Expressing Connectives in Łukasiewicz logic

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In this example we search for formulas that define any 2-place connective. We search for defining formulas containing up to 3 connectives. Negation is not definable.

The logic contains the connectives

$$\wedge, \rightarrow, \neg, \vee, \otimes, \oplus$$

and truth values

$$\mathbf{0}, \mathbf{1/2}, \mathbf{1}.$$

The truth value **1** is designated.

1 Equivalents of $\rightarrow$

Proposition 1 *The equality* $(A \rightarrow B) = (B \oplus \neg A)$ *holds.*

Proposition 2 *The equality* $(A \rightarrow B) = (\neg A \oplus B)$ *holds.*

Proposition 3 *The equality* $(A \rightarrow B) = \neg(A \otimes \neg B)$ *holds.*

Proposition 4 *The equality* $(A \rightarrow B) = \neg(\neg B \otimes A)$ *holds.*

Proposition 5 *The equality* $(A \rightarrow B) = (B \oplus \neg(A \wedge A))$ *holds.*

Proposition 6 *The equality* $(A \rightarrow B) = (B \oplus \neg(A \vee A))$ *holds.*

Proposition 7 *The equality* $(A \rightarrow B) = (B \oplus \neg(A \vee B))$ *holds.*

Proposition 8 *The equality* $(A \rightarrow B) = (B \oplus \neg(B \vee A))$ *holds.*

Proposition 9 *The equality* $(A \rightarrow B) = (\neg A \oplus (A \wedge B))$ *holds.*

Proposition 10 *The equality* $(A \rightarrow B) = (\neg A \oplus (B \wedge A))$ *holds.*

Proposition 11 *The equality* $(A \rightarrow B) = (\neg A \oplus (B \wedge B))$ *holds.*

Proposition 12 *The equality* $(A \rightarrow B) = (\neg A \oplus (B \vee B))$ *holds.*

Proposition 13 *The equality* $(A \rightarrow B) = ((A \wedge B) \oplus \neg A)$ *holds.*

Proposition 14 *The equality* $(A \rightarrow B) = ((B \wedge A) \oplus \neg A)$ *holds.*

Proposition 15 *The equality* $(A \rightarrow B) = ((B \wedge B) \oplus \neg A)$ *holds.*

Proposition 16 *The equality* $(A \rightarrow B) = ((B \vee B) \oplus \neg A)$ *holds.*

Proposition 17 *The equality* $(A \rightarrow B) = (\neg(A \wedge A) \oplus B)$ *holds.*

Proposition 18 *The equality* $(A \to B) = (\neg(A \vee A) \oplus B)$ *holds.*

Proposition 19 *The equality* $(A \to B) = (\neg(A \vee B) \oplus B)$ *holds.*

Proposition 20 *The equality* $(A \to B) = (\neg(B \vee A) \oplus B)$ *holds.*

Proposition 21 *The equality* $(A \to B) = (B \vee (B \oplus \neg A))$ *holds.*

Proposition 22 *The equality* $(A \to B) = (B \vee (\neg A \oplus B))$ *holds.*

Proposition 23 *The equality* $(A \to B) = ((B \oplus \neg A) \vee B)$ *holds.*

Proposition 24 *The equality* $(A \to B) = ((\neg A \oplus B) \vee B)$ *holds.*

2 Equivalents of $\otimes$

Proposition 25 *The equality* $(A \otimes B) = \neg(A \to \neg B)$ *holds.*

Proposition 26 *The equality* $(A \otimes B) = \neg(B \to \neg A)$ *holds.*

3 Equivalents of $\oplus$

Proposition 27 *The equality* $(A \oplus B) = (\neg A \to B)$ *holds.*

Proposition 28 *The equality* $(A \oplus B) = (\neg B \to A)$ *holds.*

Proposition 29 *The equality* $(A \oplus B) = (\neg A \to (B \wedge B))$ *holds.*

Proposition 30 *The equality* $(A \oplus B) = (\neg A \to (B \vee B))$ *holds.*

Proposition 31 *The equality* $(A \oplus B) = (\neg B \to (A \wedge A))$ *holds.*

Proposition 32 *The equality* $(A \oplus B) = (\neg B \to (A \vee A))$ *holds.*

Proposition 33 *The equality* $(A \oplus B) = (\neg(A \wedge A) \to B)$ *holds.*

Proposition 34 *The equality* $(A \oplus B) = (\neg(B \wedge B) \to A)$ *holds.*

Proposition 35 *The equality* $(A \oplus B) = (\neg(A \vee A) \to B)$ *holds.*

Proposition 36 *The equality* $(A \oplus B) = (\neg(B \vee B) \to A)$ *holds.*

Proposition 37 *The equality* $(A \oplus B) = ((A \to (A \otimes B)) \to B)$ *holds.*

Proposition 38 *The equality* $(A \oplus B) = ((A \to (B \otimes A)) \to B)$ *holds.*

Proposition 39 *The equality* $(A \oplus B) = ((A \to (B \otimes B)) \to B)$ *holds.*

Proposition 40 *The equality* $(A \oplus B) = ((B \to (A \otimes A)) \to A)$ *holds.*

Proposition 41 *The equality* $(A \oplus B) = ((B \to (A \otimes B)) \to A)$ *holds.*

Proposition 42 *The equality* $(A \oplus B) = ((B \to (B \otimes A)) \to A)$ *holds.*

Proposition 43 *The equality* $(A \oplus B) = ((A \vee \neg B) \to A)$ *holds.*

Proposition 44 *The equality* $(A \oplus B) = ((B \vee \neg A) \to B)$ *holds.*

Proposition 45 *The equality* $(A \oplus B) = ((\neg A \vee B) \to B)$ *holds.*

Proposition 46 *The equality* $(A \oplus B) = ((\neg B \vee A) \to A)$ *holds.*

Proposition 47 *The equality* $(A \oplus B) = (A \vee (\neg A \to B))$ *holds.*

Proposition 48 *The equality* $(A \oplus B) = (A \vee (\neg B \to A))$ *holds.*

Proposition 49 *The equality* $(A \oplus B) = (B \vee (\neg A \to B))$ *holds.*

Proposition 50 *The equality* $(A \oplus B) = (B \vee (\neg B \to A))$ *holds.*

Proposition 51 *The equality* $(A \oplus B) = ((\neg A \to B) \vee A)$ *holds.*

Proposition 52 *The equality* $(A \oplus B) = ((\neg A \to B) \vee B)$ *holds.*

Proposition 53 *The equality* $(A \oplus B) = ((\neg B \to A) \vee A)$ *holds.*

Proposition 54 *The equality* $(A \oplus B) = ((\neg B \to A) \vee B)$ *holds.*

4 Equivalents of $\wedge$

Proposition 55 *The equality* $(A \wedge B) = (A \otimes (A \to B))$ *holds.*

Proposition 56 *The equality* $(A \wedge B) = (B \otimes (B \to A))$ *holds.*

Proposition 57 *The equality* $(A \wedge B) = ((A \to B) \otimes A)$ *holds.*

Proposition 58 *The equality* $(A \wedge B) = ((B \to A) \otimes B)$ *holds.*

Proposition 59 *The equality* $(A \wedge B) = (A \otimes (A \to (B \vee B)))$ *holds.*

Proposition 60 *The equality* $(A \wedge B) = (A \otimes ((A \vee A) \to B))$ *holds.*

Proposition 61 *The equality* $(A \wedge B) = (A \otimes ((A \vee B) \to B))$ *holds.*

Proposition 62 *The equality* $(A \wedge B) = (A \otimes ((B \vee A) \to B))$ *holds.*

Proposition 63 *The equality* $(A \wedge B) = (A \otimes (B \oplus \neg A))$ *holds.*

Proposition 64 *The equality* $(A \wedge B) = (A \otimes (\neg A \oplus B))$ *holds.*

Proposition 65 *The equality* $(A \wedge B) = (A \otimes (B \vee (A \to B)))$ *holds.*

Proposition 66 *The equality* $(A \wedge B) = (A \otimes ((A \to B) \vee B))$ *holds.*

Proposition 67 *The equality* $(A \wedge B) = (B \otimes (B \to (A \vee A)))$ *holds.*

Proposition 68 *The equality* $(A \wedge B) = (B \otimes ((A \vee B) \to A))$ *holds.*

Proposition 69 *The equality* $(A \wedge B) = (B \otimes ((B \vee A) \to A))$ *holds.*

Proposition 70 *The equality* $(A \wedge B) = (B \otimes ((B \vee B) \to A))$ *holds.*

Proposition 71 *The equality* $(A \wedge B) = (B \otimes (A \oplus \neg B))$ *holds.*

Proposition 72 *The equality* $(A \wedge B) = (B \otimes (\neg B \oplus A))$ *holds.*

Proposition 73 *The equality* $(A \wedge B) = (B \otimes (A \vee (B \to A)))$ *holds.*

Proposition 74 *The equality* $(A \wedge B) = (B \otimes ((B \to A) \vee A))$ *holds.*

Proposition 75 *The equality* $(A \wedge B) = ((A \to B) \otimes (A \vee A))$ *holds.*

Proposition 76 *The equality* $(A \wedge B) = ((B \to A) \otimes (B \vee B))$ *holds.*

Proposition 77 *The equality* $(A \wedge B) = ((A \vee A) \otimes (A \to B))$ *holds.*

Proposition 78 *The equality* $(A \wedge B) = ((B \vee B) \otimes (B \to A))$ *holds.*

Proposition 79 *The equality* $(A \wedge B) = ((A \to (B \vee B)) \otimes A)$ *holds.*

Proposition 80 *The equality* $(A \wedge B) = ((B \to (A \vee A)) \otimes B)$ *holds.*

Proposition 81 *The equality* $(A \wedge B) = (((A \vee A) \to B) \otimes A)$ *holds.*

Proposition 82 *The equality* $(A \wedge B) = (((A \vee B) \to A) \otimes B)$ *holds.*

Proposition 83 *The equality* $(A \wedge B) = (((A \vee B) \to B) \otimes A)$ *holds.*

Proposition 84 *The equality* $(A \wedge B) = (((B \vee A) \to A) \otimes B)$ *holds.*

Proposition 85 *The equality* $(A \wedge B) = (((B \vee A) \to B) \otimes A)$ *holds.*

Proposition 86 *The equality* $(A \wedge B) = (((B \vee B) \to A) \otimes B)$ *holds.*

Proposition 87 *The equality* $(A \wedge B) = ((A \oplus \neg B) \otimes B)$ *holds.*

Proposition 88 *The equality* $(A \wedge B) = ((B \oplus \neg A) \otimes A)$ *holds.*

Proposition 89 *The equality* $(A \wedge B) = ((\neg A \oplus B) \otimes A)$ *holds.*

Proposition 90 *The equality* $(A \wedge B) = ((\neg B \oplus A) \otimes B)$ *holds.*

Proposition 91 *The equality* $(A \wedge B) = ((A \vee (B \to A)) \otimes B)$ *holds.*

Proposition 92 *The equality* $(A \wedge B) = ((B \vee (A \to B)) \otimes A)$ *holds.*

Proposition 93 *The equality* $(A \wedge B) = (((A \to B) \vee B) \otimes A)$ *holds.*

Proposition 94 *The equality* $(A \wedge B) = (((B \to A) \vee A) \otimes B)$ *holds.*

5 Equivalents of $\vee$

Proposition 95 *The equality* $(A \vee B) = ((A \to B) \to B)$ *holds.*

Proposition 96 *The equality* $(A \vee B) = ((B \to A) \to A)$ *holds.*

Proposition 97 *The equality* $(A \vee B) = ((A \to B) \to (B \wedge B))$ *holds.*

Proposition 98 *The equality* $(A \vee B) = ((B \to A) \to (A \wedge A))$ *holds.*

Proposition 99 *The equality* $(A \vee B) = ((A \to (A \wedge B)) \to B)$ *holds.*

Proposition 100 *The equality* $(A \vee B) = ((A \to (B \wedge A)) \to B)$ *holds.*

Proposition 101 *The equality* $(A \vee B) = ((A \to (B \wedge B)) \to B)$ *holds.*

Proposition 102 *The equality* $(A \vee B) = ((B \to (A \wedge A)) \to A)$ *holds.*

Proposition 103 *The equality* $(A \vee B) = ((B \to (A \wedge B)) \to A)$ *holds.*

Proposition 104 *The equality* $(A \vee B) = ((B \to (B \wedge A)) \to A)$ *holds.*

Proposition 105 *The equality* $(A \vee B) = (((A \wedge A) \to B) \to B)$ *holds.*

Proposition 106 *The equality* $(A \vee B) = (((B \wedge B) \to A) \to A)$ *holds.*

Proposition 107 *The equality* $(A \vee B) = ((A \oplus \neg B) \to A)$ *holds.*

Proposition 108 *The equality* $(A \vee B) = ((B \oplus \neg A) \to B)$ *holds.*

Proposition 109 *The equality* $(A \vee B) = ((\neg A \oplus B) \to B)$ *holds.*

Proposition 110 *The equality* $(A \vee B) = ((\neg B \oplus A) \to A)$ *holds.*

Proposition 111 *The equality* $(A \vee B) = (A \oplus \neg(B \to A))$ *holds.*

Proposition 112 *The equality* $(A \vee B) = (A \oplus (B \otimes \neg A))$ *holds.*

Proposition 113 *The equality* $(A \vee B) = (A \oplus (\neg A \otimes B))$ *holds.*

Proposition 114 *The equality* $(A \vee B) = (B \oplus \neg(A \to B))$ *holds.*

Proposition 115 *The equality* $(A \vee B) = (B \oplus (A \otimes \neg B))$ *holds.*

Proposition 116 *The equality* $(A \vee B) = (B \oplus (\neg B \otimes A))$ *holds.*

Proposition 117 *The equality* $(A \vee B) = (\neg(A \to B) \oplus B)$ *holds.*

Proposition 118 *The equality* $(A \vee B) = (\neg(B \to A) \oplus A)$ *holds.*

Proposition 119 *The equality* $(A \vee B) = ((A \otimes \neg B) \oplus B)$ *holds.*

Proposition 120 *The equality* $(A \vee B) = ((B \otimes \neg A) \oplus A)$ *holds.*

Proposition 121 *The equality* $(A \vee B) = ((\neg A \otimes B) \oplus A)$ *holds.*

Proposition 122 *The equality* $(A \vee B) = ((\neg B \otimes A) \oplus B)$ *holds.*

6 Program listing: ex_lukasiewicz3.pl

```
% Test file to find definitions of operators operators

% make sure MUltseq is loaded
:- ensure_loaded('../multseq/multseq').

% load the rules
:- load_logic('lukasiewicz.msq').

% define standard Omap
:- setOmap([(neg)/(-),imp/(>),and/(/\),or/(\/),equiv/(=)]).

% check all properties and write report to out.tex

:- set_option(tex_output(terse)).
:- set_option(tex_failure(off)).

:- start_logging(ex_lukasiewicz3,'.tex').

:- print_tex(tex_title("Expressing␣Connectives␣in␣\\L␣ukasiewicz␣logic")).

:- print_tex(tex_paragraph(["In␣this␣example␣we␣search␣for␣formulas␣that␣define␣any␣2-place␣
    connective.␣We␣search␣for␣defining␣formulas␣containing␣up␣to␣3␣connectives.␣Negation␣is␣
    not␣definable."])).

:- print_tex(tex_logic).

:- bagof(X, A^operator(X, A), Ops),
  ( operator(Op,2),
    print_tex(tex_section(["Equivalents␣of␣$", tex_conn(Op), "$"])),
    between(0, 3, N),
    subtract(Ops, [Op], ToRep),
    instantiate(+ : a+b @ formulas(2, ToRep, N), X),
    F =.. [Op, a, b],
    equality(F, X),
    fail)
  ; true.

:- print_tex(tex_listing("ex_lukasiewicz3.pl")).

:- stop_logging.
```