Analytic Proof Systems for Shramko-Wansing logic $\mathbf{SIXTEEN}_{2,des}$

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Abstract

We give sequent calculus, analytic tableaux, natural deduction, and clause translation systems for resolution for Shramko-Wansing logic $\mathbf{SIXTEEN}_{2,des}$.

1 Introduction

In this paper we present calculi for Shramko-Wansing logic $\mathbf{SIXTEEN}_{2,des}$ [12]. $\mathbf{SIXTEEN}_{2,des}$ has sixteen truth values $\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}$ (with $\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}$ designated), and connectives $\neg_t$, $\neg_f$, $\wedge_t$, $\vee_t$, $\wedge_f$, $\vee_f$. Its syntax and semantics is detailed in section 2.

We first present a 16-sided sequent calculus in section 3. The fundamental idea for many-sided sequent calculi for finite-valued logics goes back to Schröter [11], Rousseau [8], Takahashi [14]. We follow the method given by Baaz, Fermüller, and Zach [4] and Zach [15] for constructing inference rules. This guarantees that our system automatically has soundness and completness theorems, cut-elimination theorem and Maehara lemma (interpolation). For proofs of these results see [4, 15].

Signed tableau systems for finite-valued logics were proposed by Surma [13] and Carnielli [6], and generalized by Hähnle [7]. In section 4, we present a signed tableau system for Shramko-Wansing logic.

Many-valued natural deduction systems for finite-valued logics have been investigated by Baaz, Fermüller, and Zach [3] and Zach [15]. We give the introduction and elimination rules for the natural deduction system for $\mathbf{SIXTEEN}_{2,des}$ in section 5.

In addition to Hähnle's work on tableaux-based theorem proving for finite-valued logic, Baaz and Fermüller [1] have studied resolution calculi for clauses

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See appendices A and B for the specification of Shramko-Wansing logic.

of signed literals. In order for these calculi to be used to prove that formulas of Shramko-Wansing logic are valid or follow from some others, it is necessary to produce sets of signed clauses. In section 6, we present a translation calculus that yields a set of clauses from a set of formulas.

The rules we provide are optimal in each case, and use the algorithms developed by Salzer [9, 10].

2 Syntax and semantics

Definition 1. The *propositional language* $\mathscr{L}$ for Shramko-Wansing logic consists of

1. propositional variables: $x_0, x_1, x_2, \ldots$
2. propositional connectives, arity given in parenthesis: $\neg_t$ (1) , $\neg_f$ (1), $\wedge_t$ (2), $\vee_t$ (2), $\wedge_f$ (2), and $\vee_f$ (2)
3. auxiliary symbols: "(", ")" and ","

Formulas are defined inductively:

1. Every propositional variable is a formula.
2. If A is a formula, so is $\neg_t A$.
3. If A is a formula, so is $\neg_f A$.
4. If A and B are formulas, so is $(A \wedge_t B)$.
5. If A and B are formulas, so is $(A \vee_t B)$.
6. If A and B are formulas, so is $(A \wedge_f B)$.
7. If A and B are formulas, so is $(A \vee_f B)$.

As a notational convention, lowercase letters will be used to denote variables, possibly indexed. Uppercase letters $A, B, C, \ldots$ will stand for formulas, greek letters $\Gamma, \Delta, \Lambda, \ldots$ for sequences and sets of formulas. The symbol $\Box$ stands for general propositional connectives.

The connectives $\wedge_t$ and $\vee_t$ of $\mathbf{SIXTEEN}_{2,des}$ are defined as the inf and sup of the $\leq_t$ ordering given in fig. 1. The connectives $\wedge_f$ and $\vee_f$ correspond to inf and sup of $\leq_f$. For simplicity, we leave out the $\leq_i$ ordering and other operators and inversions defined in [12].

Definition 2. The *matrix* for Shramko-Wansing logic is given by:

1. the set of *truth values* $V = \{\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}\}$,
2. the set $V^+ = \{\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\} \subseteq V$ of *designated truth values*,

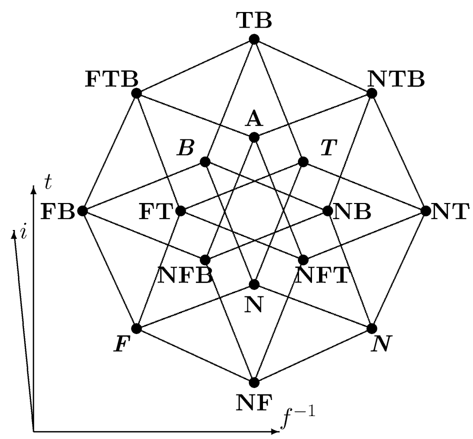

Figure 1: The $\leq_t$ and $\leq_f$ orderings

3. the truth functions for connectives $\neg_t$, $\neg_f$, $\wedge_t$, $\vee_t$, $\wedge_f$ and $\vee_f$, as given below;

The set of *undesignated values* is $V^- = V \setminus V^+ = \{\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\}$.

The truth functions for connectives $\neg_t$, $\neg_f$, $\wedge_t$, $\vee_t$, $\wedge_f$ and $\vee_f$ are defined by

$\widetilde{\neg}_t$	
N	**N**
N	***T***
F	***B***
T	***N***
B	***F***
NF	**TB**
NT	**NT**
FT	**NB**
NB	**FT**
FB	**FB**
TB	**NF**
NFT	**NTB**
NFB	**FTB**
NTB	**NFT**
FTB	**NFB**
A	**A**

$\widetilde{\neg}_f$	
N	**N**
N	***F***
F	***N***
T	***B***
B	***T***
NF	**NF**
NT	**FB**
FT	**NB**
NB	**FT**
FB	**NT**
TB	**TB**
NFT	**NFB**
NFB	**NFT**
NTB	**FTB**
FTB	**NTB**
A	**A**

$\widetilde{\wedge}_t$	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
N	N	*N*	*F*	N	N	NF	*N*	*F*	*N*	*F*	N	NF	NF	*N*	*F*	NF
N	*N*	*N*	NF	*N*	*N*	NF	*N*	NF	*N*	NF	*N*	NF	NF	*N*	NF	NF
F	*F*	NF	*F*	*F*	*F*	NF	NF	*F*	NF	*F*	*F*	NF	NF	NF	*F*	NF
T	N	*N*	*F*	*T*	N	NF	NT	FT	*N*	*F*	*T*	NFT	NF	NT	FT	NFT
B	N	*N*	*F*	N	*B*	NF	*N*	*F*	NB	FB	*B*	NF	NFB	NB	FB	NFB
NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF	NF
NT	*N*	*N*	NF	NT	*N*	NF	NT	NFT	*N*	NF	NT	NFT	NF	NT	NFT	NFT
FT	*F*	NF	*F*	FT	*F*	NF	NFT	FT	NF	*F*	FT	NFT	NF	NFT	FT	NFT
NB	*N*	*N*	NF	*N*	NB	NF	*N*	NF	NB	NFB	NB	NF	NFB	NB	NFB	NFB
FB	*F*	NF	*F*	*F*	FB	NF	NF	*F*	NFB	FB	FB	NF	NFB	NFB	FB	NFB
TB	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
NFT	NF	NF	NF	NFT	NF	NF	NFT	NFT	NF	NF	NFT	NFT	NF	NFT	NFT	NFT
NFB	NF	NF	NF	NF	NFB	NF	NF	NF	NFB	NFB	NFB	NF	NFB	NFB	NFB	NFB
NTB	*N*	*N*	NF	NT	NB	NF	NT	NFT	NB	NFB	NTB	NFT	NFB	NTB	A	A
FTB	*F*	NF	*F*	FT	FB	NF	NFT	FT	NFB	FB	FTB	NFT	NFB	A	FTB	A
A	NF	NF	NF	NFT	NFB	NF	NFT	NFT	NFB	NFB	A	NFT	NFB	A	A	A

$\widetilde{\vee}_t$	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
N	N	N	N	*T*	*B*	N	*T*	*T*	*B*	*B*	TB	*T*	*B*	TB	TB	TB
N	N	*N*	N	*T*	*B*	*N*	NT	*T*	NB	*B*	TB	NT	NB	NTB	TB	NTB
F	N	N	*F*	*T*	*B*	*F*	*T*	FT	*B*	FB	TB	FT	FB	TB	FTB	FTB
T	*T*	*T*	*T*	*T*	TB	*T*	*T*	*T*	TB	TB	TB	*T*	TB	TB	TB	TB
B	*B*	*B*	*B*	TB	*B*	*B*	TB	TB	*B*	*B*	TB	TB	*B*	TB	TB	TB
NF	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
NT	*T*	NT	*T*	*T*	TB	NT	NT	*T*	NTB	TB	TB	NT	NTB	NTB	TB	NTB
FT	*T*	*T*	FT	*T*	TB	FT	*T*	FT	TB	FTB	TB	FT	FTB	TB	FTB	FTB
NB	*B*	NB	*B*	TB	*B*	NB	NTB	TB	NB	*B*	TB	NTB	NB	NTB	TB	NTB
FB	*B*	*B*	FB	TB	*B*	FB	TB	FTB	*B*	FB	TB	FTB	FB	TB	FTB	FTB
TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB	TB
NFT	*T*	NT	FT	*T*	TB	NFT	NT	FT	NTB	FTB	TB	NFT	A	NTB	FTB	A
NFB	*B*	NB	FB	TB	*B*	NFB	NTB	FTB	NB	FB	TB	A	NFB	NTB	FTB	A
NTB	TB	NTB	TB	TB	TB	NTB	NTB	TB	NTB	TB	TB	NTB	NTB	NTB	TB	NTB
FTB	TB	TB	FTB	TB	TB	FTB	TB	FTB	TB	FTB	TB	FTB	FTB	TB	FTB	FTB
A	TB	NTB	FTB	TB	TB	A	NTB	FTB	NTB	FTB	TB	A	A	NTB	FTB	A

$\widetilde{\wedge}_f$	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
N	N	*N*	N	*T*	N	*N*	NT	*T*	*N*	N	*T*	NT	*N*	NT	*T*	NT
N	*N*	*N*	*N*	NT	*N*	*N*	NT	NT	*N*	*N*	NT	NT	*N*	NT	NT	NT
F	N	*N*	*F*	*T*	N	NF	NT	FT	*N*	*F*	*T*	NFT	NF	NT	FT	NFT
T	*T*	NT	*T*	*T*	*T*	NT	NT	*T*	NT	*T*	*T*	NT	NT	NT	*T*	NT
B	N	*N*	N	*T*	*B*	*N*	NT	*T*	NB	*B*	TB	NT	NB	NTB	TB	NTB
NF	*N*	*N*	NF	NT	*N*	NF	NT	NFT	*N*	NF	NT	NFT	NF	NT	NFT	NFT
NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT	NT
FT	*T*	NT	FT	*T*	*T*	NFT	NT	FT	NT	FT	*T*	NFT	NFT	NT	FT	NFT
NB	*N*	*N*	*N*	NT	NB	*N*	NT	NT	NB	NB	NTB	NT	NB	NTB	NTB	NTB
FB	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
TB	*T*	NT	*T*	*T*	TB	NT	NT	*T*	NTB	TB	TB	NT	NTB	NTB	TB	NTB
NFT	NT	NT	NFT	NT	NT	NFT	NT	NFT	NT	NFT	NT	NFT	NFT	NT	NFT	NFT
NFB	*N*	*N*	NF	NT	NB	NF	NT	NFT	NB	NFB	NTB	NFT	NFB	NTB	A	A
NTB	NT	NT	NT	NT	NTB	NT	NT	NT	NTB	NTB	NTB	NT	NTB	NTB	NTB	NTB
FTB	*T*	NT	FT	*T*	TB	NFT	NT	FT	NTB	FTB	TB	NFT	A	NTB	FTB	A
A	NT	NT	NFT	NT	NTB	NFT	NT	NFT	NTB	A	NTB	NFT	A	NTB	A	A

$\widetilde{\vee}_f$	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
N	N	N	*F*	N	*B*	*F*	N	*F*	*B*	FB	*B*	*F*	FB	*B*	FB	FB
N	N	*N*	*F*	N	*B*	NF	*N*	*F*	NB	FB	*B*	NF	NFB	NB	FB	NFB
F	*F*	*F*	*F*	*F*	FB	*F*	*F*	*F*	FB	FB	FB	*F*	FB	FB	FB	FB
T	N	N	*F*	*T*	*B*	*F*	*T*	FT	*B*	FB	TB	FT	FB	TB	FTB	FTB
B	*B*	*B*	FB	*B*	*B*	FB	*B*	FB	*B*	FB	*B*	FB	FB	*B*	FB	FB
NF	*F*	NF	*F*	*F*	FB	NF	NF	*F*	NFB	FB	FB	NF	NFB	NFB	FB	NFB
NT	N	*N*	*F*	*T*	*B*	NF	NT	FT	NB	FB	TB	NFT	NFB	NTB	FTB	A
FT	*F*	*F*	*F*	FT	FB	*F*	FT	FT	FB	FB	FTB	FT	FB	FTB	FTB	FTB
NB	*B*	NB	FB	*B*	*B*	NFB	NB	FB	NB	FB	*B*	NFB	NFB	NB	FB	NFB
FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB	FB
TB	*B*	*B*	FB	TB	*B*	FB	TB	FTB	*B*	FB	TB	FTB	FB	TB	FTB	FTB
NFT	*F*	NF	*F*	FT	FB	NF	NFT	FT	NFB	FB	FTB	NFT	NFB	A	FTB	A
NFB	FB	NFB	FB	FB	FB	NFB	NFB	FB	NFB	FB	FB	NFB	NFB	NFB	FB	NFB
NTB	*B*	NB	FB	TB	*B*	NFB	NTB	FTB	NB	FB	TB	A	NFB	NTB	FTB	A
FTB	FB	FB	FB	FTB	FB	FB	FTB	FTB	FB	FB	FTB	FTB	FB	FTB	FTB	FTB
A	FB	NFB	FB	FTB	FB	NFB	A	FTB	NFB	FB	FTB	A	NFB	A	FTB	A

Definition 3. Let A be a formula and $x_0, x_1, \ldots, x_k$ the variables occurring in A. Then an *interpretation* $\mathfrak{I}$ of A is an assignment of truth values to the variables.

Definition 4. Given an interpretation $\mathfrak{I}$, we define the *valuation* $\mathrm{val}_{\mathfrak{I}}$ for formulas A to truth values as follows:

1. If A is atomic, then $\mathrm{val}_{\mathfrak{I}}(A)$ simply is the interpretation of A.

2. If $A = \Box(A_1, \ldots, A_n)$, where $A_1, \ldots, A_n$ are formulas, and $\widetilde{\Box}$ is the associated truth function to $\Box$, then $\mathrm{val}_{\mathfrak{I}}(A) = \widetilde{\Box}\big(\mathrm{val}_{\mathfrak{I}}(A_1), \ldots, \mathrm{val}_{\mathfrak{I}}(A_n)\big)$.

Definition 5. An interpretation $\mathfrak{I}$ *satisfies* a formula A, in symbols: $\mathfrak{I} \models A$, iff $\mathrm{val}_{\mathfrak{I}}(A) \in V^+$.

Definition 6. Δ *entails* A iff $\mathfrak{I} \models A$ for every interpretation $\mathfrak{I}$ such that $\mathfrak{I} \models B$ for all $B \in \Delta$. A is a *tautology* iff it is satisfied by every interpretation $\mathfrak{I}$.

3 Sequent calculus for Shramko-Wansing logic

Definition 7 (Syntax of Sequents)**.** A *sequent* Γ is a 16-tuple

$$\Gamma_{\mathbf{N}} \mid \Gamma_{\boldsymbol{N}} \mid \Gamma_{\boldsymbol{F}} \mid \Gamma_{\boldsymbol{T}} \mid \Gamma_{\boldsymbol{B}} \mid \Gamma_{\mathbf{NF}} \mid \Gamma_{\mathbf{NT}} \mid \Gamma_{\mathbf{FT}} \mid \Gamma_{\mathbf{NB}} \mid \Gamma_{\mathbf{FB}} \mid \Gamma_{\mathbf{TB}} \mid \Gamma_{\mathbf{NFT}} \mid \Gamma_{\mathbf{NFB}} \mid \Gamma_{\mathbf{NTB}} \mid \Gamma_{\mathbf{FTB}} \mid \Gamma_{\mathbf{A}}$$

of finite sequences Γ_v of formulas, where $v \in V$. The Γ_v are called the *components* of Γ.

For a sequence of formulas Δ, and $W \subseteq V$, let $[W\colon \Delta]$ denote the sequent whose component Γ_v is Δ if $v \in W$ and empty otherwise. For $[\{w_1, \ldots, w_k\}\colon \Delta]$ we also write $[w_1, \ldots, w_k\colon \Delta]$. If Γ and Γ' are sequents, then Γ, Γ' denotes the component-wise union, i.e., the v-component of Γ, Γ' is Γ_v, Γ'_v.

Definition 8. Let $\mathfrak{I}$ be an interpretation. $\mathfrak{I}$ *satisfies* a sequent Γ iff there is a $v \in V$ so that for some formula $A \in \Gamma_v$, $\mathrm{val}_{\mathfrak{I}}(F) = v$. $\mathfrak{I}$ is called a *model* of Γ, in symbols $\mathfrak{I} \models \Gamma$.

Γ is called *satisfiable* iff there is an interpretation $\mathfrak{I}$ so that $\mathfrak{I} \models \Gamma$ and *valid* iff for every interpretation $\mathfrak{I}$, $\mathfrak{I} \models \Gamma$.

Proposition 9. $\Delta \models A$ *iff the sequent* $[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \Delta], [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]$ *is valid.*

Definition 10. The *sequent calculus* for Shramko-Wansing logic is given by:

1. axiom schemas of the form $[V\colon A]$,

2. weakening rules for every truth value v:

$$\frac{\Gamma}{\Gamma, [v\colon A]}\ \text{W}{:}v$$

3. exchange rules for every truth value v:

$$\frac{\Gamma, [v\colon A, B], \Delta}{\Gamma, [v\colon B, A], \Delta}\ \text{X}{:}v$$

4. contraction rules for every truth value v:

$$\frac{\Gamma, [v\colon A, A]}{\Gamma, [v\colon A]}\ \text{C}{:}v$$

5. cut rules for every two truth values $v \neq w$:

$$\frac{\Gamma, [v\colon A] \quad \Delta, [w\colon A]}{\Gamma, \Delta}\ \text{CUT}{:}vw$$

6. an introduction rule $\Box\colon v$ for every connective $\Box$ and every truth value v, as specified below.

(2)–(5) are called *structural rules*. (6) are called *logical rules*.

The introduction rules for connective $\neg_t$ are given by

$$\frac{\Gamma, [\mathbf{N}\colon A]}{\Gamma, [\mathbf{N}\colon \neg_t A]}\ \neg_t{:}\mathbf{N} \qquad \frac{\Gamma, [\boldsymbol{T}\colon A]}{\Gamma, [\boldsymbol{N}\colon \neg_t A]}\ \neg_t{:}\boldsymbol{N} \qquad \frac{\Gamma, [\boldsymbol{B}\colon A]}{\Gamma, [\boldsymbol{F}\colon \neg_t A]}\ \neg_t{:}\boldsymbol{F}$$

$$\frac{\Gamma, [\boldsymbol{N}\colon A]}{\Gamma, [\boldsymbol{T}\colon \neg_t A]}\ \neg_t{:}\boldsymbol{T} \qquad \frac{\Gamma, [\boldsymbol{F}\colon A]}{\Gamma, [\boldsymbol{B}\colon \neg_t A]}\ \neg_t{:}\boldsymbol{B} \qquad \frac{\Gamma, [\mathbf{TB}\colon A]}{\Gamma, [\mathbf{NF}\colon \neg_t A]}\ \neg_t{:}\mathbf{NF}$$

$$\frac{\Gamma, [\mathbf{NT}\colon A]}{\Gamma, [\mathbf{NT}\colon \neg_t A]}\ \neg_t{:}\mathbf{NT} \qquad \frac{\Gamma, [\mathbf{NB}\colon A]}{\Gamma, [\mathbf{FT}\colon \neg_t A]}\ \neg_t{:}\mathbf{FT} \qquad \frac{\Gamma, [\mathbf{FT}\colon A]}{\Gamma, [\mathbf{NB}\colon \neg_t A]}\ \neg_t{:}\mathbf{NB}$$

$$\frac{\Gamma, [\mathbf{FB}\colon A]}{\Gamma, [\mathbf{FB}\colon \neg_t A]}\ \neg_t{:}\mathbf{FB} \qquad \frac{\Gamma, [\mathbf{NF}\colon A]}{\Gamma, [\mathbf{TB}\colon \neg_t A]}\ \neg_t{:}\mathbf{TB} \qquad \frac{\Gamma, [\mathbf{NTB}\colon A]}{\Gamma, [\mathbf{NFT}\colon \neg_t A]}\ \neg_t{:}\mathbf{NFT}$$

$$\frac{\Gamma, [\mathbf{FTB}\colon A]}{\Gamma, [\mathbf{NFB}\colon \neg_t A]}\ \neg_t{:}\mathbf{NFB} \qquad \frac{\Gamma, [\mathbf{NFT}\colon A]}{\Gamma, [\mathbf{NTB}\colon \neg_t A]}\ \neg_t{:}\mathbf{NTB}$$

$$\frac{\Gamma, [\mathbf{NFB}\colon A]}{\Gamma, [\mathbf{FTB}\colon \neg_t A]}\ \neg_t{:}\mathbf{FTB} \qquad \frac{\Gamma, [\mathbf{A}\colon A]}{\Gamma, [\mathbf{A}\colon \neg_t A]}\ \neg_t{:}\mathbf{A}$$

The introduction rules for connective $\neg_f$ are given by

$$\frac{\Gamma, [\mathbf{N}\colon A]}{\Gamma, [\mathbf{N}\colon \neg_f A]}\ \neg_f{:}\mathbf{N} \qquad \frac{\Gamma, [\boldsymbol{F}\colon A]}{\Gamma, [\boldsymbol{N}\colon \neg_f A]}\ \neg_f{:}\boldsymbol{N} \qquad \frac{\Gamma, [\boldsymbol{N}\colon A]}{\Gamma, [\boldsymbol{F}\colon \neg_f A]}\ \neg_f{:}\boldsymbol{F}$$

$$\frac{\Gamma, [\boldsymbol{B}\colon A]}{\Gamma, [\boldsymbol{T}\colon \neg_f A]}\ \neg_f{:}\boldsymbol{T} \qquad \frac{\Gamma, [\boldsymbol{T}\colon A]}{\Gamma, [\boldsymbol{B}\colon \neg_f A]}\ \neg_f{:}\boldsymbol{B} \qquad \frac{\Gamma, [\mathbf{NF}\colon A]}{\Gamma, [\mathbf{NF}\colon \neg_f A]}\ \neg_f{:}\mathbf{NF}$$

$$\frac{\Gamma, [\mathbf{FB}\colon A]}{\Gamma, [\mathbf{NT}\colon \neg_f A]}\ \neg_f{:}\mathbf{NT} \qquad \frac{\Gamma, [\mathbf{NB}\colon A]}{\Gamma, [\mathbf{FT}\colon \neg_f A]}\ \neg_f{:}\mathbf{FT} \qquad \frac{\Gamma, [\mathbf{FT}\colon A]}{\Gamma, [\mathbf{NB}\colon \neg_f A]}\ \neg_f{:}\mathbf{NB}$$

$$\frac{\Gamma, [\mathbf{NT}\colon A]}{\Gamma, [\mathbf{FB}\colon \neg_f A]}\ \neg_f{:}\mathbf{FB} \qquad \frac{\Gamma, [\mathbf{TB}\colon A]}{\Gamma, [\mathbf{TB}\colon \neg_f A]}\ \neg_f{:}\mathbf{TB} \qquad \frac{\Gamma, [\mathbf{NFB}\colon A]}{\Gamma, [\mathbf{NFT}\colon \neg_f A]}\ \neg_f{:}\mathbf{NFT}$$

$$\frac{\Gamma, [\mathbf{NFT}\colon A]}{\Gamma, [\mathbf{NFB}\colon \neg_f A]}\ \neg_f{:}\mathbf{NFB} \qquad \frac{\Gamma, [\mathbf{FTB}\colon A]}{\Gamma, [\mathbf{NTB}\colon \neg_f A]}\ \neg_f{:}\mathbf{NTB}$$

$$\frac{\Gamma, [\mathbf{NTB}\colon A]}{\Gamma, [\mathbf{FTB}\colon \neg_f A]}\ \neg_f{:}\mathbf{FTB} \qquad \frac{\Gamma, [\mathbf{A}\colon A]}{\Gamma, [\mathbf{A}\colon \neg_f A]}\ \neg_f{:}\mathbf{A}$$

The introduction rules for connective $\wedge_t$ are given by

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{TB}\colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{T}\colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{B}\colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{TB}\colon A]}{\Gamma, [\mathbf{N}\colon A \wedge_t B]}\ \wedge_t{:}\mathbf{N}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NT}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB} \colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \mathbf{NT} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB} \colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NT}, \mathbf{NB}, \mathbf{NTB} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NT}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB} \colon A]}{\Gamma, [\boldsymbol{N} \colon A \wedge_t B]} \wedge_t{:}\boldsymbol{N}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{FT} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FB} \colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon A]}{\Gamma, [\boldsymbol{F} \colon A \wedge_t B]} \wedge_t{:}\boldsymbol{F}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{TB} \colon B] \quad \Gamma, [\boldsymbol{T} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{TB} \colon A]}{\Gamma, [\boldsymbol{T} \colon A \wedge_t B]} \wedge_t{:}\boldsymbol{T}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{TB} \colon B] \quad \Gamma, [\boldsymbol{B} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{TB} \colon A]}{\Gamma, [\boldsymbol{B} \colon A \wedge_t B]} \wedge_t{:}\boldsymbol{B}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB} \colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A} \colon A, B]}{\Gamma, [\mathbf{NF} \colon A \wedge_t B]} \wedge_t{:}\mathbf{NF}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB} \colon B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT} \colon A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NTB} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB} \colon A]}{\Gamma, [\mathbf{NT} \colon A \wedge_t B]} \wedge_t{:}\mathbf{NT}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{FT}, \mathbf{TB}, \mathbf{FTB} \colon B] \quad \Gamma, [\boldsymbol{T}, \mathbf{FT} \colon A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{FTB} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{FT}, \mathbf{TB}, \mathbf{FTB} \colon A]}{\Gamma, [\mathbf{FT} \colon A \wedge_t B]} \wedge_t{:}\mathbf{FT}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB} \colon B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB} \colon A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NTB} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB} \colon A]}{\Gamma, [\mathbf{NB} \colon A \wedge_t B]} \wedge_t{:}\mathbf{NB}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon B] \quad \Gamma, [\boldsymbol{B}, \mathbf{FB} \colon A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{FTB} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon A]}{\Gamma, [\mathbf{FB} \colon A \wedge_t B]} \wedge_t{:}\mathbf{FB}$$

$$\frac{\Gamma, [\mathbf{TB} \colon B] \quad \Gamma, [\mathbf{TB} \colon A]}{\Gamma, [\mathbf{TB} \colon A \wedge_t B]} \wedge_t{:}\mathbf{TB}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT} \colon A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon A]}{\Gamma, [\mathbf{NFT} \colon A \wedge_t B]} \wedge_t{:}\mathbf{NFT}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB} \colon A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon A]}{\Gamma, [\mathbf{NFB} \colon A \wedge_t B]} \wedge_t{:}\mathbf{NFB}$$

$$\frac{\Gamma, [\mathbf{TB}, \mathbf{NTB} \colon B] \quad \Gamma, [\mathbf{NTB} \colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{NTB} \colon A]}{\Gamma, [\mathbf{NTB} \colon A \wedge_t B]} \wedge_t{:}\mathbf{NTB}$$

$$\frac{\Gamma, [\mathbf{TB}, \mathbf{FTB} \colon B] \quad \Gamma, [\mathbf{FTB} \colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{FTB} \colon A]}{\Gamma, [\mathbf{FTB} \colon A \wedge_t B]} \wedge_t{:}\mathbf{FTB}$$

$$\frac{\Gamma, [\mathbf{TB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon B] \quad \Gamma, [\mathbf{NTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{FTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon A]}{\Gamma, [\mathbf{A} \colon A \wedge_t B]} \wedge_t{:}\mathbf{A}$$

The introduction rules for connective $\vee_t$ are given by

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF} \colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF} \colon A]}{\Gamma, [\mathbf{N} \colon A \vee_t B]} \vee_t{:}\mathbf{N}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NF} \colon B] \quad \Gamma, [\boldsymbol{N} \colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF} \colon A]}{\Gamma, [\boldsymbol{N} \colon A \vee_t B]} \vee_t{:}\boldsymbol{N}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{NF}: B] \quad \Gamma, [\boldsymbol{F}: A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}: A]}{\Gamma, [\boldsymbol{F}: A \vee_t B]} \; \vee_t{:}\boldsymbol{F}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT}: B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \mathbf{NT}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{FT}: A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT}: A]}{\Gamma, [\boldsymbol{T}: A \vee_t B]} \; \vee_t{:}\boldsymbol{T}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}: B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FB}: A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}: A]}{\Gamma, [\boldsymbol{B}: A \vee_t B]} \; \vee_t{:}\boldsymbol{B}$$

$$\frac{\Gamma, [\mathbf{NF}: B] \quad \Gamma, [\mathbf{NF}: A]}{\Gamma, [\mathbf{NF}: A \vee_t B]} \; \vee_t{:}\mathbf{NF}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NFT}: B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NT}: A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NFT}: A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NFT}: A]}{\Gamma, [\mathbf{NT}: A \vee_t B]} \; \vee_t{:}\mathbf{NT}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}: B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FT}: A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{NFT}: A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}: A]}{\Gamma, [\mathbf{FT}: A \vee_t B]} \; \vee_t{:}\mathbf{FT}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}: B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NB}: A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NFB}: A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}: A]}{\Gamma, [\mathbf{NB}: A \vee_t B]} \; \vee_t{:}\mathbf{NB}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FB}, \mathbf{NFB}: B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FB}: A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{NFB}: A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FB}, \mathbf{NFB}: A]}{\Gamma, [\mathbf{FB}: A \vee_t B]} \; \vee_t{:}\mathbf{FB}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NT}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB}: A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}: A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}: A, B]}{\Gamma, [\mathbf{TB}: A \vee_t B]} \; \vee_t{:}\mathbf{TB}$$

$$\frac{\Gamma, [\mathbf{NF}, \mathbf{NFT}: B] \quad \Gamma, [\mathbf{NFT}: A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFT}: A]}{\Gamma, [\mathbf{NFT}: A \vee_t B]} \; \vee_t{:}\mathbf{NFT}$$

$$\frac{\Gamma, [\mathbf{NF}, \mathbf{NFB}: B] \quad \Gamma, [\mathbf{NFB}: A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFB}: A]}{\Gamma, [\mathbf{NFB}: A \vee_t B]} \; \vee_t{:}\mathbf{NFB}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A}: B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NT}, \mathbf{NB}, \mathbf{NTB}: A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{A}: A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A}: A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A}: A]}{\Gamma, [\mathbf{NTB}: A \vee_t B]} \; \vee_t{:}\mathbf{NTB}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB}: A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}: A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: A]}{\Gamma, [\mathbf{FTB}: A \vee_t B]} \; \vee_t{:}\mathbf{FTB}$$

$$\frac{\Gamma, [\mathbf{NF}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}: B] \quad \Gamma, [\mathbf{NFT}, \mathbf{A}: A, B] \quad \Gamma, [\mathbf{NFB}, \mathbf{A}: A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}: A]}{\Gamma, [\mathbf{A}: A \vee_t B]} \; \vee_t{:}\mathbf{A}$$

The introduction rules for connective $\wedge_f$ are given by

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FB}: B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{B}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FB}: A]}{\Gamma, [\mathbf{N}: A \wedge_f B]} \; \wedge_f{:}\mathbf{N}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}: B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}: A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}: A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}: A]}{\Gamma, [\boldsymbol{N}: A \wedge_f B]} \; \wedge_f{:}\boldsymbol{N}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{FB}: B] \quad \Gamma, [\boldsymbol{F}: A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FB}: A]}{\Gamma, [\boldsymbol{F}: A \wedge_f B]} \; \wedge_f{:}\boldsymbol{F}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{FT} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{TB} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{FT}, \mathbf{TB}, \mathbf{FTB} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon A]}{\Gamma, [\boldsymbol{T} \colon A \wedge_f B]} \wedge_f{:}\boldsymbol{T}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{FB} \colon B] \quad \Gamma, [\boldsymbol{B} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{FB} \colon A]}{\Gamma, [\boldsymbol{B} \colon A \wedge_f B]} \wedge_f{:}\boldsymbol{B}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FB}, \mathbf{NFB} \colon B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF} \colon A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFB} \colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FB}, \mathbf{NFB} \colon A]}{\Gamma, [\mathbf{NF} \colon A \wedge_f B]} \wedge_f{:}\mathbf{NF}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NT}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB} \colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon A, B]}{\Gamma, [\mathbf{NT} \colon A \wedge_f B]} \wedge_f{:}\mathbf{NT}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB} \colon B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FT} \colon A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{FTB} \colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB} \colon A]}{\Gamma, [\mathbf{FT} \colon A \wedge_f B]} \wedge_f{:}\mathbf{FT}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB} \colon B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB} \colon A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NFB} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB} \colon A]}{\Gamma, [\mathbf{NB} \colon A \wedge_f B]} \wedge_f{:}\mathbf{NB}$$

$$\frac{\Gamma, [\mathbf{FB} \colon B] \quad \Gamma, [\mathbf{FB} \colon A]}{\Gamma, [\mathbf{FB} \colon A \wedge_f B]} \wedge_f{:}\mathbf{FB}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon B] \quad \Gamma, [\boldsymbol{B}, \mathbf{TB} \colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{FTB} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB} \colon A]}{\Gamma, [\mathbf{TB} \colon A \wedge_f B]} \wedge_f{:}\mathbf{TB}$$

$$\frac{\Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A} \colon B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT} \colon A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A} \colon A]}{\Gamma, [\mathbf{NFT} \colon A \wedge_f B]} \wedge_f{:}\mathbf{NFT}$$

$$\frac{\Gamma, [\mathbf{FB}, \mathbf{NFB} \colon B] \quad \Gamma, [\mathbf{NFB} \colon A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{NFB} \colon A]}{\Gamma, [\mathbf{NFB} \colon A \wedge_f B]} \wedge_f{:}\mathbf{NFB}$$

$$\frac{\Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB} \colon A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A} \colon A]}{\Gamma, [\mathbf{NTB} \colon A \wedge_f B]} \wedge_f{:}\mathbf{NTB}$$

$$\frac{\Gamma, [\mathbf{FB}, \mathbf{FTB} \colon B] \quad \Gamma, [\mathbf{FTB} \colon A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{FTB} \colon A]}{\Gamma, [\mathbf{FTB} \colon A \wedge_f B]} \wedge_f{:}\mathbf{FTB}$$

$$\frac{\Gamma, [\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A} \colon B] \quad \Gamma, [\mathbf{NFB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{FTB}, \mathbf{A} \colon A, B] \quad \Gamma, [\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A} \colon A]}{\Gamma, [\mathbf{A} \colon A \wedge_f B]} \wedge_f{:}\mathbf{A}$$

The introduction rules for connective $\vee_f$ are given by

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \mathbf{NT} \colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{T} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \mathbf{NT} \colon A]}{\Gamma, [\mathbf{N} \colon A \vee_f B]} \vee_f{:}\mathbf{N}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NT} \colon B] \quad \Gamma, [\boldsymbol{N} \colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NT} \colon A]}{\Gamma, [\boldsymbol{N} \colon A \vee_f B]} \vee_f{:}\boldsymbol{N}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT} \colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{FT} \colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT} \colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{T}, \mathbf{NF}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT} \colon A]}{\Gamma, [\boldsymbol{F} \colon A \vee_f B]} \vee_f{:}\boldsymbol{F}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{NT} \colon B] \quad \Gamma, [\boldsymbol{T} \colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT} \colon A]}{\Gamma, [\boldsymbol{T} \colon A \vee_f B]} \vee_f{:}\boldsymbol{T}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NT}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB}\colon B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}\colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{TB}\colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB}\colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{NT}, \mathbf{NB}, \mathbf{TB}, \mathbf{NTB}\colon A]}{\Gamma, [\boldsymbol{B}\colon A \vee_f B]} \vee_f{:}\boldsymbol{B}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NFT}\colon B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}\colon A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFT}\colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NFT}\colon A]}{\Gamma, [\mathbf{NF}\colon A \vee_f B]} \vee_f{:}\mathbf{NF}$$

$$\frac{\Gamma, [\mathbf{NT}\colon B] \quad \Gamma, [\mathbf{NT}\colon A]}{\Gamma, [\mathbf{NT}\colon A \vee_f B]} \vee_f{:}\mathbf{NT}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT}\colon B] \quad \Gamma, [\boldsymbol{T}, \mathbf{FT}\colon A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{NFT}\colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{NFT}\colon A]}{\Gamma, [\mathbf{FT}\colon A \vee_f B]} \vee_f{:}\mathbf{FT}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NT}, \mathbf{NB}, \mathbf{NTB}\colon B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NB}\colon A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NTB}\colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NT}, \mathbf{NB}, \mathbf{NTB}\colon A]}{\Gamma, [\mathbf{NB}\colon A \vee_f B]} \vee_f{:}\mathbf{NB}$$

$$\frac{\Gamma, [\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}\colon A, B] \quad \Gamma, [\mathbf{N}, \boldsymbol{F}, \boldsymbol{T}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{TB}, \mathbf{FTB}\colon A, B] \quad \Gamma, [\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A, B] \quad \Gamma, [\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{TB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}\colon A, B]}{\Gamma, [\mathbf{FB}\colon A \vee_f B]} \vee_f{:}\mathbf{FB}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon B] \quad \Gamma, [\boldsymbol{T}, \mathbf{TB}\colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{NTB}\colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]}{\Gamma, [\mathbf{TB}\colon A \vee_f B]} \vee_f{:}\mathbf{TB}$$

$$\frac{\Gamma, [\mathbf{NT}, \mathbf{NFT}\colon B] \quad \Gamma, [\mathbf{NFT}\colon A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NFT}\colon A]}{\Gamma, [\mathbf{NFT}\colon A \vee_f B]} \vee_f{:}\mathbf{NFT}$$

$$\frac{\Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A}\colon B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}\colon A, B] \quad \Gamma, [\mathbf{NF}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon A, B] \quad \Gamma, [\mathbf{NB}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A}\colon A, B] \quad \Gamma, [\boldsymbol{N}, \mathbf{NF}, \mathbf{NT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{NTB}, \mathbf{A}\colon A]}{\Gamma, [\mathbf{NFB}\colon A \vee_f B]} \vee_f{:}\mathbf{NFB}$$

$$\frac{\Gamma, [\mathbf{NT}, \mathbf{NTB}\colon B] \quad \Gamma, [\mathbf{NTB}\colon A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NTB}\colon A]}{\Gamma, [\mathbf{NTB}\colon A \vee_f B]} \vee_f{:}\mathbf{NTB}$$

$$\frac{\Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}\colon B] \quad \Gamma, [\boldsymbol{T}, \mathbf{FT}, \mathbf{TB}, \mathbf{FTB}\colon A, B] \quad \Gamma, [\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon A, B] \quad \Gamma, [\mathbf{TB}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}\colon A, B] \quad \Gamma, [\boldsymbol{T}, \mathbf{NT}, \mathbf{FT}, \mathbf{TB}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{FTB}, \mathbf{A}\colon A]}{\Gamma, [\mathbf{FTB}\colon A \vee_f B]} \vee_f{:}\mathbf{FTB}$$

$$\frac{\Gamma, [\mathbf{NT}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{A}\colon B] \quad \Gamma, [\mathbf{NFT}, \mathbf{A}\colon A, B] \quad \Gamma, [\mathbf{NTB}, \mathbf{A}\colon A, B] \quad \Gamma, [\mathbf{NT}, \mathbf{NFT}, \mathbf{NTB}, \mathbf{A}\colon A]}{\Gamma, [\mathbf{A}\colon A \vee_f B]} \vee_f{:}\mathbf{A}$$

Definition 11. An upward tree of sequents is called a *proof* in the sequent calculus iff every leaf is an axiom, and all other sequents in it are obtained from the ones standing immediately above it by applying one of the rules. The sequent at the root of P is called its *end-sequent*. A sequent Γ is called *provable* iff it is the end-sequent of some proof.

Theorem 12 (Soundness and Completeness)**.** *A sequent is provable if and only if it is valid.*

Proof. See Theorems 3.1 and 3.2 of Baaz et al. [4] or Theorems 3.3.8 and 3.3.10 of Zach [15]. ☐

Corollary 13. *In Shramko-Wansing logic,* $\models A$ *iff* $[\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]$ *has a sequent proof, and* $\Delta \models A$ *iff* $[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \Delta], [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]$ *has a proof.*

Theorem 14 (Cut-elimination)**.** *The cut rule is eliminable in the sequent calculus for Shramko-Wansing logic.*

Proof. See Theorem 4.1 of Baaz et al. [4] or Theorem 3.5.3 of Zach [15]. □

Theorem 15 (Maehara lemma)**.** *The Maehara lemma holds for the sequent calculus for Shramko-Wansing logic.*

Proof. See Theorem 3.8.1 of Zach [15]. □

4 Tableaux for Shramko-Wansing logic

Although the method of Surma [13] and Carnielli [6] for obtaining signed analytic tableaux systems applies to Shramko-Wansing logic, it has a drawback. As Hähnle [7] pointed out, to show that a formula is valid, it is required to provide as many closed tableaux as there are non-designated values. This is usually not desirable; the generalized approach by Hähnle [7] solves this problem. Below we give a tableau system for Shramko-Wansing logic using the sets of signs $V \setminus \{v\}$, i.e., the tableau system exactly dual to that of Carnielli (in the sense of [2]).

Definition 16. A *signed formula* is an expression of the form $v\colon A$ where $v \in V$ and A is a formula.

Definition 17. A *tableau* for a set of signed formulas Δ is a downward rooted tree of signed formulas where each one is either an element of Δ or results from a signed formula in the branch above it by a branch expansion rule. A tableau is *closed* if every branch contains, for some formula A, the signed formulas $v\colon A$ for all $v \in V$, or a signed formula $v\colon A$ with a branch expansion rule that explicitly closes the branch ($\otimes$).

The branch expansion rules for connective $\neg_t$ are given by

$$\frac{\mathbf{N}\colon \neg_t A}{\mathbf{N}\colon A} \quad \frac{\boldsymbol{N}\colon \neg_t A}{\boldsymbol{T}\colon A} \quad \frac{\boldsymbol{F}\colon \neg_t A}{\boldsymbol{B}\colon A} \quad \frac{\boldsymbol{T}\colon \neg_t A}{\boldsymbol{N}\colon A} \quad \frac{\boldsymbol{B}\colon \neg_t A}{\boldsymbol{F}\colon A} \quad \frac{\mathbf{NF}\colon \neg_t A}{\mathbf{TB}\colon A}$$

$$\frac{\mathbf{NT}\colon \neg_t A}{\mathbf{NT}\colon A} \quad \frac{\mathbf{FT}\colon \neg_t A}{\mathbf{NB}\colon A} \quad \frac{\mathbf{NB}\colon \neg_t A}{\mathbf{FT}\colon A} \quad \frac{\mathbf{FB}\colon \neg_t A}{\mathbf{FB}\colon A} \quad \frac{\mathbf{TB}\colon \neg_t A}{\mathbf{NF}\colon A} \quad \frac{\mathbf{NFT}\colon \neg_t A}{\mathbf{NTB}\colon A}$$

$$\frac{\mathbf{NFB}\colon \neg_t A}{\mathbf{FTB}\colon A} \quad \frac{\mathbf{NTB}\colon \neg_t A}{\mathbf{NFT}\colon A} \quad \frac{\mathbf{FTB}\colon \neg_t A}{\mathbf{NFB}\colon A} \quad \frac{\mathbf{A}\colon \neg_t A}{\mathbf{A}\colon A}$$

The branch expansion rules for connective $\neg_f$ are given by

$$\frac{\mathbf{N}\colon \neg_f A}{\mathbf{N}\colon A} \quad \frac{\boldsymbol{N}\colon \neg_f A}{\boldsymbol{F}\colon A} \quad \frac{\boldsymbol{F}\colon \neg_f A}{\boldsymbol{N}\colon A} \quad \frac{\boldsymbol{T}\colon \neg_f A}{\boldsymbol{B}\colon A} \quad \frac{\boldsymbol{B}\colon \neg_f A}{\boldsymbol{T}\colon A} \quad \frac{\mathbf{NF}\colon \neg_f A}{\mathbf{NF}\colon A}$$

$$\frac{\mathbf{NT}\colon \neg_f A}{\mathbf{FB}\colon A} \quad \frac{\mathbf{FT}\colon \neg_f A}{\mathbf{NB}\colon A} \quad \frac{\mathbf{NB}\colon \neg_f A}{\mathbf{FT}\colon A} \quad \frac{\mathbf{FB}\colon \neg_f A}{\mathbf{NT}\colon A} \quad \frac{\mathbf{TB}\colon \neg_f A}{\mathbf{TB}\colon A}$$

$$\frac{\mathbf{NFT}\colon \neg_f A}{\mathbf{NFB}\colon A} \quad \frac{\mathbf{NFB}\colon \neg_f A}{\mathbf{NFT}\colon A} \quad \frac{\mathbf{NTB}\colon \neg_f A}{\mathbf{FTB}\colon A} \quad \frac{\mathbf{FTB}\colon \neg_f A}{\mathbf{NTB}\colon A} \quad \frac{\mathbf{A}\colon \neg_f A}{\mathbf{A}\colon A}$$

The branch expansion rules for connective $\wedge_t$ are given by

$$
\begin{array}{cccc}
\multicolumn{4}{c}{\mathbf{N}\colon A \wedge_t B} \\
\hline
\mathbf{N}\colon B & \mathbf{N}\colon A & \mathbf{N}\colon A & \mathbf{N}\colon A \\
\boldsymbol{T}\colon B & \boldsymbol{T}\colon A & \boldsymbol{B}\colon A & \boldsymbol{T}\colon A \\
\boldsymbol{B}\colon B & \mathbf{N}\colon B & \mathbf{N}\colon B & \boldsymbol{B}\colon A \\
\mathbf{TB}\colon B & \boldsymbol{T}\colon B & \boldsymbol{B}\colon B & \mathbf{TB}\colon A
\end{array}
$$

$$
\begin{array}{ccccc}
\multicolumn{5}{c}{\boldsymbol{N}\colon A \wedge_t B} \\
\hline
\mathbf{N}\colon B & \mathbf{N}\colon A & \mathbf{N}\colon A & \boldsymbol{N}\colon A & \mathbf{N}\colon A \\
\boldsymbol{N}\colon B & \boldsymbol{N}\colon A & \boldsymbol{N}\colon A & \mathbf{NT}\colon A & \boldsymbol{N}\colon A \\
\boldsymbol{T}\colon B & \boldsymbol{T}\colon A & \boldsymbol{B}\colon A & \mathbf{NB}\colon A & \boldsymbol{T}\colon A \\
\boldsymbol{B}\colon B & \mathbf{NT}\colon A & \mathbf{NB}\colon A & \mathbf{NTB}\colon A & \boldsymbol{B}\colon A \\
\mathbf{NT}\colon B & \mathbf{N}\colon B & \mathbf{N}\colon B & \boldsymbol{N}\colon B & \mathbf{NT}\colon A \\
\mathbf{NB}\colon B & \boldsymbol{N}\colon B & \boldsymbol{N}\colon B & \mathbf{NT}\colon B & \mathbf{NB}\colon A \\
\mathbf{TB}\colon B & \boldsymbol{T}\colon B & \boldsymbol{B}\colon B & \mathbf{NB}\colon B & \mathbf{TB}\colon A \\
\mathbf{NTB}\colon B & \mathbf{NT}\colon B & \mathbf{NB}\colon B & \mathbf{NTB}\colon B & \mathbf{NTB}\colon A
\end{array}
$$

$$
\begin{array}{ccccc}
\multicolumn{5}{c}{\boldsymbol{F}\colon A \wedge_t B} \\
\hline
\mathbf{N}\colon B & \mathbf{N}\colon A & \mathbf{N}\colon A & \boldsymbol{F}\colon A & \mathbf{N}\colon A \\
\boldsymbol{F}\colon B & \boldsymbol{F}\colon A & \boldsymbol{F}\colon A & \mathbf{FT}\colon A & \boldsymbol{F}\colon A \\
\boldsymbol{T}\colon B & \boldsymbol{T}\colon A & \boldsymbol{B}\colon A & \mathbf{FB}\colon A & \boldsymbol{T}\colon A \\
\boldsymbol{B}\colon B & \mathbf{FT}\colon A & \mathbf{FB}\colon A & \mathbf{FTB}\colon A & \boldsymbol{B}\colon A \\
\mathbf{FT}\colon B & \mathbf{N}\colon B & \mathbf{N}\colon B & \boldsymbol{F}\colon B & \mathbf{FT}\colon A \\
\mathbf{FB}\colon B & \boldsymbol{F}\colon B & \boldsymbol{F}\colon B & \mathbf{FT}\colon B & \mathbf{FB}\colon A \\
\mathbf{TB}\colon B & \boldsymbol{T}\colon B & \boldsymbol{B}\colon B & \mathbf{FB}\colon B & \mathbf{TB}\colon A \\
\mathbf{FTB}\colon B & \mathbf{FT}\colon B & \mathbf{FB}\colon B & \mathbf{FTB}\colon B & \mathbf{FTB}\colon A
\end{array}
$$

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\boldsymbol{T}\colon A \wedge_t B} \\
\hline
\boldsymbol{T}\colon B & \boldsymbol{T}\colon A & \boldsymbol{T}\colon A \\
\mathbf{TB}\colon B & \boldsymbol{T}\colon B & \mathbf{TB}\colon A
\end{array}
\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{\boldsymbol{B}\colon A \wedge_t B} \\
\hline
\boldsymbol{B}\colon B & \boldsymbol{B}\colon A & \boldsymbol{B}\colon A \\
\mathbf{TB}\colon B & \boldsymbol{B}\colon B & \mathbf{TB}\colon A
\end{array}
$$

$\mathbf{NF}\colon A \wedge_t B$

$\mathbf{N}\colon A$	$\mathbf{N}\colon A$	$\boldsymbol{N}\colon A$	$\boldsymbol{F}\colon A$
$\boldsymbol{N}\colon A$	$\boldsymbol{N}\colon A$	$\mathbf{NF}\colon A$	$\mathbf{NF}\colon A$
$\boldsymbol{F}\colon A$	$\boldsymbol{F}\colon A$	$\mathbf{NT}\colon A$	$\mathbf{FT}\colon A$
$\boldsymbol{T}\colon A$	$\boldsymbol{B}\colon A$	$\mathbf{NB}\colon A$	$\mathbf{FB}\colon A$
$\mathbf{NF}\colon A$	$\mathbf{NF}\colon A$	$\mathbf{NFT}\colon A$	$\mathbf{NFT}\colon A$
$\mathbf{NT}\colon A$	$\mathbf{NB}\colon A$	$\mathbf{NFB}\colon A$	$\mathbf{NFB}\colon A$
$\mathbf{FT}\colon A$	$\mathbf{FB}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{FTB}\colon A$
$\mathbf{NFT}\colon A$	$\mathbf{NFB}\colon A$	$\mathbf{A}\colon A$	$\mathbf{A}\colon A$
$\mathbf{N}\colon B$	$\mathbf{N}\colon B$	$\boldsymbol{N}\colon B$	$\boldsymbol{F}\colon B$
$\boldsymbol{N}\colon B$	$\boldsymbol{N}\colon B$	$\mathbf{NF}\colon B$	$\mathbf{NF}\colon B$
$\boldsymbol{F}\colon B$	$\boldsymbol{F}\colon B$	$\mathbf{NT}\colon B$	$\mathbf{FT}\colon B$
$\boldsymbol{T}\colon B$	$\boldsymbol{B}\colon B$	$\mathbf{NB}\colon B$	$\mathbf{FB}\colon B$
$\mathbf{NF}\colon B$	$\mathbf{NF}\colon B$	$\mathbf{NFT}\colon B$	$\mathbf{NFT}\colon B$
$\mathbf{NT}\colon B$	$\mathbf{NB}\colon B$	$\mathbf{NFB}\colon B$	$\mathbf{NFB}\colon B$
$\mathbf{FT}\colon B$	$\mathbf{FB}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{FTB}\colon B$
$\mathbf{NFT}\colon B$	$\mathbf{NFB}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon B$

$\mathbf{NT}\colon A \wedge_t B$

$\boldsymbol{T}\colon B$	$\boldsymbol{T}\colon A$	$\mathbf{NT}\colon A$	$\boldsymbol{T}\colon A$
$\mathbf{NT}\colon B$	$\mathbf{NT}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{TB}\colon B$	$\boldsymbol{T}\colon B$	$\mathbf{NT}\colon B$	$\mathbf{TB}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{NT}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$

$\mathbf{FT}\colon A \wedge_t B$

$\boldsymbol{T}\colon B$	$\boldsymbol{T}\colon A$	$\mathbf{FT}\colon A$	$\boldsymbol{T}\colon A$
$\mathbf{FT}\colon B$	$\mathbf{FT}\colon A$	$\mathbf{FTB}\colon A$	$\mathbf{FT}\colon A$
$\mathbf{TB}\colon B$	$\boldsymbol{T}\colon B$	$\mathbf{FT}\colon B$	$\mathbf{TB}\colon A$
$\mathbf{FTB}\colon B$	$\mathbf{FT}\colon B$	$\mathbf{FTB}\colon B$	$\mathbf{FTB}\colon A$

$\mathbf{NB}\colon A \wedge_t B$

$\boldsymbol{B}\colon B$	$\boldsymbol{B}\colon A$	$\mathbf{NB}\colon A$	$\boldsymbol{B}\colon A$
$\mathbf{NB}\colon B$	$\mathbf{NB}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{NB}\colon A$
$\mathbf{TB}\colon B$	$\boldsymbol{B}\colon B$	$\mathbf{NB}\colon B$	$\mathbf{TB}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{NB}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$

$\mathbf{FB}\colon A \wedge_t B$

$\boldsymbol{B}\colon B$	$\boldsymbol{B}\colon A$	$\mathbf{FB}\colon A$	$\boldsymbol{B}\colon A$
$\mathbf{FB}\colon B$	$\mathbf{FB}\colon A$	$\mathbf{FTB}\colon A$	$\mathbf{FB}\colon A$
$\mathbf{TB}\colon B$	$\boldsymbol{B}\colon B$	$\mathbf{FB}\colon B$	$\mathbf{TB}\colon A$
$\mathbf{FTB}\colon B$	$\mathbf{FB}\colon B$	$\mathbf{FTB}\colon B$	$\mathbf{FTB}\colon A$

$\mathbf{TB}\colon A \wedge_t B$

$\mathbf{TB}\colon B$	$\mathbf{TB}\colon A$

NFT: $A \wedge_t B$

T: B	***T***: A	**NT**: A	**FT**: A	***T***: A
NT: B	**NT**: A	**NFT**: A	**NFT**: A	**NT**: A
FT: B	**FT**: A	**NTB**: A	**FTB**: A	**FT**: A
TB: B	**NFT**: A	**A**: A	**A**: A	**TB**: A
NFT: B	***T***: B	**NT**: B	**FT**: B	**NFT**: A
NTB: B	**NT**: B	**NFT**: B	**NFT**: B	**NTB**: A
FTB: B	**FT**: B	**NTB**: B	**FTB**: B	**FTB**: A
A: B	**NFT**: B	**A**: B	**A**: B	**A**: A

NFB: $A \wedge_t B$

B: B	***B***: A	**NB**: A	**FB**: A	***B***: A
NB: B	**NB**: A	**NFB**: A	**NFB**: A	**NB**: A
FB: B	**FB**: A	**NTB**: A	**FTB**: A	**FB**: A
TB: B	**NFB**: A	**A**: A	**A**: A	**TB**: A
NFB: B	***B***: B	**NB**: B	**FB**: B	**NFB**: A
NTB: B	**NB**: B	**NFB**: B	**NFB**: B	**NTB**: A
FTB: B	**FB**: B	**NTB**: B	**FTB**: B	**FTB**: A
A: B	**NFB**: B	**A**: B	**A**: B	**A**: A

NTB: $A \wedge_t B$

TB: B	**NTB**: A	**TB**: A
NTB: B	**NTB**: B	**NTB**: A

FTB: $A \wedge_t B$

TB: B	**FTB**: A	**TB**: A
FTB: B	**FTB**: B	**FTB**: A

A: $A \wedge_t B$

TB: B	**NTB**: A	**FTB**: A	**TB**: A
NTB: B	**A**: A	**A**: A	**NTB**: A
FTB: B	**NTB**: B	**FTB**: B	**FTB**: A
A: B	**A**: B	**A**: B	**A**: A

The branch expansion rules for connective $\vee_t$ are given by

N: $A \vee_t B$

N: B	**N**: A	**N**: A	**N**: A
N: B	***N***: A	***F***: A	***N***: A
F: B	**N**: B	**N**: B	***F***: A
NF: B	***N***: B	***F***: B	**NF**: A

N: $A \vee_t B$

N: B	***N***: A	***N***: A
NF: B	***N***: B	**NF**: A

F: $A \vee_t B$

F: B	***F***: A	***F***: A
NF: B	***F***: B	**NF**: A

T: $A \vee_t B$

N: B	**N**: A	**N**: A	***T***: A	**N**: A
N: B	***N***: A	***F***: A	**NT**: A	***N***: A
F: B	***T***: A	***T***: A	**FT**: A	***F***: A
T: B	**NT**: A	**FT**: A	**NFT**: A	***T***: A
NF: B	**N**: B	**N**: B	***T***: B	**NF**: A
NT: B	***N***: B	***F***: B	**NT**: B	**NT**: A
FT: B	***T***: B	***T***: B	**FT**: B	**FT**: A
NFT: B	**NT**: B	**FT**: B	**NFT**: B	**NFT**: A

B: $A \vee_t B$

N: B	**N**: A	**N**: A	***B***: A	**N**: A
N: B	***N***: A	***F***: A	**NB**: A	***N***: A
F: B	***B***: A	***B***: A	**FB**: A	***F***: A
B: B	**NB**: A	**FB**: A	**NFB**: A	***B***: A
NF: B	**N**: B	**N**: B	***B***: B	**NF**: A
NB: B	***N***: B	***F***: B	**NB**: B	**NB**: A
FB: B	***B***: B	***B***: B	**FB**: B	**FB**: A
NFB: B	**NB**: B	**FB**: B	**NFB**: B	**NFB**: A

NF: $A \vee_t B$

NF: B	**NF**: A

NT: $A \vee_t B$

N: B	***N***: A	**NT**: A	***N***: A
NF: B	**NT**: A	**NFT**: A	**NF**: A
NT: B	***N***: B	**NT**: B	**NT**: A
NFT: B	**NT**: B	**NFT**: B	**NFT**: A

FT: $A \vee_t B$

F: B	***F***: A	**FT**: A	***F***: A
NF: B	**FT**: A	**NFT**: A	**NF**: A
FT: B	***F***: B	**FT**: B	**FT**: A
NFT: B	**FT**: B	**NFT**: B	**NFT**: A

NB: $A \vee_t B$

N: B	***N***: A	**NB**: A	***N***: A
NF: B	**NB**: A	**NFB**: A	**NF**: A
NB: B	***N***: B	**NB**: B	**NB**: A
NFB: B	**NB**: B	**NFB**: B	**NFB**: A

FB: $A \vee_t B$

F: B	***F***: A	**FB**: A	***F***: A
NF: B	**FB**: A	**NFB**: A	**NF**: A
FB: B	***F***: B	**FB**: B	**FB**: A
NFB: B	**FB**: B	**NFB**: B	**NFB**: A

$\mathbf{TB}: A \vee_t B$			
$\mathbf{N}: A$	$\mathbf{N}: A$	$\boldsymbol{T}: A$	$\boldsymbol{B}: A$
$\boldsymbol{N}: A$	$\boldsymbol{F}: A$	$\mathbf{NT}: A$	$\mathbf{NB}: A$
$\boldsymbol{T}: A$	$\boldsymbol{T}: A$	$\mathbf{FT}: A$	$\mathbf{FB}: A$
$\boldsymbol{B}: A$	$\boldsymbol{B}: A$	$\mathbf{TB}: A$	$\mathbf{TB}: A$
$\mathbf{NT}: A$	$\mathbf{FT}: A$	$\mathbf{NFT}: A$	$\mathbf{NFB}: A$
$\mathbf{NB}: A$	$\mathbf{FB}: A$	$\mathbf{NTB}: A$	$\mathbf{NTB}: A$
$\mathbf{TB}: A$	$\mathbf{TB}: A$	$\mathbf{FTB}: A$	$\mathbf{FTB}: A$
$\mathbf{NTB}: A$	$\mathbf{FTB}: A$	$\mathbf{A}: A$	$\mathbf{A}: A$
$\mathbf{N}: B$	$\mathbf{N}: B$	$\boldsymbol{T}: B$	$\boldsymbol{B}: B$
$\boldsymbol{N}: B$	$\boldsymbol{F}: B$	$\mathbf{NT}: B$	$\mathbf{NB}: B$
$\boldsymbol{T}: B$	$\boldsymbol{T}: B$	$\mathbf{FT}: B$	$\mathbf{FB}: B$
$\boldsymbol{B}: B$	$\boldsymbol{B}: B$	$\mathbf{TB}: B$	$\mathbf{TB}: B$
$\mathbf{NT}: B$	$\mathbf{FT}: B$	$\mathbf{NFT}: B$	$\mathbf{NFB}: B$
$\mathbf{NB}: B$	$\mathbf{FB}: B$	$\mathbf{NTB}: B$	$\mathbf{NTB}: B$
$\mathbf{TB}: B$	$\mathbf{TB}: B$	$\mathbf{FTB}: B$	$\mathbf{FTB}: B$
$\mathbf{NTB}: B$	$\mathbf{FTB}: B$	$\mathbf{A}: B$	$\mathbf{A}: B$

$\mathbf{NFT}: A \vee_t B$		
$\mathbf{NF}: B$	$\mathbf{NFT}: A$	$\mathbf{NF}: A$
$\mathbf{NFT}: B$	$\mathbf{NFT}: B$	$\mathbf{NFT}: A$

$\mathbf{NFB}: A \vee_t B$		
$\mathbf{NF}: B$	$\mathbf{NFB}: A$	$\mathbf{NF}: A$
$\mathbf{NFB}: B$	$\mathbf{NFB}: B$	$\mathbf{NFB}: A$

$\mathbf{NTB}: A \vee_t B$				
$\boldsymbol{N}: B$	$\boldsymbol{N}: A$	$\mathbf{NT}: A$	$\mathbf{NB}: A$	$\boldsymbol{N}: A$
$\mathbf{NF}: B$	$\mathbf{NT}: A$	$\mathbf{NFT}: A$	$\mathbf{NFB}: A$	$\mathbf{NF}: A$
$\mathbf{NT}: B$	$\mathbf{NB}: A$	$\mathbf{NTB}: A$	$\mathbf{NTB}: A$	$\mathbf{NT}: A$
$\mathbf{NB}: B$	$\mathbf{NTB}: A$	$\mathbf{A}: A$	$\mathbf{A}: A$	$\mathbf{NB}: A$
$\mathbf{NFT}: B$	$\boldsymbol{N}: B$	$\mathbf{NT}: B$	$\mathbf{NB}: B$	$\mathbf{NFT}: A$
$\mathbf{NFB}: B$	$\mathbf{NT}: B$	$\mathbf{NFT}: B$	$\mathbf{NFB}: B$	$\mathbf{NFB}: A$
$\mathbf{NTB}: B$	$\mathbf{NB}: B$	$\mathbf{NTB}: B$	$\mathbf{NTB}: B$	$\mathbf{NTB}: A$
$\mathbf{A}: B$	$\mathbf{NTB}: B$	$\mathbf{A}: B$	$\mathbf{A}: B$	$\mathbf{A}: A$

$\mathbf{FTB}: A \vee_t B$				
$\boldsymbol{F}: B$	$\boldsymbol{F}: A$	$\mathbf{FT}: A$	$\mathbf{FB}: A$	$\boldsymbol{F}: A$
$\mathbf{NF}: B$	$\mathbf{FT}: A$	$\mathbf{NFT}: A$	$\mathbf{NFB}: A$	$\mathbf{NF}: A$
$\mathbf{FT}: B$	$\mathbf{FB}: A$	$\mathbf{FTB}: A$	$\mathbf{FTB}: A$	$\mathbf{FT}: A$
$\mathbf{FB}: B$	$\mathbf{FTB}: A$	$\mathbf{A}: A$	$\mathbf{A}: A$	$\mathbf{FB}: A$
$\mathbf{NFT}: B$	$\boldsymbol{F}: B$	$\mathbf{FT}: B$	$\mathbf{FB}: B$	$\mathbf{NFT}: A$
$\mathbf{NFB}: B$	$\mathbf{FT}: B$	$\mathbf{NFT}: B$	$\mathbf{NFB}: B$	$\mathbf{NFB}: A$
$\mathbf{FTB}: B$	$\mathbf{FB}: B$	$\mathbf{FTB}: B$	$\mathbf{FTB}: B$	$\mathbf{FTB}: A$
$\mathbf{A}: B$	$\mathbf{FTB}: B$	$\mathbf{A}: B$	$\mathbf{A}: B$	$\mathbf{A}: A$

A: $A \vee_t B$			
NF: B	**NFT**: A	**NFB**: A	**NF**: A
NFT: B	**A**: A	**A**: A	**NFT**: A
NFB: B	**NFT**: B	**NFB**: B	**NFB**: A
A: B	**A**: B	**A**: B	**A**: A

The branch expansion rules for connective $\wedge_f$ are given by

N: $A \wedge_f B$			
N: B	**N**: A	**N**: A	**N**: A
F: B	***F***: A	***B***: A	***F***: A
B: B	**N**: B	**N**: B	***B***: A
FB: B	***F***: B	***B***: B	**FB**: A

N: $A \wedge_f B$				
N: B	**N**: A	**N**: A	***N***: A	**N**: A
N: B	***N***: A	***N***: A	**NF**: A	***N***: A
F: B	***F***: A	***B***: A	**NB**: A	***F***: A
B: B	**NF**: A	**NB**: A	**NFB**: A	***B***: A
NF: B	**N**: B	**N**: B	***N***: B	**NF**: A
NB: B	***N***: B	***N***: B	**NF**: B	**NB**: A
FB: B	***F***: B	***B***: B	**NB**: B	**FB**: A
NFB: B	**NF**: B	**NB**: B	**NFB**: B	**NFB**: A

F: $A \wedge_f B$		
F: B	***F***: A	***F***: A
FB: B	***F***: B	**FB**: A

T: $A \wedge_f B$				
N: B	**N**: A	**N**: A	***T***: A	**N**: A
F: B	***F***: A	***T***: A	**FT**: A	***F***: A
T: B	***T***: A	***B***: A	**TB**: A	***T***: A
B: B	**FT**: A	**TB**: A	**FTB**: A	***B***: A
FT: B	**N**: B	**N**: B	***T***: B	**FT**: A
FB: B	***F***: B	***T***: B	**FT**: B	**FB**: A
TB: B	***T***: B	***B***: B	**TB**: B	**TB**: A
FTB: B	**FT**: B	**TB**: B	**FTB**: B	**FTB**: A

B: $A \wedge_f B$		
B: B	***B***: A	***B***: A
FB: B	***B***: B	**FB**: A

NF: $A \wedge_f B$			
F: B	***F***: A	**NF**: A	***F***: A
NF: B	**NF**: A	**NFB**: A	**NF**: A
FB: B	***F***: B	**NF**: B	**FB**: A
NFB: B	**NF**: B	**NFB**: B	**NFB**: A

$\mathbf{NT}: A \wedge_f B$			
$\mathbf{N}: A$	$\mathbf{N}: A$	$\boldsymbol{N}: A$	$\boldsymbol{T}: A$
$\boldsymbol{N}: A$	$\boldsymbol{N}: A$	$\mathbf{NF}: A$	$\mathbf{NT}: A$
$\boldsymbol{F}: A$	$\boldsymbol{T}: A$	$\mathbf{NT}: A$	$\mathbf{FT}: A$
$\boldsymbol{T}: A$	$\boldsymbol{B}: A$	$\mathbf{NB}: A$	$\mathbf{TB}: A$
$\mathbf{NF}: A$	$\mathbf{NT}: A$	$\mathbf{NFT}: A$	$\mathbf{NFT}: A$
$\mathbf{NT}: A$	$\mathbf{NB}: A$	$\mathbf{NFB}: A$	$\mathbf{NTB}: A$
$\mathbf{FT}: A$	$\mathbf{TB}: A$	$\mathbf{NTB}: A$	$\mathbf{FTB}: A$
$\mathbf{NFT}: A$	$\mathbf{NTB}: A$	$\mathbf{A}: A$	$\mathbf{A}: A$
$\mathbf{N}: B$	$\mathbf{N}: B$	$\boldsymbol{N}: B$	$\boldsymbol{T}: B$
$\boldsymbol{N}: B$	$\boldsymbol{N}: B$	$\mathbf{NF}: B$	$\mathbf{NT}: B$
$\boldsymbol{F}: B$	$\boldsymbol{T}: B$	$\mathbf{NT}: B$	$\mathbf{FT}: B$
$\boldsymbol{T}: B$	$\boldsymbol{B}: B$	$\mathbf{NB}: B$	$\mathbf{TB}: B$
$\mathbf{NF}: B$	$\mathbf{NT}: B$	$\mathbf{NFT}: B$	$\mathbf{NFT}: B$
$\mathbf{NT}: B$	$\mathbf{NB}: B$	$\mathbf{NFB}: B$	$\mathbf{NTB}: B$
$\mathbf{FT}: B$	$\mathbf{TB}: B$	$\mathbf{NTB}: B$	$\mathbf{FTB}: B$
$\mathbf{NFT}: B$	$\mathbf{NTB}: B$	$\mathbf{A}: B$	$\mathbf{A}: B$

$\mathbf{FT}: A \wedge_f B$			
$\boldsymbol{F}: B$	$\boldsymbol{F}: A$	$\mathbf{FT}: A$	$\boldsymbol{F}: A$
$\mathbf{FT}: B$	$\mathbf{FT}: A$	$\mathbf{FTB}: A$	$\mathbf{FT}: A$
$\mathbf{FB}: B$	$\boldsymbol{F}: B$	$\mathbf{FT}: B$	$\mathbf{FB}: A$
$\mathbf{FTB}: B$	$\mathbf{FT}: B$	$\mathbf{FTB}: B$	$\mathbf{FTB}: A$

$\mathbf{NB}: A \wedge_f B$			
$\boldsymbol{B}: B$	$\boldsymbol{B}: A$	$\mathbf{NB}: A$	$\boldsymbol{B}: A$
$\mathbf{NB}: B$	$\mathbf{NB}: A$	$\mathbf{NFB}: A$	$\mathbf{NB}: A$
$\mathbf{FB}: B$	$\boldsymbol{B}: B$	$\mathbf{NB}: B$	$\mathbf{FB}: A$
$\mathbf{NFB}: B$	$\mathbf{NB}: B$	$\mathbf{NFB}: B$	$\mathbf{NFB}: A$

$\mathbf{FB}: A \wedge_f B$	
$\mathbf{FB}: B$	$\mathbf{FB}: A$

$\mathbf{TB}: A \wedge_f B$			
$\boldsymbol{B}: B$	$\boldsymbol{B}: A$	$\mathbf{TB}: A$	$\boldsymbol{B}: A$
$\mathbf{FB}: B$	$\mathbf{TB}: A$	$\mathbf{FTB}: A$	$\mathbf{FB}: A$
$\mathbf{TB}: B$	$\boldsymbol{B}: B$	$\mathbf{TB}: B$	$\mathbf{TB}: A$
$\mathbf{FTB}: B$	$\mathbf{TB}: B$	$\mathbf{FTB}: B$	$\mathbf{FTB}: A$

$\mathbf{NFT}: A \wedge_f B$				
$\boldsymbol{F}: B$	$\boldsymbol{F}: A$	$\mathbf{NF}: A$	$\mathbf{FT}: A$	$\boldsymbol{F}: A$
$\mathbf{NF}: B$	$\mathbf{NF}: A$	$\mathbf{NFT}: A$	$\mathbf{NFT}: A$	$\mathbf{NF}: A$
$\mathbf{FT}: B$	$\mathbf{FT}: A$	$\mathbf{NFB}: A$	$\mathbf{FTB}: A$	$\mathbf{FT}: A$
$\mathbf{FB}: B$	$\mathbf{NFT}: A$	$\mathbf{A}: A$	$\mathbf{A}: A$	$\mathbf{FB}: A$
$\mathbf{NFT}: B$	$\boldsymbol{F}: B$	$\mathbf{NF}: B$	$\mathbf{FT}: B$	$\mathbf{NFT}: A$
$\mathbf{NFB}: B$	$\mathbf{NF}: B$	$\mathbf{NFT}: B$	$\mathbf{NFT}: B$	$\mathbf{NFB}: A$
$\mathbf{FTB}: B$	$\mathbf{FT}: B$	$\mathbf{NFB}: B$	$\mathbf{FTB}: B$	$\mathbf{FTB}: A$
$\mathbf{A}: B$	$\mathbf{NFT}: B$	$\mathbf{A}: B$	$\mathbf{A}: B$	$\mathbf{A}: A$

$$\dfrac{\mathbf{NFB}\colon A \wedge_f B}{\begin{array}{rrr} \mathbf{FB}\colon B & \mathbf{NFB}\colon A & \mathbf{FB}\colon A \\ \mathbf{NFB}\colon B & \mathbf{NFB}\colon B & \mathbf{NFB}\colon A \end{array}}$$

$$\dfrac{\mathbf{NTB}\colon A \wedge_f B}{\begin{array}{rrrrr} \boldsymbol{B}\colon B & \boldsymbol{B}\colon A & \mathbf{NB}\colon A & \mathbf{TB}\colon A & \boldsymbol{B}\colon A \\ \mathbf{NB}\colon B & \mathbf{NB}\colon A & \mathbf{NFB}\colon A & \mathbf{NTB}\colon A & \mathbf{NB}\colon A \\ \mathbf{FB}\colon B & \mathbf{TB}\colon A & \mathbf{NTB}\colon A & \mathbf{FTB}\colon A & \mathbf{FB}\colon A \\ \mathbf{TB}\colon B & \mathbf{NTB}\colon A & \mathbf{A}\colon A & \mathbf{A}\colon A & \mathbf{TB}\colon A \\ \mathbf{NFB}\colon B & \boldsymbol{B}\colon B & \mathbf{NB}\colon B & \mathbf{TB}\colon B & \mathbf{NFB}\colon A \\ \mathbf{NTB}\colon B & \mathbf{NB}\colon B & \mathbf{NFB}\colon B & \mathbf{NTB}\colon B & \mathbf{NTB}\colon A \\ \mathbf{FTB}\colon B & \mathbf{TB}\colon B & \mathbf{NTB}\colon B & \mathbf{FTB}\colon B & \mathbf{FTB}\colon A \\ \mathbf{A}\colon B & \mathbf{NTB}\colon B & \mathbf{A}\colon B & \mathbf{A}\colon B & \mathbf{A}\colon A \end{array}}$$

$$\dfrac{\mathbf{FTB}\colon A \wedge_f B}{\begin{array}{rrr} \mathbf{FB}\colon B & \mathbf{FTB}\colon A & \mathbf{FB}\colon A \\ \mathbf{FTB}\colon B & \mathbf{FTB}\colon B & \mathbf{FTB}\colon A \end{array}}$$

$$\dfrac{\mathbf{A}\colon A \wedge_f B}{\begin{array}{rrrr} \mathbf{FB}\colon B & \mathbf{NFB}\colon A & \mathbf{FTB}\colon A & \mathbf{FB}\colon A \\ \mathbf{NFB}\colon B & \mathbf{A}\colon A & \mathbf{A}\colon A & \mathbf{NFB}\colon A \\ \mathbf{FTB}\colon B & \mathbf{NFB}\colon B & \mathbf{FTB}\colon B & \mathbf{FTB}\colon A \\ \mathbf{A}\colon B & \mathbf{A}\colon B & \mathbf{A}\colon B & \mathbf{A}\colon A \end{array}}$$

The branch expansion rules for connective $\vee_f$ are given by

$$\dfrac{\mathbf{N}\colon A \vee_f B}{\begin{array}{rrrr} \mathbf{N}\colon B & \mathbf{N}\colon A & \mathbf{N}\colon A & \mathbf{N}\colon A \\ \boldsymbol{N}\colon B & \boldsymbol{N}\colon A & \boldsymbol{T}\colon A & \boldsymbol{N}\colon A \\ \boldsymbol{T}\colon B & \mathbf{N}\colon B & \mathbf{N}\colon B & \boldsymbol{T}\colon A \\ \mathbf{NT}\colon B & \boldsymbol{N}\colon B & \boldsymbol{T}\colon B & \mathbf{NT}\colon A \end{array}} \qquad \dfrac{\boldsymbol{N}\colon A \vee_f B}{\begin{array}{rrr} \boldsymbol{N}\colon B & \boldsymbol{N}\colon A & \boldsymbol{N}\colon A \\ \mathbf{NT}\colon B & \boldsymbol{N}\colon B & \mathbf{NT}\colon A \end{array}}$$

$$\dfrac{\boldsymbol{F}\colon A \vee_f B}{\begin{array}{rrrrr} \mathbf{N}\colon B & \mathbf{N}\colon A & \mathbf{N}\colon A & \boldsymbol{F}\colon A & \mathbf{N}\colon A \\ \boldsymbol{N}\colon B & \boldsymbol{N}\colon A & \boldsymbol{F}\colon A & \mathbf{NF}\colon A & \boldsymbol{N}\colon A \\ \boldsymbol{F}\colon B & \boldsymbol{F}\colon A & \boldsymbol{T}\colon A & \mathbf{FT}\colon A & \boldsymbol{F}\colon A \\ \boldsymbol{T}\colon B & \mathbf{NF}\colon A & \mathbf{FT}\colon A & \mathbf{NFT}\colon A & \boldsymbol{T}\colon A \\ \mathbf{NF}\colon B & \mathbf{N}\colon B & \mathbf{N}\colon B & \boldsymbol{F}\colon B & \mathbf{NF}\colon A \\ \mathbf{NT}\colon B & \boldsymbol{N}\colon B & \boldsymbol{F}\colon B & \mathbf{NF}\colon B & \mathbf{NT}\colon A \\ \mathbf{FT}\colon B & \boldsymbol{F}\colon B & \boldsymbol{T}\colon B & \mathbf{FT}\colon B & \mathbf{FT}\colon A \\ \mathbf{NFT}\colon B & \mathbf{NF}\colon B & \mathbf{FT}\colon B & \mathbf{NFT}\colon B & \mathbf{NFT}\colon A \end{array}}$$

$$\dfrac{\boldsymbol{T}\colon A \vee_f B}{\begin{array}{rrr} \boldsymbol{T}\colon B & \boldsymbol{T}\colon A & \boldsymbol{T}\colon A \\ \mathbf{NT}\colon B & \boldsymbol{T}\colon B & \mathbf{NT}\colon A \end{array}}$$

B: $A \vee_f B$				
N: B	**N**: A	**N**: A	***B***: A	**N**: A
N: B	***N***: A	***T***: A	**NB**: A	***N***: A
T: B	***B***: A	***B***: A	**TB**: A	***T***: A
B: B	**NB**: A	**TB**: A	**NTB**: A	***B***: A
NT: B	**N**: B	**N**: B	***B***: B	**NT**: A
NB: B	***N***: B	***T***: B	**NB**: B	**NB**: A
TB: B	***B***: B	***B***: B	**TB**: B	**TB**: A
NTB: B	**NB**: B	**TB**: B	**NTB**: B	**NTB**: A

NF: $A \vee_f B$			
N: B	***N***: A	**NF**: A	***N***: A
NF: B	**NF**: A	**NFT**: A	**NF**: A
NT: B	***N***: B	**NF**: B	**NT**: A
NFT: B	**NF**: B	**NFT**: B	**NFT**: A

NT: $A \vee_f B$	
NT: B	**NT**: A

FT: $A \vee_f B$			
T: B	***T***: A	**FT**: A	***T***: A
NT: B	**FT**: A	**NFT**: A	**NT**: A
FT: B	***T***: B	**FT**: B	**FT**: A
NFT: B	**FT**: B	**NFT**: B	**NFT**: A

NB: $A \vee_f B$			
N: B	***N***: A	**NB**: A	***N***: A
NT: B	**NB**: A	**NTB**: A	**NT**: A
NB: B	***N***: B	**NB**: B	**NB**: A
NTB: B	**NB**: B	**NTB**: B	**NTB**: A

FB: $A \vee_f B$			
N: A	**N**: A	***F***: A	***B***: A
N: A	***F***: A	**NF**: A	**NB**: A
F: A	***T***: A	**FT**: A	**FB**: A
B: A	***B***: A	**FB**: A	**TB**: A
NF: A	**FT**: A	**NFT**: A	**NFB**: A
NB: A	**FB**: A	**NFB**: A	**NTB**: A
FB: A	**TB**: A	**FTB**: A	**FTB**: A
NFB: A	**FTB**: A	**A**: A	**A**: A
N: B	**N**: B	***F***: B	***B***: B
N: B	***F***: B	**NF**: B	**NB**: B
F: B	***T***: B	**FT**: B	**FB**: B
B: B	***B***: B	**FB**: B	**TB**: B
NF: B	**FT**: B	**NFT**: B	**NFB**: B
NB: B	**FB**: B	**NFB**: B	**NTB**: B
FB: B	**TB**: B	**FTB**: B	**FTB**: B
NFB: B	**FTB**: B	**A**: B	**A**: B

$$\mathbf{TB}\colon A \vee_f B$$

$\boldsymbol{T}\colon B$	$\boldsymbol{T}\colon A$	$\mathbf{TB}\colon A$	$\boldsymbol{T}\colon A$
$\mathbf{NT}\colon B$	$\mathbf{TB}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{TB}\colon B$	$\boldsymbol{T}\colon B$	$\mathbf{TB}\colon B$	$\mathbf{TB}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{TB}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$

$$\mathbf{NFT}\colon A \vee_f B$$

$\mathbf{NT}\colon B$	$\mathbf{NFT}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{NFT}\colon B$	$\mathbf{NFT}\colon B$	$\mathbf{NFT}\colon A$

$$\mathbf{NFB}\colon A \vee_f B$$

$\boldsymbol{N}\colon B$	$\boldsymbol{N}\colon A$	$\mathbf{NF}\colon A$	$\mathbf{NB}\colon A$	$\boldsymbol{N}\colon A$
$\mathbf{NF}\colon B$	$\mathbf{NF}\colon A$	$\mathbf{NFT}\colon A$	$\mathbf{NFB}\colon A$	$\mathbf{NF}\colon A$
$\mathbf{NT}\colon B$	$\mathbf{NB}\colon A$	$\mathbf{NFB}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{NB}\colon B$	$\mathbf{NFB}\colon A$	$\mathbf{A}\colon A$	$\mathbf{A}\colon A$	$\mathbf{NB}\colon A$
$\mathbf{NFT}\colon B$	$\boldsymbol{N}\colon B$	$\mathbf{NF}\colon B$	$\mathbf{NB}\colon B$	$\mathbf{NFT}\colon A$
$\mathbf{NFB}\colon B$	$\mathbf{NF}\colon B$	$\mathbf{NFT}\colon B$	$\mathbf{NFB}\colon B$	$\mathbf{NFB}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{NB}\colon B$	$\mathbf{NFB}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$
$\mathbf{A}\colon B$	$\mathbf{NFB}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon A$

$$\mathbf{NTB}\colon A \vee_f B$$

$\mathbf{NT}\colon B$	$\mathbf{NTB}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$

$$\mathbf{FTB}\colon A \vee_f B$$

$\boldsymbol{T}\colon B$	$\boldsymbol{T}\colon A$	$\mathbf{FT}\colon A$	$\mathbf{TB}\colon A$	$\boldsymbol{T}\colon A$
$\mathbf{NT}\colon B$	$\mathbf{FT}\colon A$	$\mathbf{NFT}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{FT}\colon B$	$\mathbf{TB}\colon A$	$\mathbf{FTB}\colon A$	$\mathbf{FTB}\colon A$	$\mathbf{FT}\colon A$
$\mathbf{TB}\colon B$	$\mathbf{FTB}\colon A$	$\mathbf{A}\colon A$	$\mathbf{A}\colon A$	$\mathbf{TB}\colon A$
$\mathbf{NFT}\colon B$	$\boldsymbol{T}\colon B$	$\mathbf{FT}\colon B$	$\mathbf{TB}\colon B$	$\mathbf{NFT}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{FT}\colon B$	$\mathbf{NFT}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$
$\mathbf{FTB}\colon B$	$\mathbf{TB}\colon B$	$\mathbf{FTB}\colon B$	$\mathbf{FTB}\colon B$	$\mathbf{FTB}\colon A$
$\mathbf{A}\colon B$	$\mathbf{FTB}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon A$

$$\mathbf{A}\colon A \vee_f B$$

$\mathbf{NT}\colon B$	$\mathbf{NFT}\colon A$	$\mathbf{NTB}\colon A$	$\mathbf{NT}\colon A$
$\mathbf{NFT}\colon B$	$\mathbf{A}\colon A$	$\mathbf{A}\colon A$	$\mathbf{NFT}\colon A$
$\mathbf{NTB}\colon B$	$\mathbf{NFT}\colon B$	$\mathbf{NTB}\colon B$	$\mathbf{NTB}\colon A$
$\mathbf{A}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon B$	$\mathbf{A}\colon A$

Definition 18. An interpretation $\mathfrak{I}$ *satisfies* a signed formula $v\colon A$ iff $\mathrm{val}_{\mathfrak{I}}(A) \neq v$. A set of signed formulas is *satisfiable* if some interpretation $\mathfrak{I}$ satisfies all signed formulas in it.

Theorem 19. *A set of signed formulas is unsatisfiable iff it has a closed tableau.*

Proof. Apply Theorems 4.14 and 4.21 of Hähnle [7]; interpreting $v\colon A$ as $\mathsf{S}\ A$ where $\mathsf{S} = V \setminus \{v\}$. □

Corollary 20. *In Shramko-Wansing logic,* $\models A$ *iff* $\{v: A \mid v \in V^+\}$ *has a closed tableau.* $\Delta \models A$ *iff* $\{v: B \mid v \in V^-, B \in \Delta\} \cup \{v: A \mid v \in V^+\}$ *has a closed tableau.*

5 Natural deduction for Shramko-Wansing logic

Let Γ be a (set) sequent, $V^+ \subseteq V$ the set of *designated truth values*. The set of non-designated truth values is then $V^- = V \setminus V^+$. We divide the sequent Γ into its designated part Γ^+ and its non-designated part Γ^- in the obvious way:

$$\begin{array}{rcl} \Gamma^+ & := & \langle \Gamma_v \mid v \in V^+ \rangle \\ \Gamma^- & := & \langle \Gamma_v \mid v \in V^- \rangle \end{array}$$

Definition 21. The *natural deduction rules* for Shramko-Wansing logic are given by:

1. A weakening rule for all $v \in V^+$:

$$\frac{\Gamma^+}{\Gamma^+, [v: A]}\ \text{W:}v$$

2. For every connective $\Box$ and every truth value v an introduction rule $\Box\text{I:}v$ (if $v \in V^+$) or an elimination rule $\Box\text{E:}v$ (if $v \in V^-$).

The introduction and elimination rules for connective $\neg_t$ are given by

$$\frac{\begin{array}{cc} \Gamma_0^-, \lceil[\boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: \neg_t A]\rceil & \Gamma_1^-, \lceil[\mathbf{N}: A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}: \neg_t A] & \Gamma_1^+ \end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\text{E}: \mathbf{N}$$

$$\frac{\begin{array}{cc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: \neg_t A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}: \neg_t A] & \Gamma_1^+, [\boldsymbol{T}: A] \end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\text{E}: \boldsymbol{N}$$

$$\frac{\begin{array}{cc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: \neg_t A]\rceil & \Gamma_1^-, \lceil[\boldsymbol{B}: A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}: \neg_t A] & \Gamma_1^+ \end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\text{E}: \boldsymbol{F}$$

$$\frac{\begin{array}{c} \Gamma_1^-, \lceil[\boldsymbol{N}: A]\rceil \\ \Gamma_1^+ \end{array}}{\Gamma_1^+, [\boldsymbol{T}: \neg_t A]}\ \neg_t\text{I}: \boldsymbol{T}$$

$$\frac{\begin{array}{cc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}: \neg_t A]\rceil & \Gamma_1^-, \lceil[\boldsymbol{F}: A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}: \neg_t A] & \Gamma_1^+ \end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\text{E}: \boldsymbol{B}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-\\ \Gamma_1^+, [\mathbf{TB}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{NF}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-\\ \Gamma_1^+, [\mathbf{NT}\colon A]\end{array}}{\Gamma_1^+, [\mathbf{NT}\colon \neg_t A]}\ \neg_t\mathrm{I}\colon \mathbf{NT}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{NB}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{FT}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FT}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{NB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FB}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{FB}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{NF}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{TB}\colon \neg_t A]}\ \neg_t\mathrm{I}\colon \mathbf{TB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-\\ \Gamma_1^+, [\mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{NFT}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FTB}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{NFB}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{NFT}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{NTB}\colon \neg_t A]}\ \neg_t\mathrm{I}\colon \mathbf{NTB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{NFB}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{FTB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}\colon \neg_t A]\rceil\\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_t A]\end{array} \qquad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{A}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_t\mathrm{E}\colon \mathbf{A}$$

The introduction and elimination rules for connective $\neg_f$ are given by

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^-, \lceil[\mathbf{N}\colon A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{N}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^-, \lceil[\boldsymbol{F}\colon A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \boldsymbol{N}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^-, \lceil[\boldsymbol{N}\colon A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \boldsymbol{F}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-, \lceil[\boldsymbol{B}\colon A]\rceil \\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\boldsymbol{T}\colon \neg_f A]}\ \neg_f\mathrm{I}\colon \boldsymbol{T}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+, [\boldsymbol{T}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \boldsymbol{B}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^-, \lceil[\mathbf{NF}\colon A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{NF}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FB}\colon A]\rceil \\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{NT}\colon \neg_f A]}\ \neg_f\mathrm{I}\colon \mathbf{NT}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^-, \lceil[\mathbf{NB}\colon A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{FT}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^-, \lceil[\mathbf{FT}\colon A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{NB}$$

$$\dfrac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon \neg_f A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon \neg_f A] & \Gamma_1^+, [\mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{FB}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-\\ \Gamma_1^+,[\mathbf{TB}\colon A]\end{array}}{\Gamma_1^+,[\mathbf{TB}\colon \neg_f A]}\ \neg_f\mathrm{I}\colon \mathbf{TB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon \neg_f A]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon \neg_f A]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{NFB}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+,\Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{NFT}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{FTB},\mathbf{A}\colon \neg_f A]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon \neg_f A]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{NFT}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+,\Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{NFB}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-,\lceil[\mathbf{FTB}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_1^+,[\mathbf{NTB}\colon \neg_f A]}\ \neg_f\mathrm{I}\colon \mathbf{NTB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{A}\colon \neg_f A]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon \neg_f A]\end{array}\quad \begin{array}{c}\Gamma_1^-\\ \Gamma_1^+,[\mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+,\Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{FTB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB}\colon \neg_f A]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon \neg_f A]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{A}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_0^+,\Gamma_1^+}\ \neg_f\mathrm{E}\colon \mathbf{A}$$

The introduction and elimination rules for connective $\wedge_t$ are given by

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_t B]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_t B]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{B}\colon B]\rceil\\ \Gamma_1^+,[\boldsymbol{T},\mathbf{TB}\colon B]\end{array}\quad \begin{array}{c}\Gamma_2^-,\lceil[\mathbf{N}\colon A,B]\rceil\\ \Gamma_2^+,[\boldsymbol{T}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_3^-,\lceil[\mathbf{N},\boldsymbol{B}\colon A,B]\rceil\\ \Gamma_3^+\end{array}\quad \begin{array}{c}\Gamma_4^-,\lceil[\mathbf{N},\boldsymbol{B}\colon A]\rceil\\ \Gamma_4^+,[\boldsymbol{T},\mathbf{TB}\colon A]\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \wedge_t\mathrm{E}\colon \mathbf{N}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_t B]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_t B]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NB}\colon B]\rceil\\ \Gamma_1^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon B]\end{array}\quad \begin{array}{c}\Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{N}\colon A,B]\rceil\\ \Gamma_2^+,[\boldsymbol{T},\mathbf{NT}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_3^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NB}\colon A,B]\rceil\\ \Gamma_3^+\end{array}\quad \begin{array}{c}\Gamma_4^-,\lceil[\boldsymbol{N},\mathbf{NB}\colon A,B]\rceil\\ \Gamma_4^+,[\mathbf{NT},\mathbf{NTB}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_5^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NB}\colon A]\rceil\\ \Gamma_5^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+,\ldots,\Gamma_5^+}\ \wedge_t\mathrm{E}\colon \boldsymbol{N}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_t B]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_t B]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon B]\rceil\\ \Gamma_1^+,[\boldsymbol{T},\mathbf{TB}\colon B]\end{array}\quad \begin{array}{c}\Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{F},\mathbf{FT}\colon A,B]\rceil\\ \Gamma_2^+,[\boldsymbol{T}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_3^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FB}\colon A,B]\rceil\\ \Gamma_3^+\end{array}\quad \begin{array}{c}\Gamma_4^-,\lceil[\boldsymbol{F},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon A,B]\rceil\\ \Gamma_4^+\end{array}\quad \begin{array}{c}\Gamma_5^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon A]\rceil\\ \Gamma_5^+,[\boldsymbol{T},\mathbf{TB}\colon A]\end{array}}{\Gamma_0^+,\ldots,\Gamma_5^+}\ \wedge_t\mathrm{E}\colon \boldsymbol{F}$$

$$\dfrac{\begin{array}{c}\Gamma_1^-\\ \Gamma_1^+,[\boldsymbol{T},\mathbf{TB}\colon B]\end{array}\quad \begin{array}{c}\Gamma_2^-\\ \Gamma_2^+,[\boldsymbol{T}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_3^-\\ \Gamma_3^+,[\boldsymbol{T},\mathbf{TB}\colon A]\end{array}}{\Gamma_1^+,\Gamma_2^+,\Gamma_3^+,[\boldsymbol{T}\colon A\wedge_t B]}\ \wedge_t\mathrm{I}\colon \boldsymbol{T}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_t B]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_t B]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\boldsymbol{B}\colon B]\rceil\\ \Gamma_1^+,[\mathbf{TB}\colon B]\end{array}\quad \begin{array}{c}\Gamma_2^-,\lceil[\boldsymbol{B}\colon A,B]\rceil\\ \Gamma_2^+\end{array}\quad \begin{array}{c}\Gamma_3^-,\lceil[\boldsymbol{B}\colon A]\rceil\\ \Gamma_3^+,[\mathbf{TB}\colon A]\end{array}}{\Gamma_0^+,\ldots,\Gamma_3^+}\ \wedge_t\mathrm{E}\colon \boldsymbol{B}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_t B]\rceil\\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_t B]\end{array}\quad \begin{array}{c}\Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{NFT}\colon A,B]\rceil\\ \Gamma_1^+,[\boldsymbol{T},\mathbf{NT}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{NB},\mathbf{FB},\mathbf{NFB}\colon A,B]\rceil\\ \Gamma_2^+\end{array}\quad \begin{array}{c}\Gamma_3^-,\lceil[\boldsymbol{N},\mathbf{NF},\mathbf{NB},\mathbf{NFT},\mathbf{NFB},\mathbf{A}\colon A,B]\rceil\\ \Gamma_3^+,[\mathbf{NT},\mathbf{NTB}\colon A,B]\end{array}\quad \begin{array}{c}\Gamma_4^-,\lceil[\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A,B]\rceil\\ \Gamma_4^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \wedge_t\mathrm{E}\colon \mathbf{NF}$$

$$\frac{\begin{array}{cccc} \Gamma_1^- & \Gamma_2^- & \Gamma_3^- & \Gamma_4^- \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, B] & \Gamma_2^+, [\boldsymbol{T}, \mathbf{NT}{:}\, A, B] & \Gamma_3^+, [\mathbf{NT}, \mathbf{NTB}{:}\, A, B] & \Gamma_4^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_1^+, \ldots, \Gamma_4^+, [\mathbf{NT}{:}\, A \wedge_t B]} \ \wedge_t\mathrm{I}{:}\ \mathbf{NT}$$

$$\frac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{FT}, \mathbf{FTB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{FT}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FT}, \mathbf{FTB}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{FT}, \mathbf{FTB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\boldsymbol{T}, \mathbf{TB}{:}\, B] & \Gamma_2^+, [\boldsymbol{T}{:}\, A, B] & \Gamma_3^+ & \Gamma_4^+, [\boldsymbol{T}, \mathbf{TB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{FT}$$

$$\frac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{B}, \mathbf{NB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{B}, \mathbf{NB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NB}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{B}, \mathbf{NB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\mathbf{TB}, \mathbf{NTB}{:}\, B] & \Gamma_2^+ & \Gamma_3^+, [\mathbf{NTB}{:}\, A, B] & \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{NB}$$

$$\frac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{B}, \mathbf{FB}, \mathbf{FTB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{B}, \mathbf{FB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FB}, \mathbf{FTB}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{B}, \mathbf{FB}, \mathbf{FTB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\mathbf{TB}{:}\, B] & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+, [\mathbf{TB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{FB}$$

$$\frac{\begin{array}{cc} \Gamma_1^- & \Gamma_2^- \\ \Gamma_1^+, [\mathbf{TB}{:}\, B] & \Gamma_2^+, [\mathbf{TB}{:}\, A] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{TB}{:}\, A \wedge_t B]} \ \wedge_t\mathrm{I}{:}\ \mathbf{TB}$$

$$\frac{\begin{array}{cccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{FT}, \mathbf{NFT}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NFT}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_5^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, B] & \Gamma_2^+, [\boldsymbol{T}, \mathbf{NT}{:}\, A, B] & \Gamma_3^+, [\mathbf{NT}, \mathbf{NTB}{:}\, A, B] & \Gamma_4^+ & \Gamma_5^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_5^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{NFT}$$

$$\frac{\begin{array}{cccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NB}, \mathbf{NFB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_5^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\mathbf{TB}, \mathbf{NTB}{:}\, B] & \Gamma_2^+ & \Gamma_3^+, [\mathbf{NTB}{:}\, A, B] & \Gamma_4^+ & \Gamma_5^+, [\mathbf{TB}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_5^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{NFB}$$

$$\frac{\begin{array}{ccc} \Gamma_1^- & \Gamma_2^- & \Gamma_3^- \\ \Gamma_1^+, [\mathbf{TB}, \mathbf{NTB}{:}\, B] & \Gamma_2^+, [\mathbf{NTB}{:}\, A, B] & \Gamma_3^+, [\mathbf{TB}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{NTB}{:}\, A \wedge_t B]} \ \wedge_t\mathrm{I}{:}\ \mathbf{NTB}$$

$$\frac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{FTB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{FTB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FTB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\mathbf{TB}{:}\, B] & \Gamma_2^+ & \Gamma_3^+, [\mathbf{TB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{FTB}$$

$$\frac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}{:}\, A \wedge_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{FTB}, \mathbf{A}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{A}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{FTB}, \mathbf{A}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \wedge_t B] & \Gamma_1^+, [\mathbf{TB}, \mathbf{NTB}{:}\, B] & \Gamma_2^+, [\mathbf{NTB}{:}\, A, B] & \Gamma_3^+ & \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \wedge_t\mathrm{E}{:}\ \mathbf{A}$$

The introduction and elimination rules for connective $\vee_t$ are given by

$$\frac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{N}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{N}, \boldsymbol{F}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \vee_t\mathrm{E}{:}\ \mathbf{N}$$

$$\frac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NF}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{N}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\boldsymbol{N}, \mathbf{NF}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+} \ \vee_t\mathrm{E}{:}\ \boldsymbol{N}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{F}, \mathbf{NF}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{F}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\boldsymbol{F}, \mathbf{NF}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \vee_t\mathrm{E}{:}\, \boldsymbol{F}$$

$$\dfrac{\begin{array}{ccccc} \Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{N}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{N}, \boldsymbol{F}, \mathbf{FT}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{FT}, \mathbf{NFT}{:}\, A, B]\rceil & \Gamma_5^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}{:}\, A]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}{:}\, B] & \Gamma_2^+, [\boldsymbol{T}, \mathbf{NT}{:}\, A, B] & \Gamma_3^+, [\boldsymbol{T}{:}\, A, B] & \Gamma_4^+, [\boldsymbol{T}, \mathbf{NT}{:}\, A, B] & \Gamma_5^+, [\boldsymbol{T}, \mathbf{NT}{:}\, A] \end{array}}{\Gamma_1^+, \ldots, \Gamma_5^+, [\boldsymbol{T}{:}\, A \vee_t B]}\ \vee_t\mathrm{I}{:}\, \boldsymbol{T}$$

$$\dfrac{\begin{array}{cccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FB}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}{:}\, A, B]\rceil & \Gamma_5^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ & \Gamma_5^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_5^+}\ \vee_t\mathrm{E}{:}\, \boldsymbol{B}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{NF}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{NF}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \vee_t\mathrm{E}{:}\, \mathbf{NF}$$

$$\dfrac{\begin{array}{cccc} \Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NFT}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{N}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NFT}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NFT}{:}\, A]\rceil \\ \Gamma_1^+, [\mathbf{NT}{:}\, B] & \Gamma_2^+, [\mathbf{NT}{:}\, A, B] & \Gamma_3^+, [\mathbf{NT}{:}\, A, B] & \Gamma_4^+, [\mathbf{NT}{:}\, A] \end{array}}{\Gamma_1^+, \ldots, \Gamma_4^+, [\mathbf{NT}{:}\, A \vee_t B]}\ \vee_t\mathrm{I}{:}\, \mathbf{NT}$$

$$\dfrac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{F}, \mathbf{FT}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FT}, \mathbf{NFT}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+}\ \vee_t\mathrm{E}{:}\, \mathbf{FT}$$

$$\dfrac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{N}, \mathbf{NB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NB}, \mathbf{NFB}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+}\ \vee_t\mathrm{E}{:}\, \mathbf{NB}$$

$$\dfrac{\begin{array}{ccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FB}, \mathbf{NFB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{F}, \mathbf{FB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FB}, \mathbf{NFB}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FB}, \mathbf{NFB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+}\ \vee_t\mathrm{E}{:}\, \mathbf{FB}$$

$$\dfrac{\begin{array}{cccc} \Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}{:}\, A, B]\rceil & \Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A, B] & \Gamma_2^+, [\boldsymbol{T}, \mathbf{TB}{:}\, A, B] & \Gamma_3^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A, B] & \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}{:}\, A, B] \end{array}}{\Gamma_1^+, \ldots, \Gamma_4^+, [\mathbf{TB}{:}\, A \vee_t B]}\ \vee_t\mathrm{I}{:}\, \mathbf{TB}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{NF}, \mathbf{NFT}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{NFT}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NF}, \mathbf{NFT}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \vee_t\mathrm{E}{:}\, \mathbf{NFT}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\mathbf{NF}, \mathbf{NFB}{:}\, B]\rceil & \Gamma_2^-, \lceil[\mathbf{NFB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NF}, \mathbf{NFB}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \vee_t\mathrm{E}{:}\, \mathbf{NFB}$$

$$\dfrac{\begin{array}{ccccc} \Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{N}, \mathbf{NB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{NFT}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{NB}, \mathbf{NFB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_5^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}{:}\, A]\rceil \\ \Gamma_1^+, [\mathbf{NT}, \mathbf{NTB}{:}\, B] & \Gamma_2^+, [\mathbf{NT}, \mathbf{NTB}{:}\, A, B] & \Gamma_3^+, [\mathbf{NT}, \mathbf{NTB}{:}\, A, B] & \Gamma_4^+, [\mathbf{NTB}{:}\, A, B] & \Gamma_5^+, [\mathbf{NT}, \mathbf{NTB}{:}\, A] \end{array}}{\Gamma_1^+, \ldots, \Gamma_5^+, [\mathbf{NTB}{:}\, A \vee_t B]}\ \vee_t\mathrm{I}{:}\, \mathbf{NTB}$$

$$\dfrac{\begin{array}{cccccc} \Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}{:}\, A \vee_t B]\rceil & \Gamma_1^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, B]\rceil & \Gamma_2^-, \lceil[\boldsymbol{F}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB}{:}\, A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_4^-, \lceil[\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A, B]\rceil & \Gamma_5^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}{:}\, A]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}{:}\, A \vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ & \Gamma_5^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_5^+}\ \vee_t\mathrm{E}{:}\, \mathbf{FTB}$$

$$\dfrac{\begin{array}{ccccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB}\colon A\vee_t B]\rceil & \Gamma_1^-,\lceil[\mathbf{NF},\mathbf{NFT},\mathbf{NFB},\mathbf{A}\colon B]\rceil & \Gamma_2^-,\lceil[\mathbf{NFT},\mathbf{A}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{NFB},\mathbf{A}\colon A,B]\rceil & \Gamma_4^-,\lceil[\mathbf{NF},\mathbf{NFT},\mathbf{NFB},\mathbf{A}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\vee_t B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \vee_t\mathrm{E}\colon\mathbf{A}$$

The introduction and elimination rules for connective $\wedge_f$ are given by

$$\dfrac{\begin{array}{ccccc}\Gamma_0^-,\lceil[\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FB}\colon B]\rceil & \Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{F}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{N},\boldsymbol{B}\colon A,B]\rceil & \Gamma_4^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \wedge_f\mathrm{E}\colon\mathbf{N}$$

$$\dfrac{\begin{array}{cccccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{NB},\mathbf{FB},\mathbf{NFB}\colon B]\rceil & \Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\mathbf{NF}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NB}\colon A,B]\rceil & \Gamma_4^-,\lceil[\boldsymbol{N},\mathbf{NF},\mathbf{NB},\mathbf{NFB}\colon A,B]\rceil & \Gamma_5^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{NB},\mathbf{FB},\mathbf{NFB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ & \Gamma_5^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_5^+}\ \wedge_f\mathrm{E}\colon\boldsymbol{N}$$

$$\dfrac{\begin{array}{cccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\boldsymbol{F},\mathbf{FB}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{F}\colon A,B]\rceil & \Gamma_3^-,\lceil[\boldsymbol{F},\mathbf{FB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_3^+}\ \wedge_f\mathrm{E}\colon\boldsymbol{F}$$

$$\dfrac{\begin{array}{ccccc}\Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon B]\rceil & \Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{F},\mathbf{FT}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{N},\boldsymbol{B}\colon A,B]\rceil & \Gamma_4^-,\lceil[\mathbf{FT},\mathbf{FTB}\colon A,B]\rceil & \Gamma_5^-,\lceil[\mathbf{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon A]\rceil \\ \Gamma_1^+,[\boldsymbol{T},\mathbf{TB}\colon B] & \Gamma_2^+,[\boldsymbol{T}\colon A,B] & \Gamma_3^+,[\boldsymbol{T},\mathbf{TB}\colon A,B] & \Gamma_4^+,[\boldsymbol{T},\mathbf{TB}\colon A,B] & \Gamma_5^+,[\boldsymbol{T},\mathbf{TB}\colon A]\end{array}}{\Gamma_1^+,\ldots,\Gamma_5^+,[\boldsymbol{T}\colon A\wedge_f B]}\ \wedge_f\mathrm{I}\colon\boldsymbol{T}$$

$$\dfrac{\begin{array}{cccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\boldsymbol{B},\mathbf{FB}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{B}\colon A,B]\rceil & \Gamma_3^-,\lceil[\boldsymbol{B},\mathbf{FB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_3^+}\ \wedge_f\mathrm{E}\colon\boldsymbol{B}$$

$$\dfrac{\begin{array}{ccccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\boldsymbol{F},\mathbf{NF},\mathbf{FB},\mathbf{NFB}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{F},\mathbf{NF}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{NF},\mathbf{NFB}\colon A,B]\rceil & \Gamma_4^-,\lceil[\boldsymbol{F},\mathbf{NF},\mathbf{FB},\mathbf{NFB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \wedge_f\mathrm{E}\colon\mathbf{NF}$$

$$\dfrac{\begin{array}{cccc}\Gamma_1^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{NFT}\colon A,B]\rceil & \Gamma_2^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{B},\mathbf{NB}\colon A,B]\rceil & \Gamma_3^-,\lceil[\boldsymbol{N},\mathbf{NF},\mathbf{NB},\mathbf{NFT},\mathbf{NFB},\mathbf{A}\colon A,B]\rceil & \Gamma_4^-,\lceil[\mathbf{FT},\mathbf{NFT},\mathbf{FTB},\mathbf{A}\colon A,B]\rceil \\ \Gamma_1^+,[\boldsymbol{T},\mathbf{NT}\colon A,B] & \Gamma_2^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A,B] & \Gamma_3^+,[\mathbf{NT},\mathbf{NTB}\colon A,B] & \Gamma_4^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A,B]\end{array}}{\Gamma_1^+,\ldots,\Gamma_4^+,[\mathbf{NT}\colon A\wedge_f B]}\ \wedge_f\mathrm{I}\colon\mathbf{NT}$$

$$\dfrac{\begin{array}{ccccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{NB},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\boldsymbol{F},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{F},\mathbf{FT}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{FT},\mathbf{FTB}\colon A,B]\rceil & \Gamma_4^-,\lceil[\boldsymbol{F},\mathbf{FT},\mathbf{FB},\mathbf{FTB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \wedge_f\mathrm{E}\colon\mathbf{FT}$$

$$\dfrac{\begin{array}{ccccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\boldsymbol{B},\mathbf{NB},\mathbf{FB},\mathbf{NFB}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{B},\mathbf{NB}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{NB},\mathbf{NFB}\colon A,B]\rceil & \Gamma_4^-,\lceil[\boldsymbol{B},\mathbf{NB},\mathbf{FB},\mathbf{NFB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_4^+}\ \wedge_f\mathrm{E}\colon\mathbf{NB}$$

$$\dfrac{\begin{array}{ccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\mathbf{FB}\colon B]\rceil & \Gamma_2^-,\lceil[\mathbf{FB}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+\end{array}}{\Gamma_0^+,\Gamma_1^+,\Gamma_2^+}\ \wedge_f\mathrm{E}\colon\mathbf{FB}$$

$$\dfrac{\begin{array}{cccc}\Gamma_1^-,\lceil[\boldsymbol{B},\mathbf{FB},\mathbf{FTB}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{B}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{FTB}\colon A,B]\rceil & \Gamma_4^-,\lceil[\boldsymbol{B},\mathbf{FB},\mathbf{FTB}\colon A]\rceil \\ \Gamma_1^+,[\mathbf{TB}\colon B] & \Gamma_2^+,[\mathbf{TB}\colon A,B] & \Gamma_3^+,[\mathbf{TB}\colon A,B] & \Gamma_4^+,[\mathbf{TB}\colon A]\end{array}}{\Gamma_1^+,\ldots,\Gamma_4^+,[\mathbf{TB}\colon A\wedge_f B]}\ \wedge_f\mathrm{I}\colon\mathbf{TB}$$

$$\dfrac{\begin{array}{cccccc}\Gamma_0^-,\lceil[\mathbf{N},\boldsymbol{N},\boldsymbol{F},\boldsymbol{B},\mathbf{NF},\mathbf{FT},\mathbf{NB},\mathbf{FB},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A\wedge_f B]\rceil & \Gamma_1^-,\lceil[\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon B]\rceil & \Gamma_2^-,\lceil[\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{NFT}\colon A,B]\rceil & \Gamma_3^-,\lceil[\mathbf{NF},\mathbf{NFT},\mathbf{NFB},\mathbf{A}\colon A,B]\rceil & \Gamma_4^-,\lceil[\mathbf{FT},\mathbf{NFT},\mathbf{FTB},\mathbf{A}\colon A,B]\rceil & \Gamma_5^-,\lceil[\boldsymbol{F},\mathbf{NF},\mathbf{FT},\mathbf{FB},\mathbf{NFT},\mathbf{NFB},\mathbf{FTB},\mathbf{A}\colon A]\rceil \\ \Gamma_0^+,[\boldsymbol{T},\mathbf{NT},\mathbf{TB},\mathbf{NTB}\colon A\wedge_f B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ & \Gamma_4^+ & \Gamma_5^+\end{array}}{\Gamma_0^+,\ldots,\Gamma_5^+}\ \wedge_f\mathrm{E}\colon\mathbf{NFT}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon A \wedge_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \wedge_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FB}, \mathbf{NFB}\colon B]\rceil \\ \Gamma_1^+\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{NFB}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{FB}, \mathbf{NFB}\colon A]\rceil \\ \Gamma_3^+\end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+} \ \wedge_f\mathrm{E}\colon \mathbf{NFB}$$

$$\frac{\begin{array}{c}\Gamma_1^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{TB}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\boldsymbol{B}, \mathbf{NB}\colon A, B]\rceil \\ \Gamma_2^+, [\mathbf{TB}, \mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{NB}, \mathbf{NFB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_3^+, [\mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{FTB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_5^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A]\rceil \\ \Gamma_5^+, [\mathbf{TB}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_1^+, \ldots, \Gamma_5^+, [\mathbf{NTB}\colon A \wedge_f B]} \ \wedge_f\mathrm{I}\colon \mathbf{NTB}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon A \wedge_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \wedge_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FB}, \mathbf{FTB}\colon B]\rceil \\ \Gamma_1^+\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{FTB}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{FB}, \mathbf{FTB}\colon A]\rceil \\ \Gamma_3^+\end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+} \ \wedge_f\mathrm{E}\colon \mathbf{FTB}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}\colon A \wedge_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \wedge_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon B]\rceil \\ \Gamma_1^+\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{NFB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{FTB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_3^+\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A]\rceil \\ \Gamma_4^+\end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \wedge_f\mathrm{E}\colon \mathbf{A}$$

The introduction and elimination rules for connective $\vee_f$ are given by

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}\colon B]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{N}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{N}\colon A, B]\rceil \\ \Gamma_3^+, [\boldsymbol{T}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{N}, \boldsymbol{N}\colon A]\rceil \\ \Gamma_4^+, [\boldsymbol{T}, \mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \vee_f\mathrm{E}\colon \mathbf{N}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\boldsymbol{N}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\boldsymbol{N}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\boldsymbol{N}\colon A]\rceil \\ \Gamma_3^+, [\mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+} \ \vee_f\mathrm{E}\colon \boldsymbol{N}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}\colon B]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{N}, \boldsymbol{F}, \mathbf{FT}\colon A, B]\rceil \\ \Gamma_3^+, [\boldsymbol{T}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}\colon A, B]\rceil \\ \Gamma_4^+\end{array} \quad \begin{array}{c}\Gamma_5^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NFT}\colon A]\rceil \\ \Gamma_5^+, [\boldsymbol{T}, \mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_5^+} \ \vee_f\mathrm{E}\colon \boldsymbol{F}$$

$$\frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^- \\ \Gamma_2^+, [\boldsymbol{T}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^- \\ \Gamma_3^+, [\boldsymbol{T}, \mathbf{NT}\colon A]\end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\boldsymbol{T}\colon A \vee_f B]} \ \vee_f\mathrm{I}\colon \boldsymbol{T}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}\colon B]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{N}, \boldsymbol{B}\colon A, B]\rceil \\ \Gamma_3^+, [\boldsymbol{T}, \mathbf{TB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\boldsymbol{B}, \mathbf{NB}\colon A, B]\rceil \\ \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_5^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{B}, \mathbf{NB}\colon A]\rceil \\ \Gamma_5^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_5^+} \ \vee_f\mathrm{E}\colon \boldsymbol{B}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NFT}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\boldsymbol{N}, \mathbf{NF}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{NF}, \mathbf{NFT}\colon A, B]\rceil \\ \Gamma_3^+\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NFT}\colon A]\rceil \\ \Gamma_4^+, [\mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \vee_f\mathrm{E}\colon \mathbf{NF}$$

$$\frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^- \\ \Gamma_2^+, [\mathbf{NT}\colon A]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{NT}\colon A \vee_f B]} \ \vee_f\mathrm{I}\colon \mathbf{NT}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FT}, \mathbf{NFT}\colon B]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{FT}\colon A, B]\rceil \\ \Gamma_2^+, [\boldsymbol{T}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{FT}, \mathbf{NFT}\colon A, B]\rceil \\ \Gamma_3^+\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{FT}, \mathbf{NFT}\colon A]\rceil \\ \Gamma_4^+, [\boldsymbol{T}, \mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \vee_f\mathrm{E}\colon \mathbf{FT}$$

$$\frac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NB}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{NT}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\boldsymbol{N}, \mathbf{NB}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{NB}\colon A, B]\rceil \\ \Gamma_3^+, [\mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\boldsymbol{N}, \mathbf{NB}\colon A]\rceil \\ \Gamma_4^+, [\mathbf{NT}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_4^+} \ \vee_f\mathrm{E}\colon \mathbf{NB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}\colon A, B]\rceil \\ \Gamma_1^+\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{FT}, \mathbf{FB}, \mathbf{FTB}\colon A, B]\rceil \\ \Gamma_2^+, [\boldsymbol{T}, \mathbf{TB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\boldsymbol{F}, \mathbf{NF}, \mathbf{FT}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_3^+\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\boldsymbol{B}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}\colon A, B]\end{array}}{\Gamma_0^+, \dots, \Gamma_4^+} \; \vee_f\mathrm{E}\colon \mathbf{FB}$$

$$\dfrac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^- \\ \Gamma_2^+, [\boldsymbol{T}, \mathbf{TB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^- \\ \Gamma_3^+, [\mathbf{TB}, \mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^- \\ \Gamma_4^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_1^+, \dots, \Gamma_4^+, [\mathbf{TB}\colon A \vee_f B]} \; \vee_f\mathrm{I}\colon \mathbf{TB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFB}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{NFT}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{NT}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{NFT}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{NFT}\colon A]\rceil \\ \Gamma_3^+, [\mathbf{NT}\colon A]\end{array}}{\Gamma_0^+, \dots, \Gamma_3^+} \; \vee_f\mathrm{E}\colon \mathbf{NFT}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{NT}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFB}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{NF}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_3^+\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{NB}, \mathbf{NFB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_4^+, [\mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_5^-, \lceil[\boldsymbol{N}, \mathbf{NF}, \mathbf{NB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon A]\rceil \\ \Gamma_5^+, [\mathbf{NT}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+, \dots, \Gamma_5^+} \; \vee_f\mathrm{E}\colon \mathbf{NFB}$$

$$\dfrac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\mathbf{NT}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^- \\ \Gamma_2^+, [\mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^- \\ \Gamma_3^+, [\mathbf{NT}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{NTB}\colon A \vee_f B]} \; \vee_f\mathrm{I}\colon \mathbf{NTB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{A}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon B]\rceil \\ \Gamma_1^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{FT}, \mathbf{FTB}\colon A, B]\rceil \\ \Gamma_2^+, [\boldsymbol{T}, \mathbf{TB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_3^+\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{FTB}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_4^+, [\mathbf{TB}, \mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_5^-, \lceil[\mathbf{FT}, \mathbf{NFT}, \mathbf{FTB}, \mathbf{A}\colon A]\rceil \\ \Gamma_5^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+, \dots, \Gamma_5^+} \; \vee_f\mathrm{E}\colon \mathbf{FTB}$$

$$\dfrac{\begin{array}{c}\Gamma_0^-, \lceil[\mathbf{N}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{B}, \mathbf{NF}, \mathbf{FT}, \mathbf{NB}, \mathbf{FB}, \mathbf{NFT}, \mathbf{NFB}, \mathbf{FTB}\colon A \vee_f B]\rceil \\ \Gamma_0^+, [\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\colon A \vee_f B]\end{array} \quad \begin{array}{c}\Gamma_1^-, \lceil[\mathbf{NFT}, \mathbf{A}\colon B]\rceil \\ \Gamma_1^+, [\mathbf{NT}, \mathbf{NTB}\colon B]\end{array} \quad \begin{array}{c}\Gamma_2^-, \lceil[\mathbf{NFT}, \mathbf{A}\colon A, B]\rceil \\ \Gamma_2^+\end{array} \quad \begin{array}{c}\Gamma_3^-, \lceil[\mathbf{A}\colon A, B]\rceil \\ \Gamma_3^+, [\mathbf{NTB}\colon A, B]\end{array} \quad \begin{array}{c}\Gamma_4^-, \lceil[\mathbf{NFT}, \mathbf{A}\colon A]\rceil \\ \Gamma_4^+, [\mathbf{NT}, \mathbf{NTB}\colon A]\end{array}}{\Gamma_0^+, \dots, \Gamma_4^+} \; \vee_f\mathrm{E}\colon \mathbf{A}$$

Definition 22. A *natural deduction derivation* is defined inductively as follows:

1. Let A be any formula. Then

$$\dfrac{[V^-\colon A]}{[V^+\colon A]}$$

   is a derivation of A from the assumption $[V^-\colon A]$ (an *initial derivation*).

2. If D_k are derivations of Γ_k^+, Δ_k^+ from the assumptions $\Gamma_k^-, \hat{\Delta}_k^-$, and

$$\dfrac{\left\langle \begin{array}{c}\Gamma_k^-, \lceil\Delta_k^-\rceil \\ \Gamma_k^+, \Delta_k^+\end{array} \right\rangle_{k \in K}}{\Pi^+}$$

   is an instance of a deduction rule with $\hat{\Delta}_k^-$ a subsequent of Δ_k^-, and all eigenvariable conditions are satisfied, then

$$\dfrac{\langle D_k \rangle_{k \in K}}{\Pi^+}$$

   is a derivation of Π^+ from the assumptions $\bigcup_{k \in K} \Gamma_k^-$. The formulas in $\hat{\Delta}_k^-$ which do not occur in $\bigcup_{k \in K} \Gamma_k^-$ are said to be *discharged* at this inference.

Theorem 23. *A partial sequent Γ^+ can be derived from the assumptions Γ^- in the natural deduction system for Shramko-Wansing logic iff, for every interpretation $\mathfrak{I}$, either some formula in Γ^-_v ($v \in V^-$) evaluates to the truth value v, or there is a $w \in V^+$ and a formula in Γ^+_w that evaluates to w.*

Proof. See Theorems 4.7 and 5.4 of Baaz et al. [3] or Theorems 4.2.8 and 4.3.4 of Zach [15]. □

Corollary 24. *$\Gamma \models A$ iff there is a natural deduction derivation of $[V^+\colon A]$ from $[V^-\colon \Gamma]$.*

6 Resolution and clause formation rules for Shramko-Wansing logic

The many-valued resolution calculus of Baaz and Fermüller [1] applies to Shramko-Wansing logic. We present the framework here, as well as a language preserving clause translation system for Shramko-Wansing logic.

Definition 25 (Signed formula)**.** A *signed formula* is an expression of the form A^v, where A is a formula and $v \in V$. If A is a propositonal variable, A^v is a *signed atom.*

Definition 26 (Signed clause)**.** A (signed) *clause* $C = \{A_1^{v_1}, \dots, A_n^{v_n}\}$ is a finite set of signed atoms (proper clause). The empty clause is denoted by □.

An *extended clause* is a finite set of signed formulas.

Definition 27 (Semantics of clause sets)**.** Let $\mathfrak{I}$ be an interpretation. $\mathfrak{I}$ *satisfies* a clause C iff there is some signed formula $A^v \in C$, so that $\mathrm{val}_{\mathfrak{I}}(A) = v$. $\mathfrak{I}$ satisfies a clause set $\mathscr{C}$ iff it satisfies every clause in $\mathscr{C}$. $\mathscr{C}$ is called *satisfiable* iff some structure satisfies it, and *unsatisfiable* otherwise.

The clause formation rules for connective $\neg_t$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{N}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{N}}\}\}}\ \neg_t{:}\mathbf{N} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\boldsymbol{N}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{T}}\}\}}\ \neg_t{:}\boldsymbol{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\boldsymbol{F}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{B}}\}\}}\ \neg_t{:}\boldsymbol{F} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\boldsymbol{T}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{N}}\}\}}\ \neg_t{:}\boldsymbol{T}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\boldsymbol{B}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{F}}\}\}}\ \neg_t{:}\boldsymbol{B} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{NF}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{TB}}\}\}}\ \neg_t{:}\mathbf{NF}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{NT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NT}}\}\}}\ \neg_t{:}\mathbf{NT} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{FT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NB}}\}\}}\ \neg_t{:}\mathbf{FT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{NB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{FT}}\}\}}\ \neg_t{:}\mathbf{NB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{FB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{FB}}\}\}}\ \neg_t{:}\mathbf{FB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{TB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NF}}\}\}}\ \neg_t{:}\mathbf{TB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{NFT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NTB}}\}\}}\ \neg_t{:}\mathbf{NFT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{NFB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{FTB}}\}\}}\ \neg_t{:}\mathbf{NFB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{NTB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NFT}}\}\}}\ \neg_t{:}\mathbf{NTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{FTB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NFB}}\}\}}\ \neg_t{:}\mathbf{FTB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_t A)^{\mathbf{A}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{A}}\}\}}\ \neg_t{:}\mathbf{A}$$

The clause formation rules for connective $\neg_f$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{N}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{N}}\}\}}\ \neg_f{:}\mathbf{N} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\boldsymbol{N}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{F}}\}\}}\ \neg_f{:}\boldsymbol{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\boldsymbol{F}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{N}}\}\}}\ \neg_f{:}\boldsymbol{F} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\boldsymbol{T}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{B}}\}\}}\ \neg_f{:}\boldsymbol{T}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\boldsymbol{B}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\boldsymbol{T}}\}\}}\ \neg_f{:}\boldsymbol{B} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{NF}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NF}}\}\}}\ \neg_f{:}\mathbf{NF}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{NT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{FB}}\}\}}\ \neg_f{:}\mathbf{NT} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{FT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NB}}\}\}}\ \neg_f{:}\mathbf{FT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{NB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{FT}}\}\}}\ \neg_f{:}\mathbf{NB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{FB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NT}}\}\}}\ \neg_f{:}\mathbf{FB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{TB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{TB}}\}\}}\ \neg_f{:}\mathbf{TB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{NFT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NFB}}\}\}}\ \neg_f{:}\mathbf{NFT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{NFB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NFT}}\}\}}\ \neg_f{:}\mathbf{NFB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{NTB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{FTB}}\}\}}\ \neg_f{:}\mathbf{NTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{FTB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{NTB}}\}\}}\ \neg_f{:}\mathbf{FTB} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_f A)^{\mathbf{A}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{A}}\}\}}\ \neg_f{:}\mathbf{A}$$

The clause formation rules for connective $\wedge_t$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_t B)^{\mathbf{N}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{TB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{T}}, B^{\mathbf{N}}, B^{\boldsymbol{T}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{B}}, B^{\mathbf{N}}, B^{\boldsymbol{B}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{TB}}\}\}}\ \wedge_t{:}\mathbf{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_t B)^{\boldsymbol{N}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{T}}, A^{\mathbf{NT}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{T}}, B^{\mathbf{NT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{B}}, A^{\mathbf{NB}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{NTB}}, B^{\boldsymbol{N}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}\}\}}\ \wedge_t{:}\boldsymbol{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_t B)^{\boldsymbol{F}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\mathbf{FT}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\mathbf{FT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{B}}, A^{\mathbf{FB}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{B}}, B^{\mathbf{FB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{FTB}}, B^{\boldsymbol{F}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}\}\}}\ \wedge_t{:}\boldsymbol{F}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_t B)^{\boldsymbol{T}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{T}}, B^{\mathbf{TB}}\},\ C \cup \{A^{\boldsymbol{T}}, B^{\boldsymbol{T}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{TB}}\}\}}\ \wedge_t{:}\boldsymbol{T}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\boldsymbol{B}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{B}},B^{\mathbf{TB}}\},\ C\cup\{A^{\boldsymbol{B}},B^{\boldsymbol{B}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{TB}}\}\}}\ \wedge_t{:}\boldsymbol{B}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{NF}}\}\}}{\mathscr{C}\cup\{C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{F}},A^{\boldsymbol{T}},A^{\mathbf{NF}},A^{\mathbf{NT}},A^{\mathbf{FT}},A^{\mathbf{NFT}},B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{F}},B^{\boldsymbol{T}},B^{\mathbf{NF}},B^{\mathbf{NT}},B^{\mathbf{FT}},B^{\mathbf{NFT}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{F}},A^{\boldsymbol{B}},A^{\mathbf{NF}},A^{\mathbf{NB}},A^{\mathbf{FB}},A^{\mathbf{NFB}},B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{F}},B^{\boldsymbol{B}},B^{\mathbf{NF}},B^{\mathbf{NB}},B^{\mathbf{FB}},B^{\mathbf{NFB}}\},\ C\cup\{A^{\boldsymbol{N}},A^{\mathbf{NF}},A^{\mathbf{NT}},A^{\mathbf{NB}},A^{\mathbf{NFT}},A^{\mathbf{NFB}},A^{\mathbf{NTB}},A^{\mathbf{A}},B^{\boldsymbol{N}},B^{\mathbf{NF}},B^{\mathbf{NT}},B^{\mathbf{NB}},B^{\mathbf{NFT}},B^{\mathbf{NFB}},B^{\mathbf{NTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\boldsymbol{F}},A^{\mathbf{NF}},A^{\mathbf{FT}},A^{\mathbf{FB}},A^{\mathbf{NFT}},A^{\mathbf{NFB}},A^{\mathbf{FTB}},A^{\mathbf{A}},B^{\boldsymbol{F}},B^{\mathbf{NF}},B^{\mathbf{FT}},B^{\mathbf{FB}},B^{\mathbf{NFT}},B^{\mathbf{NFB}},B^{\mathbf{FTB}},B^{\mathbf{A}}\}\}}\ \wedge_t{:}\mathbf{NF}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{NT}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{T}},B^{\mathbf{NT}},B^{\mathbf{TB}},B^{\mathbf{NTB}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{NT}},B^{\boldsymbol{T}},B^{\mathbf{NT}}\},\ C\cup\{A^{\mathbf{NT}},A^{\mathbf{NTB}},B^{\mathbf{NT}},B^{\mathbf{NTB}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{NT}},A^{\mathbf{TB}},A^{\mathbf{NTB}}\}\}}\ \wedge_t{:}\mathbf{NT}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{FT}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{T}},B^{\mathbf{FT}},B^{\mathbf{TB}},B^{\mathbf{FTB}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{FT}},B^{\boldsymbol{T}},B^{\mathbf{FT}}\},\ C\cup\{A^{\mathbf{FT}},A^{\mathbf{FTB}},B^{\mathbf{FT}},B^{\mathbf{FTB}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{FT}},A^{\mathbf{TB}},A^{\mathbf{FTB}}\}\}}\ \wedge_t{:}\mathbf{FT}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{NB}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{B}},B^{\mathbf{NB}},B^{\mathbf{TB}},B^{\mathbf{NTB}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{NB}},B^{\boldsymbol{B}},B^{\mathbf{NB}}\},\ C\cup\{A^{\mathbf{NB}},A^{\mathbf{NTB}},B^{\mathbf{NB}},B^{\mathbf{NTB}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{NB}},A^{\mathbf{TB}},A^{\mathbf{NTB}}\}\}}\ \wedge_t{:}\mathbf{NB}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{FB}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{B}},B^{\mathbf{FB}},B^{\mathbf{TB}},B^{\mathbf{FTB}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{FB}},B^{\boldsymbol{B}},B^{\mathbf{FB}}\},\ C\cup\{A^{\mathbf{FB}},A^{\mathbf{FTB}},B^{\mathbf{FB}},B^{\mathbf{FTB}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{FB}},A^{\mathbf{TB}},A^{\mathbf{FTB}}\}\}}\ \wedge_t{:}\mathbf{FB}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{TB}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{TB}}\},\ C\cup\{A^{\mathbf{TB}}\}\}}\ \wedge_t{:}\mathbf{TB}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{NFT}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{T}},B^{\mathbf{NT}},B^{\mathbf{FT}},B^{\mathbf{TB}},B^{\mathbf{NFT}},B^{\mathbf{NTB}},B^{\mathbf{FTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{NT}},A^{\mathbf{FT}},A^{\mathbf{NFT}},B^{\boldsymbol{T}},B^{\mathbf{NT}},B^{\mathbf{FT}},B^{\mathbf{NFT}}\},\ C\cup\{A^{\mathbf{NT}},A^{\mathbf{NFT}},A^{\mathbf{NTB}},A^{\mathbf{A}},B^{\mathbf{NT}},B^{\mathbf{NFT}},B^{\mathbf{NTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\mathbf{FT}},A^{\mathbf{NFT}},A^{\mathbf{FTB}},A^{\mathbf{A}},B^{\mathbf{FT}},B^{\mathbf{NFT}},B^{\mathbf{FTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{NT}},A^{\mathbf{FT}},A^{\mathbf{TB}},A^{\mathbf{NFT}},A^{\mathbf{NTB}},A^{\mathbf{FTB}},A^{\mathbf{A}}\}\}}\ \wedge_t{:}\mathbf{NFT}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{NFB}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{B}},B^{\mathbf{NB}},B^{\mathbf{FB}},B^{\mathbf{TB}},B^{\mathbf{NFB}},B^{\mathbf{NTB}},B^{\mathbf{FTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{NB}},A^{\mathbf{FB}},A^{\mathbf{NFB}},B^{\boldsymbol{B}},B^{\mathbf{NB}},B^{\mathbf{FB}},B^{\mathbf{NFB}}\},\ C\cup\{A^{\mathbf{NB}},A^{\mathbf{NFB}},A^{\mathbf{NTB}},A^{\mathbf{A}},B^{\mathbf{NB}},B^{\mathbf{NFB}},B^{\mathbf{NTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\mathbf{FB}},A^{\mathbf{NFB}},A^{\mathbf{FTB}},A^{\mathbf{A}},B^{\mathbf{FB}},B^{\mathbf{NFB}},B^{\mathbf{FTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{NB}},A^{\mathbf{FB}},A^{\mathbf{TB}},A^{\mathbf{NFB}},A^{\mathbf{NTB}},A^{\mathbf{FTB}},A^{\mathbf{A}}\}\}}\ \wedge_t{:}\mathbf{NFB}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{NTB}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{TB}},B^{\mathbf{NTB}}\},\ C\cup\{A^{\mathbf{NTB}},B^{\mathbf{NTB}}\},\ C\cup\{A^{\mathbf{TB}},A^{\mathbf{NTB}}\}\}}\ \wedge_t{:}\mathbf{NTB}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{FTB}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{TB}},B^{\mathbf{FTB}}\},\ C\cup\{A^{\mathbf{FTB}},B^{\mathbf{FTB}}\},\ C\cup\{A^{\mathbf{TB}},A^{\mathbf{FTB}}\}\}}\ \wedge_t{:}\mathbf{FTB}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\wedge_t B)^{\mathbf{A}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{TB}},B^{\mathbf{NTB}},B^{\mathbf{FTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\mathbf{NTB}},A^{\mathbf{A}},B^{\mathbf{NTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\mathbf{FTB}},A^{\mathbf{A}},B^{\mathbf{FTB}},B^{\mathbf{A}}\},\ C\cup\{A^{\mathbf{TB}},A^{\mathbf{NTB}},A^{\mathbf{FTB}},A^{\mathbf{A}}\}\}}\ \wedge_t{:}\mathbf{A}$$

The clause formation rules for connective $\vee_t$ are given by

$$\frac{\mathscr{C}\cup\{C\cup\{(A\vee_t B)^{\mathbf{N}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{F}},B^{\mathbf{NF}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},B^{\mathbf{N}},B^{\boldsymbol{N}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{F}},B^{\mathbf{N}},B^{\boldsymbol{F}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{F}},A^{\mathbf{NF}}\}\}}\ \vee_t{:}\mathbf{N}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\vee_t B)^{\boldsymbol{N}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{N}},B^{\mathbf{NF}}\},\ C\cup\{A^{\boldsymbol{N}},B^{\boldsymbol{N}}\},\ C\cup\{A^{\boldsymbol{N}},A^{\mathbf{NF}}\}\}}\ \vee_t{:}\boldsymbol{N}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\vee_t B)^{\boldsymbol{F}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\boldsymbol{F}},B^{\mathbf{NF}}\},\ C\cup\{A^{\boldsymbol{F}},B^{\boldsymbol{F}}\},\ C\cup\{A^{\boldsymbol{F}},A^{\mathbf{NF}}\}\}}\ \vee_t{:}\boldsymbol{F}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\vee_t B)^{\boldsymbol{T}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{F}},B^{\boldsymbol{T}},B^{\mathbf{NF}},B^{\mathbf{NT}},B^{\mathbf{FT}},B^{\mathbf{NFT}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{T}},A^{\mathbf{NT}},B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{T}},B^{\mathbf{NT}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{F}},A^{\boldsymbol{T}},A^{\mathbf{FT}},B^{\mathbf{N}},B^{\boldsymbol{F}},B^{\boldsymbol{T}},B^{\mathbf{FT}}\},\ C\cup\{A^{\boldsymbol{T}},A^{\mathbf{NT}},A^{\mathbf{FT}},A^{\mathbf{NFT}},B^{\boldsymbol{T}},B^{\mathbf{NT}},B^{\mathbf{FT}},B^{\mathbf{NFT}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{F}},A^{\boldsymbol{T}},A^{\mathbf{NF}},A^{\mathbf{NT}},A^{\mathbf{FT}},A^{\mathbf{NFT}}\}\}}\ \vee_t{:}\boldsymbol{T}$$

$$\frac{\mathscr{C}\cup\{C\cup\{(A\vee_t B)^{\boldsymbol{B}}\}\}}{\mathscr{C}\cup\{C\cup\{B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{F}},B^{\boldsymbol{B}},B^{\mathbf{NF}},B^{\mathbf{NB}},B^{\mathbf{FB}},B^{\mathbf{NFB}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{B}},A^{\mathbf{NB}},B^{\mathbf{N}},B^{\boldsymbol{N}},B^{\boldsymbol{B}},B^{\mathbf{NB}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{F}},A^{\boldsymbol{B}},A^{\mathbf{FB}},B^{\mathbf{N}},B^{\boldsymbol{F}},B^{\boldsymbol{B}},B^{\mathbf{FB}}\},\ C\cup\{A^{\boldsymbol{B}},A^{\mathbf{NB}},A^{\mathbf{FB}},A^{\mathbf{NFB}},B^{\boldsymbol{B}},B^{\mathbf{NB}},B^{\mathbf{FB}},B^{\mathbf{NFB}}\},\ C\cup\{A^{\mathbf{N}},A^{\boldsymbol{N}},A^{\boldsymbol{F}},A^{\boldsymbol{B}},A^{\mathbf{NF}},A^{\mathbf{NB}},A^{\mathbf{FB}},A^{\mathbf{NFB}}\}\}}\ \vee_t{:}\boldsymbol{B}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{NF}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NF}}\},\ C \cup \{A^{\mathbf{NF}}\}\}}\ \vee_t{:}\mathbf{NF}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{NT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NT}}, B^{\boldsymbol{N}}, B^{\mathbf{NT}}\},\ C \cup \{A^{\mathbf{NT}}, A^{\mathbf{NFT}}, B^{\mathbf{NT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{NFT}}\}\}}\ \vee_t{:}\mathbf{NT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{FT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FT}}, B^{\boldsymbol{F}}, B^{\mathbf{FT}}\},\ C \cup \{A^{\mathbf{FT}}, A^{\mathbf{NFT}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FT}}, A^{\mathbf{NFT}}\}\}}\ \vee_t{:}\mathbf{FT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{NB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NB}}, B^{\boldsymbol{N}}, B^{\mathbf{NB}}\},\ C \cup \{A^{\mathbf{NB}}, A^{\mathbf{NFB}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NB}}, A^{\mathbf{NFB}}\}\}}\ \vee_t{:}\mathbf{NB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{FB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FB}}, B^{\boldsymbol{F}}, B^{\mathbf{FB}}\},\ C \cup \{A^{\mathbf{FB}}, A^{\mathbf{NFB}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FB}}, A^{\mathbf{NFB}}\}\}}\ \vee_t{:}\mathbf{FB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{TB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{NT}}, A^{\mathbf{FT}}, A^{\mathbf{TB}}, A^{\mathbf{NFT}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\boldsymbol{T}}, B^{\mathbf{NT}}, B^{\mathbf{FT}}, B^{\mathbf{TB}}, B^{\mathbf{NFT}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\}\}}\ \vee_t{:}\mathbf{TB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{NFT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NF}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{NFT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFT}}\}\}}\ \vee_t{:}\mathbf{NFT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{NFB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NF}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{NFB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFB}}\}\}}\ \vee_t{:}\mathbf{NFB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{NTB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{NTB}}, B^{\boldsymbol{N}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{NT}}, A^{\mathbf{NFT}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}, B^{\mathbf{NT}}, B^{\mathbf{NFT}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NB}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}\}\}}\ \vee_t{:}\mathbf{NTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{FTB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{FTB}}, B^{\boldsymbol{F}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{FT}}, A^{\mathbf{NFT}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{FB}}, A^{\mathbf{NFB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}\}\}}\ \vee_t{:}\mathbf{FTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_t B)^{\mathbf{A}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NF}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NFT}}, A^{\mathbf{A}}, B^{\mathbf{NFT}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NFB}}, A^{\mathbf{A}}, B^{\mathbf{NFB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{A}}\}\}}\ \vee_t{:}\mathbf{A}$$

The clause formation rules for connective $\wedge_f$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{N}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{B}}, B^{\mathbf{FB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{B}}, B^{\mathbf{N}}, B^{\boldsymbol{B}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{B}}, A^{\mathbf{FB}}\}\}}\ \wedge_f{:}\mathbf{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\boldsymbol{N}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{B}}, B^{\mathbf{NF}}, B^{\mathbf{NB}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{F}}, A^{\mathbf{NF}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{F}}, B^{\mathbf{NF}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{B}}, A^{\mathbf{NB}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NB}}, A^{\mathbf{NFB}}, B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{B}}, A^{\mathbf{NF}}, A^{\mathbf{NB}}, A^{\mathbf{FB}}, A^{\mathbf{NFB}}\}\}}\ \wedge_f{:}\boldsymbol{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\boldsymbol{F}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{FB}}\},\ C \cup \{A^{\boldsymbol{F}}, B^{\boldsymbol{F}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FB}}\}\}}\ \wedge_f{:}\boldsymbol{F}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\boldsymbol{T}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\mathbf{FT}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\mathbf{FT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{TB}}, B^{\mathbf{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{TB}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{FT}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}, B^{\boldsymbol{T}}, B^{\mathbf{FT}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}\}\}}\ \wedge_f{:}\boldsymbol{T}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\boldsymbol{B}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{B}}, B^{\mathbf{FB}}\},\ C \cup \{A^{\boldsymbol{B}}, B^{\boldsymbol{B}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{FB}}\}\}}\ \wedge_f{:}\boldsymbol{B}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{NF}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, B^{\boldsymbol{F}}, B^{\mathbf{NF}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFB}}, B^{\mathbf{NF}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FB}}, A^{\mathbf{NFB}}\}\}}\ \wedge_f{:}\mathbf{NF}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{NT}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{FT}}, A^{\mathbf{NFT}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}, B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{NT}}, A^{\mathbf{FT}}, A^{\mathbf{TB}}, A^{\mathbf{NFT}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\boldsymbol{T}}, B^{\mathbf{NT}}, B^{\mathbf{FT}}, B^{\mathbf{TB}}, B^{\mathbf{NFT}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\}\}}\ \wedge_f{:}\mathbf{NT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{FT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FT}}, B^{\boldsymbol{F}}, B^{\mathbf{FT}}\},\ C \cup \{A^{\mathbf{FT}}, A^{\mathbf{FTB}}, B^{\mathbf{FT}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{FTB}}\}\}}\ \wedge_f{:}\mathbf{FT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{NB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{B}}, B^{\mathbf{NB}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}\},\ C \cup \{A^{\mathbf{NB}}, A^{\mathbf{NFB}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, A^{\mathbf{FB}}, A^{\mathbf{NFB}}\}\}}\ \wedge_f{:}\mathbf{NB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{FB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{FB}}\},\ C \cup \{A^{\mathbf{FB}}\}\}}\ \wedge_f{:}\mathbf{FB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{TB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{B}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{TB}}, B^{\boldsymbol{B}}, B^{\mathbf{TB}}\},\ C \cup \{A^{\mathbf{TB}}, A^{\mathbf{FTB}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}\}\}}\ \wedge_f{:}\mathbf{TB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{NFT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FT}}, A^{\mathbf{NFT}}, B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{A}}, B^{\mathbf{NF}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{FT}}, A^{\mathbf{NFT}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}\}\}}\ \wedge_f{:}\mathbf{NFT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{NFB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{NFB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{FB}}, A^{\mathbf{NFB}}\}\}}\ \wedge_f{:}\mathbf{NFB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{NTB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{B}}, B^{\mathbf{NB}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{NB}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{TB}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}\}\}}\ \wedge_f{:}\mathbf{NTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{FTB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{FB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{FTB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{FB}}, A^{\mathbf{FTB}}\}\}}\ \wedge_f{:}\mathbf{FTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge_f B)^{\mathbf{A}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{FB}}, B^{\mathbf{NFB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NFB}}, A^{\mathbf{A}}, B^{\mathbf{NFB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{FB}}, A^{\mathbf{NFB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}\}\}}\ \wedge_f{:}\mathbf{A}$$

The clause formation rules for connective $\vee_f$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{N}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{T}}, B^{\mathbf{NT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{T}}, B^{\mathbf{N}}, B^{\boldsymbol{T}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{T}}, A^{\mathbf{NT}}\}\}}\ \vee_f{:}\mathbf{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\boldsymbol{N}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NT}}\},\ C \cup \{A^{\boldsymbol{N}}, B^{\boldsymbol{N}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NT}}\}\}}\ \vee_f{:}\boldsymbol{N}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\boldsymbol{F}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{F}}, A^{\mathbf{NF}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{F}}, B^{\mathbf{NF}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\mathbf{FT}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\mathbf{FT}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FT}}, A^{\mathbf{NFT}}, B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{FT}}, A^{\mathbf{NFT}}\}\}}\ \vee_f{:}\boldsymbol{F}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\boldsymbol{T}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{T}}, B^{\mathbf{NT}}\},\ C \cup \{A^{\boldsymbol{T}}, B^{\boldsymbol{T}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{NT}}\}\}}\ \vee_f{:}\boldsymbol{T}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\boldsymbol{B}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{B}}, A^{\mathbf{NB}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{TB}}, B^{\mathbf{N}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{TB}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}\}\}}\ \vee_f{:}\boldsymbol{B}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{NF}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, B^{\boldsymbol{N}}, B^{\mathbf{NF}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFT}}, B^{\mathbf{NF}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{NFT}}\}\}}\ \vee_f{:}\mathbf{NF}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{NT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NT}}\},\ C \cup \{A^{\mathbf{NT}}\}\}}\ \vee_f{:}\mathbf{NT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{FT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{T}}, B^{\mathbf{NT}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{FT}}, B^{\boldsymbol{T}}, B^{\mathbf{FT}}\},\ C \cup \{A^{\mathbf{FT}}, A^{\mathbf{NFT}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{NT}}, A^{\mathbf{FT}}, A^{\mathbf{NFT}}\}\}}\ \vee_f{:}\mathbf{FT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{NB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NB}}, B^{\boldsymbol{N}}, B^{\mathbf{NB}}\},\ C \cup \{A^{\mathbf{NB}}, A^{\mathbf{NTB}}, B^{\mathbf{NB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{NTB}}\}\}}\ \vee_f{:}\mathbf{NB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{FB}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{B}}, A^{\mathbf{NF}}, A^{\mathbf{NB}}, A^{\mathbf{FB}}, A^{\mathbf{NFB}}, B^{\mathbf{N}}, B^{\boldsymbol{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{B}}, B^{\mathbf{NF}}, B^{\mathbf{NB}}, B^{\mathbf{FB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{N}}, A^{\boldsymbol{F}}, A^{\boldsymbol{T}}, A^{\boldsymbol{B}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}, B^{\mathbf{N}}, B^{\boldsymbol{F}}, B^{\boldsymbol{T}}, B^{\boldsymbol{B}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\boldsymbol{F}}, A^{\mathbf{NF}}, A^{\mathbf{FT}}, A^{\mathbf{FB}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\boldsymbol{F}}, B^{\mathbf{NF}}, B^{\mathbf{FT}}, B^{\mathbf{FB}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{B}}, A^{\mathbf{NB}}, A^{\mathbf{FB}}, A^{\mathbf{TB}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\boldsymbol{B}}, B^{\mathbf{NB}}, B^{\mathbf{FB}}, B^{\mathbf{TB}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\}\}}\ \vee_f{:}\mathbf{FB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{TB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{T}}, B^{\mathbf{NT}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{TB}}, B^{\boldsymbol{T}}, B^{\mathbf{TB}}\},\ C \cup \{A^{\mathbf{TB}}, A^{\mathbf{NTB}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{NT}}, A^{\mathbf{TB}}, A^{\mathbf{NTB}}\}\}}\ \vee_f{:}\mathbf{TB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{NFT}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{NFT}}, B^{\mathbf{NFT}}\},\ C \cup \{A^{\mathbf{NT}}, A^{\mathbf{NFT}}\}\}}\ \vee_f{:}\mathbf{NFT}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{NFB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NT}}, B^{\mathbf{NB}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NB}}, A^{\mathbf{NFB}}, B^{\boldsymbol{N}}, B^{\mathbf{NF}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}\},\ C \cup \{A^{\mathbf{NF}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{A}}, B^{\mathbf{NF}}, B^{\mathbf{NFT}}, B^{\mathbf{NFB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NB}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}, B^{\mathbf{NB}}, B^{\mathbf{NFB}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{N}}, A^{\mathbf{NF}}, A^{\mathbf{NT}}, A^{\mathbf{NB}}, A^{\mathbf{NFT}}, A^{\mathbf{NFB}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}\}\}}\ \vee_f{:}\mathbf{NFB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{NTB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NT}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{NTB}}, B^{\mathbf{NTB}}\},\ C \cup \{A^{\mathbf{NT}}, A^{\mathbf{NTB}}\}\}}\ \vee_f{:}\mathbf{NTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{FTB}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\boldsymbol{T}}, B^{\mathbf{NT}}, B^{\mathbf{FT}}, B^{\mathbf{TB}}, B^{\mathbf{NFT}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{FT}}, A^{\mathbf{TB}}, A^{\mathbf{FTB}}, B^{\boldsymbol{T}}, B^{\mathbf{FT}}, B^{\mathbf{TB}}, B^{\mathbf{FTB}}\},\ C \cup \{A^{\mathbf{FT}}, A^{\mathbf{NFT}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{FT}}, B^{\mathbf{NFT}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{TB}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}, B^{\mathbf{TB}}, B^{\mathbf{NTB}}, B^{\mathbf{FTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\boldsymbol{T}}, A^{\mathbf{NT}}, A^{\mathbf{FT}}, A^{\mathbf{TB}}, A^{\mathbf{NFT}}, A^{\mathbf{NTB}}, A^{\mathbf{FTB}}, A^{\mathbf{A}}\}\}}\ \vee_f{:}\mathbf{FTB}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee_f B)^{\mathbf{A}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{NT}}, B^{\mathbf{NFT}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NFT}}, A^{\mathbf{A}}, B^{\mathbf{NFT}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NTB}}, A^{\mathbf{A}}, B^{\mathbf{NTB}}, B^{\mathbf{A}}\},\ C \cup \{A^{\mathbf{NT}}, A^{\mathbf{NFT}}, A^{\mathbf{NTB}}, A^{\mathbf{A}}\}\}}\ \vee_f{:}\mathbf{A}$$

Theorem 28. *Let $T(\mathscr{C})$ be the result of exhaustively applying the translation rules to a clause set $\mathscr{C}$. Then $T(\mathscr{C})$ is a set of proper clauses, i.e., $T(\mathscr{C})$ contains only signed atoms (all connectives are eliminated). Furthermore, $T(\mathscr{C})$ is satisfiable iff $\mathscr{C}$ is.*

Proposition 29. *Let A be a sentence and Δ be a set of sentences. Then*

1. $\models A$ *iff* $\{\{A^v \mid v \in V^-\}\}$ *is unsatisfiable.*
2. $\Delta \models A$ *iff*

$$\{\{B^w \mid w \in V^+\} \mid B \in \Delta\} \cup \{\{A^v \mid v \in V^-\}\}$$

is unsatisfiable.

Definition 30. A clause R is a *resolvent* of clauses C_1, C_2 if $R = (C_1 \setminus \{A^{v_1}\}) \cup (C_2 \setminus \{A^{v_2}\})$ where $v_1 \neq v_2$

Definition 31. A *resolution refutation* of a clause set $\mathscr{C}$ is a sequence of clauses $C_1, \ldots, C_n$ so that for every i, $C_i \in \mathscr{C}$ or C_i is a resolvent of C_j, C_k with $j, k < i$, and $C_n = \emptyset$.

Theorem 32. *A clause set $\mathscr{C}$ is unsatisfiable iff it has a resolution refutation.*

Proof. See Theorems 3.14 and 3.19 of Baaz and Fermüller [1] or Theorems 2.5.5 and 2.5.8 of Zach [15]. □

Corollary 33. $\Delta \models A$ *iff*

$$T(\{\{B^w \mid w \in V^+\} \mid B \in \Delta\} \cup \{\{A^v \mid v \in V^-\}\})$$

has a resolution refutation.

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A shramko-wansing.lgc – specification of Shramko-Wansing logic

```
logic "Shramko-Wansing".
truth_values { e , n , f , t , b , nf , nt , ft , nb , fb , tb ,
    nft , nfb , ntb , ftb , a }.
designated_truth_values { t, nt, tb, ntb }.
operator(negt/1, mapping {
          (e  ) : e  ,
          (n  ) : t  ,
          (f  ) : b  ,
          (t  ) : n  ,
          (b  ) : f  ,
          (nf ) : tb ,
          (nt ) : nt ,
          (nb ) : ft ,
          (ft ) : nb ,
          (fb ) : fb ,
          (tb ) : nf ,
          (nft) : ntb,
          (nfb) : ftb,
          (ntb) : nft,
          (ftb) : nfb,
          (a  ) : a
         }
).
operator(negf/1, mapping {
          (e  ) : e  ,
          (n  ) : f  ,
          (f  ) : n  ,
          (t  ) : b  ,
          (b  ) : t  ,
          (nf ) : nf ,
          (nt ) : fb ,
          (nb ) : ft ,
          (ft ) : nb ,
          (fb ) : nt ,
          (tb ) : tb ,
          (nft) : nfb,
          (nfb) : nft,
          (ntb) : ftb,
```

```
          (ftb) : ntb,
          (a  ) : a
         }
).
ordering(truth, "nf < { f < { fb < {ftb < tb, b < tb}, ft < {ftb, t
     < tb}, e  < {b,t}}, nfb < { fb, a  < {ftb, ntb}, nb < {b, ntb
    }}, nft < { ft, a, nt < {t, ntb < tb}}, n < {e, nb, nt}}").
ordering(falsity, "nt < { n < { nf < {f < fb, nfb < fb}, e < {f, b
    < fb}, nb  < {nfb, b}}, nft < { nf, ft  < {f, ftb}, a < {nfb,
    ftb}}, t < { e, ft, tb < {b, ftb < fb}}, ntb < {nb, a, tb}}").
operator(andt  /2, inf(truth)).
operator(ort   /2, sup(truth)).
operator(andf  /2, inf(falsity)).
operator(orf   /2, sup(falsity)).
```

B shramko-wansing.cfg – LaTeX translations

```
texName(n, "\\boldsymbol{N}").
texName(f, "\\boldsymbol{F}").
texName(t, "\\boldsymbol{T}").
texName(b, "\\boldsymbol{B}").
texName(e, "\\mathbf{N}").
texName(a, "\\mathbf{A}").
texName(nf, "\\mathbf{NF}").
texName(nt, "\\mathbf{NT}").
texName(nb, "\\mathbf{NB}").
texName(ft, "\\mathbf{FT}").
texName(fb, "\\mathbf{FB}").
texName(tb, "\\mathbf{TB}").
texName(nft, "\\mathbf{NFT}").
texName(nfb, "\\mathbf{NFB}").
texName(ntb, "\\mathbf{NTB}").
texName(ftb, "\\mathbf{FTB}").

texName(andt,     "\\land_t"  ).
texName(ort,      "\\lor_t"   ).
texName(negt,     "\\lnot_t"  ).
texInfix(andt).
texInfix(ort).
texPrefix(negt).
texName(andf,     "\\land_f"  ).
texName(orf,      "\\lor_f"   ).
texName(negf,     "\\lnot_f"  ).
texInfix(andf).
texInfix(orf).
texPrefix(negf).

texName(forall, "\\forall_t").
texName(exists, "\\exists_t").

texExtra("ShortName", "\\mathbf{SIXTEEN}_{2,\\mathit{des}}").
texExtra("Semantics", "The connectives $\\land_t$ and $\\lor_t$ of
    $\\ShortName$ are defined as the inf and sup of the $\\le_t$
    ordering given in \\cref{fig}. The connectives $\\land_f$ and $
    \\lor_f$ correspond to inf and sup of~$\\le_f$. For simplicity,
     we leave out the $\\le_i$ ordering and other operators and
```

```
    inversions defined in \\cite{ml}.\\begin{figure}\\centering\\
    includegraphics[width=.5\\textwidth]{shramko-wansing.png}\\
    caption{The $\\le_t$ and $\\le_f$ orderings}\\label{fig}\\end{
    figure}").
texExtra("Preamble","\\usepackage{graphicx,amsbsy}").
texExtra("Link","https://logic.at/multlog/shramko-wansing.pdf").
```