$ over FL_{SI} by Lemma 3.15.

Example 4.6. The noncommutative version of the weak nilpotent minimum axiom

$$1 \leq \neg(xy) \nabla ((x \wedge y) \backslash (xy))$$

is equivalent to the disjunctive clause

$$1 \leq \neg(xy) \text{ or } 1 \leq (x \wedge y) \backslash (xy). \qquad (wnm_1)$$

STEP 2

By Lemma 2.7, each $\mathcal{N}_2$ term t is equivalent to $\bigwedge_{1 \leq i \leq m} l_i \backslash u_i / r_i$ where u_i is either 0 or a $\mathcal{P}_1$ term and l_i, r_i are products of $\mathcal{N}_1$ terms. Hence given a disjunctive clause $C \cup \{1 \leq t\}$ (expressed as a set), we may replace it with the following set of disjunctive clauses:

$$C \cup \{l_1 r_1 \leq u_1\}, \cdots, C \cup \{l_m r_m \leq u_m\}.$$

By repeating this argument we end up with a set $\mathcal{C}$ of disjunctive clauses such that each $C \in \mathcal{C}$ consists of two types of equations:

$$t_1 \cdots t_n \leq 0, \qquad t_1 \cdots t_n \leq u,$$

where $t_1, \ldots, t_n$ are $\mathcal{N}_1$ terms and u is a $\mathcal{P}_1$ term.

Example 4.7. (wnm_1) is equivalent to

$$xy \leq 0 \text{ or } x \wedge y \leq xy. \qquad (wnm_2)$$

STEP 3

Now let us focus on each disjunctive clause $\emptyset \Longrightarrow C_0$ and transform it step-by-step as follows. Given a clause $P \Longrightarrow C \cup \{t_1 \cdots t_n \leq 0\}$, we replace it with

$$P \cup \{x_1 \leq t_1, \ldots, x_n \leq t_n\} \Longrightarrow C \cup \{x_1 \ldots x_n \leq 0\}$$

where $x_1, \ldots, x_n$ are distinct fresh variables. Likewise, given a clause $P \Longrightarrow C \cup \{t_1 \cdots t_n \leq u\}$, we replace it with

$$P \cup \{x_1 \leq t_1, \ldots, x_n \leq t_n, u \leq y\} \Longrightarrow C \cup \{x_1 \ldots x_n \leq y\}$$

where $x_1, \ldots, x_n, y$ are distinct fresh variables. The resulting clause is equivalent to the original one by Lemma 4.3.

By repetition, we obtain a clause $P \Longrightarrow C$ where P consists of equations of the form $x \leq t$ (with $t \in \mathcal{N}_1$) or $u \leq y$ (with $u \in \mathcal{P}_1$), and C consists of structural equations (of the form $x_1 \cdots x_n \leq 0$ or $x_1 \cdots x_n \leq y$).

Example 4.8. (wnm_2) is equivalent to

$$z_1 \leq x \text{ and } z_2 \leq y \text{ and } z_3 \leq x \wedge y \text{ and } xy \leq z_4 \Longrightarrow z_1 z_2 \leq 0 \text{ or } z_3 \leq z_4. \qquad (wnm_3)$$

STEP 4

We further transform the premise set to obtain a fully structural clause.

By Lemma 2.7, each $\mathcal{N}_1$ term t is equivalent to $\bigwedge_{1 \leq i \leq m} l_i \backslash u_i / r_i$ where u_i is either 0 or a variable and l_i, r_i are products of variables. Hence we may replace a clause $P \cup \{x \leq t\} \Longrightarrow C$ with

$$P \cup \{l_1 x r_1 \leq u_1, \cdots, l_m x r_m \leq u_m\} \Longrightarrow C.$$

Likewise, any equation of the form $u \leq y$ (with $u \in \mathcal{P}_1$) in the premise set can be replaced by a set of structural equations.

Example 4.9. (wnm_3) is equivalent to

$$z_1 \leq x \text{ and } z_2 \leq y \text{ and } z_3 \leq x \text{ and } z_3 \leq y \text{ and } xy \leq z_4 \Longrightarrow z_1 z_2 \leq 0 \text{ or } z_3 \leq z_4. \qquad (wnm_4)$$

It is clear that the resulting set of structural clauses is normal. To see this, think of equations as formulas and clauses as rules. If a step transforms $\mathcal{R}$ into $\mathcal{R}'$, then

$$\mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R})} \alpha \quad \Longleftrightarrow \quad \mathcal{F} \vdash_{\mathbf{HFL}(\mathcal{R}')} \alpha$$

holds for every set $\mathcal{F} \cup \{\alpha\}$ of *formulas* (not hypersequents). We have thus established:

Theorem 4.10. *Every equation in $\mathcal{P}_3^\flat$ is equivalent to a finite set of structural clauses.*

4.3. From structural clauses to analytic clauses

Our transformation procedure is not yet complete. In particular, the premises of the clauses obtained so far may contain variables that do not appear in the conclusion, so their translation into structural (hypersequent) rules would lead to rules that do not enjoy the subformula property. As shown in this section all structural clauses satisfying the acyclicity condition, defined below, can be transformed into *analytic clauses*, that do enjoy the subformula property.

Definition 4.11. Given a structural clause (q)

$$P \Longrightarrow C$$

we build its *dependency graph* $D(q)$ in the following way:

- The vertices of $D(q)$ are the variables occurring in P (we do not distinguish occurrences).
- There is a directed edge $x \longrightarrow y$ in $D(q)$ if and only if there is a premise of the form $lxr \leq y$ in P.

A clause (q) is said to be *acyclic* if the graph $D(q)$ is acyclic (no directed cycles or loops). A $\mathcal{P}_3^\flat$ equation ε is said to be *acyclic* if applying the above procedure to ε results in a set of acyclic clauses.

Acyclicity is a sufficient condition to transform a given structural clause into an analytic one.

Definition 4.12. Given a structural clause $(q) : P \Longrightarrow C$ with $P = \{t_1 \leq u_1, \ldots, t_m \leq u_m\}$ and $C = \{t_{m+1} \leq u_{m+1}, \ldots, t_n \leq u_n\}$, we call the variables occurring in $t_{m+1}, \ldots, t_n$ *left* variables, and those in $u_{m+1}, \ldots, u_n$ *right* variables. The set of left (resp. right) variables is denoted by $L(q)$ (resp. $R(q)$). (q) is said to be *analytic* if it satisfies the following conditions[4]:

Linearity Each $x \in L(q) \cup R(q)$ occurs exactly once in $t_{m+1}, u_{m+1}, \ldots, t_n, u_n$.

Inclusion Each of $t_1, \ldots, t_m$ is a product of variables in $L(q)$ (here repetition is allowed), while each of $u_1, \ldots, u_m$ is either 0 or a variable in $R(q)$.

Let us describe the remaining steps of the transformation procedure.

STEP 5

Suppose that an acyclic clause (q) is given. It is easy to transform (q) into another one which satisfies linearity, while preserving acyclicity. Indeed, we may apply Step 3 to all conclusions, so that all variables in the conclusions are replaced with fresh distinct variables. Incidentally, this results in a structural clause, which we still call (q), satisfying the additional property:

[4]The linearity condition formulated below subsumes what we called the separation condition in [14].

Exclusion None of variables in $L(q)$ appear in $u_1, \ldots, u_m$, and none of variables in $R(q)$ appear in $t_1, \ldots, t_m$.

Hence all what we have to do is to eliminate the *redundant* variables, namely those occurring in the premises that do not occur in the conclusions.

Example 4.13. (wnm_4) already satisfies linearity and exclusion, hence there is no need to apply Step 5. The redundant variables are x and y.

STEP 6

Let x be a redundant variable of $(q) : P \Longrightarrow C$. There are three cases.

- x occurs only on the right-hand sides of premises. Then there is a set $I \subseteq \{1, \ldots, m\}$ and (q) can be written as
$$\{t_i \leq x : i \in I\} \cup P' \Longrightarrow C,$$
so that x does not occur in $P' \cup C$. We claim that (q) is equivalent to $(q') : P' \Longrightarrow C$. Indeed, (q') implies (q) since $P' \subseteq P$. Conversely, (q) implies (q') since by instantiating x with $\bigvee_{i \in I} t_i$, the premises $\{t_i \leq x : i \in I\}$ trivially hold, while it does not affect P' and C.

- x occurs only on the left hand sides of premises. Then there is a set $J \subseteq \{1, \ldots, m\}$ and (q) can be written as
$$\{l_j x r_j \leq u_j : j \in J\} \cup P' \Longrightarrow C,$$
so that x does not occur in $P' \cup C$. It may occur in l_j and r_j, but this causes no problem. As before, (q) is equivalent to $P' \Longrightarrow C$. This time the relevant instantiation is $\sigma(x) := \bigwedge_{j \in J}(l_j \backslash u_j / r_j) \wedge x$. We then have
$$\sigma(l_j x r_j) = \sigma(l_j)\sigma(x)\sigma(r_j) \leq l_j(l_j \backslash u_j / r_j) r_j \leq u_j = \sigma(u_j),$$
so the instantiation makes the premises $\{l_j x r_j \leq u_j : j \in J\}$ trivial.

- x occurs both on the left and right hand sides. Then there are $I, J \subseteq \{1, \ldots, m\}$ such that (q) is
$$\{t_i \leq x : i \in I\} \cup \{s_j(x, \ldots, x) \leq u_j : j \in J\} \cup P' \Longrightarrow C,$$
so that x does not occur in $P' \cup C$ and all occurrences of x are indicated. By acyclicity I and J are disjoint. Let P_{IJ} be the set
$$\{s_j(t_{k_1}, \ldots, t_{k_l}) \leq u_j : j \in J, k_1, \ldots, k_l \in I\}.$$
We claim that (q) is equivalent to $P_{IJ} \cup P' \Longrightarrow C$. Indeed, the latter implies (q) since P implies $P_{IJ} \cup P'$ by transitivity. Conversely, by instantiation $\sigma(x) := \bigvee_{i \in I} t_i$ each $t_i \leq x$ $(i \in I)$ trivially holds, and each $s_j(x, \ldots, x) \leq u_j$ follows from P_{IJ}. Indeed, we have $s_j(t_{k_1}, \ldots, t_{k_l}) \leq u_j$ in P_{IJ} for all $k_1, \ldots, k_l \in I$, so $s_j(\sigma(x), \ldots, \sigma(x)) \leq u_j$.

Observe that acyclicity and exclusion are preserved by the above transformations. Hence by repetition, we can remove all redundant variables, and the resulting clause satisfies inclusion.

Example 4.14. (wnm_4) is equivalent to

$$z_1 z_2 \leq z_4 \text{ and } z_1 z_3 \leq z_4 \text{ and } z_3 z_2 \leq z_4 \text{ and } z_3 z_3 \leq z_4 \Longrightarrow z_1 z_2 \leq 0 \text{ or } z_3 \leq z_4. \tag{wnm_6}$$

Theorem 4.15. *Every acyclic structural clause is equivalent to an analytic one. The same holds for an arbitrary structural clause in presence of integrality* $x \leq 1$.

Proof. The first claim has just been verified. For the second claim, let $(q) : P \Longrightarrow C$ be any structural clause. Step 5 works as before, so that we may suppose that (q) satisfies linearity and exclusion. For Step 6, there may be a redundant variable that occur both on the left and right hand sides of the same equation. Namely, P may contain $lxr \leq x$. Since such equation trivially holds by integrality, it may be ignored. □

4.4. From structural clauses to structural rules

We now turn to proof theory and show how to transform structural clauses into structural hypersequent rules. This, in combination with the algorithm outlined in the previous section, leads to a procedure for transforming each *acyclic* $\mathcal{P}_3^\flat$ axiom into *analytic rules*. These rules will be shown in Section 6 to preserve strong analyticity when added to the hypersequent calculus **HFL**.

Recall that we identify a formula α with the equation $1 \leq \alpha$. This allows us to define the set of *acyclic* $\mathcal{P}_3^\flat$ formulas in an obvious way. Acyclic formulas can be transformed into analytic clauses as described above; the latter are further transformed into structural hypersequent rules. This is done by carefully associating suitable metavariables to each variable in the clause.

Definition 4.16. Let

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \dots \text{ or } t_n \leq u_n \tag{q}$$

be an analytic clause. We define a structural rule (q°) corresponding to (q) as follows. Let $L(q) = \{x_1, \dots, x_k\}$ and $R(q) = \{y_1, \dots, y_l\}$. By the linearity condition $L(q)$ and $R(q)$ are disjoint. Let $\Sigma_1, \dots, \Sigma_k$, $\Gamma_1, \dots, \Gamma_l$, $\Delta_1, \dots, \Delta_l$ be distinct metavariables for formula sequences, and $\Pi_1, \dots, \Pi_l$ distinct metavariables for stoups. We associate to each equation $t_p \leq u_p$ $(1 \leq p \leq n)$ a sequent Θ_p as follows.

$$\begin{array}{rcl} x_{i_1} \cdots x_{i_q} \leq y_j & \mapsto & \Gamma_j, \Sigma_{i_1}, \dots, \Sigma_{i_q}, \Delta_j \Rightarrow \Pi_j \\ x_{i_1} \cdots x_{i_q} \leq 0 & \mapsto & \Sigma_{i_1}, \dots, \Sigma_{i_q} \Rightarrow \end{array}$$

The rule (q°) is defined to be

$$\frac{\Xi \mid \Theta_1 \quad \cdots \quad \Xi \mid \Theta_m}{\Xi \mid \Theta_{m+1} \mid \cdots \mid \Theta_n} \ (q^\circ).$$

We call a structural rule obtained in this way *analytic*.

Notice that to each right variable y_j we associate a *triple* $(\Gamma_j, \Delta_j, \Pi_j)$ of metavariables. This is important for obtaining a structural rule preserving cut-admissibility (see [16]).

Theorem 4.17. *Every acyclic structural rule is equivalent to an analytic structural rule. The same holds for an arbitrary structural rule in presence of left weakening* $\alpha \to 1$.

Example 4.18. From (wnm_6) we obtain the analytic rule

$$\frac{\begin{array}{cc}\Xi \mid \Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi & \Xi \mid \Gamma, \Sigma_3, \Sigma_2, \Delta \Rightarrow \Pi \\ \Xi \mid \Gamma, \Sigma_1, \Sigma_3, \Delta \Rightarrow \Pi & \Xi \mid \Gamma, \Sigma_3, \Sigma_3, \Delta \Rightarrow \Pi\end{array}}{\Xi \mid \Sigma_1, \Sigma_2 \Rightarrow \quad \mid \Gamma, \Sigma_3, \Delta \Rightarrow \Pi}\ (wnm)$$

Below are further examples of equivalent axioms and rules (see Figure 4 for the latter):

$$\begin{array}{lcl}
\alpha \nabla \neg\alpha & \Leftrightarrow & (em) \\
\neg\alpha \nabla \neg\neg\alpha & \Leftrightarrow & (lq) \\
(\alpha \to \beta)\nabla(\beta \to \alpha) & \Leftrightarrow & (com) \\
\alpha_0 \nabla(\alpha_0 \to \alpha_1)\nabla \dots \nabla(\alpha_0 \wedge \dots \wedge \alpha_{k-1} \to \alpha_k) & \Leftrightarrow & (Bc_k) \\
(\alpha_0 \to \bigvee_{0 \neq j} \alpha_j)\nabla \dots \nabla(\alpha_k \to \bigvee_{k \neq j} \alpha_j) & \Leftrightarrow & (Bw_k)
\end{array}$$

Remark 4.19. The above correspondence does not apply, in general, between *single* formulas and rules. Consider, for example, the formula $(\alpha\backslash\beta) \vee (\beta\backslash\alpha)$. This formula might be of interest because in the commutative and integral case $(\alpha \to \beta) \vee (\beta \to \alpha)$ axiomatizes precisely the semilinear residuated lattices, namely the variety generated by linear (commutative and integral) residuated lattices. However, the variety generated by linear (not-necessarily-commutative) residuated lattices is *not* axiomatized by $(\alpha\backslash\beta) \vee (\beta\backslash\alpha)$; it is axiomatized by $(\alpha\backslash\beta)\nabla(\beta\backslash\alpha)$. Our analysis captures this in a native way by identifying the hypersequent $(\alpha \Rightarrow \beta)|(\beta \Rightarrow \alpha)$ as the correct axiom for the hypersequent calculus and also presents the step-by-step transformation it should undergo in order to give rise to the communication rule (com), the addition of which preserves the cut-elimination property.

5. Residuated hyperframes

As shown in [14], proving that a substructural logic defined by $\mathcal{N}_2$ axioms admits a strongly analytic sequent calculus is essentially the same as proving that the corresponding variety is closed under MacNeille completions. The common essence between these two notions (one proof theoretic and the other algebraic) is captured by the residuated frames of [20]. These come with a construction of a complete FL-algebra and a (quasi)embedding into it, but in our case they also provide a key insight into the fact that analytic quasiequations are preserved by this dual (complete) algebra construction.

In this section we introduce the tools needed to extend the above success story to the richer framework of hypersequents, $\mathcal{P}_3^\flat$ axioms, and a new algebraic completion which we call hyper-MacNeille completion. To do that we begin by extending residuated frames to residuated hyperframes and developing the necessary machinery.

In detail, we review the basics of residuated frames in Section 5.1 and define the residuated hyperframes in Section 5.2. In Section 5.3 we introduce two ways of defining validity of a structural clause in a residuated hyperframe (*pointwise* validity and *setwise* validity) and show that they coincide for analytic clauses, thus allowing for proving the persistence of the validity of equations through the algebraic completion. Finally in Section 5.4 we provide a set of conditions, called Gentzen rules, ensuring the existence of a (quasi)homomorphism used for the proof of both strong analyticity and the algebraic completion.

5.1. Preliminaries on residuated frames

Introduced in [20, 14], residuated frames consist of two sets W and W', a binary operation $\circ$ on W and a binary relation N between W and W'. We explain the motivations behind these ingredients by mentioning connections to proof-theory, relational semantics and algebra.

The sets W and W'. Under a proof-theoretic interpretation W is the set of all left-hand sides of sequents and W' is the set of right-hand sides. Algebraically speaking, W corresponds to the set of join-irreducible and W' to the set of meet-irreducible elements of a (finite) non-distributive lattice. In relation to the latter, note that (finite) *distributive* lattices are determined simply by the set of their join-irreducible elements, which corresponds to the set W of possible worlds in the setting, say, of Kripke frames for intuitionistic logic. However, for non-distributive logics, including **FL**, the description of the algebraic models requires also a second set W' of meet-irreducible elements, hence the need for extensions of Kripke-type frames to a setting with two sets of worlds W and W'. The two sets collapse[5] for Kripke frames for distributive logics such as intuitionistic and relevance logics.

The (functional) ternary relation $\circ$. According to the proof-theoretic interpretation, W consists of all possible left-hand sides of sequents, and thus carries a monoid structure under the comma separator and the empty sequence. We thus stipulate in the definition of a residuated frame that W comes equipped with a monoidal binary operation and unit constant, which we denote by $\circ$ and ε, respectively. This also models the multiplication operation on an FL-algebra. Even in Kripke frames of distributive logics, where W and W' are identified, such as the ones for relevance logic, in order to capture the multiplication on the dual algebra a ternary accessibility relation is needed on W. However, in the simplified case of Kripke frames for intuitionistic logic the ternary accessibility relation is hidden as part of the (binary) order accessibility relation. One then

[5]This corresponds to the algebraic fact that the posets of join and of meet-irreducibles are isomorphic.

uses the (unary) modality provided by the latter to modalize classical implication, which is coincidentally available in the ambient setting, and thus obtain the desired intuitionistic (binary) implication (essentially along the Gödel translation of intuitionistic logic into **S4**). General residuated frames as defined in [20] allow $\circ$ to be a ternary relation, but for our proof-theoretic applications we can restrict to the case where this relation is functional, namely a binary operation on W.

The relation N. Actually, the (binary) accessibility relation in Kripke frames serves a second role in combination with the identification of W and W' (given by the relation $\not\geq$), namely that of providing a binary relation between W and W', which we call N in the setting of residuated frames. Proof-theoretically the relation N holds when the sequent formed by the two sides is provable, and algebraically N is simply the ordering relation of the FL-algebra.

In all the three different (though connected) motivations and interpretations of a residuated frame the binary relation N and the functional accessibility relation (aka, monoid operation) $\circ$ turn out to be connected by the *nuclearity* condition.

The above ingredients, stripped of their proof-theoretic, algebraic and duality-theoretic intuitions, are abstracted and presented in the following definition.

Definition 5.1. A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, where

- W and W' are sets and N is a binary relation between W and W',
- $(W, \circ, \varepsilon)$ is a monoid, $\epsilon \in W'$, and
- for all $x, y \in W$ and $z \in W'$, there are elements $x \backslash\!\backslash z$ and $z /\!/ y$ in W' such that
$$x \circ y \; N \; z \iff y \; N \; x \backslash\!\backslash z \iff x \; N \; z /\!/ y.$$

We refer to the last property by saying that the relation N is *nuclear.*

Note that we have been typically using the symbol ε for a generic equation, but hereafter we will use it for the monoid unit.

Residuated frames support a construction of a complete FL-algebra. Actually, it is well known that the (W, W', N) part of a residuated frame yields a complete lattice along the following lines. We first define for $X \subseteq W$ and $Z \subseteq W'$,

$$\begin{aligned} X^{\rhd} &:= \{z \in W' : \forall x \in X.\ x \; N \; z\}, \\ Z^{\lhd} &:= \{x \in W : \forall z \in Z.\ x \; N \; z\}, \end{aligned}$$

and write $x^{\rhd}$ for $\{x\}^{\rhd}$ and $z^{\lhd}$ for $\{z\}^{\lhd}$. The pair $({}^{\rhd}, {}^{\lhd})$ forms a *Galois connection*
$$X \subseteq Z^{\lhd} \iff X^{\rhd} \supseteq Z,$$
which induces a closure operator $\gamma(X) = X^{\rhd\lhd}$ on the powerset $\mathcal{P}(W)$. We say that $X \subseteq W$ is *Galois-closed* if $X = \gamma(X)$, or equivalently if there is $Z \subseteq W'$

such that $X = Z^{\lhd}$. The set of Galois-closed sets is denoted by $\gamma[\mathcal{P}(W)]$. Then $(\gamma[\mathcal{P}(W)], \cap, \cup_\gamma)$ is a complete lattice, where $X \cup_\gamma Y := \gamma(X \cup Y)$.

In the setting of a residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, we extend this construction by first defining for $X, Y \subseteq W$,

$$X \circ Y \quad := \quad \{x \circ y : x \in X,\ y \in Y\}$$

and observing that the closure operator γ satisfies $\gamma(X) \circ \gamma(Y) \subseteq \gamma(X \circ Y)$. This map is called a *nucleus* on $\mathcal{P}(W)$, see [20]. We further define the *dual algebra* of $\mathbf{W}$ by

$$\mathbf{W}^+ := (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \circ_\gamma, \backslash, /, \varepsilon^{\rhd\lhd}, \epsilon^{\lhd}),$$

where

$$\begin{aligned} X \circ_\gamma Y &:= \gamma(X \circ Y), \\ X \cup_\gamma Y &:= \gamma(X \cup Y), \\ X \backslash Y &:= \{y : \forall x \in X.\ xy \in Y\}, \\ Y / X &:= \{y : \forall x \in X.\ yx \in Y\}. \end{aligned}$$

Lemma 5.2. *If $\mathbf{W}$ is a residuated frame, then $\mathbf{W}^+$ is a complete FL-algebra.*

Thus residuated frames provide a handy way of producing a complete algebra. Below is a characteristic feature of $\mathbf{W}^+$.

Lemma 5.3. *Let $\mathbf{W}$ be a residuated frame, $X \subseteq W$ and $Z \subseteq W'$:*

$$\gamma(X) = \bigcup\nolimits_\gamma \{x^{\rhd\lhd} : x \in X\}, \qquad Z^{\lhd} = \bigcap \{z^{\lhd} : z \in Z\}.$$

Example 5.4. Given an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$, we define the residuated frame $\mathbf{W_A} := (A, A, \leq, \cdot, 1, 0)$. Note that the nuclearity condition is exactly the residuation condition of the algebra. Its dual algebra $\mathbf{W_A^+}$ together with the embedding $e(a) := a^{\lhd} = a^{\rhd\lhd}$ is nothing but the MacNeille completion of $\mathbf{A}$ (see Theorem 2.5). Indeed, the join-density and meet-density are direct consequences of Lemma 5.3, and the definition of $(\cdot, \backslash, /)$ in $\mathbf{W}^+$ conforms to Theorem 2.5.

Example 5.5. The second motivating example comes from proof theory and the sequent calculus for $\mathbf{FL}$. Define $\mathbf{W_{FL}} := (W, W', N, \cdot, \varepsilon, \epsilon)$ as follows:

- W is the set of formula sequences.
- W' is the set of *contexts* of the form $(\Gamma, \_\,, \Delta \Rightarrow \Pi)$. If $C = (\Gamma, \_\,, \Delta \Rightarrow \Pi) \in W'$, then $C[\Sigma]$ denotes the sequent $\Gamma, \Sigma, \Delta \Rightarrow \Pi$.
- $\Sigma\ N\ C \iff \vdash_{\mathbf{FL}} C[\Sigma]$.
- $\Gamma \circ \Delta := \Gamma, \Delta$ (concatenation of sequences).
- $\varepsilon :=$ the empty sequence.

- $\epsilon := (\_ \Rightarrow \ \ )$.

Note that the naive definition of W' as the collection of all right-hand sides Π would not allow for N to be nuclear. By contrast, under the above augmented definition of W', N becomes nuclear for purely syntactical reasons:

$$\begin{array}{rcl} \Sigma_1 \circ \Sigma_2 \ N \ (\Gamma, \_ , \Delta \Rightarrow \Pi) & \iff & \vdash_{\mathbf{FL}} \Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi \\ & \iff & \Sigma_2 \ N \ (\Gamma, \Sigma_1, \_ , \Delta \Rightarrow \Pi) \\ & \iff & \Sigma_1 \ N \ (\Gamma, \_ , \Sigma_2, \Delta \Rightarrow \Pi). \end{array}$$

The dual algebra $\mathbf{W}^+_{\mathbf{FL}}$ together with the valuation $e(\alpha) := \alpha^{\rhd\lhd} = \alpha^{\lhd}$ leads to the completeness theorem:

$$\models_{\mathbf{FL}} \alpha \quad \Longrightarrow \quad \mathbf{W}^+_{\mathbf{FL}}, e \models 1 \leq \alpha \quad \Longrightarrow \quad \varepsilon \in \alpha^{\lhd} \quad \Longrightarrow \quad \vdash_{\mathbf{FL}} \alpha.$$

Finally, by replacing $\vdash_{\mathbf{FL}}$ above with $\vdash^{cf}_{\mathbf{FL}}$ (the cut-free derivability relation) we obtain a residuated frame $\mathbf{W}^{cf}_{\mathbf{FL}}$. Although $\alpha^{\rhd\lhd} = \alpha^{\lhd}$ is not ensured *a priori*, the dual algebra $(\mathbf{W}^{cf}_{\mathbf{FL}})^+$ is nevertheless useful for an algebraic proof of cut-admissibility (cf. [20]):

$$\vdash_{\mathbf{FL}} \alpha \quad \Longrightarrow \quad \models_{\mathbf{FL}} \alpha \quad \Longrightarrow \quad \varepsilon \in \alpha^{\lhd} \quad \Longrightarrow \quad \vdash^{cf}_{\mathbf{FL}} \alpha.$$

Remark 5.6. We may write $x \rightharpoonup z$ for $(x, z) \in W \times W'$. Then N can be viewed as the collection of "valid sequents." The perspective that N selects some valid objects, rather than linking two elements, will be useful in the definition of a residuated hyperframe that follows.

5.2. Residuated hyperframes

As illustrated by Example 5.5, residuated frames are intimately connected with sequents; to capture hypersequents we define below residuated hyperframes. Residuated hyperframes also have a double motivation. In the setting of proof-theory they reflect the structure of hypersequents, just as residuated frames reflect the structure of sequents. In the algebraic setting they reflect the behavior of ∇.

Given a set X, we write X^* for the free commutative monoid $(X^*, \mid, \emptyset)$ generated by X; notice that here we use symbol $\mid$ for the multiplication.

Definition 5.7. A *residuated hyperframe* is a structure of the form $\mathbf{H} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$, where

- W and W' are sets and $\Vdash \subseteq H$, where $H := (W \times W')^*$. We write $x \rightharpoonup y$ for $(x, y) \in W \times W'$ and $\Vdash h$ when $h \in \Vdash$ holds.
- $(W, \circ, \varepsilon)$ is a monoid and $\epsilon \in W'$.
- For all $x, y \in W$ and $z \in W'$ there exist elements $x \backslash\!\!\backslash z, z /\!\!/ y \in W'$ such that for any $h \in H$,

$$\Vdash h \mid x \circ y \rightharpoonup z \iff \Vdash h \mid y \rightharpoonup x \backslash\!\!\backslash z \iff \Vdash h \mid x \rightharpoonup z /\!\!/ y.$$

- $\Vdash h$ implies $\Vdash h \mid g$ for any $h, g \in H$.
- $\Vdash h \mid g \mid g$ implies $\Vdash h \mid g$ for any $h, g \in H$.

Each element h of H is of the form $x_1 \rightharpoonup y_1 \mid \cdots \mid x_n \rightharpoonup y_n$. This is obviously an analogue of a hypersequent, where each component $x_i \rightharpoonup y_i \in W \times W'$ corresponds to a sequent. Also, the last two rules are frame analogues of the external structural rules $(EW), (EC)$.

Example 5.8. Given an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$, we construct a residuated hyperframe $\mathbf{H_A} := (A, A, \Vdash, \cdot, 1, 0)$, where $\Vdash$ is defined by:

$$\Vdash x_1 \rightharpoonup y_1 | \ldots | x_n \rightharpoonup y_n \iff 1 \leq (x_1 \backslash y_1) \nabla \cdots \nabla (x_n \backslash y_n).$$

This is a natural construction, since in the case when $\mathbf{A}$ is subdirectly irreducible, we have:

$$\Vdash x_1 \rightharpoonup y_1 | \ldots | x_n \rightharpoonup y_n \iff x_1 \leq y_1 \text{ or } \cdots \text{ or } x_n \leq y_n. \tag{2}$$

More generally, let $\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i$ be a subdirect representation with canonical projections $e_i : \mathbf{A} \longrightarrow \mathbf{A}_i$ $(i \in I)$. Then,

$$\Vdash x_1 \rightharpoonup y_1 | \ldots | x_n \rightharpoonup y_n \iff \forall i \in I.\ e_i(x_1) \leq e_i(y_1) \text{ or } \cdots \text{ or } e_i(x_n) \leq e_i(y_n).$$

Example 5.9. For another example, we may build $\mathbf{H_{FL}} := (W, W', \Vdash, \circ, \varepsilon, \epsilon)$, which is the same as $\mathbf{W_{FL}}$ except that $\Vdash$ is defined by:

$$\Vdash \Sigma_1 \rightharpoonup C_1 \mid \ldots \mid \Sigma_n \rightharpoonup C_n \iff \vdash_{\mathbf{HFL}} C_1[\Sigma_1] \mid \cdots \mid C_n[\Sigma_n].$$

By replacing $\vdash_{\mathbf{HFL}}$ with $\vdash^{cf}_{\mathbf{HFL}}$ (the cut-free derivability relation) as before, we obtain a residuated hyperframe $\mathbf{H}^{cf}_{\mathbf{FL}}$.

The above examples reveal that the notion of residuated hyperframe is applicable to both algebra and proof theory, as was the notion of residuated frame. However, the success of residuated frames in Algebraic Proof Theory comes from the ability to construct an algebraic model, the dual algebra, and we wish to do the same starting from a residuated hyperframe.

We first observe that residuated hyperframes can be considered as a special class of residuated frames. Given a residuated hyperframe $\mathbf{H} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$, we obtain a residuated frame $r(\mathbf{H}) := (HW, HW', N, \circ, (\emptyset; \varepsilon), (\emptyset; \epsilon))$, where $H := (W \times W')^*$, $HW := H \times W$, $HW' := H \times W'$ and

$$\begin{array}{rcl}(h_x; x) \circ (h_y; y) & = & (h_x | h_y; x \circ y), \\ (h_x; x) \mathbin{\backslash\!\backslash} (h_z; z) & = & (h_x | h_z; x \mathbin{\backslash\!\backslash} z), \\ (h_z; z) \mathbin{/\!/} (h_x; x) & = & (h_z | h_x; z \mathbin{/\!/} x), \\ (h_x; x)\ N\ (h_z; z) & \iff & \Vdash h_x \mid h_z \mid x \rightharpoonup z.\end{array}$$

The nuclearity of N can be easily verified. Hence each residuated hyperframe $\mathbf{H} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$ leads to a complete FL-algebra $\mathbf{H}^+ := r(\mathbf{H})^+$, called the *dual algebra* of $\mathbf{H}$, by Lemma 5.2.

For the later purpose, let us give a more concrete description to the dual algebra. Given $X, Y \subseteq HW$ and $Z \subseteq HW'$, we have

$$\begin{array}{rcl} X^{\rhd} & = & \{(h_z;\, z) \in HW' : \forall (h_x;\, x) \in X.\ \Vdash h_x \mid h_z \mid x \rightharpoonup z\}, \\ Z^{\lhd} & = & \{(h_x;\, x) \in HW : \forall (h_z;\, z) \in Z.\ \Vdash h_x \mid h_z \mid x \rightharpoonup z\}. \end{array}$$

As before, the pair $(^{\rhd}, ^{\lhd})$ forms a Galois connection, and induces a nucleus $\gamma(X) := X^{\rhd\lhd}$ on $\mathcal{P}(HW)$. The dual algebra $\mathbf{H}^+$ has the following structure:

$$\mathbf{H}^+ = (\gamma[\mathcal{P}(HW)], \cap, \cup_\gamma, \circ_\gamma, \backslash, /, (\emptyset;\, \varepsilon)^{\rhd\lhd}, (\emptyset;\, \epsilon)^{\lhd}),$$

where

$$\begin{array}{rcl} X \circ Y & = & \{(h_x|h_y;\, x \circ y) : (h_x;\, x) \in X,\ (h_y;\, y) \in Y\}, \\ X \backslash Y & = & \{(h_y;\, y) : \forall (h_x;\, x) \in X.\ (h_x|h_y;\, x \circ y) \in Y\}, \\ Y/X & = & \{(h_y;\, y) : \forall (h_x;\, x) \in X.\ (h_y|h_x;\, y \circ x) \in Y\}. \end{array}$$

The need for such complicated definition will be justified below in the proof of preservation of structural clauses by the construction of the dual algebra, which in turn can be derived by a link between a pointwise and a setwise interpretation of a structural clause in a residuated hyperframe (Theorem 5.15).

The starting point for the latter is an interesting fact on residuated hyperframes that the hypersequent structure $\Vdash x_1 \rightharpoonup y_1 \mid \cdots \mid x_n \rightharpoonup y_n$ defined on points of W, W' propagates to a higher level structure defined on Galois-closed sets. Given $X, Y \subseteq HW$ and $G_1, G_2 \subseteq H$, we define:

$$\begin{array}{rcll} X \rightharpoonup Y & = & \{h_x|h_y|x \rightharpoonup y : (h_x;\, x) \in X, (h_y;\, y) \in Y^{\rhd}\} & \subseteq H, \\ X \rightharpoonup \emptyset & = & \{h|x \rightharpoonup \epsilon : (h;\, x) \in X\} & \subseteq H, \\ G_1 \mid G_2 & = & \{h_1|h_2 : h_1 \in G_1,\ h_2 \in G_2\} & \subseteq H, \\ \Vdash G_1 & \iff & \Vdash h \text{ for every } h \in G_1. & \end{array}$$

The following lemma plays a fundamental role, connecting the higher level hypersequent structure with the FL-algebra structure $\mathbf{H}^+$.

Lemma 5.10. *For every Galois-closed sets $X, Y \subseteq HW$ and $G \subseteq H$,*

$$\begin{array}{rclcl} \Vdash G \mid X \rightharpoonup Y & \iff & G \times \{\varepsilon\} \subseteq X \backslash Y & \iff & G \times \{\varepsilon\} \subseteq Y/X \\ \Vdash G \mid X \rightharpoonup \emptyset & \iff & G \times \{\varepsilon\} \subseteq X \backslash (\emptyset;\, \epsilon)^{\lhd} & \iff & G \times \{\varepsilon\} \subseteq (\emptyset;\, \epsilon)^{\lhd}/X. \end{array}$$

Hence $\Vdash X \rightharpoonup Y \iff \Vdash \{\emptyset\} \mid X \rightharpoonup Y \iff (\emptyset;\, \varepsilon) \in X \backslash Y \iff X \subseteq Y$. *Also,* $\Vdash X \rightharpoonup \emptyset \iff X \subseteq (\emptyset;\, \epsilon)^{\lhd}$.

Proof. Suppose $\Vdash G \mid X \rightharpoonup Y$. Then for every $g \in G$, $(h_x;\, x) \in X$ and $(h_y;\, y) \in Y^{\rhd}$ we have $\Vdash g \mid h_x \mid h_y \mid x \rightharpoonup y$. Since it holds for every $(h_y;\, y) \in Y^{\rhd}$, we have $(g|h_x;\, x) \in Y^{\rhd\lhd} = Y$. Since it holds for every $(h_x;\, x) \in X$, we obtain $(g;\, \varepsilon) \in X \backslash Y, Y/X$. The converse direction is similar. □

This leads to a soundness theorem for the higher level hypersequent structure, which is completely different from Lemma 3.11. Recall that a valuation f

into $\mathbf{H}^+$ assigns to each propositional variable p a Galois-closed set $f(p) \subseteq HW$. This is homomorphically extended to all formulas, to all sequents and further to all hypersequents:

$$\begin{array}{lll} f(\alpha_1, \ldots, \alpha_n \Rightarrow \beta) & := & f(\alpha_1) \circ \cdots \circ f(\alpha_n) \rightharpoonup f(\beta), \\ f(\alpha_1, \ldots, \alpha_n \Rightarrow \ \ ) & := & f(\alpha_1) \circ \cdots \circ f(\alpha_n) \rightharpoonup \emptyset, \\ f(\Theta_1 \mid \cdots \mid \Theta_n) & := & f(\Theta_1) \mid \cdots \mid f(\Theta_n). \end{array}$$

Notice that $f(\Xi) \subseteq H$ for every hypersequent Ξ. We refer to the above as the *setwise* interpretation of a hypersequent. A hypersequent Ξ is *setwise satisfied* in a residuated hyperframe $\mathbf{H}$ under a valuation f if $\Vdash f(\Xi)$. Below we will show soundness with respect to this notion of satisfaction. Note that for a sequent $\Theta = (\Gamma \Rightarrow \beta)$, Θ is setwise satisfied by $(\mathbf{H}, f)$ iff $\Vdash f(\Gamma) \rightharpoonup f(\beta)$ iff $(\emptyset, \varepsilon) \in f(\Gamma) \backslash f(\beta)$ iff $f(\Gamma) \subseteq f(\beta)$ iff $\mathbf{H}^+, f \models \Theta$. This establishes the second statement of the following lemma.

Theorem 5.11 (Soundness). *Let $\mathbf{H}$ be a residuated hyperframe, f a valuation into $\mathbf{H}^+$ and $\mathcal{H} \cup \{\Xi_0\}$ a set of hypersequents. If $\mathcal{H} \vdash_{\mathbf{HFL}} \Xi_0$, then $\Vdash f(\Xi)$ for all $\Xi \in \mathcal{H}$ implies $\Vdash f(\Xi_0)$. In particular when Ξ_0 is a sequent Θ_0, we have $\mathbf{H}^+, f \models \Theta_0$.*

Proof. By induction on the length of the derivation of $\mathcal{H} \vdash_{\mathbf{HFL}} \Xi_0$. For instance, consider the $(\wedge r)$ rule

$$\frac{\Xi \mid \Gamma \Rightarrow \alpha \quad \Xi \mid \Gamma \Rightarrow \beta}{\Xi \mid \Gamma \Rightarrow \alpha \wedge \beta} \ (\wedge r)$$

and suppose that $\Vdash f(\Xi \mid \Gamma \Rightarrow \alpha)$ and $\Vdash f(\Xi \mid \Gamma \Rightarrow \beta)$. By Lemma 5.10, we have $f(\Xi) \times \{\varepsilon\} \subseteq f(\Gamma) \backslash f(\alpha)$ and $f(\Xi) \times \{\varepsilon\} \subseteq f(\Gamma) \backslash f(\beta)$. Hence

$$f(\Xi) \times \{\varepsilon\} \subseteq f(\Gamma) \backslash (f(\alpha) \cap f(\beta)) = f(\Gamma) \backslash f(\alpha \wedge \beta).$$

Therefore $\Vdash f(\Xi \mid \Gamma \Rightarrow \alpha \wedge \beta)$ by Lemma 5.10 again.

The other right rules are treated similarly. For the left rules, an essential observation is that the element $Z := f(\Xi) \times \{\varepsilon\}$ is *central*, in the sense that $X \circ Z = Z \circ X$ holds for every $X \subseteq HW$.

Now, for the rule

$$\frac{\Xi \mid \Gamma, \alpha, \Delta \Rightarrow \Pi \quad \Xi \mid \Gamma, \beta, \Delta \Rightarrow \Pi}{\Xi \mid \Gamma, \alpha \vee \beta, \Delta \Rightarrow \Pi} \ (\vee l)$$

we assume $\Vdash f(\Xi \mid \Gamma, \alpha, \Delta \Rightarrow \Pi)$ and $\Vdash f(\Xi \mid \Gamma, \beta, \Delta \Rightarrow \Pi)$, which yield $f(\Xi) \times \{\varepsilon\} \subseteq (f(\Gamma) \circ f(\alpha) \circ f(\Delta)) \backslash f(\Pi)$ and $f(\Xi) \times \{\varepsilon\} \subseteq (f(\Gamma) \circ f(\beta) \circ f(\Delta)) \backslash f(\Pi)$, by Lemma 5.10. By letting $X := f(\Gamma) \backslash f(\Pi) / f(\Delta)$, $Z := f(\Xi) \times \{\varepsilon\}$ and taking the centrality of Z into account, we obtain $f(\alpha) \subseteq Z \backslash X$ and $f(\beta) \subseteq Z \backslash X$, so $f(\alpha \vee \beta) \subseteq Z \backslash X$, hence $Z \subseteq (f(\Gamma) \circ f(\alpha \vee \beta) \circ f(\Delta)) \backslash f(\Pi)$. Therefore $\Vdash f(\Xi \mid \Gamma, \alpha \vee \beta, \Delta \Rightarrow \Pi)$.

For

$$\frac{\Xi \mid \Sigma \Rightarrow \alpha \quad \Xi \mid \Gamma, \beta, \Delta \Rightarrow \Pi}{\Xi \mid \Gamma, \Sigma, \alpha\backslash\beta, \Delta \Rightarrow \Pi} \ (\backslash l)$$

we assume $\Vdash f(\Xi \mid \Sigma \Rightarrow \alpha)$ and $\Vdash f(\Xi \mid \Gamma, \beta, \Delta \Rightarrow \Pi)$, namely $Z \subseteq f(\Sigma)\backslash f(\alpha)$ and $Z \subseteq f(\beta)\backslash X$, where $Z := f(\Xi) \times \{\varepsilon\}$ and $X := f(\Gamma)\backslash f(\Pi)/f(\Delta)$. As a consequence, we obtain $Z \circ Z \circ f(\Sigma) \circ f(\alpha\backslash\beta) \subseteq X$, so $Z \circ Z \subseteq (f(\Gamma) \circ f(\Sigma) \circ f(\alpha\backslash\beta) \circ f(\Delta))\backslash f(\Pi)$. By noting that $Z \circ Z = f(\Xi \mid \Xi) \times \{\varepsilon\}$, we have $\Vdash f(\Xi \mid \Xi \mid \Gamma, \Sigma, \alpha\backslash\beta, \Delta \Rightarrow \Pi)$, that implies $\Vdash f(\Xi \mid \Gamma, \Sigma, \alpha\backslash\beta, \Delta \Rightarrow \Pi)$. □

5.3. Preservation of analytic clauses

Note that given a residuated hyperframe $\mathbf{H}$ there are two possible ways to interpret hypersequents, namely as elements of H and as subsets of H, each with an associated form coming from the operations we allow at each level. Thus, corresponding to a typical atomic hypersequent of **HFL** of the form $\Gamma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Gamma_n \Rightarrow \Pi_n$, where each Γ_i is a list of variables, and each Π is a variable or empty, the general form of a *point-hypersequent* (or *first-order hypersequent*) of $\mathbf{H}$ is $x_1 \rightharpoonup y_1 \mid \cdots \mid x_n \rightharpoonup y_n$, where each $x_i \in W$ and each $y_i \in W'$, while the general form of a *set-hypersequent* (or *second-order hypersequent*) of $\mathbf{H}$ is $X_1 \rightharpoonup Y_1 \mid \cdots \mid X_n \rightharpoonup Y_n$, where X_i, Y_i are Galois-closed subsets of HW.

Accordingly, there are two ways, namely *pointwise* and *setwise*, to interpret a structural clause in $\mathbf{H}$. After defining the two interpretations, we show that they coincide for analytic clauses. This will be later used for establishing strong analyticity and extend soundness to hypersequent calculi extending **HFL** with additional structural rules.

We begin with an example, illustrating the two interpretations.

Example 5.12. Let $\mathbf{H} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$ be a residuated hyperframe, and consider the analytic clause

$$x \cdot y \leq z \Longrightarrow x \leq 0 \text{ or } y \leq z, \qquad (em)$$

which corresponds to the excluded middle axiom. Its *pointwise* interpretation in $\mathbf{H}$ is:

$$\frac{\Vdash g \mid x \circ y \rightharpoonup z}{\Vdash g \mid x \rightharpoonup \epsilon \mid y \rightharpoonup z} \ (em^0)$$

for all $x, y \in W$, $z \in W'$ and $g \in H$. The interpretation is obtained by replacing $\cdot$ with $\circ$, 0 with ϵ, $\leq$ with $\rightharpoonup$, and by adding a new variable g. The *setwise* interpretation is:

$$\frac{\Vdash G \mid X \circ Y \rightharpoonup Z}{\Vdash G \mid X \rightharpoonup \emptyset \mid Y \rightharpoonup Z} \ (em^+)$$

for all Galois-closed sets $X, Y, Z \subseteq HW$ and $G \subseteq H$.

The general definition is as follows.

Definition 5.13. Let (q) be an analytic clause

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \dots \text{ or } t_n \leq u_n. \qquad (q)$$

By replacing $(\cdot, 1, 0)$ in each $t_i \leq u_i$ with $(\circ, \varepsilon, \epsilon)$, we obtain $t_i^0 \rightharpoonup u_i^0$. We may then obtain a pointwise interpretation of (q):

$$\frac{\Vdash g \mid t_1^0 \rightharpoonup u_1^0 \quad \cdots \quad \Vdash g \mid t_m^0 \rightharpoonup u_m^0}{\Vdash g \mid t_{m+1}^0 \rightharpoonup u_{m+1}^0 \mid \dots \mid t_n^0 \rightharpoonup u_n^0} \ (q^0)$$

We say that (q) is *pointwise valid* in $\mathbf{H}$ if (q^0) holds for every interpretation of left variables in W, right variables in W' and $g \in H$.

We may also obtain a setwise interpretation:

$$\frac{\Vdash G \mid t_1 \rightharpoonup u_1 \quad \cdots \quad \Vdash G \mid t_m \rightharpoonup u_m}{\Vdash G \mid t_{m+1} \rightharpoonup u_{m+1} \mid \dots \mid t_n \rightharpoonup u_n} \ (q^+)$$

by interpreting each t_i, u_i in the algebra $\mathbf{H}^+$ so that each of t_i, u_i denotes a Galois-closed set. We say that (q) is *setwise valid* in $\mathbf{H}$ if (q^+) holds for every valuation f of variables into $\mathbf{H}^+$ and for every $G \subseteq H$.

Example 5.14. Continuing Example 5.12, we prove that the two interpretations coincide.

(Pointwise $\Rightarrow$ setwise) Assume $\Vdash G \mid X \circ Y \rightharpoonup Z$ (the premise of (em^+)). Our goal is to show that $\Vdash G \mid X \rightharpoonup \emptyset \mid Y \rightharpoonup Z$. So let $g \in G$, $(h_x; x) \in X$, $(h_y; y) \in Y$ and $(h_z; z) \in Z^{\triangleright}$. Then we have $h|x \circ y \rightharpoonup z \in X \circ Y \rightharpoonup Z$, where $h := h_x|h_y|h_z$, so $\Vdash g \mid h \mid x \circ y \rightharpoonup z$ by the assumption. By (em^0) we have $\Vdash g \mid h \mid x \rightharpoonup \epsilon \mid y \rightharpoonup z$. Therefore $\Vdash G \mid X \rightharpoonup \emptyset \mid Y \rightharpoonup Z$.

(Setwise $\Rightarrow$ pointwise) Assume $\Vdash g \mid x \circ y \rightharpoonup z$ (the premise of (em^0)). This means

$$(g; \varepsilon) \in \{(\emptyset; x)\} \circ \{(\emptyset; y)\} \backslash \{(\emptyset; z)\}^{\triangleleft},$$

hence $G \times \{\varepsilon\} \subseteq X \circ Y \backslash Z$, where

$$X := (\emptyset; x)^{\triangleright\triangleleft}, \quad Y := (\emptyset; y)^{\triangleright\triangleleft}, \quad Z := (\emptyset; z)^{\triangleleft}, \quad G := \{g\},$$

so $\Vdash G \mid X \circ Y \rightharpoonup Z$ by Lemma 5.10. By (em^+), we have $\Vdash G \mid X \rightharpoonup \emptyset \mid Y \rightharpoonup Z$, from which we easily derive $\Vdash g \mid x \rightharpoonup \epsilon \mid y \rightharpoonup z$ (the conclusion of (em^0)).

By generalizing the above example, we can prove:

Theorem 5.15. *Let (q) be an analytic clause and $\mathbf{H}$ a residuated hyperframe. Then (q) is pointwise valid in $\mathbf{H}$ if and only if it is setwise valid.*

This theorem allows us to extend the soundness theorem (Theorem 5.11) to hypersequent calculi with additional *analytic* structural rules.

Theorem 5.16. *Let $\mathcal{R}$ be a set of analytic structural rules, $\mathbf{H}$ a residuated hyperframe and f a valuation into $\mathbf{H}^+$. Suppose that all rules in $\mathcal{R}$ are pointwise valid in $\mathbf{H}$. If $\mathcal{H} \vdash_{\mathbf{HFL}(\mathcal{R})} \Xi_0$, then $\Vdash f(\Xi)$ for all $\Xi \in \mathcal{H}$ implies $\Vdash f(\Xi_0)$.*

In particular when Ξ_0 is a sequent Θ_0, we have $\mathbf{H}^+, f \models \Theta_0$.

$$\frac{\Vdash h \mid x \rightharpoonup \iota'(a) \quad \Vdash h \mid \iota(a) \rightharpoonup z}{\Vdash h \mid x \rightharpoonup z}\ (\text{Cut}) \qquad \frac{}{\Vdash \iota(a) \rightharpoonup \iota'(a)}\ (\text{Id})$$

$$\frac{\Vdash h \mid x \rightharpoonup \iota'(a) \quad \Vdash h \mid \iota(b) \rightharpoonup z}{\Vdash h \mid \iota(a\backslash b) \rightharpoonup x \mathbin{\backslash\!\!\backslash} z}\ (\backslash\text{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota(a) \mathbin{\backslash\!\!\backslash} \iota'(b)}{\Vdash h \mid x \rightharpoonup \iota'(a\backslash b)}\ (\backslash\text{R})$$

$$\frac{\Vdash h \mid x \rightharpoonup \iota'(a) \quad \Vdash h \mid \iota(b) \rightharpoonup z}{\Vdash h \mid \iota(b/a) \rightharpoonup z \mathbin{/\!\!/} x}\ (/\text{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota'(b) \mathbin{/\!\!/} \iota(a)}{\Vdash h \mid x \rightharpoonup \iota'(b/a)}\ (/\text{R})$$

$$\frac{\Vdash h \mid \iota(a) \circ \iota(b) \rightharpoonup z}{\Vdash h \mid \iota(a \cdot b) \rightharpoonup z}\ (\cdot\text{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota'(a) \quad \Vdash h \mid y \rightharpoonup \iota'(b)}{\Vdash h \mid x \circ y \rightharpoonup \iota'(a \cdot b)}\ (\cdot\text{R})$$

$$\frac{\Vdash h \mid \iota(a_i) \rightharpoonup z}{\Vdash h \mid \iota(a_1 \wedge a_2) \rightharpoonup z}\ (\wedge\text{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota'(a) \quad \Vdash h \mid x \rightharpoonup \iota'(b)}{\Vdash h \mid x \rightharpoonup \iota'(a \wedge b)}\ (\wedge\text{R})$$

$$\frac{\Vdash h \mid \iota(a) \rightharpoonup z \quad \Vdash h \mid \iota(b) \rightharpoonup z}{\Vdash h \mid \iota(a \vee b) \rightharpoonup z}\ (\vee\text{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota'(a_i)}{\Vdash h \mid x \rightharpoonup \iota'(a_1 \vee a_2)}\ (\vee\text{R})$$

$$\frac{\Vdash h \mid \varepsilon \rightharpoonup z}{\Vdash h \mid \iota(1) \rightharpoonup z}\ (1\text{L}) \qquad \frac{}{\Vdash \varepsilon \rightharpoonup \iota'(1)}\ (1\text{R})$$

$$\frac{}{\Vdash \iota(0) \rightharpoonup \epsilon}\ (0\text{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \epsilon}{\Vdash h \mid x \rightharpoonup \iota'(0)}\ (0\text{R})$$

Figure 5: Gentzen rules

5.4. Gentzen hyperframes

We have seen how residuated hyperframes yield a complete algebra. We will now obtain an embedding $e : \mathbf{A} \longrightarrow \mathbf{H}_{\mathbf{A}}^{+}$ and a valuation f into $(\mathbf{H}_{\mathbf{FL}}^{cf})^{+}$ such that $1 \leq f(\alpha)$ implies $\vdash_{\mathbf{HFL}}^{cf} \alpha$. To ensure the existence of a suitable (quasi)homomorphism (Definition 5.19 below) we need to impose further conditions on residuated hyperframes.

Definition 5.17. A *Gentzen hyperframe* is a tuple $(\mathbf{H}, \mathbf{A}, \iota, \iota')$ where

- $\mathbf{H} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$ is a residuated hyperframe,
- $\mathbf{A}$ is an algebra in the language of $\mathbf{FL}$,
- $\iota : A \longrightarrow W$ and $\iota' : A \longrightarrow W'$ are functions,
- $\Vdash$ satisfies the *Gentzen rules* in (Figure 5) for all $a, b, a_1, a_2 \in A$, $x, y \in W$, $z \in W'$ and $h \in H = (W \times W')^*$.

A *cut-free Gentzen hyperframe* is defined in the same way, but it is not assumed to satisfy the (Cut) rule.

Example 5.18. Consider the frame $\mathbf{H_{FL}} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$ in Example 5.9. Define $\iota : Fm \longrightarrow W$ and $\iota' : Fm \longrightarrow W'$ by

$$\iota(\alpha) := \alpha, \qquad \iota'(\alpha) := (\_ \Rightarrow \alpha).$$

Then $(\mathbf{H_{FL}}, \mathbf{Fm}, \iota, \iota')$ is a Gentzen hyperframe, and $(\mathbf{H^{cf}_{FL}}, \mathbf{Fm}, \iota, \iota')$ is a cut-free Gentzen hyperframe. To see this, just notice that (\ L) and (\ R) can be alternatively presented as

$$\frac{\Vdash h \mid x \rightharpoonup \iota'(a) \quad \Vdash h \mid \iota(b) \rightharpoonup z}{\Vdash h \mid x \circ \iota(a\backslash b) \rightharpoonup z}\ (\backslash\mathrm{L}) \qquad \frac{\Vdash h \mid \iota(a) \circ x \rightharpoonup \iota'(b)}{\Vdash h \mid x \rightharpoonup \iota'(a\backslash b)}\ (\backslash\mathrm{R})$$

which is nothing but the hypersequent rules for \, when x is instantiated with a formula sequence Γ, z with a context C, and a, b with formulas α, β. This illustrates that Gentzen rules are just inference rules of the hypersequent calculus under disguise.

Gentzen rules ensure the existence of a (quasi)homomorphism.

Definition 5.19. Given two algebras $\mathbf{A}$ and $\mathbf{B}$ in the language of $\mathbf{FL}$, a *quasi-homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $F : A \longrightarrow \mathcal{P}(B)$ such that

$$\begin{array}{rcll} c_{\mathbf{B}} & \in & F(c_{\mathbf{A}}) & \text{for } c \in \{0, 1\}, \\ F(a) \star_{\mathbf{B}} F(b) & \subseteq & F(a \star_{\mathbf{A}} b) & \text{for } \star \in \{\cdot, \backslash, /, \wedge, \vee\}, \end{array}$$

where $X \star_{\mathbf{B}} Y = \{x \star_{\mathbf{B}} y \mid x \in X, y \in Y\}$ for any $X, Y \subseteq B$.

It is equivalent to the standard notion of homomorphism when $F(a)$ is a singleton for every $a \in A$.

We have finally reached the main property of Gentzen hyperframes.

Theorem 5.20.

1. *If $(\mathbf{H}, \mathbf{A}, \iota, \iota')$ is a Gentzen hyperframe, then*

$$f(a) := (\emptyset;\ \iota(a))^{\rhd\lhd} = (\emptyset;\ \iota'(a))^{\lhd}$$

defines a homomorphism from $\mathbf{A}$ to $\mathbf{H}^+$. Moreover, if ι, ι' are "injective" in the sense that $\Vdash \iota(a) \rightharpoonup \iota'(b)$ implies $a \leq b$ for every $a, b \in A$, then f is an embedding.

2. *If $(\mathbf{H}, \mathbf{A}, \iota, \iota')$ is a cut-free Gentzen hyperframe, then*

$$F(a) = \{X \in \gamma[\mathcal{P}(HW)] : (\emptyset;\ \iota(a)) \in X \subseteq (\emptyset;\ \iota'(a))^{\lhd}\}$$

is a quasihomomorphism from $\mathbf{A}$ to $\mathbf{H}^+$.

Proof. This is actually a corollary of the main theorem of Gentzen frames proved in [20]. Indeed, if $(\mathbf{H}, \mathbf{A}, \iota, \iota')$ is a (cut-free) Gentzen hyperframe, then $(r(\mathbf{H}), \mathbf{A}, r\iota, r\iota')$, where $r\iota(a) := (\emptyset;\ \iota(a))$ and $r\iota'(a) := (\emptyset;\ \iota'(a))$, is a (cut-free) Gentzen frame in the sense of [13].

We nevertheless outline part of the proof to convey the reader some flavor of the argument (originally due to [33] and later developed by [5]). Let us focus on 2 and prove that F is a quasihomomorphism with respect to the connectives $\backslash$ and $\wedge$.

(Case $\backslash$) Our goal is to show that $F(a)\backslash F(b) \subseteq F(a\backslash b)$ for every $a, b \in A$. So let us take $X \in F(a)$ and $Y \in F(b)$. We have

$$(\emptyset;\ \iota(a)) \in X \subseteq (\emptyset;\ \iota'(a))^{\triangleleft}, \qquad (\emptyset;\ \iota(b)) \in Y \subseteq (\emptyset;\ \iota'(b))^{\triangleleft}. \tag{3}$$

To prove $X\backslash Y \in F(a\backslash b)$, we need to show two things: (i) $(\emptyset;\ \iota(a\backslash b)) \in X\backslash Y$ and (ii) $X\backslash Y \subseteq (\emptyset;\ \iota'(a\backslash b))^{\triangleleft}$.

For (i), suppose that $(h_x;\ x) \in X$ and $(h_y;\ y) \in Y^{\triangleright}$. By (3) we have $(h_x;\ x) \in (\emptyset;\ \iota'(a))^{\triangleleft}$ and $(\emptyset;\ \iota(b)) \in Y$, which imply $\Vdash h_x \mid x \rightharpoonup \iota'(a)$ and $\Vdash h_y \mid \iota(b) \rightharpoonup y$. Hence we have $\Vdash h_x \mid h_y \mid x \circ \iota(a\backslash b) \rightharpoonup y$ by external weakening and ($\backslash$ L). Since it holds for every $(h_y;\ y) \in Y^{\triangleright}$, we have $(h_x;\ x \circ \iota(a\backslash b)) \in Y^{\triangleright\triangleleft} = Y$. Since it holds for every $(h_x;\ x) \in X$, we conclude that $(\emptyset;\ \iota(a\backslash b)) \in X\backslash Y$.

For (ii), suppose that $(h;\ y) \in X\backslash Y$. By (3), we have $(\emptyset;\ \iota(a)) \in X$ and $Y \subseteq (\emptyset;\ \iota'(b))^{\triangleleft}$, hence $(h;\ \iota(a) \circ y) \in (\emptyset;\ \iota'(b))^{\triangleleft}$, namely $\Vdash h \mid \iota(a) \circ y \rightharpoonup \iota'(b)$. By ($\backslash$ R), we have $\Vdash h \mid y \rightharpoonup \iota'(a\backslash b)$. This proves that $(h;\ y) \in (\emptyset;\ \iota'(a\backslash b))^{\triangleleft}$.

(Case $\wedge$) Our goal is to show that $F(a) \wedge F(b) \subseteq F(a \wedge b)$ for every $a, b \in A$. So let us take $X \in F(a)$ and $Y \in F(b)$. We then have (3) again. To prove $X \cap Y \in F(a \wedge b)$, we need to show two things: (i) $(\emptyset;\ \iota(a \wedge b)) \in X \cap Y$ and (ii) $X \cap Y \subseteq (\emptyset;\ \iota'(a \wedge b))^{\triangleleft}$.

For (i), suppose that $(h;\ x) \in X^{\triangleright}$. We have $\Vdash h \mid \iota(a) \rightharpoonup x$ by (3). Hence $\Vdash h \mid \iota(a \wedge b) \rightharpoonup x$ by rule ($\wedge$ L). Since it holds for every $(h;\ x) \in X^{\triangleright}$, we have $(\emptyset;\ \iota(a \wedge b)) \in X^{\triangleright\triangleleft} = X$. Likewise $(\emptyset;\ \iota(a \wedge b)) \in Y$. We therefore conclude that $(\emptyset;\ \iota(a \wedge b)) \in X \cap Y$.

For (ii), suppose that $(h;\ z) \in X \cap Y$. Then by (3) $(h;\ z) \in (\emptyset;\ \iota'(a))^{\triangleleft} \cap (\emptyset;\ \iota'(b))^{\triangleleft}$, namely $\Vdash h \mid z \rightharpoonup \iota'(a)$ and $\Vdash h \mid z \rightharpoonup \iota'(b)$. Hence $\Vdash h \mid z \rightharpoonup \iota'(a \wedge b)$ by rule ($\wedge$ R), from which we conclude $(h;\ z) \in (\emptyset;\ \iota'(a \wedge b))^{\triangleleft}$.

A final remark on 1: If the (Cut) rule is further satisfied, then it results in $(\emptyset;\ \iota(a))^{\triangleright\triangleleft} = (\emptyset;\ \iota'(a))^{\triangleleft}$ so that $F(a)$ of (2) becomes a singleton for every $a \in A$. Indeed, the forward inclusion holds by rule (Id). For the backward inclusion, suppose that $(h_x;\ x) \in (\emptyset;\ \iota'(a))^{\triangleleft}$ and $(h_y;\ y) \in (\emptyset;\ \iota(a))^{\triangleright}$. It means that $\Vdash h_x \mid x \rightharpoonup \iota'(a)$ and $\Vdash h_y \mid \iota(a) \rightharpoonup y$. Hence $\Vdash h_x \mid h_y \mid x \rightharpoonup y$ by external weakening and (Cut). Since it holds for every $(h_y;\ y) \in (\emptyset;\ \iota(a))^{\triangleright}$, we conclude that $(h_x;\ x) \in (\emptyset;\ \iota(a))^{\triangleright\triangleleft}$. □

6. Strong analyticity and hyper-MacNeille completions

The general theory of residuated hyperframes introduced in the previous section is applied here to prove two important results. The first is an algebraic, uniform proof of the strong analyticity of hypersequent calculi defined by analytic rules (Section 6.1). The second is the introduction of a new type of completions, called hyper-MacNeille completions (Section 6.2). We will show that every variety defined by acyclic $\mathcal{P}_3^\flat$ equations admits hyper-MacNeille completions; although the argument here is considerably more complicated than the argument in [13], the benefit of the current approach is that the new completion method behaves well with respect to regularity, i.e., preservation of existing joins and meets.

6.1. Strong analyticity

Let $\mathcal{R}$ be any set of analytic structural rules, we show that the hypersequent calculus $\mathbf{HFL}(\mathcal{R})$ is strongly analytic (cf. Definition 2.1 referring to hypersequents). Our proof encompasses many ad-hoc proofs of cut-admissibility that work for specific (families of) hypersequent calculi, e.g. [3, 12, 32, 31].

First of all, recall that an analytic rule (q°) is obtained from an analytic clause (q) as described in Definition 4.16. Henceforth we identify the clause (q) with the rule (q°), so given a set $\mathcal{R}$ of analytic clauses, we write $\mathbf{HFL}(\mathcal{R})$ to denote the system obtained by adding to $\mathbf{HFL}$ the analytic rules $\{(q^\circ) : (q) \in \mathcal{R}\}$.

To prove strong analyticity, we build a suitable residuated hyperframe. Given an elementary set $\mathcal{S}$ of sequents (cf. Definition 2.1), we define the residuated hyperframe $\mathbf{H}_{\mathcal{R},\mathcal{S}} = (W, W', \Vdash, \circ, \varepsilon, \epsilon)$ as follows:

- $(W, W', \circ, \varepsilon, \epsilon)$ is the same as in $\mathbf{H}_{\mathbf{FL}}$ (Example 5.9).
- $\Vdash (\Sigma_1, C_1) \mid \ldots \mid (\Sigma_n, C_n) \iff \mathcal{S} \vdash^{cf}_{\mathbf{HFL}(\mathcal{R})} C_1[\Sigma_1] \mid \cdots \mid C_n[\Sigma_n]$.

Lemma 6.1. *$(\mathbf{H}_{\mathcal{R},\mathcal{S}}, \mathbf{Fm}, \iota, \iota')$, where $\iota(\alpha) := \alpha \in W$ and $\iota'(\alpha) := (\_ \Rightarrow \alpha) \in W'$, is a cut-free Gentzen hyperframe in which all rules in $\mathcal{R}$ are pointwise valid.*

Proof. $(\mathbf{H}_{\mathcal{R},\mathcal{S}}, \mathbf{Fm}, \iota, \iota')$ is obviously a cut-free Gentzen hyperframe. The following example illustrates that $\mathcal{R}$ is pointwise valid in $\mathbf{H}_{\mathcal{R},\mathcal{S}}$.

Suppose that $\mathcal{R}$ contains the analytic clause

$$x \cdot y \leq z \Longrightarrow x \leq 0 \text{ or } y \leq z. \qquad (em)$$

We need to verify that

$$\frac{\Vdash g \mid x \circ y \rightharpoonup z}{\Vdash g \mid x \rightharpoonup \epsilon \mid y \rightharpoonup z}\ (em^0)$$

holds for every $x, y \in W$, $z \in W'$ and $g \in H = (W \times W')^*$. Notice that each $x \in W$ is a formula sequence Σ, each z is a context of the form $(\Gamma, \_, \Delta \Rightarrow \Pi)$, and each g is a hypersequent Ξ. Hence (em^0) just amounts to the analytic rule corresponding to (em) (Figure 4):

$$\frac{\Xi \mid \Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi}{\Xi \mid \Sigma_1 \Rightarrow\ \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi}\ (em^\circ)$$

Hence it is trivial that (em) is pointwise valid in $\mathbf{H}_{R,S}$. $\square$

We now define a valuation f. For every propositional variable p, let

$$S(p) := \{(\emptyset;\, \Gamma) \in HW : (\Gamma \Rightarrow p) \in \mathcal{S}\} \cup \{(\emptyset;\, p)\}$$

and define $f : \mathbf{Fm} \longrightarrow \mathbf{H}^{+}_{R,S}$ by $f(p) := S(p)^{\rhd\lhd}$ and homomorphically extending it to all formulas.

Lemma 6.2. *For any formula α, $(\emptyset;\, \alpha) \in f(\alpha) \subseteq (\emptyset;\, \_ \Rightarrow \alpha)^{\lhd}$. Moreover, $\Vdash f(\Theta)$ holds for every sequent $\Theta \in \mathcal{S}$.*

Proof. The first claim is proved by induction on the structure of α. For the base case, observe that $(\emptyset;\, p) \in f(p) \subseteq (\emptyset;\, \_ \Rightarrow p)^{\lhd}$ holds by definition. For the induction step, notice that the claim to be proved just amounts to $f(\alpha) \in F(\alpha)$ where F is the quasihomomorphism mentioned in Theorem 5.20.2. Now the induction hypothesis implies

$$f(\alpha \star \beta) = f(\alpha) \star f(\beta) \in F(\alpha) \star F(\beta) \subseteq F(\alpha \star \beta)$$

for every $\star \in \{\wedge, \vee, \cdot, \backslash, /\}$. The last inclusion is due to Theorem 5.20.2.

For the second claim, suppose that Θ is of the form $p_1, \dots, p_n \Rightarrow q$ (the case when it is of the form $p_1, \dots, p_n \Rightarrow$ is similar). Since $\mathcal{S}$ is closed under cuts, we obviously have $S(p_1) \circ \cdots \circ S(p_n) \subseteq S(q)$, so $f(p_1) \circ \cdots \circ f(p_n) \subseteq f(q)$. Therefore $\Vdash f(\Theta)$ by Lemma 5.10. $\square$

We are now ready to prove:

Theorem 6.3. *If $\mathcal{R}$ is a set of analytic clauses, then $\mathbf{HFL}(\mathcal{R})$ is strongly analytic.*

Proof. Let $\mathcal{S}$ be an elementary set and suppose that $\mathcal{S} \vdash_{\mathbf{HFL}(\mathcal{R})} \Xi$ holds for some hypersequent Ξ. We build a cut-free Gentzen hyperframe $(\mathbf{H}_{\mathcal{R},\mathcal{S}}, \mathbf{Fm}, \iota, \iota')$ as above. Since $\mathcal{R}$ is pointwise valid in $\mathbf{H}_{\mathcal{R},\mathcal{S}}$ by Lemma 6.1, and also since the valuation f satisfies all sequents in $\mathcal{S}$ by Lemma 6.2, Theorem 5.16 implies that $\Vdash f(\Xi)$.

If Ξ consists of a single sequent $\Rightarrow \alpha$, then $\Vdash f(\Xi)$ means $\mathbf{H}^{+}_{\mathcal{R},\mathcal{S}}, f \models \alpha$, so $(\emptyset;\, \varepsilon) \in f(\alpha) \subseteq (\emptyset;\, \_ \Rightarrow \alpha)^{\lhd}$ by Lemma 6.2. Hence $\mathcal{S} \vdash^{cf}_{\mathbf{HFL}(\mathcal{R})} \Xi$.

The general case can be best understood by means of an example. Suppose that Ξ is of the form $\alpha, \beta \Rightarrow \gamma \mid \gamma \Rightarrow$. Then $\Vdash f(\Xi)$ means that

$$\Vdash f(\alpha) \circ f(\beta) \rightharpoonup f(\gamma) \mid f(\gamma) \rightharpoonup \emptyset.$$

We have $(\emptyset;\, \alpha) \in f(\alpha)$, $(\emptyset;\, \beta) \in f(\beta)$, $(\emptyset;\, \gamma) \in f(\gamma)$ and $(\emptyset;\, \_ \Rightarrow \gamma) \in f(\gamma)^{\rhd}$. Altogether, they imply $\Vdash \alpha, \beta \rightharpoonup \gamma \mid \gamma \rightharpoonup \epsilon$, namely $\mathcal{S} \vdash^{cf}_{\mathbf{HFL}(\mathcal{R})} \alpha, \beta \Rightarrow \gamma \mid \gamma \Rightarrow$. $\square$

6.2. Hyper-MacNeille completions

As another application of residuated hyperframes, we address here the issue of completions. A simple argument that the variety of FL-algebras defined by acyclic $\mathcal{P}_3^\flat$ equations is closed under completions is already contained in [13]. However, the completions there are not regular (namely they do not necessarily preserve existing joins and meets), as they are obtained by combining subdirect representations and MacNeille completions:

$$\mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i \longrightarrow \prod_{i\in I} \mathbf{W}^+_{\mathbf{A}_i},$$

and the former is not regular.

Regular completions are important, for instance, to prove algebraic completeness of a predicate logic (i.e., completeness with respect to complete algebras) [34, 35]. However, it is not always easy to prove that a whole variety is closed under regular completions, especially when the variety is not closed under MacNeille completions. For instance, it takes a 22-page paper [23] to prove that the variety of Heyting algebras generated by the 3-element algebra admits regular completions.

Our purpose here is to apply the methodology developed so far to the issue of regular completions for the variety of FL-algebras defined by acyclic $\mathcal{P}_3^\flat$ equations. However, we will not be as ambitious as [23] and will not try to prove that *all* members of a variety admit regular completions. Instead, we show that *externally distributive* members of a given variety admit regular completions.

We begin with a basic observation on the residuated hyperframes $\mathbf{H}_\mathbf{A}$ defined in Section 5.2.

Lemma 6.4. *Let $\mathbf{A}$ be an FL-algebra and $id : A \longrightarrow A$ the identity map. Then, $(\mathbf{H}_\mathbf{A}, \mathbf{A}, id, id)$ is a Gentzen hyperframe and $\Vdash a \rightharpoonup b$ implies $a \leq b$.*

Proof. Let $e : \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ be a subdirect representation. We have:

$$\begin{array}{lll} \Vdash a_1 \rightharpoonup b_1 \mid \cdots \mid a_n \rightharpoonup b_n & \Longleftrightarrow & 1 \leq \flat(a_1\backslash b_1) \vee \cdots \vee \flat(a_n\backslash b_n) \\ & \Longleftrightarrow & \forall i \in I.\ e_i(a_1) \leq e_i(b_1) \text{ or } \cdots \text{ or } e_i(a_n) \leq e_i(b_n), \end{array}$$

where $e_i : \mathbf{A} \longrightarrow \mathbf{A}_i$ is the canonical projection map. This allows us to verify the Gentzen rules component-wise, in a straightforward way. $\square$

As a consequence of Theorem 5.20, we obtain an embedding $f : \mathbf{A} \longrightarrow \mathbf{H}^+_\mathbf{A}$ and we call $(\mathbf{H}^+_\mathbf{A}, f)$ the *hyper-MacNeille completion* of $\mathbf{A}$.

We first observe that the hyper-MacNeille completion of $\mathbf{A}$ coincides with the MacNeille completion when $\mathbf{A}$ is subdirectly irreducible.

Lemma 6.5. *Let $\mathbf{A}$ be a subdirectly irreducible FL-algebra and $X \subseteq HW$ a Galois-closed set in $\mathbf{H}_\mathbf{A}$. Then $(h;\, a) \in X$ if and only if $(\emptyset;\, a) \in X$ or $\Vdash h$.*

Proof. ($\Leftarrow$) Let $(g;\, c) \in X^\rhd$. If $(\emptyset;\, a) \in X$, then we obtain $\Vdash g \mid a \rightharpoonup c$, so $\Vdash g \mid h \mid a \rightharpoonup c$ by external weakening. On the other hand, $\Vdash h$ immediately implies $\Vdash g \mid h \mid a \rightharpoonup c$. Hence $(h;\, a) \in X$.

($\Rightarrow$) Suppose that $(h;\, a) \in X$ and $\not\Vdash h$. For every $(g;\, c) \in X^\rhd$, we have $\Vdash h \mid g \mid a \rightharpoonup c$, namely $\Vdash h$ or $\Vdash g$ or $a \leq c$ by (2) of Example 5.8. Since the first case never holds by assumption, we have $\Vdash g$ or $a \leq c$, namely $\Vdash g \mid a \rightharpoonup c$ for every $(g;\, c) \in X^\rhd$. This shows that $(\emptyset;\, a) \in X$. $\square$

Proposition 6.6. *Let* $\mathbf{A}$ *be a subdirectly irreducible FL-algebra. Then the hyper-MacNeille completion* $(\mathbf{H}^+_{\mathbf{A}}, f)$ *is isomorphic to the MacNeille completion of* $\mathbf{A}$*.*

Proof. Given a Galois-closed set X, let $X_0 := \{(\emptyset;\, a) : (\emptyset;\, a) \in X\}$ and $X_1 := \{(\emptyset;\, c) : (\emptyset;\, c) \in X^{\rhd}\}$. We claim that

$$X = X_0^{\rhd\lhd} = X_1^{\lhd}.$$

It is straightforward to verify $X \supseteq X_0^{\rhd\lhd}$ and $X \subseteq X_1^{\lhd}$.

To show $X \subseteq X_0^{\rhd\lhd}$, let $(h;\, a) \in X$. Then either $(\emptyset;\, a) \in X$ or $\Vdash h$ by Lemma 6.5. In both cases, $(h;\, a)$ belongs to $X_0^{\rhd\lhd}$ by Lemma 6.5 again.

To show $X_1^{\lhd} \subseteq X$, let $(h;\, a) \in X_1^{\lhd}$ and $(g;\, c) \in X^{\rhd}$. Similarly to Lemma 6.5, we can show that either $\Vdash g$ or $(\emptyset;\, c) \in X^{\rhd}$ holds. In the former case, we have $\Vdash g \mid h \mid a \rightharpoonup c$ by external weakening. In the latter case, we have $\Vdash h \mid a \rightharpoonup c$, hence $\Vdash g \mid h \mid a \rightharpoonup c$. This proves that $(h;\, a) \in X^{\rhd\lhd} = X$.

By Lemma 5.3 and by recalling that the embedding $f : \mathbf{A} \longrightarrow \mathbf{H}^+_{\mathbf{A}}$ is defined by $f(a) := (\emptyset;\, a)^{\rhd\lhd} = (\emptyset;\, a)^{\lhd}$, we obtain

$$\begin{array}{rcccl} X & = & \gamma[X_0] & = & \bigcup_\gamma \{f(a) : (\emptyset;\, a) \in X_0\}, \\ & = & X_1^{\lhd} & = & \bigcap \{f(c) : (\emptyset;\, c) \in X_1\}. \end{array}$$

Hence the completion $(\mathbf{H}^+_{\mathbf{A}}, f)$ is join-dense and meet-dense. The definition of $(\cdot, \backslash, /)$ in $\mathbf{H}^+_{\mathbf{A}}$ also conforms to Theorem 2.5. $\square$

This establishes the equivalence of hyper-MacNeille and MacNeille completions for subdirectly irreducible FL-algebras. On the other hand, the forthcoming theorem shows that they are in general quite different.

Lemma 6.7. *Let* $\mathbf{A}$ *be an FL-algebra and* $e : \mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i$ *a subdirect representation. Also, let* (q) *be an analytic clause. If* $\mathbf{A}_i \models (q)$ *for every* $i \in I$*, then* (q) *is pointwise valid in* $\mathbf{H}_{\mathbf{A}}$*.*

Proof. We again work on an example. Assume that (q) is

$$x \cdot y \leq z \Longrightarrow x \leq 0 \text{ or } y \leq z. \qquad (em)$$

We need to verify that

$$\frac{\Vdash g \mid x \circ y \rightharpoonup z}{\Vdash g \mid x \rightharpoonup \epsilon \mid y \rightharpoonup z}\ (em^0)$$

holds for every $x, y \in W$, $z \in W'$ and $g \in H = (W \times W')^*$. We assume that g is of the form $x_1 \rightharpoonup y_1 \mid \cdots \mid x_n \rightharpoonup y_n$ and write $e_i(g)$ for the disjunctive clause $e_i(x_1) \leq e_i(y_1)$ or $\cdots$ or $e_i(x_n) \leq e_i(y_n)$. Then $\Vdash g \mid x \circ y \rightharpoonup z$ means that $e_i(g)$ or $e_i(xy) \leq e_i(z)$ holds in every $\mathbf{A}_i$ $(i \in I)$. Since $\mathbf{A}_i$ satisfies (em) we obtain $e_i(g)$ or $e_i(x) \leq 0$ or $e_i(y) \leq e_i(z)$. This shows $\Vdash g \mid x \rightharpoonup \epsilon \mid y \rightharpoonup z$ (recall that $\epsilon = 0$ in $\mathbf{H}_{\mathbf{A}}$). $\square$

Theorem 6.8. *Let $\mathcal{E}$ be a set of equations equivalent to a set $\mathcal{R}$ of analytic structural clauses. Then the variety $\mathsf{FL}(\mathcal{E})$ admits hyper-MacNeille completions.*

Proof. Let $\mathbf{A} \in \mathsf{FL}(\mathcal{E})$ and $\mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ be a subdirect representation. We have $\mathbf{A}_i \in \mathsf{FL}(\mathcal{E})_{SI}$ for every $i \in I$. Hence by the definition of equivalence (see Corollary 3.19), every $\mathbf{A}_i$ satisfies $\mathcal{R}$. Hence $\mathcal{R}$ is pointwise valid in $\mathbf{H_A}$ by Lemma 6.7.

On the other hand, by thinking of $\mathcal{E}$ as a set of formulas, we have $\mathcal{E} \subseteq \mathbf{L}(\mathcal{E}) = \mathbf{L}(\mathcal{R})$ by Corollary 3.19. Therefore by Theorem 5.16, we have $\mathbf{H}^+_\mathbf{A} \models \alpha$ for every $\alpha \in \mathcal{E}$. We have thus obtained the hyper-MacNeille completion $\mathbf{H}^+_\mathbf{A}$ of $\mathbf{A}$, which belongs to $\mathsf{FL}(\mathcal{E})$. $\square$

Remark 6.9. This should be contrasted with a deep result of [7], which shows that there are exactly three varieties of Heyting algebras which admit MacNeille completions: the whole variety HA, the variety BA of Boolean algebras and the trivial variety. On the other hand, there are infinitely many different $\mathcal{P}_3^\flat$ equations (= $\mathcal{P}_3$ equations in presence of commutativity and integrality) that define an intermediate variety between BA and HA. Our result states that all such varieties admit hyper-MacNeille completions.

The above argument is more complicated than that in [13] for the subdirect MacNeille completion. The advantage of hyper-MacNeille completions is a better behavior with respect to regularity.

Definition 6.10. An FL-algebra $\mathbf{A}$ is said to be *externally distributive* if for every $a, b \in A$ and every set $C \subseteq A$ such that $\bigwedge C$ and $\bigvee C$ exist in $\mathbf{A}$,

$$\begin{array}{rcl} 1 \le (a\backslash c)\nabla b \text{ for every } c \in C & \Longrightarrow & 1 \le (a\backslash (\bigwedge C))\nabla b, \\ 1 \le (c\backslash a)\nabla b \text{ for every } c \in C & \Longrightarrow & 1 \le ((\bigvee C)\backslash a)\nabla b. \end{array}$$

External distributivity turns out to be a sufficient condition for regularity.

Theorem 6.11. *If $\mathbf{A}$ is an externally distributive FL-algebra, then $\mathbf{H}^+_\mathbf{A}$ is a regular completion.*

Proof. If external distributivity holds for $C \subseteq A$, then the Gentzen hyperframe $\mathbf{H_A}$ satisfies:

$$\frac{\Vdash h \mid \iota(c) \rightharpoonup z, \quad \text{for some } c \in C}{\Vdash h \mid \iota(\bigwedge C) \rightharpoonup z}\ (\textstyle\bigwedge \mathrm{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota'(c) \quad \text{for every } c \in C}{\Vdash h \mid x \rightharpoonup \iota'(\bigwedge C)}\ (\textstyle\bigwedge \mathrm{R})$$

$$\frac{\Vdash h \mid \iota(c) \rightharpoonup z \qquad \text{for every } c \in C}{\Vdash h \mid \iota(\bigvee C) \rightharpoonup z}\ (\textstyle\bigvee \mathrm{L}) \qquad \frac{\Vdash h \mid x \rightharpoonup \iota'(c) \quad \text{for some } c \in C}{\Vdash h \mid x \rightharpoonup \iota'(\bigvee C)}\ (\textstyle\bigvee \mathrm{R})$$

These are just infinitary variants of the Gentzen rules for $\wedge$ and $\vee$. By inspecting the proof of Theorem 5.20, we can confirm that the argument goes through even if we replace binary $\wedge$ with infinitary $\bigwedge$ and binary $\vee$ with infinitary $\bigvee$. Hence we obtain an embedding f which preserves $\bigwedge$ and $\bigvee$. $\square$

For instance, every FL-chain $\mathbf{A}$ is externally distributive, so that the hyper-MacNeille completion $\mathbf{H}^+_\mathbf{A}$ is regular.

7. Summing up

In the previous section we proved that, when added to **HFL**, analytic rules yield strongly analytic hypersequent calculi. We now prove the converse direction.

Theorem 7.1. *Let $\mathcal{R}$ be a set of structural rules. If* $\mathbf{HFL}(\mathcal{R})$ *is strongly analytic, then $\mathcal{R}$ is equivalent to a set $\mathcal{R}'$ of analytic structural rules.*

Proof. Let

$$\frac{\Xi \mid \Theta_1 \quad \cdots \quad \Xi \mid \Theta_m}{\Xi \mid \Theta_{m+1} \mid \cdots \mid \Theta_n}\ (r)$$

be a structural rule in $\mathcal{R}$ and consider its *atomic instance*, in which $\Xi = \emptyset$ and each distinct metavariable is instantiated by a new propositional variable. We still denote it by (r). We may assume that (r) satisfies (the syntactic analogues of) linearity and exclusion (Section 4.3). To transform (r) into an analytic rule, we need to remove the redundant variables from the premises (i.e., those which do not occur in the conclusion). Let us write Ξ_0 for the conclusion $\Theta_{m+1} \mid \cdots \mid \Theta_n$, and P for the premise set $\{\Theta_1, \ldots, \Theta_m\}$. Let P^+ be the least elementary set of sequents that includes P (cf. Definition 2.1). We then have $P^+ \vdash_{\mathbf{HFL}(\mathcal{R})} \Xi_0$. Hence the strong analyticity of $\mathbf{HFL}(\mathcal{R})$ implies that Ξ_0 has a derivation from a finite subset $\{\Theta'_1, \ldots, \Theta'_k\}$ of P^+, and furthermore none of Θ'_l contains a variable that does not occur in Ξ_0 (recall that the definition of strong analyticity includes the subformula property). Thus all the redundant variables, if any, have been removed from the premises. Now by substituting back the metavariables Γ, Π, etc. for the propositional variables (and associating triples $(\Gamma_j, \Delta_j, \Pi_j)$ for each variable on the right hand side), we obtain a structural rule

$$\frac{\Xi \mid \Theta'_1 \quad \cdots \quad \Xi \mid \Theta'_k}{\Xi \mid \Theta_{m+1} \mid \cdots \mid \Theta_n}\ (r')$$

which is analytic (see Definition 4.16) and derivable in $\mathbf{HFL}(\mathcal{R})$. Moreover, it is easy to see that the obtained rule (r') implies the original one (r). If we do this transformation for all $(r) \in \mathcal{R}$, we obtain a set $\mathcal{R}'$ of analytic rules equivalent to $\mathcal{R}$. $\square$

In the previous section, we also proved that if a set $\mathcal{E}$ of equations is equivalent to a set of analytic clauses, then $\mathsf{FL}(\mathcal{E})$ admits (hyper-MacNeille) completions. We now prove the converse direction *under the assumption that $\mathcal{E}$ implies commutativity* (i.e. $xy \leq yx$).

Theorem 7.2. *Let $\mathcal{E}$ be a set of equations which implies commutativity. If $\mathcal{E}$ is equivalent to a set $\mathcal{R}$ of structural clauses and $\mathsf{FL}(\mathcal{E})$ admits completions, then $\mathcal{E}$ is equivalent to a set $\overline{\mathcal{R}}$ of analytic structural clauses.*

In short, admitting completions implies analyticity in the commutative case.

Proof. Let $(q) : P \Longrightarrow C$ be a structural clause in $\mathcal{R}$, where P is a set of premises and C is a disjunctive clause $t_1 \le u_1$ or $\cdots$ or $t_n \le u_n$. We have

$$P \models_{\mathsf{FL}(\mathcal{E})} C^\flat, \tag{4}$$

where $C^\flat := (t_1\backslash u_1)\nabla \cdots \nabla(t_n\backslash u_n)$. Indeed, $\mathsf{FL}(\mathcal{E})_{SI} = \mathsf{FL}(\mathcal{R})_{SI}$ implies $P \models_{\mathsf{FL}(\mathcal{E})_{SI}} C$, so $P \models_{\mathsf{FL}(\mathcal{E})_{SI}} C^\flat$ by Lemma 3.15, and $P \models_{\mathsf{FL}(\mathcal{E})} C^\flat$ by Corollary 3.14.

We transform (q) into an analytic clause $(\overline{q})$ such that $(\overline{q})$ implies (q) over FL and $\models_{\mathsf{FL}(\mathcal{E})_{SI}} (\overline{q})$. This will be sufficient to establish that $\mathcal{E}$ and $\overline{\mathcal{R}} := \{(\overline{q}) : (q) \in \mathcal{R}\}$ are equivalent.

We basically follow the procedure described in Section 4.3. Step 5 can be fulfilled without any problem. So we may assume that (q) satisfies linearity and exclusion. We just have to remove all redundant variables (those which do not occur in the conclusions) from the premises.

Suppose that (q) contains a redundant variable x. Let P^+ be the least set of equations such that $P \subseteq P^+$ and

$$t \le x,\ lxr \le u\ \in P^+ \quad\Longrightarrow\quad ltr \le u\ \in P^+.$$

Let $P^\pm$ be the subset of P^+ that consists of equations which do not contain x. We claim:

$$P^\pm \models_{\mathsf{FL}(\mathcal{E})} C^\flat. \tag{5}$$

Once this claim has been established, the rest will be easy. Indeed, we may regard $C^\flat$ as a *single* formula $(t_1\backslash u_1 \wedge 1) \vee \cdots \vee (t_n\backslash u_n \wedge 1)$ due to commutativity. Hence there is a finite subset $P_0^\pm$ of $P^\pm$ such that $P_0^\pm \models_{\mathsf{FL}(\mathcal{E})} C^\flat$ by the compactness theorem.

We thus obtain a new clause $P_0^\pm \Longrightarrow C$ which holds in $\mathsf{FL}(\mathcal{E})_{SI}$ and implies (q) since all premises in $P_0^\pm$ are derivable from P. Moreover, $P_0^\pm$ does not contain the redundant variable x. Hence by repetition we end up with an analytic clause $(\overline{q})$ with the desired property.

Now let us prove the claim (5). Let $\mathbf{A} \in \mathsf{FL}(\mathcal{E})$ and f a valuation into $\mathbf{A}$ which satisfies all equations in $P^\pm$:

$$\mathbf{A}, f \models P^\pm. \tag{6}$$

By assumption, there is a completion $\mathbf{A}'$ of $\mathbf{A}$ which belongs to $\mathsf{FL}(\mathcal{E})$. We now extend f to a valuation f' into $\mathbf{A}'$ so that $f'(y) = f(y)$ for any variable y different from x. To define $f'(x)$, let T_x be the set of terms defined by

$$T_x := \{t : (t \le x) \in P^+,\ t \text{ does not contain } x\},$$

and let $f'(x) := \bigvee\{f(t) : t \in T_x\}$. It is well defined since $\mathbf{A}'$ is complete. We claim:

$$\mathbf{A}', f' \models P. \tag{7}$$

Once this has been proved, we obtain $\mathbf{A}', f' \models C^\flat$ by (4). Since x does not occur in C, it implies $\mathbf{A}, f \models C^\flat$, thus the claim (5) holds.

So it remains to prove (7).

- Let $s(x, \ldots, x) \leq u$ be an equation in P, where all occurrences of x are indicated. We have:

$$f'(s(x, \ldots, x)) = \bigvee_{t_1 \in T_x} \cdots \bigvee_{t_k \in T_x} f(s(t_1, \ldots, t_k)) \leq f(u) = f'(u).$$

  The inequality holds because $s(t_1, \ldots, t_k) \leq u$ belongs to $P^{\pm}$ for every $t_1, \ldots, t_k \in T_x$, hence is satisfied by f (recall our assumption (6)).

- Let $s(x, \ldots, x) \leq x$ be an equation in P. We have:

$$f'(s(x, \ldots, x)) = \bigvee_{t_1 \in T_x} \cdots \bigvee_{t_k \in T_x} f(s(t_1, \ldots, t_k)) \leq \bigvee_{t \in T_x} f(t) = f'(x).$$

  The inequality holds because $s(t_1, \ldots, t_k) \leq x$ belongs to P^{+} for every $t_1, \ldots, t_k \in T_x$, so $s(t_1, \ldots, t_k) \in T_x$.

The above argument works even if T_x is empty. Hence we have established the remaining claim (7). □

Our main achievements can be summarized as follows:

Theorem 7.3.

1. Every $\mathcal{P}_3^{\flat}$ equation/axiom can be transformed into an equivalent set of structural clauses/rules.

2. Let $\mathcal{E}$ be a set of $\mathcal{P}_3^{\flat}$ equations/axioms. The following are equivalent:

(a) $\mathcal{E}$ is equivalent to a set of acyclic clauses.
(b) $\mathcal{E}$ is equivalent to a set of analytic clauses.
(c) $\mathcal{E}$ is equivalent to a set $\mathcal{R}$ of structural rules such that $\mathbf{HFL}(\mathcal{R})$ *is strongly analytic.*

3. (a) – (c) implies:

(d) $\mathsf{FL}(\mathcal{E})$ admits hyper-MacNeille completions.
(e) $\mathsf{FL}(\mathcal{E})$ admits completions.

4. Whenever $\mathcal{E}$ implies commutativity (exchange), (a) – (e) are all equivalent.

5. Whenever $\mathcal{E}$ implies integrality (left weakening), (a) – (e) all hold.

Proof. 1. Theorem 4.10.
2. (a) ⇒ (b): Theorem 4.15.
(b) ⇒ (c): Theorem 6.3.
(c) ⇒ (a): Theorem 7.1. Note that analytic rules are also acyclic.
3. (b) ⇒ (d): Theorem 6.8.
(d) ⇒ (e): Trivial.
4. (e) ⇒ (b): Theorem 7.2.
5. Theorem 4.15. □

The $\mathcal{N}_2$ equation $x\backslash x \le x/x$ (that also belongs to $\mathcal{P}_3^\flat$) provides a counterexample to (a) – (e) in absence of commutativity. Indeed, we have shown in [14] that the variety defined by $x\backslash x \le x/x$ does not admit any completion.

It is left open whether 4 holds without the assumption of commutativity.

8. Final observations

We conclude with some observations on the expressive power of structural hypersequent rules and on the structure of the substructural hierarchy.

8.1. Limitations of structural hypersequent rules

As seen before each $\mathcal{P}_3^\flat$ axiom can be transformed into equivalent structural (hypersequent) rules. This shows what structural rules *can express*. Here we address the converse problem, namely identifying which properties (equations over FL-algebras, or equivalently, Hilbert axioms in the language of $\mathbf{FL}_\perp$) *cannot* be expressed by structural rules. Notice that finding negative results is often more difficult than obtaining positive ones. A negative result in the formalism of display logic [6] is contained in [27] and characterizes the class of axioms that can be captured by analytic structural display rules to be added to the calculus for the tense logic **KT**; the characterization in [27] (and its generalization in [15] to all display calculi satisfying suitable conditions) is based only on the syntactic shape of the considered axioms. A semantic characterization of the expressive power of structural *sequent* rules is contained in our previous work [14], where we show that (single conclusion) structural sequent rules can only formalize properties which hold in intuitionistic logic, and, among them, only those corresponding to algebraic equations preserved by MacNeille completions in presence of integrality. Similar results can be established for structural hypersequent rules. Let **HSM** be the hypersequent calculus for three-valued Gödel logic **SM** – the strongest proper intermediate logic, semantically characterized by linearly ordered Kripke models containing two worlds. **HSM** consists of $\mathbf{HFL_{ewc}} + (com) + (Bc_2)$ (see Figure 4).

Proposition 8.1 ([12])**.** *Any structural hypersequent rule is either derivable in* **HSM** *or it derives* $\alpha \vee \neg\alpha^n$ *in* $\mathbf{HFL_{ew}}$*, for some natural number* n*.*

We denote by $\mathbf{E}_n$ the extension of $\mathbf{FL_{ew}}$ by $\alpha \vee \neg\alpha^n$ and by $\mathbf{E} = \bigcap_{n\ge1} \mathbf{E}_n$ the intersection of all these logics. Clearly $\mathbf{E}_1$ is classical logic **CL** and $\mathbf{E}_n \subseteq \mathbf{E}_m$ for $n \ge m$. The above proposition states that the logics that could be captured by extending $\mathbf{HFL_{ew}}$ by structural hypersequent rules are limited to the subregions in Figure 6 between **Fm** (the inconsistent logic) and **E** and between **SM** and $\mathbf{FL_{ew}}$.

The expressive power limitations of structural hypersequent rules are however stronger. Indeed, as shown below, only *some* of the logics in these regions can be captured by structural hypersequent rules.

Proposition 8.2. *Any equation* ε *equivalent to a structural hypersequent rule is preserved by hyper-MacNeille completions in presence of integrality.*

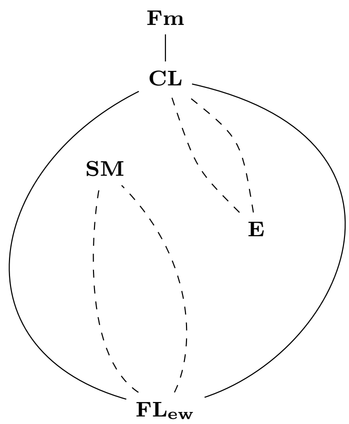

Figure 6: Limitations for structural rules extending $\mathbf{HFL_{ew}}$

Proof. Let (q) be the equivalent structural clause. Theorem 4.15 ensures that, in presence of integrality $x \leq 1$, (q) is equivalent to an analytic clause (q'). By Theorem 6.8, ε is preserved by hyper-MacNeille completions. $\square$

As a corollary we have, for instance, that there is no structural hypersequent rule equivalent to Łukasiewicz axiom $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$ since the corresponding equation is not preserved under any completion [26].

8.2. On the structure of the substructural hierarchy

Let X, Y be sets of equations. We write $X \preceq Y$ if every equation in X is equivalent to a set of equations in Y. We write $X \prec Y$ if $X \preceq Y$ but not $Y \preceq X$.

Obviously $X \subseteq Y$ implies $X \preceq Y$. Hence $\mathcal{N}_n \cup \mathcal{P}_n \preceq \mathcal{N}_{n+1} \cap \mathcal{P}_{n+1}$ holds for every $n \geq 0$.

On the other hand, we know that prelinearity belongs to $\mathcal{P}_2 \subseteq \mathcal{P}_3 \cap \mathcal{N}_3$ but not to $\mathcal{N}_2$ (see [14]). Hence

$$\mathcal{N}_2 \prec \mathcal{P}_3, \qquad \mathcal{N}_2 \prec \mathcal{N}_3.$$

We also know that the variety of MV algebras does not admit any completions [26]. Since it consists of FL_{ew}-algebras defined by an $\mathcal{N}_3 \subseteq \mathcal{N}_4 \cap \mathcal{P}_4$ equation (Łukasiewicz axiom), we obtain

$$\mathcal{P}_3 \prec \mathcal{N}_3$$

Let us also mention the trivial fact that the variety of lattice-ordered groups does not admit any completions, simply because the nontrivial ones do not have least and greatest elements. The same holds for the variety of commutative lattice-ordered groups. Since the latter is axiomatized by FL_e-algebras extended with $1 \leq x(x\backslash 1)$ and the equation is in $\mathcal{P}_3$, but not in $\mathcal{P}_3^\flat$, it follows that $\mathcal{P}_3^\flat \prec \mathcal{P}_3$.

As recently shown in [24] the substructural hierarchy collapses down to the level $\mathcal{N}_3$ in presence of commutativity. Hence a remaining open problem is whether or not the hierarchy collapses to a certain level in the general case.

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