Some Test Examples

M. Ultseq

July 23, 2001

1 Derivability of Sequents

1.1 Example s1

Problem: Is the sequent $[(A \supset B)^p, (A \vee C)^t]$ provable?

Answer: No, it is not.

Derivation of $[(A \supset B)^p, (A \vee C)^t]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{hypothesis}\\ [A^p, A^t, B^p, C^t]\end{array}}{[A^p, B^p, (A \vee C)^t]} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^t, B^f, C^t]\end{array}}{[A^t, B^f, (A \vee C)^t]}}{[(A \supset B)^p, (A \vee C)^t]}$$

List of hypotheses:

$$\begin{array}{c}[A^p, A^t, B^p, C^t]\\ [A^t, B^f, C^t]\end{array}$$

Derivation skeleton of $[(A \supset B)^p, (A \vee C)^t]$:

$$\dfrac{\dfrac{3}{2} \quad \dfrac{5}{4}}{1}$$

Table of sequents:

$$\begin{array}{rl}
1: & [(A \supset B)^p, (A \vee C)^t]\\
2: & [A^p, B^p, (A \vee C)^t]\\
3: & [A^p, A^t, B^p, C^t]\\
4: & [A^t, B^f, (A \vee C)^t]\\
5: & [A^t, B^f, C^t]
\end{array}$$

List of counter-examples:

$$\begin{array}{c}
[A^f, B^f, C^f]\\
[A^f, B^f, C^p]\\
[A^f, B^p, C^f]\\
[A^f, B^p, C^p]\\
[A^f, B^t, C^f]\\
[A^f, B^t, C^p]\\
[A^p, B^p, C^f]\\
[A^p, B^p, C^p]\\
[A^p, B^t, C^f]\\
[A^p, B^t, C^p]
\end{array}$$

1.2 Example s2

Problem: Is the sequent $[(A \supset (B \supset A))^t]$ provable?

Answer: Yes, it is.

Proof of $[(A \supset (B \supset A))^t]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f, B^p]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f]\end{array}}{[A^f, A^p, (B \supset A)^t]} \quad \dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f, B^p]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f, B^p]\end{array}}{[A^f, A^p, B^p, (B \supset A)^t]} \quad \dfrac{\begin{array}{c}\text{axiom for } B\\ [A^f, A^t, B^f, B^p, B^t]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f, B^t]\end{array}}{[A^f, B^t, (B \supset A)^t]}}{[A^f, (B \supset A)^p, (B \supset A)^t]}}{[(A \supset (B \supset A))^t]}$$

Proof skeleton of $[(A \supset (B \supset A))^t]$:

$$\dfrac{\dfrac{3 \quad 4}{2} \qquad \dfrac{\dfrac{3 \quad 3}{6} \quad \dfrac{8 \quad 9}{7}}{5}}{1}$$

Table of sequents:

1: $[(A \supset (B \supset A))^t]$
2: $[A^f, A^p, (B \supset A)^t]$
3: $[A^f, A^p, A^t, B^f, B^p]$
4: $[A^f, A^p, A^t, B^f]$
5: $[A^f, (B \supset A)^p, (B \supset A)^t]$
6: $[A^f, A^p, B^p, (B \supset A)^t]$
7: $[A^f, B^t, (B \supset A)^t]$
8: $[A^f, A^t, B^f, B^p, B^t]$
9: $[A^f, A^p, A^t, B^f, B^t]$

1.3 Example s3

Problem: Is the sequent $[(A \wedge B)^f, A^p, (A \vee B)^t]$ provable?

Answer: Yes, it is.

Proof of $[(A \wedge B)^f, A^p, (A \vee B)^t]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f, B^t]\end{array}}{[A^f, A^p, B^f, (A \vee B)^t]}}{[(A \wedge B)^f, A^p, (A \vee B)^t]}$$

Proof skeleton of $[(A \wedge B)^f, A^p, (A \vee B)^t]$:

$$\dfrac{\dfrac{3}{2}}{1}$$

Table of sequents:

1: $[(A \wedge B)^f, A^p, (A \vee B)^t]$
2: $[A^f, A^p, B^f, (A \vee B)^t]$
3: $[A^f, A^p, A^t, B^f, B^t]$

1.4 Example s4

Problem: Is the sequent $[(A \wedge B)^f, (A \wedge B)^p, B^t]$ provable?

Answer: Yes, it is.

Proof of $[(A \wedge B)^f, (A \wedge B)^p, B^t]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [A^f, A^p, B^f, B^p, B^t]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^f, A^p, A^t, B^f, B^t]\end{array} \quad \begin{array}{c}\text{axiom for } B\\ [A^f, B^f, B^p, B^t]\end{array}}{[A^f, B^f, B^t, (A \wedge B)^p]}}{[(A \wedge B)^f, (A \wedge B)^p, B^t]}$$

Proof skeleton of $[(A \wedge B)^f, (A \wedge B)^p, B^t]$:

$$\dfrac{\dfrac{3 \quad 4 \quad 5}{2}}{1}$$

Table of sequents:

1: $[(A \wedge B)^f, (A \wedge B)^p, B^t]$
2: $[A^f, B^f, B^t, (A \wedge B)^p]$
3: $[A^f, A^p, B^f, B^p, B^t]$
4: $[A^f, A^p, A^t, B^f, B^t]$
5: $[A^f, B^f, B^p, B^t]$

1.5 Example s5

Problem: Is the sequent $[(((A \supset B) \supset B) \supset ((B \supset A) \supset A))^t]$ provable?

Answer: Yes, it is.

Proof of $[(((A \supset B) \supset B) \supset ((B \supset A) \supset A))^t]$:

axiom for A | axiom for A

$[A^f, A^p, A^t, B^p, B^t, (B \supset A)^f, (B \supset A)^p]$ $\quad$ $[A^f, A^p, A^t, B^p, B^t, (B \supset A)^f]$

$[A^f, A^p, B^p, B^t, ((B \supset A) \supset A)^t]$ $\quad$ axiom for A $\quad$ $[A^f, A^p, A^t, B^f, B^p, B^t, ((B \supset A) \supset A)^t]$ $\quad$ $[A^f, A^p, A$

$[A^f, A^p, B^p, B^t, (A \supset B)^p, ((B \supset A) \supset A)^t]$

$[A^f, A^p, B^t, ((A \supset B) \supset$

Proof skeleton of $[(((A \supset B) \supset B) \supset ((B \supset A) \supset A))^t]$:

27 32

7 8 | 12 13 | 7 8 | 23 24 | 26 27 | 30 31 | 26 27 | 7 8 | 40 41 | 7 8 7

6 9 | 11 14 | 6 17 | 22 | 25 | 29 | 25 | 12 13 | 6 | 39 | 9 43 | 6

5 10 | 16 18 | 21 | 28 | 11 34 | 38 | 42 | 45

4 15 | 20 | 33 | 37

3 | 19 | 36

2

1

Table of sequents:

1: $[(((A \supset B) \supset B) \supset ((B \supset A) \supset A))^t]$

2: $[((A \supset B) \supset B)^f, ((A \supset B) \supset B)^p, ((B \supset A) \supset A)^t]$

3: $[(A \supset B)^t, ((A \supset B) \supset B)^p, ((B \supset A) \supset A)^t]$

4: $[A^f, A^p, B^t, ((A \supset B) \supset B)^p, ((B \supset A) \supset A)^t]$

5: $[A^f, A^p, B^p, B^t, (A \supset B)^p, ((B \supset A) \supset A)^t]$

6: $[A^f, A^p, B^p, B^t, ((B \supset A) \supset A)^t]$

7: $[A^f, A^p, A^t, B^p, B^t, (B \supset A)^f, (B \supset A)^p]$

8: $[A^f, A^p, A^t, B^p, B^t, (B \supset A)^f]$

9: $[A^f, A^p, A^t, B^f, B^p, B^t, ((B \supset A) \supset A)^t]$

10: $[A^f, A^p, B^f, B^t, (A \supset B)^t, ((B \supset A) \supset A)^t]$

11: $[A^f, A^p, B^f, B^t, ((B \supset A) \supset A)^t]$

12: $[A^f, A^p, A^t, B^f, B^t, (B \supset A)^f, (B \supset A)^p]$

13: $[A^f, A^p, A^t, B^f, B^t, (B \supset A)^f]$

14: $[A^f, A^p, B^f, B^p, B^t, ((B \supset A) \supset A)^t]$

15: $[A^f, B^p, B^t, ((A \supset B) \supset B)^p, ((B \supset A) \supset A)^t]$

16: $[A^f, B^p, B^t, (A \supset B)^p, ((B \supset A) \supset A)^t]$

17: $[A^f, A^t, B^f, B^p, B^t, ((B \supset A) \supset A)^t]$

18: $[A^f, B^f, B^p, B^t, (A \supset B)^t, ((B \supset A) \supset A)^t]$

19: $[B^f, ((A \supset B) \supset B)^p, ((B \supset A) \supset A)^t]$

20: $[B^f, B^p, (A \supset B)^p, ((B \supset A) \supset A)^t]$

21: $[A^p, B^f, B^p, ((B \supset A) \supset A)^t]$

22: $[A^p, A^t, B^f, B^p, (B \supset A)^f, (B \supset A)^p]$

23: $[A^p, A^t, B^f, B^p, B^t, (B \supset A)^p]$

24: $[A^f, A^p, A^t, B^f, B^p, (B \supset A)^p]$

25: $[A^p, A^t, B^f, B^p, (B \supset A)^f]$

26: $[A^p, A^t, B^f, B^p, B^t]$

27: $[A^f, A^p, A^t, B^f, B^p]$

28: $[A^t, B^f, B^p, ((B \supset A) \supset A)^t]$

29: $[A^t, B^f, B^p, (B \supset A)^f, (B \supset A)^p]$

30: $[A^t, B^f, B^p, B^t, (B \supset A)^p]$

31: $[A^f, A^t, B^f, B^p, (B \supset A)^p]$

32: $[A^f, A^t, B^f, B^p, B^t]$

33: $[B^f, (A \supset B)^t, ((B \supset A) \supset A)^t]$

34: $[A^f, B^f, B^p, B^t, ((B \supset A) \supset A)^t]$

35: $[((A \supset B) \supset B)^f, ((B \supset A) \supset A)^p, ((B \supset A) \supset A)^t]$

36: $[(A \supset B)^t, ((B \supset A) \supset A)^p, ((B \supset A) \supset A)^t]$

37: $[A^f, A^p, B^t, ((B \supset A) \supset A)^p, ((B \supset A) \supset A)^t]$

38: $[A^f, A^p, B^t, (B \supset A)^p, ((B \supset A) \supset A)^t]$

39: $[A^f, A^p, B^t, ((B \supset A) \supset A)^t]$

40: $[A^f, A^p, A^t, B^t, (B \supset A)^f, (B \supset A)^p]$

41: $[A^f, A^p, A^t, B^t, (B \supset A)^f]$

42: $[A^f, A^p, B^t, (B \supset A)^t, ((B \supset A) \supset A)^t]$

43: $[A^f, A^p, A^t, B^f, B^t, ((B \supset A) \supset A)^t]$

44: $[A^f, B^p, B^t, ((B \supset A) \supset A)^p, ((B \supset A) \supset A)^t]$

45: $[A^f, A^p, B^p, B^t, (B \supset A)^p, ((B \supset A) \supset A)^t]$

46: $[A^f, B^p, B^t, (B \supset A)^t, ((B \supset A) \supset A)^t]$

47: $[B^f, ((B \supset A) \supset A)^p, ((B \supset A) \supset A)^t]$

48: $[A^p, B^f, (B \supset A)^p, ((B \supset A) \supset A)^t]$

49: $[A^f, B^f, (B \supset A)^t, ((B \supset A) \supset A)^t]$

50: $[A^f, A^t, B^f, B^p, ((B \supset A) \supset A)^t]$

51: $[A^f, A^t, B^f, B^p, (B \supset A)^f, (B \supset A)^p]$

52: $[A^f, A^t, B^f, B^p, B^t, (B \supset A)^p]$

53: $[A^f, A^p, A^t, B^f, B^p, (B \supset A)^f]$

54: $[A^f, A^p, A^t, B^f, ((B \supset A) \supset A)^t]$

1.6 Example s6

Problem: Is the sequent $[((A \supset B) \supset B)^t]$ provable?

Answer: No, it is not.

Derivation of $[((A \supset B) \supset B)^t]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{hypothesis}}{[A^p, A^t, B^p, B^t]} \quad \overset{\text{hypothesis}}{[A^t, B^f, B^t]}}{[A^t, B^t, (A \supset B)^p]} \quad \dfrac{\overset{\text{axiom for } B}{[A^p, B^f, B^p, B^t]} \quad \overset{\text{hypothesis}}{[A^t, B^f, B^t]}}{[B^f, B^t, (A \supset B)^p]}}{[B^t, (A \supset B)^f, (A \supset B)^p]} \quad \dfrac{\overset{\text{hypothesis}}{[A^t, B^p, B^t]} \quad \overset{\text{axiom for } B}{[B^f, B^p, B^t]}}{[B^p, B^t, (A \supset B)^f]}}{[((A \supset B) \supset B)^t]}$$

List of hypotheses:

$$[A^t, B^f, B^t]$$
$$[A^t, B^p, B^t]$$

Derivation skeleton of $[((A \supset B) \supset B)^t]$:

$$\dfrac{\dfrac{\dfrac{4 \quad 5}{3} \quad \dfrac{7 \quad 5}{6}}{2} \quad \dfrac{9 \quad 10}{8}}{1}$$

Table of sequents:

$$\begin{array}{rl} 1: & [((A \supset B) \supset B)^t] \\ 2: & [B^t, (A \supset B)^f, (A \supset B)^p] \\ 3: & [A^t, B^t, (A \supset B)^p] \\ 4: & [A^p, A^t, B^p, B^t] \\ 5: & [A^t, B^f, B^t] \\ 6: & [B^f, B^t, (A \supset B)^p] \\ 7: & [A^p, B^f, B^p, B^t] \\ 8: & [B^p, B^t, (A \supset B)^f] \\ 9: & [A^t, B^p, B^t] \\ 10: & [B^f, B^p, B^t] \end{array}$$

List of counter-examples:

$$[A^f, B^f]$$
$$[A^f, B^p]$$
$$[A^p, B^f]$$
$$[A^p, B^p]$$

2 Consequence Relation on Sequents

2.1 Example cs1

Problem: Is the consequence relation

$$[A^f, A^p, B^t], [A^f, B^p, B^t] \vdash [(A \supset B)^t]$$

valid?

The problem is equivalent to proving the following sequents:

$$[A^p, A^t, (A \supset B)^t]$$
$$[A^t, B^f, (A \supset B)^t]$$
$$[B^f, B^p, (A \supset B)^t]$$

Answer: Yes, the consequence relation holds.

2.2 Example cs2

Problem: Is the consequence relation

$$[A^p, A^f, B^t], [A^f, B^p, B^t] \vdash [(A \supset B)^t]$$

valid?

The problem is equivalent to proving the following sequents:

$$[A^p, A^t, (A \supset B)^t]$$
$$[A^t, B^f, (A \supset B)^t]$$
$$[B^f, B^p, (A \supset B)^t]$$

Answer: Yes, the consequence relation holds.

2.3 Example cs3

Problem: Is the consequence relation

$$[A^f, A^p, B^t], [A^f, B^p, B^t] \vdash [(A \supset B)^p]$$

valid?

The problem is equivalent to proving the following sequents:

$$[A^p, A^t, (A \supset B)^p]$$
$$[A^t, B^f, (A \supset B)^p]$$
$$[B^f, B^p, (A \supset B)^p]$$

Answer: No, the consequence relation does not hold.

List of counter-examples:

$$[A^f, B^f]$$
$$[A^f, B^p]$$
$$[A^f, B^t]$$
$$[A^p, B^p]$$
$$[A^p, B^t]$$
$$[A^t, B^t]$$

2.4 Example cs4

Problem: Is the consequence relation

$$[A^f, A^p, B^t], [A^f, B^p, B^t] \vdash [(A \supset B)^f]$$

valid?

The problem is equivalent to proving the following sequents:

$$[A^p, A^t, (A \supset B)^f]$$
$$[A^t, B^f, (A \supset B)^f]$$
$$[B^f, B^p, (A \supset B)^f]$$

Answer: No, the consequence relation does not hold.

List of counter-examples:

$$[A^f]$$
$$[A^p, B^p]$$
$$[B^t]$$

2.5 Example cs5

Problem: Is the consequence relation

$$[(A \wedge B)^t] \vdash [A^t]$$

valid?

The problem is equivalent to proving the following sequent:

$$[A^t, (A \wedge B)^f, (A \wedge B)^p]$$

Answer: Yes, the consequence relation holds.

2.6 Example cs6

Problem: Is the consequence relation

$$[(A \vee B)^t] \vdash [A^t]$$

valid?

The problem is equivalent to proving the following sequent:

$$[A^t, (A \vee B)^f, (A \vee B)^p]$$

Answer: No, the consequence relation does not hold.

List of counter-examples:

$$[A^f, B^t]$$
$$[A^p, B^t]$$

3 Validity of Formulas

3.1 Example f1

Problem: Let $\{t\}$ be the set of designated truth values. Is the formula $(A \supset (B \supset A))$ valid?

The problem is equivalent to proving the following sequent:

$$[(A \supset (B \supset A))^t]$$

Answer: Yes, the formula is valid.

3.2 Example f2

Problem: Let $\{p\}$ be the set of designated truth values. Is the formula $(A \supset (B \supset A))$ valid?

The problem is equivalent to proving the following sequent:

$$[(A \supset (B \supset A))^p]$$

Answer: No, the formula is not valid.

List of counter-examples:

$$[A^f, B^f]$$
$$[A^f, B^p]$$
$$[A^f, B^t]$$
$$[A^p]$$
$$[A^t, B^f]$$
$$[A^t, B^p]$$
$$[A^t, B^t]$$

3.3 Example f3

Problem: Let $\{f, t\}$ be the set of designated truth values. Is the formula $((A \vee \neg A) \supset (A \wedge \neg A))$ valid?

The problem is equivalent to proving the following sequent:

$$[((A \vee \neg A) \supset (A \wedge \neg A))^f, ((A \vee \neg A) \supset (A \wedge \neg A))^t]$$

Answer: Yes, the formula is valid.

4 Consequence Relation on Formulas

4.1 Example cf1

Problem: Let $\{t\}$ be the set of designated truth values. Is the consequence relation

$$(X \supset Y), X \vdash Y$$

valid?

The problem is equivalent to proving the following sequent:

$$[X^f, X^p, Y^t, (X \supset Y)^f, (X \supset Y)^p]$$

Answer: Yes, the consequence relation holds.

4.2 Example cf2

Problem: Let $\{p\}$ be the set of designated truth values. Is the consequence relation

$$(X \supset Y), X \vdash Y$$

valid?

The problem is equivalent to proving the following sequent:

$$[X^f, X^t, Y^p, (X \supset Y)^f, (X \supset Y)^t]$$

Answer: No, the consequence relation does not hold.

List of counter-examples:

$$[X^p, Y^f]$$

4.3 Example cf3

Problem: Let $\{p, t\}$ be the set of designated truth values. Is the consequence relation

$$(X \supset Y), X \vdash Y$$

valid?

The problem is equivalent to proving the following sequent:

$$[X^f, Y^p, Y^t, (X \supset Y)^f]$$

Answer: No, the consequence relation does not hold.

List of counter-examples:

$$[X^p, Y^f]$$

4.4 Example cf4

Problem: Let $\{t\}$ be the set of designated truth values. Is the consequence relation

$$X \vdash (X \vee Y)$$

valid?

The problem is equivalent to proving the following sequent:

$$[X^f, X^p, (X \vee Y)^t]$$

Answer: Yes, the consequence relation holds.

5 Validity of Equations

5.1 Example e1

Problem: Is the equation $\neg\neg A = A$ valid?

The problem is equivalent to proving the following sequents:

$$[A^p, A^t, \neg\neg A^f]$$
$$[A^f, A^t, \neg\neg A^p]$$
$$[A^f, A^p, \neg\neg A^t]$$

Answer: Yes, the equation is valid.

6 Validity of Quasi-Equations

6.1 Example qe1

Problem: Is the quasi-equation

$$A = B, B = C \vdash A = C$$

valid?

Answer: Yes, the quasi-equation is valid.

6.2 Example qe2

Problem: Is the quasi-equation

$$A = B, C = D \vdash A = D$$

valid?

The problem is equivalent to proving the following sequents:

$$[A^f, A^t, B^f, B^t, C^p, C^t, D^p, D^t]$$
$$[A^f, A^p, B^f, B^p, C^p, C^t, D^p, D^t]$$
$$[A^p, A^t, B^p, B^t, C^f, C^t, D^f, D^t]$$
$$[A^f, A^p, B^f, B^p, C^f, C^t, D^f, D^t]$$
$$[A^p, A^t, B^p, B^t, C^f, C^p, D^f, D^p]$$
$$[A^f, A^t, B^f, B^t, C^f, C^p, D^f, D^p]$$

Answer: No, the quasi-equation is not valid.

List of counter-examples:

$$[A^f, B^f, C^p, D^p]$$
$$[A^f, B^f, C^t, D^t]$$
$$[A^p, B^p, C^f, D^f]$$
$$[A^p, B^p, C^t, D^t]$$
$$[A^t, B^t, C^f, D^f]$$
$$[A^t, B^t, C^p, D^p]$$