MUltseq: a Generic Prover for Sequents and Equations*

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Abstract. This paper presents the program `MUltseq`, which can be used to decide the validity of finitely-valued formulas, the consequence relation in finitely-valued logics, and the validity of equations and quasi-equations in finite algebras.

1 Introduction

In its core, `MUltseq` is a generic sequent prover for propositional finitely-valued logics. This means that it takes as input the rules of a many-valued sequent calculus as well as a many-sided sequent and searches – automatically or interactively – for a proof of the latter. For the sake of readability, the output of `MUltseq` is typeset as a LaTeX document.

Though the sequent rules can be entered by hand, `MUltseq` is primarily intended as a companion for `MUltlog`, a program that computes – among other calculi – optimized rules of a sequent calculus from the truth tables and distribution functions of a finitely-valued logic [1, 5, 8].

Provided the input sequent calculus is both correct and complete for the logic under consideration – which is always the case when the rules were computed by `MUltlog` – `MUltseq` serves as a decision procedure for the validity of *sequents* and *formulas*. More interestingly, `MUltseq` can also be used to decide the *consequence relations* associated with the logic and the sequent calculus. The problem of deciding whether a particular formula ϕ is true in all models satisfying a given set of formulas Δ, i.e., whether ϕ logically follows from Δ, can be reduced to the problem of proving that certain sequent that depends only on ϕ and Δ is true. Similarly, as a consequence of the *Deduction Detachment Theorem for many-valued sequents* [4, 7], the problem of finding a derivation of a sequent σ from hypotheses Σ can be reduced to proving a particular set of sequents.

From the algebraic point of view, it is an interesting problem to determine whether an *equation* or a *quasi-equation* is valid in a finite algebra. If we consider the algebra as a set of truth values and a collection of finitely-valued connectives, and use an appropriate translation of equations and quasi-equations to sequents, the problem again reduces to the provability of many-valued sequents [2, 3].

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2 Basic properties and definitions

Let $\mathcal{L}$ be a propositional language and let $\mathbf{L}$ be a finite $\mathcal{L}$-algebra with domain $L = \{v_0, \ldots, v_{m-1}\}$. The elements of L are called *truth values*. A *signed formula* is an expression F^v where F is a formula over $\mathcal{L}$ and v is a truth value. By a *(many-valued) sequent* we mean a set of signed formulas.

Definition 1. *A sequent is* $\mathbf{L}$*-true in an interpretation iff it contains a signed formula* F^v *such that* F *evaluates to* v*. A sequent is* $\mathbf{L}$*-valid iff it is true in every interpretation over* $\mathbf{L}$.

Let $D \subset L$. A *finitely-valued logic* is a pair $\langle \mathbf{L}, D \rangle$. The set D is called the set of designated truth values.

Definition 2. *A formula* F *is true in a finitely-valued logic* $\langle \mathbf{L}, D \rangle$ *iff for every interpretation, it evaluates to a truth-value in* D.

3 Examples

To illustrate the use and capabilities of the system we reproduce here some questions that can be solved by using it. More precisely we reproduce the optimized rules of the three-valued sequent calculus computed by the system `MUltlog` for the three-valued Łukasiewicz logic and some results concerning sequents, formulas, equations and quasi-equations obtained from these rules by the system `MUltseq`.

3.1 Specification of the 3-valued Łukasiewicz logic

Given the truth tables of the connectives for the 3-valued Łukasiewicz logic, the system `MUltlog` generates the following rules, which are to be used by `MUltseq`.

```
% A sequent calculus for 3-valued Lukasiewicz logic
name_of_logic('Lukasiewicz 3-valued logic').
truth_values([f,p,t]).
operators([=>,&,v,-]).
% rules
% Implication
op(800, xfx, =>).
rule((A=>B)^f, [[A^t],[B^f]], if).
rule((A=>B)^p, [[A^p,B^p],[A^t,B^f]], ip).
rule((A=>B)^t, [[A^f,A^p,B^t],[A^f,B^p,B^t]], it).
% Conjunction
op(600, yfx, &).
rule((A&B)^f, [[A^f,B^f]], af).
rule((A&B)^p, [[A^p,B^p],[A^p,A^t],[B^p,B^t]], ap).
rule((A&B)^t, [[A^t],[B^t]], at).
```

```
% Disjunction
op(700, yfx, v).
rule((A v B)^f, [[A^f],[B^f]], of).
rule((A v B)^p, [[A^p,B^p],[A^p,A^f],[B^p,B^f]], op).
rule((A v B)^t, [[A^t,B^t]], ot).
% Negation
op(500, fx, -).
rule((-A)^f, [[A^t]], nf).
rule((-A)^p, [[A^p]], np).
rule((-A)^t, [[A^f]], nt).
```

3.2 Provability of sequents

For each sequent the system computes a derivation. If hypotheses are needed the sequent is not provable and hypotheses and counter-examples are given. Moreover, a derivation of the sequent (from axioms or hypotheses) is displayed.

Problem 1. Is the sequent $[(A \wedge B)^f, A^p, (A \vee B)^t]$ provable?

Answer. Yes, it is. Proof of $[(A \wedge B)^f, A^p, (A \vee B)^t]$:

$$\dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A^f, A^p, A^t, B^f, B^t]}}{[A^f, A^p, B^f, (A \vee B)^t]}}{[(A \wedge B)^f, A^p, (A \vee B)^t]}$$

Problem 2. Is the sequent $[(A \supset B)^p, (A \vee C)^t]$ provable?

Answer. No, it is not. Derivation of $[(A \supset B)^p, (A \vee C)^t]$:

$$\dfrac{\dfrac{\dfrac{\text{hypothesis}}{[A^p, A^t, B^p, C^t]}}{[A^p, B^p, (A \vee C)^t]} \quad \dfrac{\dfrac{\text{hypothesis}}{[A^t, B^f, C^t]}}{[A^t, B^f, (A \vee C)^t]}}{[(A \supset B)^p, (A \vee C)^t]}$$

List of hypotheses: $[A^p, A^t, B^p, C^t], [A^t, B^f, C^t]$
List of counter-examples: $[A^f, B^f, C^f]$, $[A^f, B^f, C^p]$, $[A^f, B^p, C^f]$, $[A^f, B^p, C^p]$, $[A^f, B^t, C^f]$, $[A^f, B^t, C^p]$, $[A^p, B^p, C^f]$, $[A^p, B^p, C^p]$, $[A^p, B^t, C^f]$, $[A^p, B^t, C^p]$.

3.3 Validity of formulas

Given a set of designated truth values and a formula, the system determines which sequents have to be proved in order to decide if the formula is valid. If the formula is not valid counter-examples are given.

Problem 3. Let $\{p\}$ be the set of designated truth values. Is the formula $(A \supset (B \supset A))$ valid?

Answer. The problem is equivalent to proving the following sequent:

$$[(A \supset (B \supset A))^p] \ .$$

No, the formula is not valid.
List of counter-examples: $[A^f, B^f]$, $[A^f, B^p]$, $[A^f, B^t]$, $[A^p]$, $[A^t, B^f]$, $[A^t, B^p]$, $[A^t, B^t]$.

3.4 Validity of equations

Given an equation the system determines which sequents have to be proved in order to decide if the equation is valid. If the equation is not valid counter-examples are given.

Problem 4. Is the equation $\neg\neg A = A$ valid?

Answer. The problem is equivalent to proving the following sequents:

$$[A^p, A^t, \neg\neg A^f], [A^f, A^t, \neg\neg A^p], [A^f, A^p, \neg\neg A^t]$$

Yes, the equation is valid.

3.5 Validity of Quasi-Equations

Given a quasi-equation the system determines which sequents have to be proved in order to decide if the quasi-equation is valid. If it is not valid counter-examples are given.

Problem 5. Is the quasi-equation

$$A = B, B = C \vdash A = C$$

valid?

Answer. Yes, the quasi-equation is valid.

4 Conclusion

The system `MUltseq` presented in this paper is intended as a tool to gain insights and better intuitions in the study of particular finitely-valued logics, avoiding error-prone and complex computations by hand. We hope that its simplicity, and the fact that no previous knowledge (except the truth tables) of the logic is needed to experiment, make the system useful for all those researches interested in these logics. In addition, since equations and quasi-equations have also been integrated in the general framework, algebraic problems can also be addressed by the system.

`MUltseq` is developed by the authors of this abstract within a project titled *"Generic Decision Procedures for Many-Valued Logics"*. It is written in a subset of Prolog compatible with any standard Prolog interpreter. More details as well as the most recent version of `MUltseq` can be obtained on the web [6].

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