MUltseq: Sequents, Equations and Beyond

Angel Gil
Universitat Pompeu Fabra

Gernot Salzer
Technische Universität Wien

Contents

- MVLs: proof theory vs. algebra
- MUltseq
  - sequents
  - formulas
  - equations
  - quasi-equations
- Current state & future work

Proof theory of MVLs

- Generic properties of finitely valued logics

  FVL (operators + distribution quant.)
  $\Downarrow$
  optimized CNF in signed classical logic
  $\Downarrow$ $\Downarrow$
  sequent calculus ...

  labeled sequent calculus
  $\Downarrow$
  admissible cuts
  $\Downarrow$
  FVL

- MUltlog

  FVL $\Rightarrow$ "scientific" paper in LaTeX
  with optimal calculi

- Projective logics

Algebraic Logic and MVL

- Algebraizable Logics (W. Blok, D. Pigozzi)

  Logics and algebras
  Formulas and equations

- Finite algebras $\rightarrow$ Many-valued Sequent calculus

  1. Matrix semantics of the calculus: *Strong Completeness Theorem*.

  2. Translations

Sequents	$\leftrightarrow$	Formulas
Sequents	$\leftrightarrow$	Equations
Sequents	$\leftrightarrow$	Quasi-equations

  3. Decision procedures for

     – Finitely valued logics

     – Equations and quasi-equations in a finite algebra.

4. Examples: MV-algebras, Stone algebras, Pseudocomplemented distributive lattices.

- Abstract properties of sequent calculus (Algebraic logic & Proof theory):
  - Algebraizability ($\approx\Rightarrow$ cut).
  - Protoalgebraizability ($\approx\Leftrightarrow$ cut)

MUltseq

Developed within the Acción Integrada "Generic Decision Procedures for MVLs".

- Interactive generic sequent prover

  Input: m.v. sequent calculus
  formula, sequent

  Output: proof
  derivation from hypotheses

- Companion for MUltlog

- Basis for generic decision procedures

- Tool for getting better intuition on specific logics

- Test bed for optimization algorithms implemented in MUltlog

Many-valued sequents

$\mathcal{L}$ . . . propositional language
$\mathbf{L}$ . . . finite $\mathcal{L}$-algebra
$L = \{ v_0, \ldots, v_{m-1} \}$ . . . domain of $\mathbf{L}$
(finite set of truth values)

signed formula: F^v
(F . . . formula over $\mathcal{L}$, $v \in \mathbf{L}$)
sequent: set of signed formulas

A sequent is true in an interpretation iff it contains F^v such that F evaluates to v.

A sequent is valid iff it is true in every interpretation.

For every $\mathbf{L}$ there exists a complete and correct sequent calculus with the cut elimination property.

I.e.: A sequent is valid iff it is provable in the calculus.

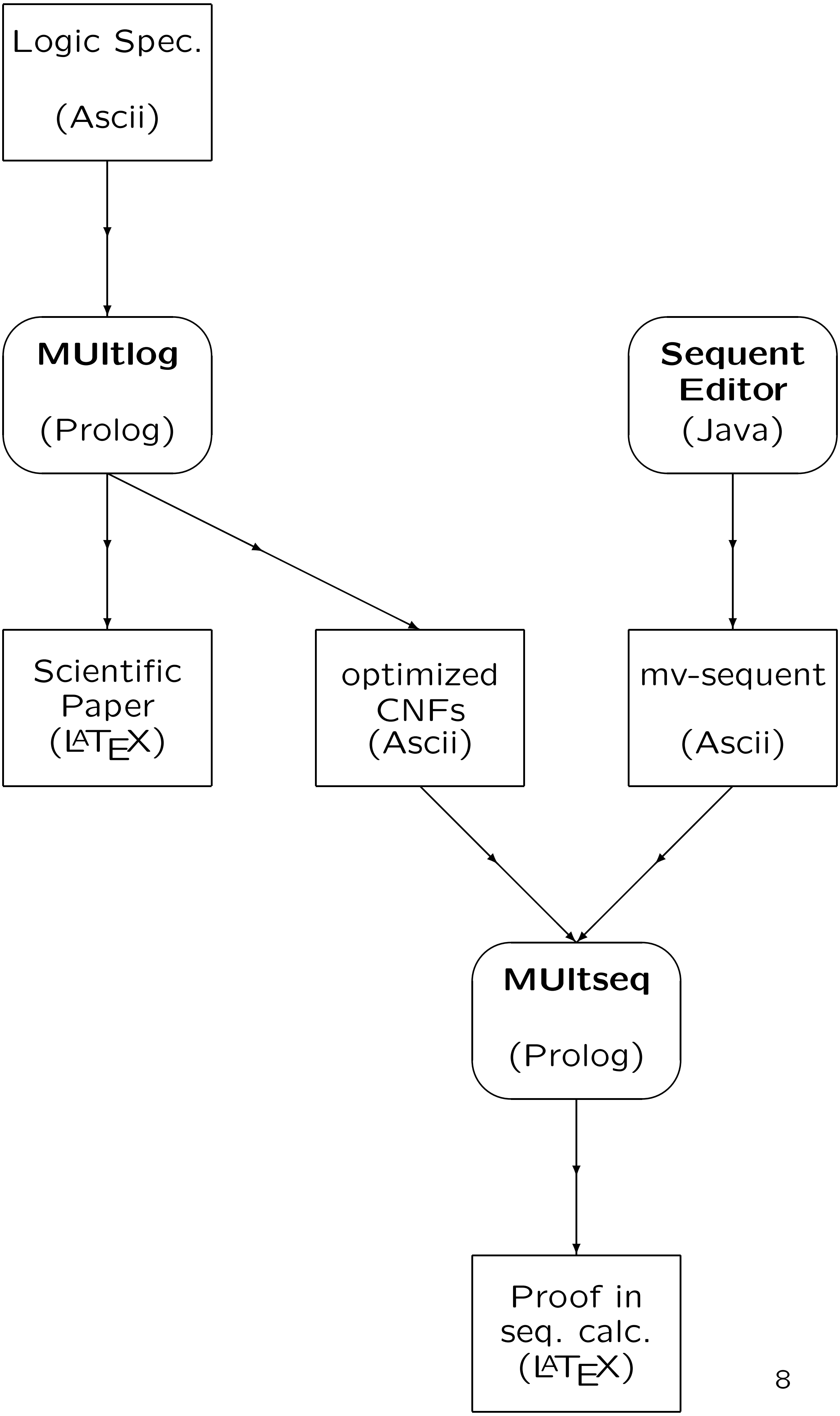
MUltseq as generic sequent prover

Problem:
Given a sequent calculus and a sequent, determine whether the sequent is provable.

Input:	Rules of calculus (from MUltlog) Sequent
Output:	Proof (in $\LaTeX$)

Options:

- Strategy: left-right, top-down,
  rule ordering, interactive
- Sequent notation: signed, multi-dimensional
- Proof style: compact, verbose, . . .
- . . .


Logic Spec.
(Ascii)
MUltlog
(Prolog)
Sequent Editor
(Java)
Scientific Paper
(LaTeX)
optimized CNFs
(Ascii)
mv-sequent
(Ascii)
MUltseq
(Prolog)
Proof in seq. calc.
(LaTeX)

```
% Seq. calculus for 3-valued Lukasiewicz logic

truth_values([f,p,t]).

% Implication
rule((A=>B)^f, [[A^t],[B^f]]).
rule((A=>B)^p, [[A^p,B^p],[A^t,B^f]]).
rule((A=>B)^t, [[A^f,A^p,B^t],[A^f,B^p,B^t]]).

% Conjunction
rule((A&B)^f, [[A^f,B^f]]).
rule((A&B)^p, [[A^p,B^p],[A^p,A^t],[B^p,B^t]]).
rule((A&B)^t, [[A^t],[B^t]]).

% Disjunction
rule((A v B)^f, [[A^f],[B^f]]).
rule((A v B)^p, [[A^p,B^p],[A^p,A^f],[B^p,B^f]]).
rule((A v B)^t, [[A^t,B^t]]).

% Negation
rule((-A)^f, [[A^t]]).
rule((-A)^p, [[A^p]]).
rule((-A)^t, [[A^f]]).
```

Sequent to prove:

```
[((a=>b)=>b)^t]
```

Output:

Derivation of $((A \supset B) \supset B)^t$:

$$
\dfrac{
\dfrac{
\dfrac{\overset{\text{hypothesis}}{A^p, A^t, B^p, B^t} \qquad \overset{\text{hypothesis}}{A^t, B^f, B^t}}{A^t, B^t, (A \supset B)^p}
\qquad
\dfrac{\overset{\text{axiom for } B}{A^p, B^f, B^p, B^t} \qquad \overset{\text{hypothesis}}{A^t, B^f, B^t}}{B^f, B^t, (A \supset B)^p}
}{B^t, (A \supset B)^f, (A \supset B)^p}
\qquad
\dfrac{\overset{\text{hypothesis}}{A^t, B^p, B^t} \qquad \overset{\text{axiom}}{B^f, B}}{B^p, B^t, (A \supset B)}
}{((A \supset B) \supset B)^t}
$$

List of hypotheses:

$$
\begin{array}{c}
A^t, B^f, B^t \\
A^t, B^p, B^t
\end{array}
$$

Derivation of $((A \supset B) \supset B)^t$:

$$
\dfrac{
\dfrac{
\dfrac{\overset{\text{hyp}}{4} \qquad \overset{\text{hyp}}{5}}{3}
\qquad
\dfrac{\overset{\text{ax } B}{7} \qquad \overset{\text{hyp}}{5}}{6}
}{2}
\qquad
\dfrac{\overset{\text{hyp}}{9} \qquad \overset{\text{ax } B}{10}}{8}
}{1}
$$

Table of sequents:

$$
\begin{array}{rl}
1: & ((A \supset B) \supset B)^t \\
2: & B^t, (A \supset B)^f, (A \supset B)^p \\
3: & A^t, B^t, (A \supset B)^p \\
4: & A^p, A^t, B^p, B^t \\
5: & A^t, B^f, B^t \\
6: & B^f, B^t, (A \supset B)^p \\
7: & A^p, B^f, B^p, B^t \\
8: & B^p, B^t, (A \supset B)^f \\
9: & A^t, B^p, B^t \\
10: & B^f, B^p, B^t
\end{array}
$$

Consequence rel. on sequents

Theorem: The consequence relation

$$\text{Set-of-Sequents} \vdash \text{Single-Sequent}$$

is decidable. The problem can be reduced to checking the validity of certain sequents.

Proof: $\vdash$ satisfies the Structural Deduction Detachment Theorem.

Example: In any 3-valued logic the relation

$$\{\, \{\, A_0^f, A_1^p, A_2^t \,\} \,\} \vdash \{\, B_0^f, B_1^p, B_2^t \,\}$$

holds iff the following sequents are provable in the calculus:

$$\{\, A_0^p, A_0^t,\ B_0^f, B_1^p, B_2^t \,\}$$
$$\{\, A_1^f, A_1^t,\ B_0^f, B_1^p, B_2^t \,\}$$
$$\{\, A_2^f, A_2^p,\ B_0^f, B_1^p, B_2^t \,\}$$

Consequence rel. on formulas

$L_t \subseteq L$ . . . designated truth values

A formula is true in an interpretation if it evaluates to a truth values in L_t.

A formula F follows from a set of formulas Γ, iff F is true for all interpretations satisfying all formulas in Γ.

Theorem: F follows from Γ iff the sequent

$$\{\, \gamma^v \mid \gamma \in \Gamma, v \in \overline{L_t} \,\} \cup \{\, F^v \mid v \in L_t \,\}$$

is provable.

Example: Let $L = \{\, f, p, t \,\}$ and $L_t = \{\, t \,\}$.
F follows from $\Gamma = \{\, A, B \,\}$ iff the sequent

$$\{\, A^f, A^p, B^f, B^p, F^t \,\}$$

is provable.
For $L_t = \{\, p, t \,\}$ we have to prove

$$\{\, A^f, B^f, F^p, F^t \,\}$$

Equations

An equation $A = B$ holds in $\mathbf{L}$ iff for all interpretations, A and B evaluate to the same value.

Theorem: The equation $A = B$ holds in $\mathbf{L}$ iff the sequent

$$\{ A^v \} \cup \{ B^{v'} \mid v' \in L, v' \neq v \}$$

is provable for all $v \in L$.

Example: $A = B$ holds in a 3-valued logic iff the sequents

$$\begin{array}{c} \{ A^f, B^p, B^t \} \\ \{ A^p, B^f, B^t \} \\ \{ A^t, B^f, B^p \} \end{array}$$

are provable.

Quasi-equations

A quasi-equation $\{ e_1, \ldots, e_n \} \vdash A = B$ holds in $\mathbf{L}$ iff for all interpretations satisfying the equations $e_1, \ldots, e_n$, A and B evaluate to the same value.

Theorem: The problem of deciding whether a quasi-equation holds in $\mathbf{L}$ is decidable.
It can be reduced to checking the validity of certain sequents.

Example: The quasi-equation

$$\{ F = G \} \vdash A = B$$

holds iff the 9 sequents

$$\left.\begin{array}{c} \{ F^p, F^t, G^p, G^t \} \\ \{ F^f, F^t, G^f, G^t \} \\ \{ F^f, F^p, G^f, G^p \} \end{array}\right\} \cup \left\{\begin{array}{c} \{ A^f, B^p, B^t \} \\ \{ A^p, B^f, B^t \} \\ \{ A^t, B^f, B^p \} \end{array}\right.$$

are provable.

MUltseq in action: Logics

```
Choose an option
  Sequents = 1.
  Logic    = 2.
  Equations= 3.
  Quit     = 4.

Option: 2.

Designated truth values: [t].

Hypotheses: [a,a=>b].
Conclusion: b.

True in this logic

*************************

Choose an option
  Sequents = 1.
  Logic    = 2.
  Equations= 3.
  Quit     = 4.

Option: 2.

Designated truth values: [p,t].

Hypotheses: [a,a=>b].
Conclusion: b.

False in this logic
```

MUltseq in action: Equations

```
Choose an option
  Sequents = 1.
  Formulas = 2.
  Equations= 3.
  Quit     = 4.

Option: 3.

Hypotheses: [].
Conclusion: a=(-(-a)).

The equation is true.

*************************

Choose an option
  Sequents = 1.
  Formulas = 2.
  Equations= 3.
  Quit     = 4.

Option: 3.

Hypotheses: [a=(b=>b)].
Conclusion: a=(a&b).

The equation is false.

Falsifiable sequent:
[a^f, a^p, (a&b)^p, (a&b)^t, (b=>b)^f, (b=>b)^p]
```

Current state

The existing prototype is able to deal with

- sequents + consequence relation
- formulas + consequence relation
- equations and quasi-equations

See `http://www.logic.at/multseq`.

To be done

- graphical user interface
- proof structuring tool
- construction of counter-examples
- more user interaction
- reuse of proofs
- improved TEX formatting