MUltseq: Sequents, Equations and Beyond

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Contents

• MVLs: proof theory vs. algebra

• MUltseq
  – sequents
  – formulas
  – equations
  – quasi-equations

• Current state & future work
Proof theory of MVLs

- Generic properties of finitely valued logics

  \[ \text{FVL (operators + distribution quant.)} \Downarrow \]
  \[ \text{optimized CNF in signed classical logic} \Downarrow \Downarrow \]
  \[ \text{sequent calculus} \Downarrow \]
  \[ \text{labeled sequent calculus} \Downarrow \]
  \[ \text{admissible cuts} \Downarrow \]
  \[ \text{FVL} \]

- MUltlog

  \[ \text{FVL} \Rightarrow \text{“scientific” paper in LATEX with optimal calculi} \]

- Projective logics
Algebraic Logic and MVL

- Algebraizable Logics (W. Blok, D. Pigozzi)

Logics and algebras
Formulas and equations

- Finite algebras $\rightarrow$ Many-valued Sequent calculus

1. Matrix semantics of the calculus: *Strong Completeness Theorem*.

2. Translations

   Sequents $\leftrightarrow$ Formulas
   Sequents $\leftrightarrow$ Equations
   Sequents $\leftrightarrow$ Quasi-equations

3. Decision procedures for
   - Finitely valued logics
   - Equations and quasi-equations in a finite algebra.

• Abstract properties of sequent calculus (Algebraic logic & Proof theory):
  
  – Algebraizability ($\approx \Rightarrow$ cut).
  
  – Protoalgebraizability ($\approx \Leftarrow \Rightarrow$ cut)
MUltseq

Developed within the Acción Integrada “Generic Decision Procedures for MVLs”.

- Interactive generic sequent prover
  
  **Input:** m.v. sequent calculus formula, sequent
  
  **Output:** proof derivation from hypotheses

- Companion for MUltlog

- Basis for generic decision procedures

- Tool for getting better intuition on specific logics

- Test bed for optimization algorithms implemented in MUltlog
Many-valued sequents

\( \mathcal{L} \ldots \) propositional language

\( \mathbf{L} \ldots \) finite \( \mathcal{L} \)-algebra

\( L = \{ v_0, \ldots, v_{m-1} \} \ldots \) domain of \( \mathbf{L} \)

(finite set of truth values)

signed formula: \( F^v \)

\((F \ldots \) formula over \( \mathcal{L} \), \( v \in \mathbf{L} \))

sequent: set of signed formulas

A sequent is true in an interpretation iff it contains \( F^v \) such that \( F \) evaluates to \( v \).

A sequent is valid iff it is true in every interpretation.

For every \( \mathbf{L} \) there exists a complete and correct sequent calculus with the cut elimination property.

I.e.: A sequent is valid iff it is provable in the calculus.
MUltseq as generic sequent prover

Problem:
Given a sequent calculus and a sequent, determine whether the sequent is provable.

Input: Rules of calculus (from MUltlog)
       Sequent

Output: Proof (in $\LaTeX$)

Options:
- Strategy: left-right, top-down, rule ordering, interactive
- Sequent notation: signed, multi-dimensional
- Proof style: compact, verbose, ...
- ...

Logic Spec. (Ascii)

MUItlog (Prolog)

Scientific Paper (\LaTeX)

MUItseq (Prolog)

Proof in seq. calc. (\LaTeX)

optimized CNFs (Ascii)

mv-sequent (Ascii)

Sequent Editor (Java)
% Seq. calculus for 3-valued Lukasiewicz logic

truth_values([f,p,t]).

% Implication
rule((A=>B)^f, [[A^t],[B^f]]).
rule((A=>B)^p, [[A^p,B^p],[A^t,B^f]]).
rule((A=>B)^t, [[A^f,A^p,B^t],[A^f,B^p,B^t]]).

% Conjunction
rule((A&B)^f, [[A^f,B^f]]).
rule((A&B)^p, [[A^p,B^p],[A^p,A^t],[B^p,B^t]]).
rule((A&B)^t, [[A^t],[B^t]]).

% Disjunction
rule((A v B)^f, [[A^f],[B^f]]).
rule((A v B)^p, [[A^p,B^p],[A^p,A^f],[B^p,B^f]]).
rule((A v B)^t, [[A^t,B^t]]).

% Negation
rule((¬A)^f, [[A^t]]).
rule((¬A)^p, [[A^p]]).
rule((¬A)^t, [[A^f]]).
Sequent to prove:

\[ \neg ((\neg a \Rightarrow b) \Rightarrow b) \] ^ t \]

Output:

Derivation of \(( (A \supset B) \supset B ) \) ^ t:

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Hypothesis</th>
<th>Axiom for B</th>
<th>Hypothesis</th>
<th>Hypothesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A, p, B, t )</td>
<td>( A, B, f, B )</td>
<td>( A, B, f, B )</td>
<td>( A, B, f, B )</td>
<td>( A, B, f, B )</td>
</tr>
<tr>
<td>( \neg A )</td>
<td>( A )</td>
<td>( \neg B )</td>
<td>( \neg B )</td>
<td>( \neg B )</td>
</tr>
<tr>
<td>( \neg B )</td>
<td>( \neg B )</td>
<td>( \neg B )</td>
<td>( \neg B )</td>
<td>( \neg B )</td>
</tr>
</tbody>
</table>

List of hypotheses:

\( A^t, B^f, B^t \)
\( A^t, B^p, B^t \)

Derivation of \(( (A \supset B) \supset B ) \) ^ t:

<table>
<thead>
<tr>
<th>Hyp</th>
<th>Hyp</th>
<th>Axi</th>
<th>Hyp</th>
<th>Ax B</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td></td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

Table of sequents:

1: \((A \supset B) \supset B)\)
2: \(B^t, (A \supset B)^f, (A \supset B)^p\)
3: \(A^t, B^t, (A \supset B)^p\)
4: \(A^p, A^t, B^p, B^t\)
5: \(A^t, B^f, B^t\)
6: \(B^f, B^t, (A \supset B)^p\)
7: \(A^p, B^f, B^p, B^t\)
8: \(B^p, B^t, (A \supset B)^f\)
9: \(A^t, B^p, B^t\)
10: \(B^f, B^p, B^t\)
Consequence rel. on sequents

**Theorem:** The consequence relation

\[ \text{Set-of-Sequents} \vdash \text{Single-Sequent} \]

is decidable. The problem can be reduced to checking the validity of certain sequents.

**Proof:** \( \vdash \) satisfies the Structural Deduction Detachment Theorem.

**Example:** In any 3-valued logic the relation

\[ \{ \{ A_{0}^{f}, A_{1}^{p}, A_{2}^{t} \} \} \vdash \{ B_{0}^{f}, B_{1}^{p}, B_{2}^{t} \} \]

holds iff the following sequents are provable in the calculus:

\[ \{ A_{0}^{p}, A_{0}^{t}, B_{0}^{f}, B_{1}^{p}, B_{2}^{t} \} \]

\[ \{ A_{1}^{f}, A_{1}^{t}, B_{0}^{f}, B_{1}^{p}, B_{2}^{t} \} \]

\[ \{ A_{2}^{f}, A_{2}^{p}, B_{0}^{f}, B_{1}^{p}, B_{2}^{t} \} \]
Consequence rel. on formulas

$L_t \subseteq L \ldots$ designated truth values

A formula is true in an interpretation if it evaluates to a truth values in $L_t$.

A formula $F$ follows from a set of formulas $\Gamma$, iff $F$ is true for all interpretations satisfying all formulas in $\Gamma$.

**Theorem:** $F$ follows from $\Gamma$ iff the sequent

$$\{ \gamma^v \mid \gamma \in \Gamma, v \in \overline{L_t} \} \cup \{ F^v \mid v \in L_t \}$$

is provable.

**Example:** Let $L = \{ f, p, t \}$ and $L_t = \{ t \}$. $F$ follows from $\Gamma = \{ A, B \}$ iff the sequent

$$\{ A^f, A^p, B^f, B^p, F^t \}$$

is provable.

For $L_t = \{ p, t \}$ we have to prove

$$\{ A^f, B^f, F^p, F^t \}$$
Equations

An equation $A = B$ holds in $\mathbb{L}$ iff for all interpretations, $A$ and $B$ evaluate to the same value.

**Theorem:** The equation $A = B$ holds in $\mathbb{L}$ iff the sequent

$$\{ A^v \} \cup \{ B^{v'} \mid v' \in L, v' \neq v \}$$

is provable for all $v \in L$.

**Example:** $A = B$ holds in a 3-valued logic iff the sequents

$$\{ A^f, B^p, B^t \}$$
$$\{ A^p, B^f, B^t \}$$
$$\{ A^t, B^f, B^p \}$$

are provable.
Quasi-equations

A quasi-equation \( \{ e_1, \ldots, e_n \} \vdash A = B \) holds in \( L \) iff for all interpretations satisfying the equations \( e_1, \ldots, e_n \), \( A \) and \( B \) evaluate to the same value.

**Theorem:** The problem of deciding whether a quasi-equation holds in \( L \) is decidable. It can be reduced to checking the validity of certain sequents.

**Example:** The quasi-equation

\[
\{ F = G \} \vdash A = B
\]

holds iff the 9 sequents

\[
\{ F^p, F^t, G^p, G^t \} \cup \{ A^f, B^p, B^t \} \\
\{ F^f, F^t, G^f, G^t \} \cup \{ A^p, B^f, B^t \} \\
\{ F^f, F^p, G^f, G^p \} \cup \{ A^t, B^f, B^p \}
\]

are provable.
**MUItseq in action: Logics**

Choose an option

Sequents = 1.
Logic = 2.
Equations = 3.
Quit = 4.

Option: 2.

Designated truth values: [t].

Hypotheses: [a,a=>b].
Conclusion: b.

True in this logic

*************************

Choose an option

Sequents = 1.
Logic = 2.
Equations = 3.
Quit = 4.

Option: 2.

Designated truth values: [p,t].

Hypotheses: [a,a=>b].
Conclusion: b.

False in this logic
MUltseq in action: Equations

Choose an option
Sequents = 1.
Formulas = 2.
Equations= 3.
Quit = 4.

Option: 3.

Hypotheses: [].
Conclusion: a=(-(-a)).

The equation is true.

*************************

Choose an option
Sequents = 1.
Formulas = 2.
Equations= 3.
Quit = 4.

Option: 3.

Hypotheses: [a=(b=>b)].
Conclusion: a=(a&b).

The equation is false.

Falsifiable sequent:
[a^f, a^p, (a&b)^p, (a&b)^t, (b=>b)^f, (b=>b)^p]
Current state

The existing prototype is able to deal with

- sequents + consequence relation
- formulas + consequence relation
- equations and quasi-equations

See http://www.logic.at/multseq.

To be done

- graphical user interface
- proof structuring tool
- construction of counter-examples
- more user interaction
- reuse of proofs
- improved \TeX\ formatting