Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions

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Abstract

Extensions of monoidal t-norm logic MTL and related fuzzy logics with truth stresser modalities such as globalization and "very true" are presented here both algebraically in the framework of residuated lattices and proof-theoretically as hypersequent calculi. Completeness with respect to standard algebras based on t-norms, embeddings between logics, decidability, and the finite embedding property are then investigated for these logics.

1 Introduction

Monoidal t-norm logic MTL, introduced by Esteva and Godo in [9], is a substructural logic underlying the most common formalizations of fuzzy logic. More precisely, it has been shown in [14] that this logic axiomatizes the tautologies of all *t-norm logics*, that is, logics whose conjunction and implication connectives are interpreted by a (left-continuous) t-norm [2] and its residuum respectively. MTL can also be viewed as the extension of affine multiplicative additive intuitionistic linear logic (sometimes known as monoidal logic [13]) with the "prelinearity" axiom

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[2] A t-norm is a commutative associative increasing binary operator on $[0, 1]$ with unit 1.

schema $((A \to B) \to C) \to (((B \to A) \to C) \to C)$. Extensions of MTL include Gödel logic, Łukasiewicz logic, and product logic, based on particular continuous t-norms, and other logics based on significant classes of t-norms. Important examples of the latter are involutive monoidal t-norm logic IMTL and strict monoidal t-norm logic SMTL, which axiomatize t-norm logics whose negations are involutive and strict respectively [8], $\mathsf{C_nMTL}$ and $\mathsf{C_nIMTL}$ which characterize t-norm logics satisfying an n-contraction property [4], and Hájek's basic fuzzy logic BL which characterizes logics based on continuous t-norms [11].

The purpose of the current work is to investigate extensions of MTL and related logics with various "truth stresser" modalities. In general, a modality is a unary connective $\Box$ that acts as a modifier of the meaning of formulas. Kripke semantics for classical modal logics can easily be generalized to obtain fuzzy modal logics that are not complete with respect to algebras with a $[0, 1]$ lattice reduct, i.e., the modalities are not truth functional (see e.g. [3,17]). However, as emphasized by Zadeh in [23], there exist also "truth stresser" modalities in fuzzy logic that capture expressions of natural language such as "very true" or "more or less true" where such completeness is desirable. For example, logics where $\Box A$ means "A is very true", admitting theorems such as $\Box A \to A$, have been axiomatized by Hájek for BL and its extensions in [12]. Also, the "globalization" (or "Delta") truth stresser modality where $\Box A$ is interpreted as "A is completely (classically) true" has been widely studied for fuzzy logics (see e.g. [2,20]).

Similarly to exponentials in linear logic [10] and modalities added to other substructural logics [22], a modality $\Box$ may specify particular properties of a certain class of formulas in a fuzzy logic. For example, the axiom schema $\Box A \to (\Box A \odot \Box A)$ permits the contraction of just boxed formulas. This allows embeddings of fuzzy logics admitting extra structural rules into weaker fuzzy logics with modalities, analogously to embeddings of intuitionistic logic into linear logic. Logics extended with certain modalities are also capable, unlike e.g. MTL, of expressing the consequence relation within the logic itself, and have been used in [19] to define multiplicative quantifiers for fuzzy logics.

In this paper, we present a general program for adding truth stresser modalities to MTL and its extensions. Syntactically, we present fuzzy logics with modalities as Hilbert-style axiomatizations and Gentzen-style proof calculi, the latter using hypersequents, a generalization of sequents to multisets of sequents introduced by Avron in [1]. For Hilbert systems, the distinctive axiom schema is $\Box(A \vee B) \to (\Box A \vee \Box B)$ which ensures completeness with respect to linearly ordered models and does not appear in classical modal logics. For hypersequent calculi, the characterizations emerge as a natural extension of the classical case: we just add hypersequent versions of standard sequent rules for modal logics. The crucial property established here is cut elimination, which provides analytic proof methods for the logics, i.e., the existence of proofs proceeding by a stepwise decomposition of the formula to be proved. Moreover, this ensures that adding these modalities is con-

(A1) $(A \to B) \to ((B \to C) \to (A \to C))$

(A2) $(A \odot B) \to A$

(A3) $(A \odot B) \to (B \odot A)$

(A4) $(A \wedge B) \to A$

(A5) $(A \wedge B) \to (B \wedge A)$

(A6) $(A \odot (A \to B)) \to (A \wedge B)$

(A7) $((A \odot B) \to C) \leftrightarrow (A \to (B \to C))$

(A8) $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

(A9) $\bot \to A$

$$\frac{A \quad A \to B}{B} \ (\text{MP})$$

Fig. 1. Monoidal t-norm logic MTL

servative: any theorem of the extended logic in the original language is a theorem of the original logic. These systems are also used to prove embedding results analogous to those of Girard [10] and Restall [22], in particular, embeddings of Gödel logic and classical logic into fuzzy logics extended with an appropriate modality.

Algebraic semantics for our fuzzy logics with truth stresser modalities are obtained by adding interior-like operators to residuated lattices as in e.g. [22]. We show that all the modal extensions considered in the paper are strongly complete with respect to a class of such (linearly ordered) algebras. For the cases where our fuzzy logics are extended by an $\mathsf{S4}$-like modality (including, e.g. globalization), we also prove so-called standard completeness with respect to algebras with lattice reduct $[0, 1]$ (i.e., where all connectives, including modalities, are interpreted as functions on the real unit interval $[0, 1]$). Finally, when in addition all boxed formulas satisfy a contraction schema, we show that the finite consequence relation is decidable using the finite embeddability property (FEP) of the corresponding class of algebras.

2 Monoidal t-norm logic and extensions

Monoidal t-norm logic MTL is based on a set of formulas Fm built in the usual way from a countably infinite set of variables $p, q, r, \ldots$ and a propositional language with binary connectives $\wedge$, $\odot$, $\to$, constant $\bot$, and defined connectives $\neg A =_{\text{def}} A \to \bot$, $A \vee B =_{\text{def}} ((A \to B) \to B) \wedge ((B \to A) \to A)$, $\top =_{\text{def}} \neg\bot$, $A \leftrightarrow B =_{\text{def}} (A \to B) \wedge (B \to A)$, $A \oplus B =_{\text{def}} \neg(\neg A \odot \neg B)$, $A^1 =_{\text{def}} A$, and $A^{n+1} =_{\text{def}} A \odot A^n$ for each $n \in \mathbb{N}^+$. An axiomatization is given in Figure 1.

A logic is a *(schematic) extension* of another logic L if it results from L by adding (finitely or infinitely many) axiom schema in the same language. In particular:

- IMTL is MTL extended with (INV) $\neg\neg A \to A$;
- SMTL is MTL extended with (NC) $\neg(A \wedge \neg A)$;
- $\mathsf{C_nMTL}$ is MTL extended with ($\mathsf{C_n}$) $A^{n-1} \to A^n$ where $n \geq 2$;
- $\mathsf{C_nIMTL}$ is $\mathsf{C_nMTL}$ extended with (INV) where $n \geq 2$;

and for convenience we let:

$$\text{Logics} = \{\mathsf{MTL}, \mathsf{IMTL}, \mathsf{SMTL}\} \cup \{\mathsf{C_nMTL} : n \geq 2\} \cup \{\mathsf{C_nIMTL} : n \geq 3\}.$$

Observe that (C_2) is just the usual contraction axiom schema $A \to (A \odot A)$, and hence that $\mathsf{C_2MTL}$ and $\mathsf{C_2IMTL}$ are Gödel logic G and Classical logic CL respectively. Since the latter is not a fuzzy logic in the usual sense (i.e., not complete with respect to algebras based on $[0, 1]$), we exclude it from the set of logics considered here. Important fuzzy logics not considered explicitly in this paper include Hájek's basic fuzzy logic BL which can be axiomatized as MTL extended with the divisibility axioms $(B \odot (B \to A)) \to (A \odot (A \to B))$, Łukasiewicz logic Ł, which is BL extended with (INV), and product logic P, which is BL extended with (NC) and the axiom schema $\neg\neg A \to ((A \to (A \odot B)) \to B)$.

An *MTL-algebra* is a prelinear bounded integral commutative residuated lattice, i.e., an algebra $\mathbf{M} = \langle M, \wedge, \vee, \odot, \to, \bot, \top\rangle$ with universe M, binary operations $\wedge$, $\vee$, $\odot$, and $\to$, and constants $\bot$ and $\top$, such that:

- $\langle M, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice;
- $\langle M, \odot, \top\rangle$ is a commutative monoid;
- $z \leq x \to y$ iff $x \odot z \leq y$ for all $x, y, z \in M$ (residuation);
- $(x \to y) \vee (y \to x) = \top$ for all $x, y \in M$.

An $\mathbf{M}$-*valuation* is a function $v : \mathrm{Fm} \to M$ satisfying $v(\bot) = \bot$ and $v(A \star B) = v(A) \star v(B)$ for $\star \in \{\wedge, \odot, \to\}$, and $A \in \mathrm{Fm}$ is $\mathbf{M}$-*valid* iff $v(A) = \top$ for all $\mathbf{M}$-valuations v. $\mathbf{M}$ is called an L-*algebra* iff all axioms of the logic L are $\mathbf{M}$-valid, and an L-*chain* iff $\mathbf{M}$ is also linearly ordered.

Theorem 1 ([9]) *For any extension* L *of* MTL*,* $\vdash_{\mathrm{L}} A$ *iff* A *is valid in all* L*-algebras (*L*-chains).*

More interesting, however, are completeness results with respect to *standard* L-*algebras* with lattice reduct $\langle [0, 1], \min, \max\rangle$, where the monoid operator $\odot$ and implication $\to$ are, respectively, a left-continuous t-norm and its residuum.

Theorem 2 ([14,8,4]) *For* $\mathrm{L} \in \text{Logics}$*,* $\vdash_{\mathrm{L}} A$ *iff* A *is valid in all standard* L*-algebras.*

Many other standard completeness results are known in the literature; e.g., Hájek's basic logic BL is complete with respect to all standard BL-algebras, or, equivalently, BL-algebras where the monoid operator is a continuous t-norm.

MTL and its extensions may also be presented "Gentzen-style" in the framework of *hypersequents*: finite multisets, denoted $\mathcal{G}$ or $\mathcal{H}$ (possibly subscripted), of the form $(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$ where each $\Gamma_i \Rightarrow \Delta_i$ is an ordered pair of finite multisets of formulas called a *sequent*, denoted S (possibly subscripted). If each Δ_i contains at most one formula, then the hypersequent is *single-conclusion*, otherwise it

Initial Sequents

$$\frac{}{\mathcal{G} \mid A \Rightarrow A}\ (\textsc{id}) \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}\ (\bot)$$

Structural Rules

$$\frac{\mathcal{G} \mid S \mid S}{\mathcal{G} \mid S}\ (\textsc{ec}) \qquad \frac{\mathcal{G}}{\mathcal{G} \mid S}\ (\textsc{ew}) \qquad \frac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1 \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}\ (\textsc{com}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{w})$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow A, \Delta_1 \quad \mathcal{G} \mid \Gamma_2, B \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}\ ({\to}{\Rightarrow}) \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}\ ({\Rightarrow}{\to})$$

$$\frac{\mathcal{G} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \odot B \Rightarrow \Delta}\ ({\odot}{\Rightarrow}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow A, \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow B, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow A \odot B, \Delta_1, \Delta_2}\ ({\Rightarrow}{\odot})$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ ({\wedge}{\Rightarrow})_1 \quad \frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ ({\wedge}{\Rightarrow})_2 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \wedge B, \Delta}\ ({\Rightarrow}{\wedge})$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta}\ ({\vee}{\Rightarrow}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B, \Delta}\ ({\Rightarrow}{\vee})_1 \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B, \Delta}\ ({\Rightarrow}{\vee})_2$$

Cut Rule

$$\frac{\mathcal{G} \mid \Gamma_1, A \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow A, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{cut})$$

Fig. 2. Hypersequent rules (GIMTL)

is *multiple-conclusion*. A *hypersequent rule* (r), typically presented schematically, is a set of ordered pairs called *instances of* (r) consisting of a hypersequent $\mathcal{G}$ called the *conclusion*, and a set of hypersequents $\mathcal{G}_1, \ldots, \mathcal{G}_n$ called the *premises*. The *single-conclusion version* of a rule restricts to its members with single-conclusion premises and conclusion. A *(single-conclusion) hypersequent calculus* is just a set of (single-conclusion) hypersequent rules, and *derivations* are defined in the usual way as trees of hypersequents constructed using the rules (see e.g. [18] for details).

Like sequent calculi, hypersequent calculi usually consist of initial sequents, logical rules, and structural rules. Logical rules for connectives are as in sequent calculi except that a "side-hypersequent" may also occur, often denoted by a meta-variable $\mathcal{G}$. Structural rules are divided into two categories. Internal rules deal with formulas within sequents and include a distinguished "cut" rule corresponding to the transitivity of deduction. External rules manipulate whole sequents. For example, external and contraction rules (EW) and (EC) add and contract sequents respectively, while the key rule for fuzzy logics is the communication rule (COM) which permits interaction between sequents.

Hypersequent calculi for the logics introduced above are defined based on the same language as for MTL but with $\vee$ taken as primitive rather than as a defined connective. Also, we write Γ, Δ and Γ, A for the multiset unions $\Gamma \uplus \Delta$ and $\Gamma \uplus [A]$, respectively, and let $\Gamma^0 =_{\text{def}} []$ and $\Gamma^{n+1} =_{\text{def}} \Gamma \uplus \Gamma^n$. GMTL and GIMTL then consist of the single-conclusion and multiple-conclusion versions of the rules in Figure 2. Referring to the additional structural rules in Figure 3, we also define:

$$\frac{\mathcal{G} \mid \Gamma, \Pi_1^n \Rightarrow \Sigma_1^n, \Delta \quad \dots \quad \mathcal{G} \mid \Gamma, \Pi_{n-1}^n \Rightarrow \Sigma_{n-1}^n, \Delta}{\mathcal{G} \mid \Gamma, \Pi_1, \dots, \Pi_{n-1} \Rightarrow \Sigma_1, \dots, \Sigma_{n-1}, \Delta}\ (\textsc{c}_\mathrm{n}) \qquad \frac{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow}\ (\textsc{wcl}) \qquad \frac{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \mid \Pi \Rightarrow \Sigma}\ (\textsc{split})$$

Fig. 3. Further structural rules

- $\mathsf{GC_nMTL}$ is GMTL extended with the single-conclusion version of $(\textsc{c}_\mathrm{n})$ $(n \geq 2)$;
- $\mathsf{GC_nIMTL}$ is GIMTL extended with $(\textsc{c}_\mathrm{n})$ $(n \geq 2)$;
- GSMTL is GMTL extended with $(\textsc{wcl})$.

$(\textsc{wcl})$ is a weak contraction rule that allows contraction only when the right hand side of the sequent is empty. $(\textsc{c}_\mathrm{n})$ is the so-called n-contraction rule, the case of $n = 2$ giving just a version of the usual (e.g. from Gentzen's LK) contraction rules. Hence the calculus $\mathsf{GC_2MTL}$ is Avron's calculus for Gödel logic G, while $\mathsf{GC_2IMTL}$ is a calculus for classical logic (where $(\textsc{com})$ is redundant in this last case). Note that alternative calculi for classical logic are obtained by adding the $(\textsc{split})$ rule of Figure 3 to any of the systems defined above.

The prelinearity axioms are derivable as follows in any calculus extending GMTL:

$$\begin{prooftree}
\AxiomC{}
\RightLabel{(\textsc{id})}
\UnaryInfC{$C \Rightarrow C \mid (B \to A) \to C \Rightarrow C$}
\AxiomC{}
\RightLabel{(\textsc{id})}
\UnaryInfC{$A \Rightarrow B \mid C \Rightarrow C$}
\AxiomC{}
\RightLabel{(\textsc{id})}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{(\textsc{id})}
\UnaryInfC{$B \Rightarrow B$}
\RightLabel{(\textsc{com})}
\BinaryInfC{$A \Rightarrow B \mid B \Rightarrow A$}
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$A \Rightarrow B \mid \Rightarrow B \to A$}
\RightLabel{$(\to\Rightarrow)$}
\BinaryInfC{$A \Rightarrow B \mid (B \to A) \to C \Rightarrow C$}
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$\Rightarrow A \to B \mid (B \to A) \to C \Rightarrow C$}
\RightLabel{$(\to\Rightarrow)$}
\BinaryInfC{$(A \to B) \to C \Rightarrow C \mid (B \to A) \to C \Rightarrow C$}
\RightLabel{(\textsc{w})}
\UnaryInfC{$(A \to B) \to C \Rightarrow C \mid (A \to B) \to C, (B \to A) \to C \Rightarrow C$}
\RightLabel{(\textsc{w})}
\UnaryInfC{$(A \to B) \to C, (B \to A) \to C \Rightarrow C \mid (A \to B) \to C, (B \to A) \to C \Rightarrow C$}
\RightLabel{(\textsc{ec})}
\UnaryInfC{$(A \to B) \to C, (B \to A) \to C \Rightarrow C$}
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$(A \to B) \to C \Rightarrow ((B \to A) \to C) \to C$}
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$\Rightarrow ((A \to B) \to C) \to (((B \to A) \to C) \to C)$}
\end{prooftree}$$

Correspondences between axiomatizations and hypersequent calculi for fuzzy logics are established by interpreting hypersequents as formulas:

(1) $i(A_1, \dots, A_n \Rightarrow B_1, \dots, B_m) =_{\mathrm{def}} (A_1 \odot \dots \odot A_n) \to (B_1 \oplus \dots \oplus B_m)$;
(2) $i(\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n) =_{\mathrm{def}} i(\Gamma_1 \Rightarrow \Delta_1) \vee \dots \vee i(\Gamma_n \Rightarrow \Delta_n)$;

where $A_1 \odot \dots \odot A_n$ is $\top$ for $n = 0$, and $B_1 \oplus \dots \oplus B_m$ is $\bot$ for $m = 0$.

It is straightforward to show that the axiom system simulates the corresponding hypersequent calculus and vice versa; the key result is rather to prove *cut elimination*, i.e., that any derivation of a hypersequent $\mathcal{G}$ in the calculus can be transformed into a derivation of $\mathcal{G}$ not using $(\textsc{cut})$.

Theorem 3 (cf. e.g. [18]) *Let* $\mathrm{L} \in$ Logics. *Then (a)* $\vdash_{\mathrm{GL}} \mathcal{G}$ *iff* $\vdash_{\mathrm{L}} i(\mathcal{G})$*; (b) cut elimination holds for* GL.

Calculi for many other fuzzy logics have been defined, e.g. by removing the weakening rules of GMTL to characterize logics complete with respect to uninorm based semantics or, for Łukasiewicz logic and product logic, by changing the logical rules and interpretation of hypersequents (see the monograph [18] for details). Moreover, an algorithm has been defined in [5] that transforms axiom systems of a given form into sequent or hypersequent calculi that are guaranteed to admit cut elimination.

3 Adding modalities

Let us now consider the addition of various "truth stresser" modalities to the above presentations of MTL and its extensions. First we extend the language of MTL with the unary operator $\Box$ (noting that we can also define a connective $\Diamond A =_{\text{def}} \neg\Box\neg A$), obtaining a set of formulas $\text{Fm}_\Box$, and define for $\text{L} \in \text{Logics}$:

- LK^r is L extended with the axiom schema:

$$(\text{K}_\Box)\;\; \Box(A \to B) \to (\Box A \to \Box B) \qquad (\vee_\Box)\;\; \Box(A \vee B) \to (\Box A \vee \Box B)$$

  and the necessitation rule: $\dfrac{A}{\Box A}$ (NEC);

- LKT^r is LK^r extended with $(\text{T}_\Box)\;\; \Box A \to A$;
- LS4^r is LKT^r extended with $(4_\Box)\;\; \Box A \to \Box\Box A$;
- L!^r is S4^r extended with $(\text{C}_\Box)\;\; \Box A \to (\Box A \odot \Box A)$;
- L^r_Δ is S4^r extended with $(\text{S}_\Box)\;\; \Box A \vee (\Box A \to \bot)$.

We also let $\text{Logics}_\Box = \{\text{LK}^\text{r}, \text{LKT}^\text{r}, \text{LS4}^\text{r}, \text{L!}^\text{r}, \text{L}^\text{r}_\Delta : \text{L} \in \text{Logics}\}$.

Such a proliferation of logics deserves some explanation. First note that $(\text{K}_\Box)$ is the standard axiom schema added to classical logic to obtain the modal logic K. However, to obtain fuzzy logics with modalities that are complete with respect to chains, this is supplemented with the "shifting law of modalities" axiom schema $(\vee_\Box)$. We emphasize this point by attaching the superscript r for each extended logic to denote the fact that the algebras for these logics (defined below) are representable as subdirect products of chains. The extensions to LKT^r and LS4^r then mimic the extension of classical logic to modal logics KT and S4, the modality $\Box$ in LKT^r matching the axiomatization of "very true" in [12]. The addition of the law of excluded middle for boxed formulas $(\text{S}_\Box)$ gives an axiomatization of fuzzy logics with the globalization (or Delta) connective, studied in e.g. [2,20]. Finally, observe that the axiom $(\text{C}_\Box)$ added to LS4^r gives the properties of a linear logic style exponential, usually written !.

We introduce algebras for the logics defined above by considering particular classes of residuated lattices where the modality is interpreted by a unary operator I. This operator satisfies not only many of the conditions of an interior operator (all in the case of $\mathrm{LS4^r}$), but also condition (2) below which ensures that the extended algebras remain representable.

For $\mathrm{L} \in \mathrm{Logics}$, an $\mathrm{LK^r}$-*algebra* is an algebra $\mathbf{M} = \langle M, \wedge, \vee, \odot, \rightarrow, \bot, \top, I\rangle$ such that $\langle M, \wedge, \vee, \odot, \rightarrow, \bot, \top\rangle$ is an L-algebra, and I is a unary operation satisfying:

(1) $I(x \rightarrow y) \leq I(x) \rightarrow I(y)$;
(2) $I(x \vee y) = I(x) \vee I(y)$;
(3) $I(\top) = \top$.

$\mathbf{M}$-*valuations* are defined as for MTL-algebras but satisfy also $v(\Box A) = I(v(A))$.

- An $\mathrm{LKT^r}$-*algebra* is an $\mathrm{LK^r}$-algebra satisfying also (4) $I(x) \leq x$.
- An $\mathrm{LS4^r}$-*algebra* is an $\mathrm{LKT^r}$-algebra satisfying also (5) $I(I(x)) = I(x)$.
- An $\mathrm{L!^r}$-*algebra* is an $\mathrm{LS4^r}$-algebra satisfying also (6) $I(x) \odot I(x) = I(x)$.
- An $\mathrm{L^r_\Delta}$-*algebra* is an $\mathrm{LS4^r}$-algebra satisfying also (7) $I(x) \vee (I(x) \rightarrow \bot) = \top$.

Before proceeding further, let us observe the following useful fact:

Lemma 4 $I(x \odot y) = I(I(x) \odot I(y))$ *holds for all MTLS4^r-algebras.*

Proof. $I(I(x) \odot I(y)) \leq I(x) \odot I(y) \leq I(x \odot y)$ follows using (4) and (1). To prove that $I(x\odot y) \leq I(I(x)\odot I(y))$, we first show that $I(x)\odot I(y) \leq I(x\odot y)$. By residuation (twice), this amounts to proving that $\top \leq I(x) \rightarrow (I(y) \rightarrow I(x \odot y))$. But since $x \rightarrow (y \rightarrow (x\odot y)) = \top$, we have $\top = I(\top) = I(x \rightarrow (y \rightarrow (x\odot y))) \leq I(x) \rightarrow I(y \rightarrow (x\odot y)) \leq I(x) \rightarrow (I(y) \rightarrow I(x\odot y))$, and the claim follows. Now replacing x by $I(x)$ and y by $I(y)$ in the formula $I(x) \odot I(y) \leq I(x \odot y)$, we get $I(I(x)) \odot I(I(y)) \leq I(I(x) \odot I(y))$. Since $I(I(x)) = I(x)$ and $I(I(y)) = I(y)$, we obtain $I(x) \odot I(y) \leq I(I(x) \odot I(y))$ as required. $\Box$

We will establish completeness results for all schematic extensions L of $\mathsf{MTLK^r}$. Given $T \subseteq \mathrm{Fm}_\Box$, the Lindenbaum algebra is defined in the usual way as $\mathbf{M}_T = \langle M_T, \wedge_T, \vee_T, \odot_T, \rightarrow_T, \bot_T, \top_T, \Box_T\rangle$ where $[A]_T = \{B \in \mathrm{Fm}_\Box : T \vdash_{\mathrm{L}} A \leftrightarrow B\}$, $M_T = \{[A]_T : A \in \mathrm{Fm}_\Box\}$, $\top_T = [\top]_T$, $\bot_T = [\bot]_T$, $\Box_T[A]_T = [\Box A]_T$, and $[A]_T \star_T [B]_T = [A \star B]_T$ for $\star \in \{\wedge, \vee, \odot, \rightarrow\}$. Then the next lemma follows from various provabilities in MTL and the extra modal axioms.

Lemma 5 $\mathbf{M}_T$ *is an* L-*algebra.*

Hence, proceeding in the standard way (see e.g [9,18]):

Theorem 6 *For any schematic extension* L *of* $\mathsf{MTLK^r}$*:*

(i) $\vdash_{\mathrm{L}} A$ *iff* A *is valid in all* L-*algebras;*
(ii) $T \vdash_{\mathrm{L}} A$ *iff for any* L-*algebra* $\mathbf{M}$ *and* $\mathbf{M}$-*valuation* v *such that* $v(B) = \top$ *for all* $B \in T$*, also* $v(A) = \top$.

We turn our attention next to completeness with respect to Ł-chains. First let us say that a *congruence filter* of an MTLK^r-algebra $\mathbf{M}$ is a set $F = \{x \in M : \exists y \leq x(y\theta\top)\}$ for some congruence θ on $\mathbf{M}$.

Lemma 7 *F is a congruence filter of an MTLK^r-algebra $\mathbf{M}$ iff (i) $\top \in F$; (ii) if x and $x \to y \in F$, then $y \in F$; (iii) if $x \in F$, then $I(x) \in F$.*

Proof. That a congruence filter must satisfy (i), (ii), and (iii) is almost immediate. We check e.g. (iii): if $x \in F$, then there is y such that $y \leq x$ and $y\theta\top$. It follows that $I(y)\theta\top$ and $I(y) \leq I(x)$, and hence that $I(x) \in F$. Conversely, let F be a congruence filter, and let θ be defined by $x\theta y$ iff $x \to y \in F$ and $y \to x \in F$. Then using a result of [15], θ is a congruence with respect to the operations of residuated lattices. We prove that θ is compatible with I. If $x\theta y$, then $x \to y \in F$ and $y \to x \in F$. So by assumption, $I(x \to y) \in F$ and $I(y \to x) \in F$. It follows that $I(x) \to I(y) \in F$ and $I(y) \to I(x) \in F$. Thus $I(x)\theta I(y)$, and θ is a congruence of MTLK^r-algebras. □

For an MTLK^r-algebra $\mathbf{M}$ and $a \in M$, let $Fg(a)$ be the smallest congruence filter containing a. We define inductively: $I_0(a) = a$; $I_{n+1}(a) = I(I_n(a)) \wedge I_n(a)$.

Lemma 8 *For every MTLK^r-algebra $\mathbf{M}$ and $a \in M$:*

$$Fg(a) = \{x \in M : \exists n \in \mathbb{N} : (I_n(a))^n \leq x\}.$$

Proof. Let $G = \{x \in M : \exists n \in \mathbb{N} : (I_n(a))^n \leq x\}$. Then $G \subseteq Fg(a)$, since $a \in Fg(a)$ and $Fg(a)$ is closed upwards and closed under I, $\odot$, and $\wedge$. For the opposite direction, since $a \in G$, it is sufficient to prove that G is a congruence filter. That $\top \in G$ and that G is closed upwards is trivial. We verify closure under detachment. If x and $x \to y \in G$, then there are m, n such that $(I_n(a))^n \leq x$ and $(I_m(a))^m \leq x \to y$. But then easily $(I_{n+m}(a))^{n+m} \leq x \odot (x \to y) \leq y$, and hence $y \in G$. Finally, G is closed under I. If $x \in G$, then there is an n such that $(I_n(a))^n \leq x$. It follows that $(I_{n+1}(a))^{n+1} \leq I\left((I_n(a))^n\right) \leq I(x)$, and $I(x) \in G$. Thus G is a filter and $a \in G$. It follows that $Fg(a) \subseteq G$. □

Theorem 9 *Every subdirectly irreducible MTLK^r-algebra is linearly ordered.*

Proof. By induction on n, we can easily show $I_n(a \vee b) = I_n(a) \vee I_n(b)$. Also, since $(a \vee b)^n = a^n \vee b^n$ holds in all MTL-algebras, $(I_n(a \vee b))^n = (I_n(a))^n \vee (I_n(b))^n$. Now suppose for a contradiction that $\mathbf{M}$ is a subdirectly irreducible MTLK^r-algebra with minimum non-trivial filter F and elements a, b such that $a \not\leq b$ and $b \not\leq a$. Then both $Fg(a \to b)$ and $Fg(b \to a)$ are non-trivial filters; hence they both contain F. Let $c \in F$ with $c < \top$. Then there are $m, n \in \mathbb{N}$ such that $(I_n(a \to b))^n \leq c$ and $(I_m(b \to a))^m \leq c$. Let $k = \max\{n, m\}$. Then $c \geq (I_k(a \to b))^k \vee (I_k(a \to b))^k = (I_k((a \to b) \vee (b \to a)))^k = (I_k(\top))^k = \top$, a contradiction. □

Hence, making use of Birkhoff's subdirect representation theorem:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box A}\ (\Box) \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta}\ (\Box\Rightarrow) \qquad \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow A}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box A}\ (\Rightarrow\Box)$$

$$\frac{\mathcal{G} \mid \Gamma, \Box A, \Box A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta}\ (\textsc{cl})_{\Box} \qquad \frac{\mathcal{G} \mid \Box\Gamma, \Pi \Rightarrow \Sigma}{\mathcal{G} \mid \Box\Gamma \Rightarrow \mid \Pi \Rightarrow \Sigma}\ (\textsc{split})_{\Box}$$

Fig. 4. Modal hypersequent rules

Corollary 10 *Every MTLK$^{\mathrm{r}}$-algebra is isomorphic to a subdirect product of a family of MTLK$^{\mathrm{r}}$-chains.*

Corollary 11 *For any schematic extension* L *of* $\mathsf{MTLK}^{\mathrm{r}}$*:*

(i) $\vdash_{\mathrm{L}} A$ *iff* A *is valid in all* L*-chains;*
(ii) $T \vdash_{\mathrm{L}} A$ *iff for any* L*-chain* $\mathbf{M}$ *and* $\mathbf{M}$*-valuation* v *such that* $v(B) = \top$ *for all* $B \in T$*, also* $v(A) = \top$.

We obtain hypersequent calculi for logics with truth stresser modalities by adding introduction rules for $\Box$, and then various structural rules characterizing its behaviour. Let us write $\Box\Gamma$ for the multiset $[\Box A : A \in \Gamma]$. Then, consulting Figure 4, we define for each $\mathrm{L} \in \mathrm{Logics}$:

- $\mathrm{GLK}^{\mathrm{r}}$ is GL extended with $(\Box)$;
- $\mathrm{GLKT}^{\mathrm{r}}$ is GL extended with $(\Box)$ and $(\Box\Rightarrow)$;
- $\mathrm{GLS4}^{\mathrm{r}}$ is GL extended with $(\Box\Rightarrow)$ and $(\Rightarrow\Box)$;
- $\mathrm{GL!}^{\mathrm{r}}$ is $\mathrm{GLS4}^{\mathrm{r}}$ extended with $(\textsc{cl})_{\Box}$;
- $\mathrm{GL}^{\mathrm{r}}_{\Delta}$ is $\mathrm{GLS4}^{\mathrm{r}}$ extended with $(\textsc{split})_{\Box}$.

$(\Box\Rightarrow)$, $(\Box)$, and $(\Rightarrow\Box)$ are hypersequent versions of rules familiar from sequent calculi for the modal logics $\mathsf{S4}$, K, and KT. However, $\Box(A \vee B) \to (\Box A \vee \Box B)$, which is not derivable in $\mathsf{S4}$, is derivable as follows in all the systems defined above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A \mid A \vee B \Rightarrow B}\ (\textsc{id}) \qquad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ (\textsc{id}) \quad \dfrac{}{B \Rightarrow B}\ (\textsc{id})}{B \Rightarrow A \mid A \Rightarrow B}\ (\textsc{com}) \qquad \dfrac{}{B \Rightarrow A \mid B \Rightarrow B}\ (\textsc{id})}{B \Rightarrow A \mid A \vee B \Rightarrow B}\ (\vee\Rightarrow)}{A \vee B \Rightarrow A \mid A \vee B \Rightarrow B}\ (\vee\Rightarrow)}{\Box(A \vee B) \Rightarrow \Box A \mid \Box(A \vee B) \Rightarrow \Box B}\ (\Box){*2}}{\Box(A \vee B) \Rightarrow \Box A \vee \Box B \mid \Box(A \vee B) \Rightarrow \Box A \vee \Box B}\ (\Rightarrow\vee){*2}}{\Box(A \vee B) \Rightarrow \Box A \vee \Box B}\ (\textsc{ec})}{\Rightarrow \Box(A \vee B) \to (\Box A \vee \Box B)}\ (\Rightarrow\to)$$

$(\textsc{cl})_{\Box}$ is a hypersequent version of a rule used for the exponential ! in linear logic [10], while $(\textsc{split})_{\Box}$ ensures that boxed formulas obey the law of excluded middle:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box A \Rightarrow \Box A}\ (\textsc{id})}{\Rightarrow \Box A \mid \Box A \Rightarrow}\ (\textsc{split})_{\Box}}{\Rightarrow \Box A \mid \Box A \Rightarrow \bot}\ (\textsc{w})}{\Rightarrow \Box A \mid \Rightarrow \Box A \to \bot}\ (\Rightarrow\to)}{\Rightarrow \Box A \vee (\Box A \to \bot) \mid \Rightarrow \Box A \vee (\Box A \to \bot)}\ (\Rightarrow\vee){*2}}{\Rightarrow \Box A \vee (\Box A \to \bot)}\ (\textsc{ec})$$

Note that it is possible to introduce a wide variety of other structural rules for $\Box$; for example, in analogy with the rule (WCL) of GSMTL, we might define:

$$\frac{\mathcal{G} \mid \Gamma, \Box\Pi, \Box\Pi \Rightarrow}{\mathcal{G} \mid \Gamma, \Box\Pi \Rightarrow} \ (\text{WCL})_\Box$$

We now turn our attention to showing that the axiomatic and hypersequent presentations really characterize the same logics. First, note that the following is easily proved using repeated applications of (CUT):

Lemma 12 (cf. e.g. [18]) *For* $\mathrm{L} \in \mathrm{Logics}_\Box$*, if* $\vdash_{\mathrm{GL}} \Rightarrow i(\mathcal{G})$*, then* $\vdash_{\mathrm{GL}} \mathcal{G}$.

Theorem 13 *For* $\mathrm{L} \in \mathrm{Logics}_\Box$*,* $\vdash_{\mathrm{GL}} \mathcal{G}$ *iff* $\vdash_{\mathrm{L}} i(\mathcal{G})$.

Proof. For the left-to-right direction we proceed by induction on the height of a derivation of $\mathcal{G}$ in GL. If $\mathcal{G}$ is an initial sequent of GL, then it is easy to check that $\vdash_{\mathrm{L}} i(\mathcal{G})$. For the inductive step, suppose that $\mathcal{G}$ follows by some rule of GL from $\mathcal{G}_1, \ldots, \mathcal{G}_n$. By the induction hypothesis n times, we have $\vdash_{\mathrm{L}} i(\mathcal{G}_1), \ldots, \vdash_{\mathrm{L}} i(\mathcal{G}_n)$. For the non-modal rules of GL (see e.g. [18] for details), it is easy to check that $\vdash_{\mathrm{L}} i(\mathcal{G}_1) \to (i(\mathcal{G}_2) \to (\ldots \to (i(\mathcal{G}_n) \to i(\mathcal{G}))\ldots)$, and that hence, by (MP) n times, $\vdash_{\mathrm{L}} i(\mathcal{G})$. For the modal rules, we check each case in turn, writing $\odot[A_1, \ldots, A_n]$ and $\oplus[A_1, \ldots, A_n]$ as shorthand for $A_1 \odot \ldots \odot A_n$ and $A_1 \oplus \ldots \oplus A_n$, respectively.

1. ($\Box$). Suppose that $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee i(\Gamma \Rightarrow A)$. By Corollary 11, it is sufficient to show that $i(\mathcal{G} \mid \Box\Gamma \Rightarrow \Box A)$ is valid in every L-chain. Consider a valuation v for such an algebra. Either $v(i(\mathcal{G})) = \top$ and hence $v(i(\mathcal{G}) \vee i(\Box\Gamma \Rightarrow \Box A)) = \top$ or $v(i(\Gamma \Rightarrow A)) = \top$. If the latter, then $I(v(i(\Gamma \Rightarrow A))) = I(\top) = \top$. But $I(v(i(\Gamma \Rightarrow A))) = v(i(\Box\Gamma \Rightarrow \Box A))$ so we are done.
2. ($\Box \Rightarrow$). Suppose that $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee ((\odot(\Gamma) \odot A) \to \oplus(\Delta))$, then by $(A7)$ $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee (A \to (\odot(\Gamma) \to \oplus(\Delta)))$, and by (NEC) and $(\vee_\Box)$, $\vdash_{\mathrm{L}} \Box i(\mathcal{G}) \vee \Box(A \to (\odot(\Gamma) \to \oplus(\Delta)))$. Using axiom $(\mathrm{K}_\Box)$ and (MP), we get $\vdash_{\mathrm{L}} \Box i(\mathcal{G}) \vee (\Box A \to \Box(\odot(\Gamma) \to \oplus(\Delta)))$ Now, using $(\mathrm{T}_\Box)$, we reach $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee (\Box A \to (\odot(\Gamma) \to \oplus(\Delta)))$, and then $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee ((\odot(\Gamma) \odot \Box A) \to \oplus(\Delta))$ as required.
3. ($\Rightarrow \Box$). If $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee (\odot(\Box\Gamma) \to A)$, then by (NEC) and $(\vee_\Box)$, $\vdash_{\mathrm{L}} \Box i(\mathcal{G}) \vee \Box(\odot(\Box\Gamma) \to A)$, and by $(\mathrm{K}_\Box)$ and (MP), $\vdash_{\mathrm{L}} \Box i(\mathcal{G}) \vee (\Box \odot (\Box\Gamma) \to \Box A)$. Using $(\mathrm{T}_\Box)$ and $(4_\Box)$, $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee (\odot(\Box\Gamma) \to \Box A)$ as required.
4. $(\text{CL})_\Box$. If $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee ((\odot(\Gamma) \odot \Box A \odot \Box A) \to \oplus(\Delta))$, then by $(\mathrm{C}_\Box)$, $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee ((\odot(\Gamma) \odot \Box A) \to \oplus(\Delta))$.
5. $(\text{SPLIT})_\Box$. Suppose that $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee ((\odot(\Box\Gamma) \odot (\Pi)) \to \oplus(\Sigma))$. Since by $(\mathrm{S}_\Box)$ $\vdash_{\mathrm{L}} \Box \odot (\Gamma) \vee (\Box \odot (\Gamma) \to \bot)$, it follows that $\vdash_{\mathrm{L}} i(\mathcal{G}) \vee ((\odot(\Pi) \to \oplus(\Sigma)) \vee (\Box \odot (\Gamma) \to \bot))$ as required.

For the right-to-left direction, we have (an easy exercise) that the axioms of L are derivable in GL. Moreover, (NEC) corresponds to ($\Box$) or ($\Rightarrow \Box$), and (MP) can be derived from $\vdash_{\mathrm{GL}} \Rightarrow A$ and $\vdash_{\mathrm{GL}} \Rightarrow A \to B$, by using (CUT) twice with $\vdash_{\mathrm{GL}} A, A \to B \Rightarrow B$. Hence, if $\vdash_{\mathrm{L}} i(\mathcal{G})$, then $\vdash_{\mathrm{GL}} \Rightarrow i(\mathcal{G})$, and so by Lemma 12, $\vdash_{\mathrm{GL}} \mathcal{G}$. $\Box$

4 Cut Elimination

The proof of completeness for the calculi considered above (Theorem 13) relies heavily on the presence of the cut rule (CUT). In this section we give a constructive proof that (CUT) can in fact be *eliminated* from derivations in GL for all $\mathrm{L} \in \mathrm{Logics}_{\Box}$. This result, known as cut elimination, implies the *subformula property* for cut-free versions of these calculi, i.e., that all formulas occurring in a cut-free derivation of GL are subformulas of the formula to be proved. Among other things, this ensures that for each logic $\mathrm{L} \in \mathrm{Logics}$, LK^{r}, $\mathrm{LKT}^{\mathrm{r}}$, $\mathrm{LS4}^{\mathrm{r}}$, $\mathrm{L!}^{\mathrm{r}}$, and $\mathrm{L}^{\mathrm{r}}_{\Delta}$ are all *conservative extensions* of L, i.e., for each formula A not containing $\Box$, A is derivable in L iff A is derivable in the extended logic.

In order to prove cut elimination for these calculi in a systematic and uniform manner, we first require a number of auxiliary concepts. Let us assume in what follows that Γ, Δ, Π, Σ, Ξ denote multisets of formulas, and λ, μ, m, n, i, j (possibly subscripted) denote natural numbers, recalling that $\Gamma^0 = []$ and $\Gamma^{n+1} = \Gamma \uplus \Gamma^n$.

The *principal formula* of an instance of a logical rule $(\star \Rightarrow)$ or $(\Rightarrow \star)$ is the formula in the conclusion with topmost connective $\star \in \{\to, \odot, \wedge, \vee, \Box\}$. A *marked hypersequent* is a hypersequent with exactly one occurrence of a formula A distinguished, written $(\mathcal{G} \mid \Gamma, \underline{A} \Rightarrow \Delta)$ or $(\mathcal{G} \mid \Gamma \Rightarrow \underline{A}, \Delta)$. A *marked rule instance* is a rule instance with the principal formula, if there is one, marked. Let us also say that a hypersequent $\mathcal{G}$ is "appropriate for a rule (r)" if it is single-conclusion when (r) is single-conclusion. We now define the result of applying (CUT) multiple times, assuming that usual notions for hypersequents apply also to marked hypersequents.

Suppose that $\mathcal{G}$ is a (possibly marked) hypersequent and $\mathcal{H}$ a marked hypersequent of the forms:

$$\mathcal{G} = (\Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n) \quad \text{and} \quad \mathcal{H} = (\mathcal{H}' \mid \Pi \Rightarrow \underline{A}, \Sigma)$$

where A does not occur unmarked in $\biguplus_{i=1}^{n} \Gamma_i$. Then $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ is the set containing, for all $0 \leq \mu_i \leq \lambda_i$ for $i = 1 \ldots n$:

$$\mathcal{H}' \mid \Gamma_1, \Pi^{\mu_1}, [A]^{\lambda_1 - \mu_1} \Rightarrow \Sigma^{\mu_1}, \Delta_1 \mid \ldots \mid \Gamma_n, \Pi^{\mu_n}, [A]^{\lambda_n - \mu_n} \Rightarrow \Sigma^{\mu_n}, \Delta_n.$$

Similarly, suppose that A does not occur unmarked in $\biguplus_{i=1}^{n} \Delta_i$ with:

$$\mathcal{G} = (\Gamma_1 \Rightarrow [A]^{\lambda_1}, \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow [A]^{\lambda_n}, \Delta_n) \quad \text{and} \quad \mathcal{H} = (\mathcal{H}' \mid \Pi, \underline{A} \Rightarrow \Sigma).$$

Then $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ contains, for all $0 \leq \mu_i \leq \lambda_i$ for $i = 1 \ldots n$:

$$\mathcal{H}' \mid \Gamma_1, \Pi^{\mu_1} \Rightarrow [A]^{\lambda_1 - \mu_1}, \Sigma^{\mu_1}, \Delta_1 \mid \ldots \mid \Gamma_n, \Pi^{\mu_n} \Rightarrow [A]^{\lambda_n - \mu_n}, \Sigma^{\mu_n}, \Delta_n.$$

A rule (r) is *substitutive* if for any:

- marked instance $\frac{\mathcal{G}_1 \ \ldots \ \mathcal{G}_n}{\mathcal{G}}$ of (r);

- marked hypersequent $\mathcal{H}$ appropriate for (r);
- $\mathcal{G}' \in \mathrm{CUT}(\mathcal{G}, \mathcal{H})$;

there exist $\mathcal{G}'_i \in \mathrm{CUT}(\mathcal{G}_i, \mathcal{H})$ for $i = 1 \ldots n$ such that $\frac{\mathcal{G}'_1 \, \ldots \, \mathcal{G}'_n}{\mathcal{G}'}$ is an instance of (r).

Substitutivity ensures that cuts over formulas that are not principal in the rule can be shifted upwards over the premises. Roughly speaking, it says that substituting occurrences of a non principal formula A with Π on the left and Σ on the right, in both the conclusion of a rule instance and suitably in its premises, gives another instance of the rule. This is easy to check for rules presented schematically; hence:

Lemma 14 (cf. e.g. [18]) *The rules of Figures 2 and 3 are substitutive.*

Substitutivity and other related conditions have been used to provide broad and uniform characterizations of sequent and hypersequent calculi admitting cut elimination in e.g. [5,18]. However, in the case of (certain) modal rules we face a problem: substitutivity fails when cutting modal formulas on the left. Consider, for example, an instance of $(\Rightarrow \Box)$ with premise $(\Box B \Rightarrow A)$ and conclusion $(\Box B \Rightarrow \Box A)$. "Cutting" the latter with $(C \Rightarrow \Box B)$ gives $(C \Rightarrow \Box A)$ but there is no way to cut the premise with $(C \Rightarrow \Box B)$ to obtain another instance of the rule. Hence we must be a bit more careful. We notice that the problem does not occur when cutting modal rules with a sequent in which all formulas are "boxed", i.e., with the conclusion of an instance of $(\Rightarrow \Box)$ (e.g. $\Box C \Rightarrow \Box B$, in the previous example). The cut-elimination proof hence proceeds by shifting a uppermost cut upwards in a specific order: first over the premise in which the cut formula appears on the right (Lemma 16) and then, when a rule introducing the cut formula is reached, shifting the cut upwards over the other premise (Lemma 15).

The *length* $|d|$ of a derivation d is (the maximal number of applications of inference rules) $+1$ occurring on any branch of d. The *complexity* $|A|$ of a formula A is the number of occurrences of its connectives. The *cut rank* $\rho(d)$ of d is (the maximal complexity of cut formulas in d) $+1$, noting that $\rho(d) = 0$ if d is cut-free.

Lemma 15 *Let* $\mathrm{L} \in \mathrm{Logics}_\Box$. *Let* d_l *and* d_r *be derivations in* GL *such that:*

(1) d_l *is a derivation of* $(\mathcal{G} \mid \Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n)$;
(2) d_r *is a derivation of* $(\mathcal{H} \mid \Sigma \Rightarrow A, \Pi)$;
(3) $\rho(d_l) \leq |A|$ *and* $\rho(d_r) \leq |A|$;
(4) A *is a compound formula and* d_r *ends with either a right logical rule or a modal rule introducing* A.

Then a derivation d *can be constructed in* GL *of* $(\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \ldots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n})$ *with* $\rho(d) \leq |A|$.

Proof. We proceed by induction on $|d_l|$. If d_l ends in an initial sequent, then we are done. Otherwise, let (r) be the last inference rule applied in d_l.

- If (r) acts only on $\mathcal{G}$, then the claim follows by the induction hypothesis and an application of (r).
- If (r) is any non-modal rule for GL not introducing A, then by Lemma 14 the claim follows by the induction hypothesis with applications of (r) and (EW).
- Suppose that (r) is a left introduction rule for A and A is $B \star C$ with $\star \in \{\wedge, \vee, \odot, \to\}$. As an example, assume that A is $B \wedge C$ and d_l ends as follows:

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ \mathcal{G} \mid \Gamma_1, [B \wedge C]^{\lambda_1 - 1}, B \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [B \wedge C]^{\lambda_n} \Rightarrow \Delta_n\end{array}}{\mathcal{G} \mid \Gamma_1, [B \wedge C]^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [B \wedge C]^{\lambda_n} \Rightarrow \Delta_n}\ (\wedge\Rightarrow)$$

  By the induction hypothesis, we obtain a derivation of $(\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma^{\lambda_1-1}, B \Rightarrow \Delta_1, \Pi^{\lambda_1 - 1} \mid \ldots \mid \Gamma_n, \Sigma^{\lambda_n}, \Rightarrow \Delta_n, \Pi^{\lambda_n})$ with cut rank $\leq |A|$. The claim follows by (CUT) with $(\mathcal{H} \mid \Sigma \Rightarrow B, \Pi)$, one of the premises of the last inference rule applied in d_r, (EW), and (EC). The resulting derivation has cut rank $\leq |A|$.
- Suppose that (r) is $(\Box \Rightarrow)$, A is $\Box B$, and the indicated occurrence of $\Box B$ is the principal formula as e.g. in:

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ \mathcal{G} \mid \Gamma_1, B, [\Box B]^{\lambda_1 - 1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [\Box B]^{\lambda_n} \Rightarrow \Delta_n\end{array}}{\mathcal{G} \mid \Gamma_1, [\Box B]^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [\Box B]^{\lambda_n} \Rightarrow \Delta_n}\ (\Box\Rightarrow)$$

  Since d_r ends either in $(\Box)$ or $(\Rightarrow \Box)$, we have $\Sigma = \Box\Sigma'$ and $\Pi = []$. Hence by applying the induction hypothesis to d_1, we obtain a derivation $\overline{d}$ of the hypersequent $(\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, B, (\Box\Sigma')^{\lambda_1 - 1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, (\Box\Sigma')^{\lambda_n} \Rightarrow \Delta_n)$ with $\rho(\overline{d}) \leq \rho(\Box B)$. The claim then follows by applying (CUT) to the conclusion of $\overline{d}$ and the premise of d_r followed by (EW), (EC), and applications of $(\Box \Rightarrow)$.
- If (r) is $(\Box \Rightarrow)$ where the principal formula is not A, then the claim follows by the induction hypothesis and an application of (r).
- Suppose that (r) is $(\mathrm{SPLIT})_\Box$, A is $\Box B$, and d_l ends as follows:

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ \mathcal{G} \mid \Gamma_1, \Box\Xi, [\Box B]^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [\Box B]^{\lambda_n} \Rightarrow \Delta_n\end{array}}{\mathcal{G} \mid \Box\Xi, [\Box B]^{\lambda} \Rightarrow \mid \Gamma_1, [\Box B]^{\lambda_1 - \lambda} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, [\Box B]^{\lambda_n} \Rightarrow \Delta_n}\ (\mathrm{SPLIT})_\Box$$

  Since d_r ends in either $(\Box)$ or $(\Rightarrow \Box)$, then $\Sigma = \Box\Sigma'$ and $\Pi = []$. Hence by applying the induction hypothesis to d_1 we obtain a derivation $\overline{d}$ of the hypersequent $(\mathcal{G} \mid \mathcal{H} \mid \Box\Xi, (\Box\Sigma')^{\lambda_1}, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, (\Box\Sigma')^{\lambda_n} \Rightarrow \Delta_n)$. The required derivation $(\mathcal{G} \mid \mathcal{H} \mid \Box\Sigma, (\Box\Sigma')^{\lambda} \Rightarrow \mid \Gamma_1, (\Box\Sigma')^{\lambda_1 - \lambda} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, (\Box\Sigma')^{\lambda_n} \Rightarrow \Delta_n)$ is then obtained by applying $(\mathrm{SPLIT})_\Box$.
- If (r) is $(\Box)$, $(\Rightarrow \Box)$, or $(\mathrm{CL})_\Box$, then the proof is similar to the previous cases. $\Box$

Lemma 16 *Let* $\mathrm{L} \in \mathrm{Logics}_\Box$. *Let* d_l *and* d_r *be derivations in* GL *such that:*

(1) d_l *is a derivation of* $(\mathcal{G} \mid \Gamma, A \Rightarrow \Delta)$*;*
(2) d_r *is a derivation of* $(\mathcal{H} \mid \Sigma_1 \Rightarrow [A]^{\lambda_1}, \Pi'_1 \mid \ldots \mid \Sigma_n \Rightarrow [A]^{\lambda_n}, \Pi'_n)$*;*

(3) $\rho(d_l) \leq |A|$ *and* $\rho(d_r) \leq |A|$.

Then a derivation d *can be constructed in* GL *of* $(\mathcal{G} \mid \mathcal{H} \mid \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi'_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi'_n, \Delta^{\lambda_n})$ *with* $\rho(d) \leq |A|$.

Proof. We proceed by induction on $|d_r|$. Let (r) be the last inference rule applied in d_r. If (r) is an initial sequent, then the claim holds trivially. Suppose that (r) acts only on $\mathcal{H}$ or is a (substitutive by Lemma 14) non-modal rule where the indicated occurrences of the cut formula A are not principal. Then the claim follows by the induction hypothesis and an application of (r). Similarly, if (r) is $(\Box\Rightarrow)$, $(\textsc{split})_\Box$, or $(\textsc{cl})_\Box$, note that the indicated occurrences of the cut formula A occur on the right and are hence unchanged by the rule application. So again the claim follows by the induction hypothesis and an application of (r).

Now suppose that (r) is $(\Rightarrow\star)$ and introduces a cut formula A of the form $B \star C$ for $\star \in \{\wedge, \vee, \odot, \to\}$. As an example, suppose that A is $B \to C$ and d_r ends with:

$$\dfrac{\begin{array}{c}\vdots\, d_1 \\ \mathcal{H} \mid \Sigma_1, B \Rightarrow (B \to C)^{\lambda_1 - 1}, C, \Pi'_1 \mid \ldots \mid \Sigma_n \Rightarrow (B \to C)^{\lambda_n}, \Pi'_n\end{array}}{\mathcal{H} \mid \Sigma_1 \Rightarrow (B \to C)^{\lambda_1}, \Pi'_1 \mid \ldots \mid \Sigma_n \Rightarrow (B \to C)^{\lambda_n}, \Pi'_n}\,(\Rightarrow\to)$$

By the induction hypothesis, we obtain a derivation of $(\mathcal{G} \mid \mathcal{H} \mid \Sigma_1, \Gamma^{\lambda_1 - 1}, B \Rightarrow C, \Pi'_1, \Delta^{\lambda_1 - 1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi'_n, \Delta^{\lambda_n})$. The claim follows by $(\Rightarrow\to)$ and Lemma 15.

Finally, suppose that (r) is $(\Rightarrow\Box)$ or $(\Box)$ and introduces an occurrence of $A = \Box B$. As an example, assume that (r) is $(\Rightarrow\Box)$ and d_r ends as follows:

$$\dfrac{\begin{array}{c}\vdots\, d_1 \\ \mathcal{H} \mid \Box\Sigma_1 \Rightarrow B \mid \ldots \mid \Sigma_n \Rightarrow [\Box B]^{\lambda_n}, \Pi'_n\end{array}}{\mathcal{H} \mid \Box\Sigma_1 \Rightarrow \Box B \mid \ldots \mid \Sigma_n \Rightarrow [\Box B]^{\lambda_n}, \Pi'_n}\,(\Rightarrow\Box)$$

By the induction hypothesis, we obtain a derivation of $(\mathcal{G} \mid \mathcal{H} \mid \Box\Sigma_1 \Rightarrow B \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi^{\lambda_n}, \Pi'_n)$. The claim then follows by $(\Rightarrow\Box)$ and Lemma 15. The case where (r) is $(\Box)$ is very similar. $\Box$

Theorem 17 (Cut Elimination) *For* $\mathrm{L} \in \mathrm{Logics}_\Box$, *cut elimination holds for* GL.

Proof. Let d be a derivation in GL with $\rho(d) > 0$. The proof proceeds by a double induction on $\langle \rho(d), n\rho(d) \rangle$, where $n\rho(d)$ is the number of applications of $(\textsc{cut})$ in d with cut rank $\rho(d)$. Consider an uppermost application of $(\textsc{cut})$ in d with cut rank $\rho(d)$. By applying Lemma 16 to its premises either $\rho(d)$ or $n\rho(d)$ decreases. $\Box$

Corollary 18 *For* $\mathrm{L} \in \mathrm{Logics}$, LK^{r}, $\mathrm{LKT}^{\mathrm{r}}$, $\mathrm{LS4}^{\mathrm{r}}$, $\mathrm{L!}^{\mathrm{r}}$, *and* $\mathrm{L}^{\mathrm{r}}_{\Delta}$ *are all conservative extensions of* L.

5 Embedding G and CL into fuzzy logics with modalities

In this section we apply the proof theory developed above to obtain embeddings of Gödel logic G and Classical logic CL into $\mathrm{L!^r}$ and $\mathrm{L^r_\Delta}$, respectively, for any $\mathrm{L} \in \mathrm{Logics}$. To this end, we consider two embeddings found in the literature on modal and substructural logics, (see e.g. [22]), noting that from now on, for both G and CL, we use a more restricted language based on $\rightarrow$, $\wedge$, and $\bot$. Let us define the following translations from formulas to modal formulas (where a is any atom):

$$\begin{aligned} a^\circ &= a & a^\Box &= \Box a \\ (A \rightarrow B)^\circ &= \Box A^\circ \rightarrow B^\circ & (A \rightarrow B)^\Box &= \Box(A^\Box \rightarrow B^\Box) \\ (A \wedge B)^\circ &= A^\circ \wedge B^\circ & (A \wedge B)^\Box &= A^\Box \wedge B^\Box. \end{aligned}$$

These two mappings are related as follows:

Lemma 19 *Let* $\mathrm{L} \in \mathrm{Logics}$. *Then (a)* $\vdash_{\mathrm{L!^r}} \Box A^\circ \leftrightarrow A^\Box$*; (b)* $\vdash_{\mathrm{L^r_\Delta}} \Box A^\circ \leftrightarrow A^\Box$.

Proof. We prove (a) and (b) by induction on $|A|$, considering just (a) since (b) is very similar. If A is atomic, then the result follows immediately. If A is $B \rightarrow C$, then $\Box A^\circ$ is $\Box(\Box B^\circ \rightarrow C^\circ)$. However, easily $\vdash_{\mathrm{L!^r}} \Box(\Box B^\circ \rightarrow C^\circ) \leftrightarrow \Box(\Box B^\circ \rightarrow \Box C^\circ)$. Hence, using the induction hypothesis twice, $\vdash_{\mathrm{L!^r}} \Box(\Box B^\circ \rightarrow \Box C^\circ) \leftrightarrow \Box(B^\Box \rightarrow C^\Box)$ as required. If A is $B \wedge C$, then $\Box A^\circ = \Box(B^\circ \wedge C^\circ)$. Again, easily $\vdash_{\mathrm{L!^r}} \Box(B^\circ \wedge C^\circ) \leftrightarrow \Box B^\circ \wedge \Box C^\circ$. Hence, using the induction hypothesis twice, $\vdash_{\mathrm{L!^r}} \Box(B^\circ \wedge C^\circ) \leftrightarrow B^\Box \wedge C^\Box$ as required. $\Box$

We prove the embedding results for CL using a hypersequent calculus GCL defined as GMTL plus the single-conclusion version of (SPLIT) that is sound and complete for classical logic and admits cut elimination (see e.g. [18]).

Theorem 20 *For* $\mathrm{L} \in \mathrm{Logics}$, $\vdash_{\mathsf{CL}} A$ *iff* $\vdash_{\mathrm{L^r_\Delta}} A^\circ$ *iff* $\vdash_{\mathrm{L^r_\Delta}} A^\Box$.

Proof. By Lemma 19, we need only show $\vdash_{\mathsf{CL}} A$ iff $\vdash_{\mathrm{L^r_\Delta}} A^\circ$. For the left-to-right direction, we show that if d is a cut-free derivation in GCL of $(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$, then $\vdash_{\mathrm{GL^r_\Delta}} \Box\Gamma_1^\circ \Rightarrow \Delta_1^\circ \mid \ldots \mid \Box\Gamma_n^\circ \Rightarrow \Delta_n^\circ$, proceeding by induction on $|d|$. The base case is straightforward, as are the cases where the last rule applied is an internal structural rule. We consider some examples from the remaining cases below, other cases being very similar.

- Suppose that d ends with (for $i \in \{1, 2\}$):

  $$\frac{\mathcal{H} \mid \Gamma_1', A_i \Rightarrow \Delta_1}{\mathcal{H} \mid \Gamma_1', A_1 \wedge A_2 \Rightarrow \Delta_1}\ (\wedge\Rightarrow)_i$$

  where $\mathcal{H} = (\Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$ and $\Gamma_1 = \Gamma_1' \uplus [A_1 \wedge A_2]$. Let $\mathcal{H}_\Box = (\Box\Gamma_2^\circ \Rightarrow \Delta_2^\circ \mid \ldots \mid \Box\Gamma_n^\circ \Rightarrow \Delta_n^\circ)$. Since $(\Box(A_1^\circ \wedge A_2^\circ) \Rightarrow \Box A_1^\circ \wedge \Box A_2^\circ)$ is derivable in $\mathrm{GL}^{\mathrm{r}}_{\Delta}$, and, by the induction hypothesis, so is $(\mathcal{H}_{\Box} \mid \Box\Gamma'^{\circ}_1, \Box A^{\circ}_i \Rightarrow \Delta^{\circ}_1)$, we obtain a derivation ending with:

$$\dfrac{\dfrac{\mathcal{H}_{\Box} \mid \Box\Gamma'^{\circ}_1, \Box A^{\circ}_i \Rightarrow \Delta^{\circ}_1}{\mathcal{H}_{\Box} \mid \Box\Gamma'^{\circ}_1, \Box A^{\circ}_1 \wedge \Box A^{\circ}_2 \Rightarrow \Delta^{\circ}_1}\ (\wedge\Rightarrow)_i \qquad \Box(A^{\circ}_1 \wedge A^{\circ}_2) \Rightarrow \Box A^{\circ}_1 \wedge \Box A^{\circ}_2}{\mathcal{H}_{\Box} \mid \Box\Gamma'^{\circ}_1, \Box(A^{\circ}_1 \wedge A^{\circ}_2) \Rightarrow \Delta^{\circ}_1}\ (\textsc{cut})$$

- Suppose that d ends with:

$$\dfrac{\mathcal{H} \mid \Gamma_1, A \Rightarrow B}{\mathcal{H} \mid \Gamma_1 \Rightarrow A \to B}\ (\Rightarrow\to)$$

where $\mathcal{H} = (\Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$ and $\Delta_1 = [A \to B]$. Let $\mathcal{H}_{\Box} = (\Box\Gamma^{\circ}_2 \Rightarrow \Delta^{\circ}_2 \mid \ldots \mid \Box\Gamma^{\circ}_n \Rightarrow \Delta^{\circ}_n)$. Since $(\mathcal{H}_{\Box} \mid \Box\Gamma^{\circ}_1, \Box A^{\circ} \Rightarrow B^{\circ})$ is derivable in $\mathrm{GL}^{\mathrm{r}}_{\Delta}$ by the induction hypothesis, we obtain a derivation ending with:

$$\dfrac{\mathcal{H}_{\Box} \mid \Box\Gamma^{\circ}_1, \Box A^{\circ} \Rightarrow B^{\circ}}{\mathcal{H}_{\Box} \mid \Box\Gamma^{\circ}_1 \Rightarrow \Box A^{\circ} \to B^{\circ}}\ (\Rightarrow\to)$$

- Suppose that d ends with:

$$\dfrac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow \mid \Gamma_2 \Rightarrow \Delta_2}\ (\textsc{split})$$

where $\mathcal{H} = (\Gamma_3 \Rightarrow \Delta_3 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$. Let $\mathcal{H}_{\Box} = (\Box\Gamma^{\circ}_3 \Rightarrow \Delta^{\circ}_3 \mid \ldots \mid \Box\Gamma^{\circ}_n \Rightarrow \Delta^{\circ}_n)$. Since, by the induction hypothesis, $(\mathcal{H}_{\Box} \mid \Box\Gamma^{\circ}_1, \Box\Gamma^{\circ}_2 \Rightarrow \Delta^{\circ}_2)$ is derivable in $\mathrm{GL}^{\mathrm{r}}_{\Delta}$, we obtain a derivation ending with:

$$\dfrac{\mathcal{H}_{\Box} \mid \Box\Gamma^{\circ}_1, \Box\Gamma^{\circ}_2 \Rightarrow \Delta^{\circ}_2}{\mathcal{H}_{\Box} \mid \Box\Gamma^{\circ}_1 \Rightarrow \mid \Box\Gamma^{\circ}_2 \Rightarrow \Delta^{\circ}_2}\ (\textsc{split})_{\Box}$$

For the right-to-left direction it is easily shown that if $d \vdash_{\mathrm{GL}^{\mathrm{r}}_{\Delta}{}^{\mathrm{cf}}} (\Box\Gamma^{\circ}_1 \Rightarrow \Delta^{\circ}_1 \mid \ldots \mid \Box\Gamma^{\circ}_n \Rightarrow \Delta^{\circ}_n)$, then $(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$ is derivable in GL + (SPLIT) (and hence also GCL), proceeding by induction on $|d|$. $\Box$

We now turn our attention to Gödel logic G, considering a multiple-conclusion calculus for this logic that corresponds (roughly speaking) to a hypersequent version of Maehara's calculus for intuitionistic logic. We begin by showing that such a calculus is equivalent to the usual one for G.

Lemma 21 *Let* GG' *be* $\mathsf{GC_2IMTL}$ *with the single-conclusion version of* $(\Rightarrow\to)$. *Then* $\vdash_{\mathsf{GG}'} \Rightarrow A$ *iff* $\vdash_{\mathsf{GC_2MTL}} \Rightarrow A$*, for any formula* A.

Proof. The right-to-left direction is almost immediate, since all the rules of $\mathsf{GC_2MTL}$ are derivable in GG'. For the left-to-right direction, we define a revised interpretation of hypersequents, i', which is exactly the same as i defined above, except that $i'(A, \ldots, A_n \Rightarrow B_1, \ldots, B_m) = (A_1 \wedge \ldots \wedge A_n) \to (B_1 \vee \ldots \vee B_m)$. It is then straightforward to show that GG' is sound with respect to this interpretation (i.e., for each rule of GG', if the interpretation of their premises is valid in G, so is its conclusion) and hence that if $\vdash_{\mathsf{GG}'} A$, then $\vdash_{\mathsf{GC_2MTL}} A$. $\Box$

Lemma 22 *For* $\mathrm{L} \in \mathrm{Logics}$, *if* $\vdash_{\mathsf{GL!^r}} \Box\Gamma^\circ \Rightarrow A^\circ$, *then there exists a derivation of* $(\Box\Gamma^\circ \Rightarrow A^\circ)$ *in* $\mathsf{GL!^r}$ *where* $(\Rightarrow\to)$ *is restricted to single-conclusion hypersequents.*

Proof. First, we note that using cut elimination, the rules $(\Rightarrow\to)$ and $(\Rightarrow\wedge)$ are cut-free invertible for $\mathsf{GL!^r}$ for each $\mathrm{L} \in \mathrm{Logics}$, that is, the conclusion of an instance is cut-free derivable iff the premises are cut-free derivable. Hence, if $\vdash_{\mathrm{GL!^r}} \Box\Gamma^\circ \Rightarrow A^\circ$, then we can construct a cut-free derivation of $(\Box\Gamma^\circ \Rightarrow A^\circ)$ in $\mathrm{GL!^r}$ where $(\Rightarrow\to)$ and $(\Rightarrow\wedge)$ are applied before all other rules. It is then easy to show inductively that for all sequents occurring in such a derivation where there is more than one formula on the right, these formulas must be either atomic or of the form $\Box A$. It follows that $(\Rightarrow\to)$ occurs only when there is just one formula on the right. $\Box$

Theorem 23 *For* $\mathrm{L} \in \mathrm{Logics}$, $\vdash_{\mathsf{G}} A$ *iff* $\vdash_{\mathrm{L!^r}} A^\circ$ *iff* $\vdash_{\mathrm{L}^{\mathrm{r}}_{\Delta}} A^\Box$.

Proof. By Lemma 19, we need only show $\vdash_{\mathsf{G}} A$ iff $\vdash_{\mathrm{L!^r}} A^\circ$. For the left-to-right direction, we show that if d is a cut-free derivation in $\mathsf{GC_2MTL}$ of $(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \Gamma_n \Rightarrow \Delta_n)$, then $\vdash_{\mathrm{GL!^r}} \Box\Gamma_1^\circ \Rightarrow \Delta_1^\circ \mid \ldots \mid \Box\Gamma_n^\circ \Rightarrow \Delta_n^\circ$, proceeding by induction on $|d|$. The proof of this claim matches almost exactly the proof of the corresponding claim in Theorem 20, except for the case of (C_2) which is replaced by an application of $(\mathrm{CL})_\Box$. For the right-to-left direction, if $\vdash_{\mathrm{L!^r}} A^\circ$, then by Lemma 22, there exists a derivation in $\mathrm{GL!^r}$ where $(\Rightarrow\to)$ is restricted to single-conclusion hypersequents. One can show inductively that if $\mathcal{G} = (\Box\Gamma_1^\circ \Rightarrow \Delta_1^\circ \mid \ldots \mid \Box\Gamma_n^\circ \Rightarrow \Delta_n^\circ)$ is derivable in $\mathrm{GL!^r}$ with the restricted use of $(\Rightarrow\to)$, then $\vdash_{\mathsf{GG'}} \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, proceeding by induction on the height of a cut-free derivation of $\mathcal{G}$. Hence, by Lemma 21, $\vdash_{\mathsf{G}} A$. $\Box$

6 Standard completeness

Completeness of each $\mathrm{L} \in \mathrm{Logics}_\Box$ is established above with respect to L-chains. In this section, we show further that in certain cases L is also *standard complete*, that is, complete with respect to standard L algebras (recall, L-algebras with lattice reduct $[0,1]$). Our proofs follow the strategy of [14,8]; namely, we construct embeddings of L-chains into dense L-chains and then into standard L-algebras. However, we remark that there exists also an alternative "proof-theoretic" strategy, used to prove standard completeness for non-modal fuzzy logics in [16] (see also [6,18]). In this approach, a density rule is first added to the logic which guarantees completeness with respect to dense L-chains, and then eliminated in similar fashion to cut elimination from proofs in hypersequent calculi.

Let us introduce some useful auxiliary concepts. Fix $\mathrm{L} \in \mathrm{Logics}$. We call an $\mathrm{LS4^r}$-algebra $\mathbf{M}$ *superstandard* if it is standard and I is left-continuous, that is, if for every $X \subseteq M$ with a supremum in M, $I(\sup(X)) = \sup(I(X))$. An *I-l-monoid* is a commutative integral bounded lattice-ordered monoid (l-monoid) $\mathbf{M}$ equipped with a unary operator I satisfying conditions (1)-(5) for LS4^r-algebras and also if $X \subseteq M$ and $\sup(X)$ exists in M, then $y \odot \sup(X) = \sup(y \odot X)$ for all $y \in M$.

Lemma 24

(i) Let $\mathbf{M}$ *be an* I*-l-monoid or MTLS4^r-algebra. Define* $O = \{x \in M : x = I(x)\}$*. Then* O *is the domain of a submonoid and sublattice* $\mathbf{O}$ *of* $\mathbf{M}$ *such that for all* $a \in M$*, the set* $O_a = \{o \in O : o \leq a\}$ *has a supremum which belongs to* O *and is equal to* $I(a)$*.*

(ii) Let $\mathbf{M}$ *be an l-monoid or MTL-algebra, and let* O *be a submonoid and sublattice of* $\mathbf{M}$ *such that for all* $a \in M$*, the set* $O_a = \{o \in O : o \leq a\}$ *has a supremum which belongs to* O*. Then, defining for all* $a \in M$*,* $I(a) = \sup(O_a)$*, the operator* I *makes* $\mathbf{M}$ *an* I*-l-monoid (an MTLS4^r-algebra respectively) where* $I(x) = x$ *iff* $x \in O$*.*

Proof. All conditions in (i) save the last follow immediately from the definitions of an MTLS4^r-algebra and I-l-monoid respectively. We hence prove that for all $a \in M$, $I(a) = \sup(O_a)$, and that therefore such a supremum exists and belongs to O, since $I(I(a)) = I(a)$. First, note that by the monotonicity of I, we have $I(a) \geq \sup(O_a)$. On the other hand, $I(a) \leq a$ and $I(a) \in O_a$ (since $I(I(a)) = I(a)$). Hence $I(a) \leq \sup(O_a)$, and the claim is proved.

Now suppose that O satisfies the conditions of (ii), and let for all $a \in M$, $I(a) = \sup(O_a)$. Then for $x \in O_a$, it holds that $x \leq a$, so $I(a) = \sup(O_a) \leq a$. Clearly also $I(x) = x$ for all $x \in O$. The remaining properties of I follow from the closure of O under the lattice operations and $\odot$. To show that O is the set of fixed points of I, it remains to prove that if $a \notin O$, then $I(a) \neq a$. By assumption, $I(a) = \sup(O_a)$ exists and is in O, hence if $a \notin O$, it is not possible that $I(a) = a$. $\square$

Hence MTLS4^r-algebras may be presented as residuated lattices with a privileged set O called an *open system* satisfying the conditions of Lemma 24. The use of open systems allows us to prove both a completion result extending the well-known completion result for residuated lattices, and also standard completeness for a number of fuzzy logics with modalities.

Lemma 25 *Let* $\mathbf{M}$ *be a dense MTLS4^r-chain. Then the following are equivalent:*

(1) the operator I *is left-continuous, i.e., if* $X \subseteq M$ *and* $\sup(X) \in M$*, then* $I(\sup(X)) = \sup(I(X))$*;*

(2) O *is densely ordered.*

Proof. First, note that if $\mathbf{M}$ is densely ordered, then the left continuity of I is equivalent to the condition that $I(x) = \sup\{I(y) : y < x\}$. Now suppose that there are $x, y \in O$ such that $x < y$ and there is no $z \in O$ such that $x < z < y$. Then for $x < z < y$ we have $I(z) = x$, whereas $I(y) = y$. Hence I is not left-continuous.

Conversely, suppose that O is densely ordered. We claim that for every $x \in O$ we have $x = \sup\{y \in O : y < x\}$. Indeed, suppose that there is $a < x$ such that for all $y \in O$, if $y < x$, then $y \leq a$. Then $I(a) = \sup\{y \in O : y \leq a\} = \sup\{y \in O : y < x\}$ is in O, and there is no $z \in O$ such that $I(a) < z < x$, contradicting the density of O. It follows that for all $x \in M$:

$$I(x) = \sup\{y \in O : y \leq x\} = \sup\{y \in O : y < x\} = \sup\{I(z) : z < x\}$$

and the claim is proved. $\square$

Theorem 26 *Let* $\mathbf{M}$ *be a linearly ordered* I*-l-monoid. Then there exists an embedding* Φ *of* $\mathbf{M}$ *into a complete MTLS4*^{r}*-chain* $\hat{\mathbf{M}}$ *which preserves the suprema and the residuals existing in* M*. Moreover:*

- *if* $\mathbf{M}$ *has no zero divisors, then* $\hat{\mathbf{M}}$ *is an SMTLS4*^{r}*-algebra;*
- *if* $\mathbf{M}$ *satisfies the axiom* $x^{n-1} = x^n$*, then* $\hat{\mathbf{M}}$ *is a C*$_n$*MTLS4*^{r}*-algebra;*
- *if* $\mathbf{M}$ *is the reduct of a IMTLS4*^{r}*-algebra, then* $\hat{\mathbf{M}}$ *is an IMTLS4*^{r}*-algebra.*

Proof. Let $Id(\mathbf{M})$ be the collection of all non-empty subsets J of M such that:

- if $\sup(J)$ exists in M, then $\sup(J) \in J$;
- if $x \leq y$ and $y \in J$, then $x \in J$.

Note that $Id(\mathbf{M})$ is closed under arbitrary intersections, and that hence the operator σ defined for all $X \subseteq M$ by $\sigma(X) = \bigcap\{J \in Id(\mathbf{M}) : X \subseteq J\}$ is a closure operator. Now let $\hat{M}$ be the family of closed subsets of M (that is, the family of all $X \subseteq M$ such that $\sigma(X) = X$) and define *the completion of* $\mathbf{M}$ to be $\hat{\mathbf{M}} = \langle \hat{M}, \star, \rightarrow, \sqcup, \sqcap, \bot', \top' \rangle$ where $\bot' = \{\bot\}$, $\top' = M$, and:

- $X \star Y = \sigma(X \odot Y)$;
- $X \rightarrow Y = \{z : z \odot X \subseteq Y\}$;
- $X \sqcup Y = \sigma(X \cup Y)$ (in fact here $X \sqcup Y = X \cup Y$, since $\mathbf{M}$ is linearly ordered);
- $X \sqcap Y = X \cap Y$.

It follows from a result of [21] that $\hat{\mathbf{M}}$ is a commutative residuated lattice and that the map Φ defined for all $a \in M$ by $\Phi(a) = \{x \in M : x \leq a\}$ is an embedding of $\mathbf{M}$ into $\hat{\mathbf{M}}$ which preserves the suprema existing in $\mathbf{M}$. Therefore, Φ also preserves the residuals existing in $\mathbf{M}$. Finally, for every element $a \in \hat{M}$:

$$a = \sup\{\Phi(x) : x \in M \text{ and } \Phi(x) \leq a\}.$$

So every element of $\hat{\mathbf{M}}$ is the supremum of the image under Φ of a subset of M. Moreover $\hat{\mathbf{M}}$ is a complete residuated lattice, while up to isomorphism $\mathbf{M}$ is both a complete sublattice and a submonoid of $\hat{\mathbf{M}}$. Furthermore, if $\mathbf{M}$ is itself a residuated lattice, then up to isomorphism it is also a residuated sublattice of $\hat{\mathbf{M}}$.

It is easily seen that the construction of $\hat{\mathbf{M}}$ preserves linearity of the order, integrality and boundedness, absence of zero-divisors and n-potency. Hence $\hat{\mathbf{M}}$ is an MTL-algebra, and if in addition $\mathbf{M}$ has no zero divisors, (satisfies $x^{n-1} = x^n$ respectively) then $\hat{\mathbf{M}}$ is a SMTL-algebra (a C$_n$MTL-algebra respectively). We verify that our construction also preserves the double negation law: let $\neg$ and $\sim$ denote the negations in $\mathbf{M}$ and $\hat{\mathbf{M}}$ respectively. Note that for $X \in \hat{M}$:

$$\sim(X) = \{z : z \odot X = \{\bot\}\} = \{z : \forall x \in X\, (x \leq \neg z)\}.$$

Hence $u \in \sim\sim(X)$ iff $v \leq \neg u$ for all $v \in\sim(X)$ iff for all v, if $x \leq \neg v$ for all $x \in X$, then $u \leq \neg v$. Since $\neg$ is involutive, it is onto, therefore we can deduce that $u \in \sim\sim(X)$ iff whenever $z = \neg v$ is an upperbound of X, then $u \leq z$. That is, iff u is a lowerbound of the set of all upper bounds of X. But this is the case iff $u \in X$. Thus if $\mathbf{M}$ is an IMTL-algebra, then so is $\hat{\mathbf{M}}$.

Now suppose that $\mathbf{M}$ is equipped with an interior operator I which makes it an I-l-lattice. We want to define an operator $\hat{I}$ on $\hat{\mathbf{M}}$ which makes $\hat{\mathbf{M}}$ an MTLS4^r-algebra in such a way that the embedding Φ of $\mathbf{M}$ into $\hat{\mathbf{M}}$ also preserves the interior operation, i.e., it satisfies the condition $\Phi(I(x)) = \hat{I}(\Phi(x))$. In the rest of the proof we identify (the interior-free reduct of) $\mathbf{M}$ with its isomorphic image under Φ. Thus we assume that $\mathbf{M}$ is a subalgebra of $\hat{\mathbf{M}}$ and that Φ is the identity embedding.

By Lemma 24, the set $O = \{x \in M : I(x) = x\}$ is closed under $\vee$, $\wedge$, and $\odot$, and for all $a \in M$, $I(a) = \sup\{x \in O : x \leq a\}$. Let $\hat{O}$ be the subset of $\hat{\mathbf{M}}$ consisting of all elements of the form $\sup(X)$ for some $X \subseteq O$. Clearly, $\hat{O}$ is closed under suprema and contains O. Note that $\hat{\mathbf{M}}$ is linearly ordered, therefore $\hat{O}$ is closed under join and meet. It is also closed under $\star$: for any $x = \sup(X)$, $y = \sup(Y)$ in $\hat{O}$, with $X, Y \subseteq O$:

$$x \star y = \sup(X) \star \sup(Y) = \sup\{x \odot y : x \in X \,\text{and}\, y \in Y\}$$

(this condition holds in any complete residuated lattice). Since $X, Y \subseteq O$ and O is closed under $\odot$, for $a \in X$ and $b \in Y$ we have $a \odot b = I(a \odot b)$, and hence:

$$\sup(X) \star \sup(Y) = \sup\{I(a \odot b) : a \in X \,\text{and}\, b \in Y\}$$

which, being the supremum of a subset of O, is in $\hat{O}$. Finally, for all $a \in \hat{M}$, $\sup\{x \in \hat{O} : x \leq a\}$ exists and is in $\hat{O}$, as $\hat{O}$ is closed under suprema. Thus $\hat{O}$ is an open system, and hence, defining for all $a \in \hat{M}$, $\hat{I}(a) = \sup\{z \in \hat{O} : z \leq a\}$, $\hat{I}$ is an interior operator, giving that $\hat{\mathbf{M}}$ is an MTLS4^r-algebra.

We prove that $\hat{I}$ extends I. Let $a \in M$, and let $O_a = \{x \in O : x \leq a\}$, and $\hat{O}_a = \{x \in \hat{O} : x \leq a\}$. Then $\sup(O_a)$ exists in $\mathbf{M}$ and belongs to $\hat{O}$. Moreover this supremum is the same in $\mathbf{M}$ and in $\hat{\mathbf{M}}$, since the suprema existing in $\mathbf{M}$ are preserved by the embedding of $\mathbf{M}$ into $\hat{\mathbf{M}}$. Also, if $x \in \hat{O}_a$, then x is the supremum of a subset X of O whose elements are $\leq a$, and therefore $x \leq I(a)$. It follows that $\hat{I}(a) = \sup(\hat{O}_a) = \sup(O_a) = I(a)$. $\square$

The superstandard completeness of $\mathsf{MTLS4^r}$, $\mathsf{SMTLS4^r}$, and $\mathsf{C_nMTLS4^r}$ is now an easy consequence of the following theorem.

Theorem 27 *For* $\mathrm{L} \in \{\mathsf{MTLS4^r}, \mathsf{SMTLS4^r}, \mathsf{C_nMTLS4^r}\}$*, every finite or countable linearly ordered* L*-algebra can be embedded into a superstandard* L*-algebra.*

Proof. We start by proving the following lemma:

Lemma 28 *For every finite or countable linearly ordered MTLS4^r-algebra* $\mathbf{S} = \langle S, \odot_S, \to_S, I_S, \leq_S, 0_S, 1_S\rangle$ *(where the lattice operations are uniquely determined by the order* $\leq_S$*), there exist a linearly and densely ordered, bounded commutative integral monoid* $\mathbf{X} = \langle X, *, \preceq, m, M\rangle$ *(where* m *and* M *are the minimum and the maximum of* $\mathbf{X}$ *respectively), a unary operation* I *on* X*, and a map* Φ *from* S *into* X *such that the following conditions hold:*

- *(a)* $*$ *is left-continuous with respect to the order topology on* $\langle X, \preceq\rangle$*;*
- *(b)* Φ *is an embedding of the structure* $\langle S, \odot, \leq_S, 0_S, 1_S\rangle$ *into* $\mathbf{X}$*. Moreover, for all* $s, t \in S$*,* $\Phi(s \to_S t)$ *is the residual of* $\Phi(s)$ *and* $\Phi(t)$ *in* $\langle X, *, \preceq, m, M\rangle$*;*
- *(c)* I *is left-continuous on* $\langle X, \preceq\rangle$*, and makes* $\mathbf{X}$ *an* I*-l-monoid;*
- *(d) for all* $a \in S$*,* $I(\Phi(a)) = \Phi(I_S(a))$*;*
- *(e) if* $\mathbf{S}$ *has no zero divisors or satisfies* $x^{n-1} = x^n$*, then the same is true of* $\mathbf{X}$*.*

Proof. Let $X = \{(s, q) : s \in S\backslash\{0_S\}, q \in \mathbb{Q}\cap]0, 1]\}\cup\{(0_S, 1)\}$. For $(s, q), (t, r) \in X$, we define:

$$(s, q) \preceq (t, r) \text{ iff either } s <_S t\text{, or } s = t \text{ and } q \leq r;$$

$$(s, q) * (t, r) = \begin{cases} \min\{(s, q), (t, r)\} & \text{if } s \odot t = \min_S\{s, t\} \\ (s \odot t, 1) & \text{otherwise.} \end{cases}$$

where $\min$ is meant with respect to $\preceq$, and $\min_S$ is meant with respect to $\leq_S$.

Properties (a) and (b) are then proved exactly as in [14]. We now define:

$$I(s, q) = \begin{cases} (s, q) & \text{if } I_S(s) = s \\ (I_S(s), 1) & \text{otherwise.} \end{cases}$$

Let O_S denote the open system of $\mathbf{S}$, i.e., the set of fixed points of I_S. Then the set O of fixed-points of I is the set:

$$\{(0_S, 1)\} \cup \{(s, q) : s \in O_s \text{ and } q \in \mathbb{Q} \cap (0, 1]\}.$$

Clearly, O is closed under joins and meets since $\langle X, \preceq\rangle$ is linearly ordered. O is also closed under $*$ since if (s, q), $(t, r) \in O$, then $s, t \in O_S$, and therefore $s \odot t \in O$. Since $(s, q) * (t, r) = (s \odot t, p)$ for some $p \in \mathbb{Q} \cap (0, 1]$, and since $s \odot t \in O_S$, $(s, q) * (t, r) \in O$. Clearly, $(0_s, 1) \in O$ and $(1_S, 1) \in O$, as 0_S and 1_S are in O_S.

Now suppose (a, p), $(b, q) \in O$ and $(a, p) \prec (b, q)$. If $a < b$, then $(b, \frac{q}{2}) \in O$ and $(a, p) \prec (b, \frac{q}{2}) \prec (b, q)$. If $a = b$ and $p < q$, then $(b, \frac{p+q}{2}) \in O$, and $(a, p) \prec (b, \frac{p+q}{2}) \prec (b, q)$. Thus O is dense. It follows that I is a left continuous interior operator which makes $\langle X, *, \preceq, (0_S, 1), (1_S, 1)\rangle$ (with lattice operations determined by the order $\preceq$) a linearly ordered, bounded and integral I-l-monoid. This proves (c). Property (d) is immediate from the definitions of Φ and I. Finally, condition (e) is easy to verify. $\square$

To conclude the proof of Theorem 27, let $\mathbf{X} = \langle X, *, \preceq, (0_S, 1), (1_S, 1), I\rangle$. Clearly $\mathbf{X}$ is countable, densely ordered, and has a maximum and a minimum. So we can assume up to isomorphism that its lattice reduct is $[0, 1] \cap \mathbb{Q}$. Now by Theorem 26, the completion $\hat{\mathbf{X}}$ of $\mathbf{X}$ is a complete MTLS4^r-algebra. Thus the lattice reduct of $\hat{\mathbf{X}}$ is $[0, 1]$. Moreover, the operator I, as well as the existing suprema and residuals are preserved, therefore Φ is an embedding of $\mathbf{S}$ into $\hat{\mathbf{X}}$. Clearly $\hat{\mathbf{X}}$ is a standard MTLS4^r-algebra, therefore it remains to prove that $\hat{I}$ is left continuous. To prove this, by Lemma 25 it suffices to prove that the open system of $\hat{\mathbf{X}}$, $\hat{O} = \{x \in [0, 1] : \exists Y \subseteq O : x = \sup(Y)\}$, is densely ordered. Now let $x < y \in \hat{O}$. If for all $o \in O$, $o \leq x$ iff $o \leq y$, then $x = y$. Hence there exists $o_0 \in O$ such that $x < o_0 \leq y$. For $q \in \mathbb{Q}$ with $x < q < o_0$, $I(q) \in O$ and $x \leq I(q) < o_0$. Since O is dense, there is $o \in O$ such that $q < o < o_0$. It then follows that $x < o < y$, and $\hat{O}$ is dense. $\square$

The standard completeness of $\mathsf{IMTLS4^r}$ and $\mathsf{C_nIMTLS4^r}$ (for $n \geq 3$) is an obvious consequence of the following theorem:

Theorem 29 *Every linearly ordered countable IMTLS4^r-algebra (C$_n$IMTLS4^r-algebra for $n \geq 3$, respectively) can be embedded into a standard IMTLS4^r-algebra (C$_n$IMTLS4^r-algebra respectively).*

Proof. Let $\mathbf{S} = \langle S, \odot, \rightarrow, \leq_S, 0_S, 1_S, I_S\rangle$ be a (finite or) countable linearly ordered IMTLS4^r-algebra, and let $\mathbf{S}^-$ be its IMTL-reduct. By [8], there is a countable linearly and densely ordered IMTL-algebra $\mathbf{Y}$ such that $\mathbf{S}^-$ embeds into $\mathbf{Y}$ by an embedding Φ. We recall the definitions of $\mathbf{Y}$ and of Φ given in [8].

Define for $x, y \in \mathbf{S}$, $Succ(x, y)$ iff $y <_\mathbf{S} x$ and there is no $u \in \mathbf{S}$ with $y <_\mathbf{S} u <_\mathbf{S} x$. Then we define:

- $Y = \{(s, 1) : s \in S\} \cup \{(s, r) : \exists s'(Succ(s, s')) \text{ and } r \in Q \cap (0, 1)\}$.
- $(s, q) \preceq (t, r)$ iff either $s <_S t$, or $s = t$ and $q \leq r$.
- In order to define $\otimes$ we first define the auxiliary operation $\circ$ as follows (cf [14]):

$$(s, q) \circ (t, r) = \begin{cases} \min_Y((s, q), (t, r)), \text{ if } s \star t = \min_S(s, t) \\ (s \star t, 1), \qquad\qquad \text{otherwise.} \end{cases}$$

where $\min_Y$ is meant with respect to $\preceq$, and $\min_S$ is meant with respect to $\leq_S$.

- Let $\neg$ denote the negation in $\mathbf{S}$. Then $\otimes$ is defined as follows:

$$(s,q)\otimes(t,r)=\begin{cases}(0_S,1), & \text{if } Succ(s,\neg t) \text{ and } q+r\leq 1\\ (s,q)\circ(t,r), & \text{otherwise.}\end{cases}$$

- $0_Y=(0,1)$, $1_Y=(1,1)$.

Note that for all $(x,q)\in Y$, there is a greatest $(y,r)=\sim(x,q)$ such that $(x,q)\otimes(y,r)=0_Y$. If $q=1$, then $\sim(x,q)=(\neg x,1)$, otherwise $\sim(x,q)=(z,1-q)$, where z is the unique element such that $Succ(z,\neg x)$. It is readily seen that $\sim$ is an involutive negation, therefore the residual of $\otimes$ is the operator $\rightarrow$ defined by $(x,q)\rightarrow(y,r)=\sim((x,q)\odot\sim(y,r))$. The embedding Φ is defined, for all $x\in\mathbf{S}$, by $\Phi(x)=(x,1)$.

Now let $O_S=\{x\in S: I_S(x)=x\}$ be the open system of $\mathbf{S}$. As in the proof of Theorem 27, we define $O=\{(x,q)\in Y: x\in O_S\}$. Clearly O is closed under the lattice operations and $\odot$. Moreover, for all $(x,q)\in Y$, $\sup\{(y,r)\in O:(y,r)\preceq(x,q)\}$ is equal to (x,q) if $x\in O_S$ and is equal to $(I_S(x),1)$ otherwise. In both cases, this supremum is in O. Therefore O is an open system, so we can associate with it an interior operator I, which makes $\mathbf{Y}$ an IMTLS4^r-algebra $\mathbf{Y}^+$. Moreover $I(\Phi(x))=(I_S(x),1)=\Phi(I_S(x))$, therefore Φ is an embedding of IMTLS4^r-algebras. Finally, by Theorem 26, $\mathbf{Y}^+$ can be in turn embedded into a standard IMTLS4^r-algebra. Since the whole construction preserves the equation $x^{n-1}=x^n$ for $n\geq 3$, we also have that every finite or countable C$_n$IMTLS4^r-algebra for $n\geq 3$ embeds into a standard C$_n$IMTLS4^r-algebra. $\square$

We remark that, unlike the case of Theorem 27, in the proof of Theorem 29, O need not be densely ordered: suppose e.g. that there are $x,y\in O_S$ such that $x<y$ and there is no $z\in O_S$ with $x<z<y$. Suppose further that there is no $z\in S$ such that $Succ(y,z)$. Then it is readily seen that there is no element in O between $(x,1)$ and $(y,1)$. As a consequence, the proof of Theorem 29 only shows standard completeness and not superstandard completeness.

We conclude this section with a proof of the standard completeness of logics with contraction for modal formulas.

Theorem 30 *For* $\mathrm{L}\in$ Logics, $\mathrm{L!^r}$ *and* $\mathrm{L^r_\Delta}$ *are standard complete.*

Proof. By Corollary 11, $\mathrm{L!^r}$ and $\mathrm{L^r_\Delta}$ are complete with respect to the class of linearly ordered L!^{r}-algebras and L$^r_\Delta$-algebras respectively. Moreover, L!^{r}-algebras are precisely those LS4^r-algebras whose open system O only consists of idempotent elements, while L$^r_\Delta$-algebras are LS4^r-algebras whose open system O consists of just two points $\top$ and $\bot$. Thus given a linearly ordered finite or countable L!^{r}-algebra $\mathbf{S}$, we can repeat the proof of Theorem 27 or Theorem 29, observing that the constructions used preserve both the idempotency of all elements of the open system, and the top and bottom elements. $\square$

7 Finite embeddability property and decidability

In this section we show that a number of the logics with truth stresser modalities introduced above have the finite embeddability property and are hence decidable. First, we recall some useful notions. Let $\mathbf{M}$ be an algebra and let $P \subseteq M$. For every n-ary function symbol f in the type of $\mathbf{M}$, let f^M denote its realization in $\mathbf{M}$. We define a partial map $f^P : P^n \mapsto P$ as follows:

$$f^P(p_1, \ldots, p_n) = \begin{cases} f^M(p_1, \ldots, p_n) & \text{if } f^M(p_1, \ldots, p_n) \in P \\ \text{undefined} & \text{otherwise.} \end{cases}$$

P equipped with all such partial operations f^P is called a *partial subalgebra* of $\mathbf{M}$, and denoted by $\mathbf{P}$.

Let $\mathbf{W}$ be an algebra of the same type as $\mathbf{M}$, and let $\mathbf{P}$ be a partial subalgebra of $\mathbf{M}$. A *partial embedding* from $\mathbf{P}$ into $\mathbf{W}$ is a one-to-one map Φ from P into W such that for every n-ary partial operation f^P of $\mathbf{P}$ and $p_1, \ldots, p_n \in P$, if $f^P(p_1, \ldots, p_n)$ is defined, then $\Phi(f^P(p_1, \ldots, p_n)) = f^W(\Phi(p_1), \ldots, \Phi(p_n))$.

A class $\mathcal{K}$ of algebras of the same type has the *finite embeddability property* (FEP for short) iff every finite partial subalgebra $\mathbf{P}$ of any $\mathbf{M} \in \mathcal{K}$ can be partially embedded into a finite algebra $\mathbf{W} \in \mathcal{K}$.

We first investigate the FEP for MTL-algebras and IMTL-algebras. Hiroakira Ono (private communication) has shown that the proof of the FEP given by Blok and Van Alten for commutative integral residuated lattices extends to these cases with a slightly simplified proof. We prove here that for MTL-algebras, not only the proof, but also the whole construction can be simplified. More precisely, we use algebras of elements of the initial algebra, rather than taking subsets of this algebra.

Lemma 31 *Suppose that $\mathcal{K}$ is a variety, and let $\mathcal{K}_{si}$ be the class of all subdirectly irreducible members of $\mathcal{K}$. If $\mathcal{K}_{si}$ has the FEP, then $\mathcal{K}$ has the FEP.*

Proof. Let $\mathbf{P}$ be a finite partial subalgebra of an algebra $\mathbf{M} \in \mathcal{K}$. Decompose $\mathbf{M}$ into a family of subdirectly irreducible members $(\mathbf{M}_i : i \in I)$. For any $p, q \in P$, $p \neq q$, choose an index $i = i(p, q) \in I$ such that $p_i \neq q_i$. Let $J = \{i(p, q) : p, q \in P, p \neq q\}$. Clearly J is finite (since P is finite), and $\mathbf{P}$ partially embeds into $\prod_{j \in J} \mathbf{M}_j$ by the embedding $\Psi : p \mapsto (p_j : j \in J)$. Let $\mathbf{P}'$ be the isomorphic image of $\mathbf{P}$ under Ψ, and let $\mathbf{P}_j$ be the j^{th} projection of $\mathbf{P}'$ for $j \in J$. Then $\mathbf{P}_j$ is a finite partial subalgebra of $\mathbf{M}_j$, and since $\mathbf{M}_j \in \mathcal{K}_{si}$, it partially embeds into a finite $\mathbf{W}_j \in \mathcal{K}$. So $\mathbf{P}$ partially embeds into $\prod_{j \in J} \mathbf{W}_j$, a finite algebra in $\mathcal{K}$. $\square$

We now recall some useful set-theoretic results.

Theorem 32 (Infinite Ramsey Theorem) *For any set X, let $[X]^2$ denote the set of unordered pairs of elements of X. Suppose that X is infinite. Then for every partition $P_1, \ldots, P_n$ of $[X]^2$, there is an infinite set $Y \subseteq X$ such that $[Y]^2$ is included in one of the P_is.*

An *inverse well quasi order* (iwqo for short) is a partial order without infinite ascending chains and without infinite antichains (an antichain is a set of mutually incomparable elements). An *inverse well order* (iwo for short) is a well quasi order which is linear (or equivalently, a linear order without infinite ascending chains).

Theorem 33 (Dickson's Lemma, cf. e.g. [7]) *The product of two iwqos is an iwqo.*

Lemma 34 *If Φ is a map from an iwqo $(X, \leq)$ onto a linear order $(X', \leq')$ such that $x \leq y$ implies $\Phi(x) \leq \Phi(y)$, then $(X', \leq')$ is an iwo.*

Proof. We proceed by contraposition. Let $y_1 <' \ldots <' y_n < \ldots$ be an infinite ascending chain in $(X', \leq')$, and let $x_1, \ldots, x_n, \ldots \in X$ be such that for all i, $\Phi(x_i) = y_i$. Clearly for $i < j$ it is not possible that $x_i \geq x_j$. Hence either $x_i < x_j$ or x_i and x_j are incomparable. Let $Z = \{x_1, \ldots, x_n, \ldots\}$. Partition the set $[Z]^2$ of unordered pairs of elements from Z into two classes: the pairs which are comparable and the class of all pairs which are incomparable. By the infinite Ramsey theorem, there is an infinite set $Y \subseteq Z$ such that all unordered pairs from Y fall in the same class. Thus all pairs from Y are either incomparable (and then Y forms an antichain) or they are all comparable (and then Y forms an infinite ascending chain). Both possibilities are impossible, and hence a contradiction is reached. $\Box$

Now consider a subdirectly irreducible (hence linearly ordered) MTL-algebra $\mathbf{M}$ and a finite partial subalgebra $\mathbf{P}$ of $\mathbf{M}$. Without loss of generality we may assume that $\perp, \top \in P$. Let us fix $\mathbf{W}$ as the submonoid of $\mathbf{M}$ generated by $\mathbf{P}$.

Lemma 35 $\mathbf{W}$ *is iwqo and residuated. Moreover, if $a, b, a \to b \in W$, then the residual of a and b in $\mathbf{W}$ is $a \to b$.*

Proof. Let $P = \{p_1, \ldots, p_n\}$. Then every element $w \in W$ has the form $p_1^{h_1} \odot \ldots \odot p_n^{h_n}$. Clearly, the map Φ sending $(h_1, \ldots, h_n)$ to $p_1^{h_1} \odot \ldots \odot p_n^{h_n}$ is an isomorphism from $(\mathbb{N}^n, +)$ into $\mathbf{W}$. Moreover, let us give $\mathbb{N}$ the inverse of the natural order. Then $\mathbb{N}$ is an iwqo, and hence $\mathbb{N}^n$ ordered component-wise is also an iwqo by Theorem 33. Finally, Φ is order-preserving. So by Lemma 34, $\mathbf{W}$ is an iwo. It follows that every non-empty subset of W has a maximum. In particular, for all $a, b \in W$ the set $\{w \in W : a \odot w \leq b\}$ has a maximum: the residual $a \Rightarrow b$ of a and b in W. Now clearly $a \Rightarrow b \leq a \to b$, since $W \subseteq P$. If in addition $a, b, a \to b \in W$, then $a \to b$ is the maximum $z \in W$ such that $z \odot a \leq b$, therefore $a \to b = a \Rightarrow b$. $\Box$

Lemma 36 *For every $p \in P$, the set $W \Rightarrow p = \{w \Rightarrow p : w \in W\}$ is finite.*

Proof. Suppose otherwise. Then since $\mathbf{W}$ is linearly ordered, $W \Rightarrow p$ contains either an infinite ascending chain or an infinite descending chain. The first case is excluded because $\mathbf{W}$ is an iwo. On the other hand, if $w_1 \Rightarrow p > w_2 \Rightarrow p > \ldots > w_n \Rightarrow p > \ldots$ is a descending chain, then $w_1 < w_2 < \ldots < w_n < \ldots$, which is impossible since $\mathbf{W}$ is an iwo. $\square$

Corollary 37 *The set $W \Rightarrow P = \{w \Rightarrow p : w \in W, p \in P\}$ is finite.*

Lemma 38 *$W \Rightarrow P$ is closed under $\Rightarrow$.*

Proof. Let $w_1 \Rightarrow p_1$, $w_2 \Rightarrow p_2 \in W \Rightarrow P$. Since $\mathbf{W}$ is residuated with respect to $\Rightarrow$, we have that $(w_1 \Rightarrow p_1) \Rightarrow (w_2 \Rightarrow p_2) \in W$. By residuation we obtain $(w_1 \Rightarrow p_1) \Rightarrow (w_2 \Rightarrow p_2) = (w_2 \odot (w_1 \Rightarrow p_1)) \Rightarrow p_2$. Since W is closed under $\odot$ and $\Rightarrow$, $w_2 \odot (w_1 \Rightarrow p_1) \in W$, and $(w_1 \Rightarrow p_1) \Rightarrow (w_2 \Rightarrow p_2) = (w_2 \odot (w_1 \Rightarrow p_1)) \Rightarrow p_2 \in W \Rightarrow P$. $\square$

To summarize, $W \Rightarrow P$ is a finite implicative subreduct of $\mathbf{W}$ (equipped with an implication $\Rightarrow$). We now define a monoid operation $*$ such that $\Rightarrow$ is the residual of $*$ in $W \Rightarrow P$. For $x, y \in W \Rightarrow P$, let:

$$x * y = \min\{z \in W \Rightarrow P : x \leq y \Rightarrow z\}.$$

Such a minimum exists since $W \Rightarrow P$ is finite and linearly ordered; moreover, $x * y \geq x \odot y$. We denote the algebra obtained in this way by $\mathbf{W} \Rightarrow \mathbf{P}$.

Lemma 39 *$*$ is a commutative and weakly increasing monoid operation, and $\Rightarrow$ is its residual in $W \Rightarrow P$. Moreover if $a, b, a \odot b \in W \Rightarrow P$, then $a * b = a \odot b$. Thus $\mathbf{W} \Rightarrow \mathbf{P}$ is an MTL-algebra and has $\mathbf{P}$ as a partial subalgebra.*

Proof. Since $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$, the definition of $*$ immediately implies that $*$ is commutative. That $*$ is weakly increasing follows by definition and the fact that $\Rightarrow$ is weakly increasing in the second argument and weakly decreasing in the first. We now prove that:

$$(\star) \qquad (x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z),$$

which immediately implies that $\Rightarrow$ is the residual of $*$. Using the residuation property in $\mathbf{W}$ and the definition of $*$:

$$\begin{aligned} u \leq x \Rightarrow (y \Rightarrow z) &\text{ iff } x \leq u \Rightarrow (y \Rightarrow z) \\ &\text{ iff } x \leq y \Rightarrow (u \Rightarrow z) \\ &\text{ iff } x * y \leq u \Rightarrow z \\ &\text{ iff } u \leq (x * y) \Rightarrow z, \end{aligned}$$

which immediately gives ($\star$). Finally, from the definition of $*$ and ($\star$):

$$\begin{aligned}(x*y)*z \leq u &\text{ iff } ((x*y)*z) \Rightarrow u = 1\\ &\text{ iff } (x*y) \Rightarrow (z \Rightarrow u) = 1\\ &\text{ iff } x \Rightarrow ((y*z) \Rightarrow u) = 1\\ &\text{ iff } (x*(y*z)) \Rightarrow u = 1\\ &\text{ iff } x*(y*z) \leq u,\end{aligned}$$

which immediately gives associativity. Finally assume that $a, b, a \odot b \in W \Rightarrow P$. Then $a \odot b \leq z$ iff $a \leq b \Rightarrow z$ iff $a * b \leq z$. Thus $a * b = a \odot b$. $\square$

We have thus shown the following:

Theorem 40 *The variety of MTL-algebras has the FEP.*

The theorem may be generalized as follows.

Theorem 41 *Let $\mathcal{V}$ be a variety of MTL-algebras possibly with operators. Suppose that any finite partial subalgebra of any subdirectly irreducible algebra $\mathbf{M} \in \mathcal{V}$ can be extended to another finite partial subalgebra $\mathbf{P}$ in such a way that the algebra $\mathbf{W} \Rightarrow \mathbf{P}$ constructed as above is closed under the operations of $\mathcal{V}$ and is in $\mathcal{V}$. Then $\mathcal{V}$ has the FEP.*

Corollary 42 *The varieties of IMTL-algebras and SMTL-algebras have the FEP.*

Proof. For SMTL-algebras, the proof of Theorem 40 works without alterations. Indeed if $\bot, \top \in P$, then for any $m \in W$, $m \Rightarrow \bot$ is either $\bot$ or $\top$, therefore the same is true in $\mathbf{W} \Rightarrow \mathbf{P}$ (see Lemmas 35 and 38). So $\mathbf{W} \Rightarrow \mathbf{P}$ is an SMTL-algebra.

For IMTL-algebras, we can assume without loss of generality that P is closed under $\neg$. (Since $\neg$ is involutive, closing under $\neg$ preserves finiteness). We construct $\mathbf{W} \Rightarrow \mathbf{P}$ as above. To conclude the proof, it is sufficient to show that $\neg$ is involutive in $\mathbf{W} \Rightarrow \mathbf{P}$. We first prove that in $\mathbf{W}$, $z \leq w \Rightarrow p$ iff $z * w * \neg p = \bot$, where $\neg p$ is the negation of p in $\mathbf{W}$ (by Lemma 35, the negations of p in $\mathbf{W}$ and $\mathbf{M}$ coincide). The left-to-right implication is trivial. For the opposite direction, if $z * w * \neg p = \bot$, then $z * w \leq \neg\neg p = p$, and finally $z \leq w \Rightarrow p$. Hence $w \Rightarrow p = \neg(w * \neg p)$. So for every $x = w \Rightarrow p \in W \Rightarrow P$, we have that x is the negation of $y = w * \neg p \in W$ and also y is the negation of x, both in $\mathbf{M}$ and in $\mathbf{W} \Rightarrow \mathbf{P}$. By Lemma 38, $W \Rightarrow P$ is closed under $\Rightarrow$ and hence under the negation of $\mathbf{M}$. So $\neg\neg x = x$ is also in $W \Rightarrow P$. $\square$

Theorem 43 *The varieties of MTL!^{r}-algebras and IMTL!^{r}-algebras have the FEP.*

Proof. Note that in any linearly ordered MTL!^{r}-algebra:

(**) $$I(a) \odot I(b) = I(a \odot b) = \min\{I(a), I(b)\}.$$

Indeed, assuming without loss of generality $I(a) \leq I(b)$, we have $I(a) \geq I(a \odot b) \geq I(I(a) \odot I(b)) = I(a) \odot I(b) \geq I(a) \odot I(a) = I(a)$. Now let us prove the FEP for MTL!^{r}-algebras. Let $\mathbf{M}$ be a subdirectly irreducible MTL!^{r}-algebra and let $\mathbf{P}$ be a finite partial subalgebra of $\mathbf{M}$. Without loss of generality, we can assume that $\perp, \top \in P$, and that P is closed under I (closing under I preserves finiteness because I is an idempotent operator). Now construct $\mathbf{W}$ and $\mathbf{W} \Rightarrow \mathbf{P}$ as above. Note that $\mathbf{W}$ is closed under I, because by (**):

$$I(p_1^{h_1} \odot \ldots \odot p_n^{h_n}) = \min\{I(p_j) : j = 1, \ldots, n \text{ and } h_j > 0\}.$$

This implies that the open system O_W of $\mathbf{W}$ is given by $\{I(p) : p \in P\}$ (so it is finite). Moreover, the interior operator on $\mathbf{W}$ induced by O_W coincides with the restriction of I to $\mathbf{W}$ (by abuse of language we still denote it by I). We claim that O_W is an open system for $\mathbf{W} \Rightarrow \mathbf{P}$. First, $O_W \subseteq P \subseteq W \Rightarrow P$. Moreover for every $x \in W \Rightarrow P$ there is a greatest element $z \in O_W$ with $z \leq x$, as O_W is finite. Finally, O_W is closed under $*$. Indeed, recalling that for $x, y \in W \Rightarrow P$, $x * y \geq x \odot y$ and using (**), we obtain, for $p, q \in P$:

$$\min\{I(p), I(q)\} \geq I(p) * I(q) \geq I(p) \odot I(q) = \min\{I(p), I(q)\}.$$

Clearly, the operator on $\mathbf{W} \Rightarrow \mathbf{P}$ induced by O_W is the restriction of I to $\mathbf{W} \Rightarrow \mathbf{P}$. It follows that $\mathbf{P}$ partially embeds into $\mathbf{W} \Rightarrow \mathbf{P}$ equipped with the operator I associated to O_W.

For IMTL!^{r}, we repeat the same proof with one exception: we start from a $\mathbf{P}$ which is closed under I and under $\neg$. If we prove that closing under such operations does not destroy finiteness, then from the proofs of the first part of the present theorem and Corollary 42, the algebra $\mathbf{W} \Rightarrow \mathbf{P}$ equipped with the interior operator induced by $O_W = \{I(p) : p \in P\}$ is an IMTL!^{r}-algebra into which $\mathbf{P}$ embeds.

We thus conclude the proof by showing that for every subdirectly irreducible IMTL!^{r}-algebra $\mathbf{M}$ and finite set $P \subseteq M$, the closure of P under $\neg$ and I is finite. Let P_1 be the closure of P under $\neg$, and P_2 the closure of P under $\neg$ and I. Let $K(x) = \neg I \neg(x)$. Then $K(x) \geq x$, and K is an idempotent and monotone operator. Moreover, $\neg I(x) = K(\neg(x))$ and $\neg K(x) = I(\neg x)$. So every element $z \in P_2$ can be represented as $z = O_1 \ldots O_n(u)$ where $u \in P_1$ and $(O_1, \ldots, O_n)$ is a sequence of operators which are either I or K, without consecutive occurrences of either I or K. We claim that for $u \in P_1$, $IKIK(u) = IK(u)$ and $KIKI(u) = KI(u)$. Since P_1 is finite, this will imply that the set of all elements of the form $O_1 \ldots O_n(u)$ as above with $u \in P_1$ is finite. We only prove the first identity, as the second one is obtained from the first by taking negations. Clearly $KIK(x) \geq IK(x)$, therefore $IKIK(x) \geq IIK(x) = IK(x)$. On the other hand, $IK(x) \leq K(x)$, therefore $KIK(x) \leq KK(x) = K(x)$. Hence $IKIK(x) \leq IK(x)$. $\square$

Corollary 44 *The universal theories of the varieties of MTL-algebras, IMTL-algebras, SMTL-algebras, MTL!^{r}-algebras, and IMTL!^{r}-algebras are decidable.*

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