Analytic Proofs for Tense Logic

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Abstract. The first algorithm to transform a proof in Nishimura's sequent calculus **GKt** for tense logic **Kt** into an analytic proof of the same sequent is presented. In an analytic proof, every rule instance is analytic i.e., each formula in every premise is a subformula of some formula in its conclusion. We call this algorithm *analytic restriction* to convey that it extends analytic cut-restriction where just the cut-rule instances are made analytic. This distinction is essential in tense logic since cut and modal rules can both cause non-analyticity. Analytic cut-restriction is itself an extension of cut-elimination so our work contributes to a broader program of transforming arbitrary sequent proofs into ones constructed from a designated set of formulas—not necessarily subformulas. As with cut-elimination, the aim is to limit the proof search space and support proof-theoretic and meta-logical investigations.

Keywords: tense logic· analytic proofs· sequent calculus· analytic restriction · cut-elimination

1 Introduction

Following Gentzen's [12] seminal work on cut-elimination for the sequent calculus in the 1930s, the primary goal of structural proof theory has been to obtain this result for the numerous non-classical logics of interest. Cut-elimination is an algorithm to remove instances of the cut-rule by transforming the given proof. In many cases it implies the subformula property (every formula in the proof is a subformula of the formula to be proved), and the latter in turn implies a significant constraint on the proof search space, aiding the establishment of metalogical results like decidability, complexity and interpolation. Of course, cut-elimination is important in its own right e.g., for the analysis of mathematical proofs [13] and as computation via the Curry-Howard (proofs-as-programs) correspondence [25].

Unfortunately, the cut-elimination result does not hold in the framework of the sequent calculus for most logics of interest. While the traditional response in structural proof theory has been the use of elaborate extensions of the sequent calculus such as hypersequent calculi [1], labeled sequent calculi [28, 19, 11], display calculi [3], this has the drawback of introducing a syntax-heavy framework that complicates or even impedes further investigation.

Ciabattoni et al. [5] propose an alternative to cut-elimination called *cut-restriction* that stays with the sequent calculus: this is a proof transformation algorithm that takes an arbitrary sequent calculus proof as input and outputs a proof where cuts are restricted to a specified and restricted set of formulas. In the case that cut is the only rule in the sequent calculus that can violate analyticity, the cut-restriction result often retains the benefits of cut-elimination, namely a significant constraint on proof search leading to applications, see e.g., [23, 5, 17]. However, a significant drawback of the argument in [5] is its reliance on cut-elimination in the richer hypersequent calculus as an intermediate step. In particular, as the hypersequent cut-elimination cannot be simulated in the sequent calculus, this approach can be said to substantially distort the original sequent proof in the process of deriving the cut-restricted version. Additionally, due to this reliance, the argument applies only to logics having a cut-free hypersequent calculus. In hindsight, Takano's [22] proof for the modal logic **S5** appears to be the first example of a cut-restriction result. Unfortunately, Takano's argument is technically intricate and not widely understood.

A general and direct—i.e., one that stays within the sequent calculus—algorithm that transforms a sequent proof into a cut-restricted sequent proof still remains beyond reach. This has motivated attempts to investigate specific cases, the idea being that techniques applicable in such cases will reveal what a general argument must incorporate. *Analytic*[5] *cut-restriction* [21], where the cut-formulas are restricted to subformulas of their conclusion, is a special case. Recently, [6] identified abstract sufficient conditions for analytic cut-restriction with transformations that stay within the sequent calculus.

In this paper, we consider Nishimura's sequent calculus **GKt** [20] for the normal basic tense logic **Kt**. As already observed in [20], cut-elimination fails in **GKt**, and hence cuts cannot be avoided. Our main result here is an algorithm for transforming any proof in **GKt** into an analytic proof. This work extends the results in [5] in a crucial way: the algorithm here is direct in the sense that the transformations stay *within* the sequent calculus (indeed, there is no cut-free hypersequent calculus for **Kt**). Moreover, the form of restriction that we obtain is a strong form, namely analytic. Note that the analytic cut-restriction algorithm in [6] only applied to a limited class of sequent calculi, those whose rules (i) introduce only one formula at a time (on either the left or right side of the sequent), and (ii) satisfy the subformula property, with the exception of the cut-rule. **GKt** satisfies neither condition, so our algorithm needs to go beyond the one given in [6]. In particular, the cut-rule *and* the modal rules in **GKt** can both cause non-analyticity, so our proof transformation shows how to restrict the rule instances of *all* these rules. To convey that this result constitutes an extension of analytic cut-restriction, we call it *analytic restriction*.

[5] The term 'analytic' follows the tradition, dating back to Leibniz, of referring to proofs which only employs notions that are contained in the statement being proved.

To encourage adaptation to other logics, we present our transformation so as to make the algorithm—rather than just the proof of its correctness—explicit.

Related literature. In Takano's paper establishing analytic cut-restriction for **S5**, it is observed [22, Digression 1.2] that every provable sequent in **GKt** and **GKt**4 has an analytic proof. Note that this is based on semantic considerations, as opposed to the proof transformation algorithm given here. Nishimura [20] presents an alternative proof calculus for **Kt** that witnesses completeness of its analytic proofs but it is not a sequent calculus. Indeed, cut-free calculi for **Kt** have been presented in several complex proof frameworks such as the display calculus [18, 29, 14] and labeled sequent systems [28, 19]; for the axiomatic extension **Kt4.3**, a cut-free hypersequent calculus has been provided in [16] but it does not contain **GKt** as subcalculus.

More generally, there is a body of work demonstrating the existence of cut-restricted proofs using semantic methods, without providing an explicit proof transformation algorithm, e.g., [10, 17, 8, 2]. We observe that this situation contrasts with that of cut-elimination, where it is the syntactic arguments that are standard, and semantic approaches (see e.g., [9, 24]) are relatively rare.

2 Preliminaries

The tense logic **Kt** extends the normal basic modal logic **K** (which has a single modal operator $\Box$) by a second modal operator $\blacksquare$ and a necessitation rule for it, and the following axiom schemes [4]. The latter two are called *converse axioms*.

$$\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B) \qquad A \to \Box\neg\blacksquare\neg A \qquad A \to \blacksquare\neg\Box\neg A$$

Kt aims to provide a minimal[6] setup for temporal reasoning, where $\Box A$ is read as 'A holds at every point in the future' and $\blacksquare A$ is read as 'A holds at every point in the past'. The corresponding diamond operators can be defined from the box operators using negation.

A *sequent* is a pair of formula multisets, written as $\Gamma \Rightarrow \Delta$. Nishimura's sequent calculus **GKt** [20] for **Kt** appears in Figure 1. The reader may observe that what is labeled here as the cut-rule is in fact Gentzen's mix (i.e., multicut) rule. This is a standard generalization of cut that is convenient for simplifying the cut-elimination argument in the presence of the contraction rule. To simplify terminology, we use cut as a shorthand for multicut throughout.

The reader may find the modal rules of **GKt** (below) somewhat unfamiliar.

$$\frac{\Gamma \Rightarrow A, \blacksquare\Delta}{\Box\Gamma \Rightarrow \Box A, \Delta}\,(\Box) \qquad\qquad \frac{\Gamma \Rightarrow A, \Box\Delta}{\blacksquare\Gamma \Rightarrow \blacksquare A, \Delta}\,(\blacksquare)$$

The above rules can be conveniently remembered as the usual **K** rule which adds a modality going from premise to conclusion, plus an additional succedent context Δ which adds the other modality but this time from conclusion to premise.

[6] Properties like transitivity/antisymmetry are not imposed on the temporal relation.

$$\frac{}{p \Rightarrow p}\ (init) \qquad \frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (w) \qquad \frac{\Sigma, \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Pi}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (c)$$

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\ (\neg_L) \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ (\neg_R)$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge_L) \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ (\wedge_R)$$

$$\frac{\Gamma \Rightarrow A, \blacksquare\Delta}{\Box\Gamma \Rightarrow \Box A, \Delta}\ (\Box) \qquad \frac{\Gamma \Rightarrow A, \Box\Delta}{\blacksquare\Gamma \Rightarrow \blacksquare A, \Delta}\ (\blacksquare)$$

$$\frac{\Gamma \Rightarrow A^k, \Delta \qquad \Sigma, A^l \Rightarrow \Pi}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (cut)\ k, l > 0$$

Fig. 1. Nishimura's sequent calculus **GKt** for **Kt**.

In the familiar Kripke semantics for **Kt**, the rule ($\Box$) asserts that if the premise $\Gamma \Rightarrow A, \blacksquare\Delta$ is valid so is the conclusion $\Box\Gamma \Rightarrow \Box A, \Delta$. This can be easily verified by reasoning contrapositively: suppose that that the conclusion is not true at some world w. That means that Γ holds at any single future step from w, A fails at some single future step (call that world w'), and every formula in Δ fails at w. From the perspective of w', we have that Γ holds and A fails and every formula in Δ fails at some single past step (this is witnessed by w). Hence the premise is not true at w'. The ($\blacksquare$) rule can be interpreted in the semantics in a similar way.

Definition 1. *A rule instance is* analytic *if every formula in every premise is a subformula of some formula in the conclusion, otherwise it is* non-analytic*.*

If a rule instance is non-analytic, any formula in the premise that is not a subformula of any formula in the conclusion is said to be *non-analytic* for that instance.

An *analytic proof* has the property that every rule instance in it is analytic. By inspection, a non-analytic proof in **GKt** must contain either an

(i) instance of ($\Box$) (resp. ($\blacksquare$)) where some context formula $\blacksquare B \in \blacksquare\Delta$ (resp. $\Box B \in \Box\Delta$) is not a subformula of the conclusion, or an
(ii) instance of (*cut*) where the cut-formula is not a subformula of the conclusion

The reason is that none of the other rules has non-analytic rule instances.

Nishimura already observed that cut-elimination fails for **GKt**, since the axiom instance $p \to \Box\neg\blacksquare\neg p$ is provable in **Kt**, but unprovable without cut. Indeed, here is a proof using cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\neg p \Rightarrow \neg p}}{\blacksquare\neg p \Rightarrow \blacksquare\neg p}}{\Rightarrow \neg\blacksquare\neg p, \blacksquare\neg p}}{\Rightarrow \Box\neg\blacksquare\neg p, \neg p} \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}}{p \Rightarrow \Box\neg\blacksquare\neg p}\ (cut)$$

This sequent cannot be proved without cut since the only rule instance with conclusion $p \Rightarrow \Box\neg\blacksquare\neg p$ is weakening (and that would certainly not lead to a proof). Observe, however, that the above proof is analytic. In particular, the cut-formula $\neg p$ is a subformula of the formula $\Box\neg\blacksquare\neg p$ in the conclusion of cut.

3 A high-level overview of the transformation

Gentzen's cut-elimination argument for **LK** and **LJ** can be summarized at a high-level as follows:

> Given a proof where only the last rule instance is cut: permute the cut upwards in each premise until it is principal in both premises of cut. Now, making use of the form of the rules which—viewed upwards—decompose the principal formula into formulas that are strictly smaller in the sense of being strictly smaller in grade (i.e., formula size), or the same grade but occurring closer to the leaves of the proof. Hence, we can transform the original proof to a proof that has strictly smaller cuts. Cut-elimination follows through an induction on the underlying measure.

In this section, in the spirit of the latter text, we give a high-level overview of the algorithm for **GKt** that transforms a proof into an analytic one. This is illustrated with a running example. Another example of a transformed proof appears at the end of Section 4. The formal argument is given in the next section.

For a start, consider a proof where only the last rule instance is non-analytic. The latter must be an instance of (i) one of the modal rules, or (ii) the cut-rule.

Case (i): Suppose that the last rule instance is modal and non-analytic. The formal version of the sketch here appears in Proposition 1. If it is $(\Box)$—the case of $(\blacksquare)$ is analogous—then the formulas that witness the non-analyticity constitute some Δ in the conclusion that becomes $\blacksquare\Delta$ in the premise.

Let us consider the following non-analytic proof as a running example.

$$
\begin{array}{c l}
p, p \Rightarrow p & \\ \hline
p \wedge p \Rightarrow p & \\ \hline
\neg p, p \wedge p \Rightarrow & \\ \hline
\neg p \Rightarrow \neg(p \wedge p) & \\ \hline
\blacksquare\neg p \Rightarrow \blacksquare\neg(p \wedge p) & (\blacksquare)\ \text{critical rule instance} \\ \hline
\Rightarrow \neg\blacksquare\neg p, \blacksquare\neg(p \wedge p) & (\neg_R) \\ \hline
\Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p) & (\Box)
\end{array}
$$

Only the $(\Box)$ rule instance is non-analytic. The witness is the $\blacksquare\neg(p \wedge p)$ in its premise; it is not a subformula of any formula in its conclusion.

The idea is to trace—using the familiar parametric ancestor relation from structural proof theory, see e.g., [3]—the formulas in Δ upwards until a modal rule instance that decomposes them is reached (this is called a *critical instance*). A formula in the *premise context of a critical instance* that is not a parametric ancestor (of some formula in Δ) is called a *critical formula*. Starting from the desired endsequent we apply cut-rules upwards using every critical formula.

In our example, there is one rule instance that is critical (i.e., a rule instance that, viewed upwards, decomposes the $\blacksquare\neg(p \wedge p)$ into $\neg(p \wedge p)$). The premise of that rule instance is $\neg p \Rightarrow \neg(p \wedge p)$. Of the two formulas in this sequent, $\neg(p \wedge p)$ is a parametric ancestor of the $\blacksquare\neg(p \wedge p)$, but $\neg p$ is not. Hence there is one critical formula: $\neg p$. Here is the cut introducing $\neg p$.

$$\dfrac{\Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p), \neg p \qquad \neg p \Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p)}{\Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p)}\ (\text{cut})$$

To prove the premises of the cuts on critical formulas that we introduced, we use portions of the original proof, either by weakening, or by adapting a subproof of the original proof. Specifically, the adaptation is to replace the ancestors of the formulas witnessing non-analyticity with critical formulas.

We utilize the original proof to obtain analytic proofs of $\Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p), \neg p$ and $\neg p \Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p)$. The latter sequent is obtained by applying weakening on $\Box\neg\blacksquare\neg p$ to a subproof of the original proof, as shown below.

$$\dfrac{\dfrac{\dfrac{\dfrac{p, p \Rightarrow p}{p \wedge p \Rightarrow p}}{\neg p, p \wedge p \Rightarrow}}{\neg p \Rightarrow \neg(p \wedge p)}}{\neg p \Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p)}\ (\text{w})$$

To obtain $\Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p), \neg p$, the idea is to reason backwards, first by removing the undesirable formula $\neg(p \wedge p)$ that witnessed the non-analyticity using (w) upwards, and then following the original proof upwards—use the introduced critical formula $\neg p$ in place of $\neg(p \wedge p)$—until the conclusion of the critical instance is reached:

$$\dfrac{\dfrac{\dfrac{\blacksquare\neg p \Rightarrow \blacksquare\neg p}{\Rightarrow \neg\blacksquare\neg p, \blacksquare\neg p}\ (\neg R)}{\Rightarrow \Box\neg\blacksquare\neg p, \neg p}\ (\Box)}{\Rightarrow \Box\neg\blacksquare\neg p, \neg(p \wedge p), \neg p}\ (\text{w})$$

The uppermost sequent in the above is a quasi-initial sequent i.e., a sequent of the form $\Gamma, A \Rightarrow \Delta, A$, easily seen to be provable (in this paper, an initial sequent is taken to have the form $p \Rightarrow p$). As we shall see, this is not a coincidence.

Case (ii): Suppose that the last rule instance is the cut-rule and non-analytic. Permute the cut upwards in the usual way by decomposing propositional connectives. The interesting case is when the cut-rule is non-analytic and principal in both premises by a modal rule. Non-analyticity implies that the cut-formula is not a subformula of the conclusion, from which we can deduce that the cut-formula must be principal in the left premise of the cut (Claim 7). Apply a standard K-style cut-reduction step to identify a cut on a formula of strictly smaller grade (yielding an analytic proof by IH), followed by an application of the modal rule. By inspection, the latter must have strictly smaller grade so the result follows from Case (i).

In general, the transformation we described might not immediately yield an analytic proof, as it may introduce new rule instances that are non-analytic although with strictly smaller grade. To account for these, we consider the multiset of grades of all the non-analytic rule instances in the proof and note that the transformation applied to topmost non-analytic rule instances leads to a strictly smaller multiset under the Dershowitz-Manna multiset ordering. Termination—and hence an analytic proof—follows.

4 Analytic restriction for tense logic

In this section we establish the main theorem. See Section 3 for the definitions of analytic rule, analytic proof, and non-analytic formula in a non-analytic rule instance. The grade of a formula defined as the number of symbols in it.

Definition 2. *The* grade *of a non-analytic rule instance is the maximal grade of a non-analytic formula in it.*

For technical reasons, it is convenient to allow only *atomic* initial sequents $p \Rightarrow p$ in **GKt**. Let us call *quasi-initial* every sequent of the form $\Gamma, A \Rightarrow A, \Delta$. The following can be proved by a simple induction:

Lemma 1 (Axiom expansion). *Every quasi-initial sequent has an analytic* **GKt***-proof.*

The *ancestor relation* between formula occurrences in **GKt**-proofs is defined in the usual way: every occurrence in the context of the conclusion of a rule instance is related to corresponding occurrences in the context of the premise(s), and an occurrence of a principal formula is related to the occurrences of its auxiliary formulas in the premise(s).

An ancestor of a formula is either a subformula or, due to the rules ($\Box$) and ($\blacksquare$), a subformula prefixed by a string of modalities. Due to contraction a formula can have multiple ancestors in the same sequent. Moreover, as we allow the cut-rule, not every formula in a proof is necessarily an ancestor of some formula in the endsequent.

The high-level intuition of Case (i) in Section 3 is formalized in the following.

Proposition 1. *Let α be a* **GKt***-proof that is analytic apart from its last rule, which is a modal rule instance of grade $k > 0$. Then there is a* **GKt***-proof α^* of the same endsequent all of whose non-analytic rule instances are of grade $< k$.*

Proof. We first construct α^* and then prove the desired reduction in grade.

1. Construction of α^*.

The construction consists of three steps: (I) Preprocessing, (II) Tracing non-analytic ancestors, and (III) Restructuring the proof.

(I) Preprocessing (remove trivial cuts). Every cut in α where the cut formula appears in the lower sequent is trivially eliminable. Such a cut must have the following form with $s + u > 0$ or $t + v > 0$.

$$\frac{A^s, \Gamma \Rightarrow A^k, \Delta, A^t \qquad A^u, \Sigma, A^l \Rightarrow \Pi, A^v}{A^{s+u}, \Gamma, \Sigma \Rightarrow \Delta, \Pi, A^{t+v}}\ (\text{cut})$$

If $s+u > 0$ and $t+v > 0$ then either premise can be used to obtain the conclusion via (w) and (c). Else, suppose that $s+u = 0$ (the case of $t+v = 0$ is analogous). Then, noting that $t + v > 0$, proceed as

$$\frac{\Gamma \Rightarrow A^k, \Delta, A^t}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, A^{t+v}}\ (\text{w}),\ (\text{c})$$

Henceforth, without loss of generality we will assume that α is preprocessed.

(II) Tracing non-analytic ancestors. The purpose of this step is to identify the rule instances and formulas that will be crucial in the next step. In particular, this step does not modify the proof.

Suppose that the last rule is ($\Box$) (the case ($\blacksquare$) is analogous). Then write α as follows, where $\blacksquare\Delta$ is the multiset of non-analytic formulas in the rule instance. This means that every formula in $\blacksquare\Pi$ is analytic i.e., a subformula of $\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$.

$$\frac{\begin{array}{c}\boxed{\alpha_0}\\[4pt] \Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi\end{array}}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi}\ (\Box)$$

Of course, every formula in $\Gamma \Rightarrow A$ is analytic as well. By hypothesis, only the last rule in α is non-analytic, so the subproof α_0 is analytic.

Since the initial sequents are on propositional variables, tracing the ancestors of any element from non-empty $\blacksquare\Delta$ upwards must lead to a weakening or a modal rule instance. More specifically, we have the following:

Claim 1. *Every lowermost modal rule instance in α_0 either*

1. *contains no ancestor of $\blacksquare\Delta$, or*
2. *contains exactly one ancestor of $\blacksquare\Delta$, which is principal in the rule instance.*

Rule instances of type 2. will be called *critical instances*.

Proof of Claim 1. Assume towards a contradiction that an ancestor of $\blacksquare\Delta$ is a context formula in the conclusion of a lowermost modal rule instance. Due to the modal rule instance being lowermost, this ancestor is unchanged from the original formula i.e., it is some $\blacksquare B \in \blacksquare\Delta$. Due to the form of the rule instances $(\Box)$ and $(\blacksquare)$, the premise of this lowermost modal rule instance contains a formula $\heartsuit\blacksquare B$, where $\heartsuit \in \{\Box, \blacksquare\}$. By analyticity of α_0, it follows that $\heartsuit\blacksquare B$ is a subformula of $\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi$ and so in particular $\blacksquare B$ is a subformula of $\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$. However, $\blacksquare\Delta$ was chosen such that every formula in it is non-analytic, so we have reached a contradiction. $\lhd$

In light of Claim 1, we write α as follows where $i \in I$ enumerates the critical instances and $\blacksquare B^i$ is the unique ancestor of $\blacksquare\Delta$ in the i-th critical instance.

$$\boxed{\beta_i}$$

$$\dfrac{\Sigma^i \Rightarrow B^i, \Box\Psi^i}{\blacksquare\Sigma^i \Rightarrow \blacksquare B^i, \Psi^i}\,(\blacksquare)^i \qquad i \in I$$

$$\boxed{core(\alpha_0)}$$

$$\dfrac{\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi}\,(\Box)$$

Following the usual terminology, a derivation of s_0 from the sequents $\{s_1, \ldots, s_n\}$ is a proof-like structure except that every leaf is either initial sequents of **GKt** or an element from $\{s_1, \ldots, s_n\}$. Then $core(\alpha_0)$ is the subderivation (of α) of $\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi$ from the sequents $\blacksquare\Sigma^i \Rightarrow \blacksquare B^i, \Psi^i$ for $i \in I$, that is, from the conclusions of the critical instances. Note that $core(\alpha_0)$ also contains all lowermost modal rule instances that are non-critical together with their subderivations but these are not shown in the picture above.

A *critical formula* is a formula that appears in some Σ^i's or $\Box\Psi^i$. i.e., some premise context of a critical instance. Let Ξ be the set of all critical formulas.

(III) Restructuring the proof. Let $\Xi = \{C_1, C_2, \ldots\}$. Starting from the desired endsequent $\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$ (this is the endsequent of α), introduce cuts bottom-up on all critical formulas, as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_1, C_2\end{matrix} \quad \begin{matrix}\vdots\\ C_2, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_1\end{matrix}}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_1}\,(cut) \quad \dfrac{\begin{matrix}\vdots\\ C_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_2\end{matrix} \quad \begin{matrix}\vdots\\ C_1, C_2, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi\end{matrix}}{C_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi}\,(cut)}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi}\,(cut)$$

First observe that every introduced cut is analytic.

Claim 2. *Every critical formula is a subformula of the endsequent* $\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$ *of* α.

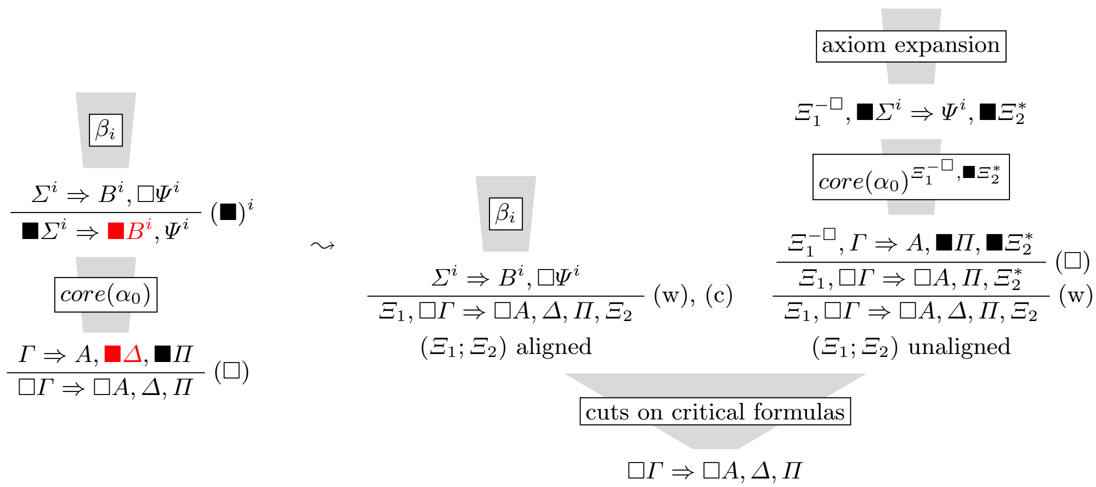

Fig. 2. From α to α^*.

Proof. Let C be a critical formula. As α_0 is analytic, C is a subformula of its endsequent $\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi$. Moreover, only the formulas in $\blacksquare\Delta$ are non-analytic in the last rule instance. So it suffices to show that $C \notin \blacksquare\Delta$. Towards a contradiction, suppose that $C \in \blacksquare\Delta$. As C is a critical formula we have $C \in \Sigma^i$ for some i (the case that $C \in \Box\Psi^i$ is impossible because $C \in \blacksquare\Delta$ begins with $\blacksquare$). As $\blacksquare\Sigma^i$ is in the conclusion of the i-th critical instance, we see that $\blacksquare C$ appears in α_0 and hence, by analyticity of α_0, it is a subformula of $\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi$. It follows that C is a subformula of $\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$, which, together with the assumption $C \in \blacksquare\Delta$, contradicts the non-analyticity of $\blacksquare\Delta$. $\lhd$

Next, observe that the above derivation is a binary tree with $2^{|\Xi|}$-many leaves, each of which is of the following form for some partition $\Xi_1 \cup \Xi_2 = \Xi$:

$$\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2$$

See Figure 2 for a pictorial representation of the transformation $\alpha \mapsto \alpha^*$.

Call a partition and its associated sequent *aligned* if there is some $i \in I$ such that Ξ_1 contains all formulas in Σ^i and Ξ_2 contains all formulas in $\Box\Psi^i$, and *unaligned* otherwise. Informally, $(\Xi_1; \Xi_2)$ aligned means that there exists $i \in I$ such that Ξ_1 contains Σ^i, and Ξ_2 contains $\Box\Psi^i$.

(a) **Proof of a sequent with an aligned partition** $(\Xi_1; \Xi_2)$
Suppose that $i \in I$ witnesses that $(\Xi_1; \Xi_2)$ is aligned. Then, noting that $B^i \in \Delta$, we can obtain a proof of the aligned sequent from the conclusion of β_i—this is the proof of the premise of the i-th critical instance, see Figure 2—using weakening and contraction.

$$\boxed{\beta_i}$$

$$\frac{\Sigma^i \Rightarrow B^i, \Box\Psi^i}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2}\ (\mathrm{w}), (\mathrm{c})$$

Contraction is required because $(\Sigma_i, \Box\Psi^i)$ is a pair of multisets, and $(\Xi_1; \Xi_2)$ a pair of sets that we read inside a sequent as multisets in the obvious way (an element is in the set iff the element has multiplicity 1 in the multiset).

(b) **Proof of a sequent with an unaligned partition** $(\Xi_1; \Xi_2)$
Working bottom-up from a unaligned sequent, consider the following derivation, where $\Xi_1^{-\Box} := \{C \mid \Box C \in \Xi_1^*\}$ and $\Xi_2^* := \Xi_2 \cap (\bigcup_i \Sigma^i)$.

$$\Xi_1^{-\Box}, \blacksquare\Sigma^i \Rightarrow \Psi^i, \blacksquare\Xi_2^*$$

$$\boxed{core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}}$$

$$\dfrac{\dfrac{\Xi_1^{-\Box}, \Gamma \Rightarrow A, \blacksquare\Pi, \blacksquare\Xi_2^*}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Pi, \Xi_2^*}\ (\Box)}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2}\ (\mathrm{w})$$

Here $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$ is obtained from $core(\alpha_0)$ by the following operation:

> In every sequent containing an ancestor of $\blacksquare\Delta$, remove all such ancestors and add $\Xi_1^{-\Box}$ in the antecedent and $\blacksquare\Xi_2^*$ in the succedent. Leave sequents without ancestors of $\blacksquare\Delta$ unchanged.

The following shows that it is well-defined.

Claim 3. *$core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$ is a derivation of $\Xi_1^{-\Box}, \Gamma \Rightarrow A, \blacksquare\Pi, \Xi_2^*$ from the premises $\Xi_1^{-\Box}, \blacksquare\Sigma^i \Rightarrow \Psi^i, \blacksquare\Xi_2^*$, for all $i \in I$.*

Proof. By Claim 1 and the definition of critical instance, the part of $core(\alpha_0)$ below the critical instances contains only propositional rule instances. Since propositional rule instances do not have side conditions, $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$ is indeed a derivation. $\lhd$

Hence we have a derivation of the unaligned sequent $\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2$ from $\{\Xi_1^{-\Box}, \blacksquare\Sigma^i \Rightarrow \Psi^i, \blacksquare\Xi_2^*\}_{i \in I}$. The following shows that every leaf of $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$ is a quasi-initial sequent (and hence has an analytic proof, by Lemma 1). Hence we obtain a proof of the unaligned sequent.

Claim 4. *If $(\Xi_1; \Xi_2)$ is unaligned, then some formula appears both in the antecedent and the succedent of $\Xi_1^{-\Box}, \blacksquare\Sigma^i \Rightarrow \Psi^i, \blacksquare\Xi_2^*$.*

Proof of Claim 4. As $(\Xi_1; \Xi_2)$ is unaligned, either Ξ_1 does not contain some formula in Σ^i or Ξ_2 does not contain some formula in $\Box\Psi^i$. Since multiset union $\Xi_1 \sqcup \Xi_2 = \Xi$, in the first case that there exists $C \in \Sigma^i \cap \Xi_2$, and hence $\blacksquare C \in \blacksquare\Sigma^i \cap \blacksquare\Xi_2^*$. In the second case, there exists $C \in \Box\Psi^i \cap \Xi_1$, which means that C is necessarily of the form $\Box D$ owing to membership in $\Box\Psi^i$, and therefore $D \in \Psi^i \cap \Xi_1^{-\Box}$. $\lhd$

This concludes the description of the construction of α^*.

2. Proof of grade reduction.
The sole non-analytic rule instance of α of grade k has been eliminated in α^*. Since non-analyticity in **GKt** may occur in any modal or cut-rule instance, and since an analytic instance may become non-analytic when formulas in it are replaced, we show that every new and modified rule instance in α^* is either analytic, or non-analytic of grade $< k$. This is shown in the following two claims.

Claim 5. *The grade of the following rule instance that appears in the proof of an unaligned sequent—see 1. Construction of α^*/step (III)/case (b)—is $< k$.*

$$\frac{\Xi_1^{-\Box}, \Gamma \Rightarrow A, \blacksquare\Pi, \blacksquare\Xi_2^*}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Pi, \Xi_2^*} (\Box)$$

Proof of Claim 5. Let $(\Box)^*$ denote the instance of $(\Box)$ in the statement.

By inspection, if a formula D in the premise $\Xi_1^{-\Box}, \Gamma \Rightarrow A, \blacksquare\Pi, \blacksquare\Xi_2^*$ of $(\Box)^*$ is not a subformula of some formula in $\Xi_1, \Box\Gamma \Rightarrow \Box A, \Pi, \Xi_2^*$—i.e., $(\Box)^*$ is non-analytic—then D must occur in either $\blacksquare\Pi$ or $\blacksquare\Xi_2^*$.

Suppose $D \in \blacksquare\Pi$. We know that D is a subformula of some formula in the conclusion $\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$ of the last rule instance in α—see Figure 2—since the non-analytic formulas in that instance are exclusively in $\blacksquare\Delta$. It follows that D is either a subformula of a formula in $\Box\Gamma \Rightarrow \Box A, \Pi$ and hence analytic in $(\Box)^*$, or a subformula of a formula in Δ, and hence of grade $< k$ (else $\blacksquare\Delta$ would contain a formula with grade $\geq k+1$ which would contradict the hypothesis).

Finally, suppose $D \in \blacksquare\Xi_2^* = \blacksquare(\Xi_2 \cap (\bigcup_i \Sigma^i))$. Hence $D \in \blacksquare\Sigma^i$ for some $i \in I$. Let D_0 be an occurrence of D in the conclusion of the i-th critical instance. Starting from D_0 in the conclusion of the i-th critical instance, we now identify a sequence $D_0, D_1, \ldots, D_{end}$ of occurrences—one in each successive sequent below it in $core(\alpha_0)$—until we reach a formula D_{end} in $\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi$ (this is the root of $core(\alpha_0)$, and the premise of the last rule instance in α; see Figure 2).

1. If D_l is the ancestor of some formula D' in the sequent below it, $D_{l+1} := D'$.
2. If D_l is not the ancestor of any formula in the sequent below it, then D_l must be the cut-formula of an analytic cut. Due to analyticity, D_l must be a subformula of some formula D' in the sequent below it. Set $D_{l+1} := D'$.

The rule instances along the sequents from which the sequence $D_0, D_1, \ldots, D_{end}$ are constructed were analytic, so D is a subformula of D_{end}. If D_{end} appears in $\Gamma \Rightarrow A$ then D is analytic in $(\Box)^*$. Else, if D_{end} appears in $\blacksquare\Pi$, then D must appear in Π or Δ (by consideration of the last rule instance in α), and hence it is analytic in $(\Box)^*$ or of grade $< k$. Else, it must be the case that D_{end} appears in $\blacksquare\Delta$. Since D_0 was not an ancestor of $\blacksquare\Delta$ in α_0—indeed, D is a critical formula so it appears in the context of the critical instance, and by Claim 1, if the conclusion of the critical instance contains an ancestor, it is unique and principal—at some index $l+1$ in the construction of $D_0, D_1, \ldots, D_{end}$, condition 2 in the construction must have been applied to obtain D_{l+1} from D_l. This implies that $D_{l+1} \neq D_l$, since otherwise the cut would be trivial, and trivial cuts were removed in the preprocessing. Thus D_0 is a *proper* subformula of $D_{end} \in \blacksquare\Delta$. It follows that the grade of D in $(\Box)^*$ is $< k$. $\lhd$

Claim 6. *The grade of every non-analytic rule instance in* $core(\alpha_0)^{\Xi_1^{-\Box},\blacksquare\Xi_2^*}$ *(the derivation that appears in the proof of an unaligned sequent) is* $< k$.

Proof. Recall that $core(\alpha_0)$ is analytic, and $core(\alpha_0)^{\Xi_1^{-\Box},\blacksquare\Xi_2^*}$ arises from $core(\alpha_0)$ by considering every sequent containing an ancestor of the $\blacksquare\Delta$ in the endsequent, removing every such ancestor from the sequent and adding $\Xi_1^{-\Box}$ in the antecedent and $\blacksquare\Xi_2^*$ in the succedent. Every rule instance in $core(\alpha_0)$ is analytic. Suppose that $core(\alpha_0)^{\Xi_1^{-\Box},\blacksquare\Xi_2^*}$ contains some non-analytic instance r' that was an analytic instance r in $core(\alpha_0)$. As r contains ancestors of $\blacksquare\Delta$ it is not a modal non-critical instance (Claim 1), and it cannot be a critical instance since $core(\alpha_0)$ does not contain those. It must therefore be a cut, since all the other rules of the calculus are analytic. A cut-rule instance is non-analytic only if the cut-formula C is not a subformula of any formula in the conclusion. Comparing the conclusion sequents of r and r', the only formulas that are in the former and not the latter are the formulas of $\blacksquare\Delta$ (since we removed these). Since r is analytic and r' is not, C must be a subformula of some $\blacksquare D \in \blacksquare\Delta$. Now C cannot be $\blacksquare D$ since r would then be a trivial cut instance since C appears in its conclusion, and trivial cut instances were removed in the preprocessing step on α. So, C must be a proper subformula of $\blacksquare D$. Hence r' has grade $< k$. $\lhd$

This concludes the proof of Proposition 1. **QED**

We are ready to prove the main theorem.

Theorem 1. *Every provable sequent in* **GKt** *has an analytic proof.*

Proof. We show that a proof where only the last rule instance is non-analytic can be transformed into an analytic proof of the same sequent. The desired result follows from this: if π_0 is a proof of sequent s in **GKt**, replacing a subproof in π_0 whose only non-analytic rule instance is the last rule with the analytic proof obtained through the latter, we obtain a proof π_1 of s that contains one less non-analytic rule; iterating this step, we ultimately reach an analytic proof of s.

Suppose that α is a proof of s where only the last rule instance r is non-analytic. We proceed by primary induction on the grade of r, and secondary induction on the sum of the heights of the premises of r.

First suppose that r is a modal rule. Applying Proposition 1, we obtain a proof α^* of s in which all non-analytic rule instances have strictly lower grade. At each subsequent step, we use the induction hypothesis to replace the subproof ending in a topmost non-analytic rule instances with analytic proofs with strictly lower grade. This process terminates by well-foundedness of the Dershowitz–Manna multiset ordering—i.e., replacing an element in the multiset by strictly smaller ones yields a strictly smaller multiset—as each application of Proposition 1 strictly decreases the multiset of grades of non-analytic rule instances. At termination we obtain an analytic proof of s.

The remaining case is that r is a non-analytic cut with analytic premises.

$$\dfrac{\begin{matrix}\alpha_1 & & \alpha_2\\ \mathit{\Gamma} \Rightarrow A^k, \mathit{\Delta} & & \mathit{\Sigma}, A^l \Rightarrow \mathit{\Pi}\end{matrix}}{\mathit{\Gamma}, \mathit{\Sigma} \Rightarrow \mathit{\Delta}, \mathit{\Pi}}\ (cut)$$

Suppose that the last rule in α_1 or α_2 is an analytic cut r_1. Permuting r above r_1 turns it into a non-analytic cut r' with smaller sum of premise heights, and hence an analytic proof is obtained by IH. The analytic cut r_1 (with cut formula A_1, say) that was permuted downwards (call it r_1') may have now become a non-analytic. Since r_1 was analytic and r_1' is not, it must be that A_1 is a subformula of A. If $A_1 = A$ then r_1 would have been a trivial cut (since A occurs in the premises of r), but these were removed in preprocessing. So A_1 is a proper subformula of A, hence we obtain an analytic proof by IH. If the last rule is a structural rule of weakening or contraction, the standard reductions apply. It remains to consider when the last rule in α_1 and α_2 is a logical rule.

If the main connective in A is not modal, then neither the last rule instance in α_1 nor α_2 is modal. Apply the usual propositional cut reduction steps leading to new cuts of smaller grade, or the same grade and smaller sum of premise height. The subproof concluding the new cuts can be made analytic by the IH, and any subsequent propositional rules that follow will, of course, preserve analyticity.

Finally, suppose that the main connective in A is modal. The last rule in α_1 (α_2) may be a propositional rule instance making some formula in $\mathit{\Gamma}, \mathit{\Delta}$ (resp. $\mathit{\Sigma}, \mathit{\Pi}$) principal. Apply the usual steps to reduce a propositional cut. We are left with the case when the last rule instance in α_1 and α_2 are modal rules.

Claim 7. *If the last rule instance in α_1 is a modal rule, then A is its principal formula and $k = 1$.*

Proof. Otherwise, one occurrence of A would be non-principal in the modal rule instance and therefore—due to the form of the modal rules—appear as $\Box A$ or $\blacksquare A$ in the premise of α_1's last rule instance. As α_1 is analytic by assumption this implies that $\Box A$ or $\blacksquare A$ is a subformula of $\mathit{\Gamma} \Rightarrow A^k, \mathit{\Delta}$ and hence of $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$, implying that the multicut on A is analytic, contrary to the assumption. $\lhd$

We have reached the following situation (an analogous argument applies when it is an instance of ($\blacksquare$)). Recall that only the last rule instance is non-analytic.

$$\dfrac{\dfrac{\mathit{\Gamma} \Rightarrow A, \blacksquare\mathit{\Delta}}{\Box\mathit{\Gamma} \Rightarrow \Box A, \mathit{\Delta}}\ (\Box) \qquad \dfrac{\mathit{\Sigma}, A^l \Rightarrow B, \blacksquare\mathit{\Pi}}{\Box\mathit{\Sigma}, \Box A^l \Rightarrow \Box B, \mathit{\Pi}}\ (\Box)}{\Box\mathit{\Gamma}, \Box\mathit{\Sigma} \Rightarrow \Box B, \mathit{\Delta}, \mathit{\Pi}}\ (\text{cut})$$

Applying a multicut to $\mathit{\Gamma} \Rightarrow A, \blacksquare\mathit{\Delta}$ and $\mathit{\Sigma}, A^l \Rightarrow B, \blacksquare\mathit{\Pi}$ we get a proof of $\mathit{\Gamma}, \mathit{\Sigma} \Rightarrow B, \blacksquare\mathit{\Delta}, \blacksquare\mathit{\Pi}$. By IH, we have an analytic proof of this sequent. Now we apply ($\Box$) to get the desired endsequent:

$$\dfrac{\mathit{\Gamma}, \mathit{\Sigma} \Rightarrow B, \blacksquare\mathit{\Delta}, \blacksquare\mathit{\Pi}}{\Box\mathit{\Gamma}, \Box\mathit{\Sigma} \Rightarrow \Box B, \mathit{\Delta}, \mathit{\Pi}}\ (\Box)$$

If the last rule instance is non-analytic, then there is a formula $\blacksquare D \in \blacksquare\mathit{\Delta} \cup \blacksquare\mathit{\Pi}$ that is not a subformula of any formula in the conclusion. Since the two rule instances of $(\Box)$ before the reduction were analytic, this can only be the case if $\blacksquare D$ is a subformula of $\Box A$. But clearly $\blacksquare D \neq \Box A$, so in fact $\blacksquare D$ is a proper subformula of $\Box A$. Hence the grade of this instance is strictly less than the grade of the original cut formula $\Box A$. By IH we obtain the desired sequent. **QED**

Example 1. Consider the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p \lor q}}{\blacksquare p \Rightarrow \blacksquare(p \lor q)}\,(\blacksquare)}{\Rightarrow \blacksquare(p \lor q), \neg\blacksquare p}}{\Rightarrow p \lor q, \Box\neg\blacksquare p}\,(\Box)$$

The rule instance of $(\Box)$ is non-analytic as $\blacksquare(p \lor q)$ is not a subformula of the endsequent. The corresponding critical instance is $(\blacksquare)$ above, with a single critical formula p. The transformed proof thus starts with a cut on p:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\blacksquare p \Rightarrow \blacksquare p}\,(\blacksquare)}{\Rightarrow \blacksquare p, \neg\blacksquare p}}{\Rightarrow p, \Box\neg\blacksquare p}\,(\Box) \qquad \dfrac{p \Rightarrow p}{p \Rightarrow p \lor q}}{\Rightarrow p \lor q, \Box\neg\blacksquare p}\,(cut)$$

Let us use this example to illustrate the semantic intuition behind the proof transformation. As in other proof systems for modal logic, derivations in **GKt** can be understood as structured searches for countermodels. The process begins with a partially constructed countermodel for the endsequent, where *formulas in the antecedent are treated as true, and those in the succedent as false.* As the proof unfolds upward, this model is incrementally extended with additional information until a contradiction is reached, which means that no countermodel is possible, and hence, the endsequent must be valid. In this dynamic view, an instance of $(\Box)$ (resp. $(\blacksquare)$) corresponds to transitioning from one world in the countermodel to a future (resp. past) world.

Looking through this lens at the proof before the transformation (we are reading upwards from the endsequent), we observe a step into the future via $(\Box)$ followed two rules later by a step back into the present via $(\blacksquare)$, with one true formula $\blacksquare p$ that was 'learned' in the future pulled back to the present as p. This additional formula is then used to derive contradiction, witnessed by $p \Rightarrow p$.

Consider now the transformed proof. Rather than moving into the future, we remain in the present and perform a case distinction (a cut) on p. In right branch, where the learned formula p is assumed true, we derive a contradiction as before. Only in the case where the learned formula is assumed false we take it into the future, where it becomes $\blacksquare p$ fails. But we already know that in the future we can learn that $\blacksquare p$ is in fact true, and this gives the desired contradiction.

5 Conclusions and Future work

Although our analytic restriction proof is carried out within **GKt**, the core idea is adaptable to related logics, by slightly modifiying the methodology. The situation is analogous to a cut-elimination strategy which, though originally developed for a specific calculus, is then extended to other calculi with similar structural characteristics—we recall Valentini's cut-elimination for provability logic [26, 15], which was subsequently adapted to various structurally similar systems [27, 7]. The proof presented in this paper can also be adapted to the logic **KB**, obtained by adding the following to the classical calculus.

$$\frac{\Gamma \Rightarrow A, \Box\Delta}{\Box\Gamma \Rightarrow \Box A, \Delta}\ (B)$$

We expect these methods to apply to some further extensions of the classical sequent calculus with non-analytic modal rules that meet the following conditions:

1. If the cut-formula is principal in both premises by the modal rule, it can be transformed to cuts on proper subformulas (reduction of principal cuts)
2. The rules are non-analytic with respect to a single type of modality, which may prefix formulas in the premise that are unmodalized in the conclusion.
3. Formulas prefixed by a modality in the conclusion (premise) are unmodalized (modalized with the same operator) in the premises (conclusion), and formulas. All formulas remain on the same side of the sequent as they appear in the conclusion, and no additional formula is introduced in the premises.

As future work, we intend to formalize these intuitive conditions and develop a general cut restriction proof applicable to a broad class of logics, and consider the algorithmic complexity and potential computational interpretations.

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