Analytic Proof Systems for Dunn And Belnap's Logic Of First Degree Entailment **FDE**

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Abstract

We give sequent calculus, analytic tableaux, natural deduction, and clause translation systems for resolution for Dunn and Belnap's logic of First Degree Entailment **FDE**.

1 Introduction

In this paper we present calculi for Dunn and Belnap's logic of First Degree Entailment **FDE**. **FDE** has four truth values $\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}$ (with $\mathbf{t}, \mathbf{b}$ designated), connectives $\sim$, $\neg_b$, $\wedge$, $\vee$, $\rightarrow_e$, $\rightarrow_b$, $\rightarrow_l$, $-$, $\otimes$, $\oplus$, $\circlearrowleft$, $\triangle$, $\square$, $\circ$, and quantifiers $\forall$, $\exists$, $\bigotimes$, $\bigoplus$. Its syntax and semantics is detailed in section 2.

FDE was introduced as a "useful four-valued logic" in [7] and [9]. The operators we consider are (most of) those discussed in Omori and Wansing's survey [14], whose notation we also follow. The sequent systems presented below are extensions of those in [5], where **FDE** was used as the running example (the $\bigotimes$ quantifier was denoted U there). The rules also improve on those of Ruet's [16] sequent calculus. 4-sided sequent systems for extensions of FDE similar to ours were also given by Wintein and Muskens [23, 24].

We first present a 4-sided sequent calculus in section 3. The fundamental idea for many-sided sequent calculi for finite-valued logics goes back to Schröter [20], Rousseau [15], Takahashi [22]. We follow the method given by Baaz et al. [5] and Zach [25] for constructing inference rules. This guarantees that our system automatically has soundness and completness theorems, cut-elimination theorem, mid-sequent theorem, and Maehara lemma (interpolation). For proofs of these results see [5, 25].

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See appendices A and B for the specification of Dunn and Belnap's logic of First Degree Entailment.

Signed tableau systems for finite-valued logics were proposed by Surma [21] and Carnielli [8], and generalized by Hähnle [13]. In section 4, we present a signed tableau system for Dunn and Belnap's logic of First Degree Entailment.

Many-valued natural deduction systems for finite-valued logics have been investigated by Baaz, Fermüller, and Zach [4] and Zach [25]. We give the introduction and elimination rules for the natural deduction system for **FDE** in section 5.

In addition to Hähnle's work on tableaux-based theorem proving for finite-valued logic, Baaz and Fermüller [2] have studied resolution calculi for clauses of signed literals. In order for these calculi to be used to prove that formulas of Dunn and Belnap's logic of First Degree Entailment are tautologies or follow from some others, it is necessary to produce sets of signed clauses. In section 6, we present a translation calculus that yields a set of clauses from a set of formulas.

The rules we provide are optimal in each case, and use the algorithms developed by Salzer [17, 18].

2 Syntax and semantics

Definition 1. The *first order language* $\mathscr{L}$ for Dunn and Belnap's logic of First Degree Entailment consists of

1. free variables: $a_0, a_1, a_2, \ldots$
2. bound variables: $x_0, x_1, x_2, \ldots$
3. function symbols of arity i ($i \in \mathbb{N}$), including constants: $f_0^i, f_1^i, f_2^i, \ldots$
4. predicate constants of arity i ($i \in \mathbb{N}$): $P_0^i, P_1^i, P_2^i, \ldots$
5. propositional connectives, arity given in parenthesis: $\sim$ (1) , $\neg_b$ (1), $\wedge$ (2), $\vee$ (2), $\to_e$ (2), $\to_b$ (2), $\to_l$ (2), $-$ (1), $\otimes$ (2), $\oplus$ (2), $\circlearrowleft$ (1), $\triangle$ (1), $\square$ (1) and $\circ$ (1)
6. quantifiers: $\forall$, $\exists$, $\bigotimes$ and $\bigoplus$
7. auxiliary symbols: "(", ")" and ","

Terms are defined inductively: Every individual constant and free variable is a term. If f^n is a function symbol of arity n, and $t_1, \ldots, t_n$ are terms, then $f^n(t_1, \ldots, t_n)$ is a term.

Formulas are also defined inductively:

1. If P^n is a predicate symbol of arity n, and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is a formula. It is called *atomic* or an *atom*.
2. If A is a formula, so is $\sim A$.
3. If A is a formula, so is $\neg_b A$.

4. If A and B are formulas, so is $(A \wedge B)$.

5. If A and B are formulas, so is $(A \vee B)$.

6. If A and B are formulas, so is $(A \rightarrow_e B)$.

7. If A and B are formulas, so is $(A \rightarrow_b B)$.

8. If A and B are formulas, so is $(A \rightarrow_l B)$.

9. If A is a formula, so is $-A$.

10. If A and B are formulas, so is $(A \otimes B)$.

11. If A and B are formulas, so is $(A \oplus B)$.

12. If A is a formula, so is $\circlearrowleft A$.

13. If A is a formula, so is $\triangle A$.

14. If A is a formula, so is $\Box A$.

15. If A is a formula, so is $\circ A$.

16. If A is a formula not containing the bound variable x, a is a free variable and Q is a quantifier, then $(\mathsf{Q}x)A(x)$, where $A(x)$ is obtained from A by replacing a by x at every occurrence of a in A, is a formula.

A formula is called *open*, if it contains free variables, and *closed* otherwise. A formula without quantifiers is called *quantifier-free*.

As a notational convention, lowercase letters from the beginning of the alphabet $(a, b, c, \ldots)$ will be used to denote free variables, $f, g, h, \ldots$ for function symbols and constants, $x, y, z, \ldots$ for bound variables, all possibly indexed. Uppercase letters $A, B, C, \ldots$ will stand for formulas, greek letters $\Gamma, \Delta, \Lambda, \ldots$ for sequences and sets of formulas, t and s for terms. The symbols $\Box$ and Q stand for general propositional connectives and quantifiers, respectively.

Definition 2. We will use α as a variable for free variables (*eigenvariable*) and τ as a variable for terms (*term variable*). A formula consisting of some formula variables, eigenvariables and term variables is called a schema.

A *pre-instance* A' of a schema A is an actual formula from the formulas of $\mathscr{L}$ which contains occurrences of the eigenvariables and term variables of A.

An *instance* A'' of A is a pre-instance A' of A, where the eigenvariables and term variables have been replaced by free variables and terms not occurring in A'.

The set of truth values of **FDE** forms a so-called bilattice (see [12]) with two orderings $\leq_t$ (truth) and $\leq_k$ (information). They are defined as $\mathbf{f} <_t \mathbf{b}, \mathbf{n} <_t \mathbf{t}$ and $\mathbf{n} <_k \mathbf{t}, \mathbf{f} <_t \mathbf{b}$. The operators $\wedge$ and $\vee$ are the inf and sup in the $\leq_t$ ordering; the operators $\otimes$ and $\oplus$ the inf and sup in the $\leq_k$ ordering. The quantifiers $\forall$, $\exists$, $\bigotimes$, and $\bigoplus$, respectively, are the quantifiers induced by these connectives (i.e., the inf and sup of the truth-value distributions of the matrix $A(x)$ of $(\mathsf{Q}x)A(x)$). The operator $\sim$ is the usual negation of **FDE**, while $\neg_b$ is the Boolean negation, and $-$ is the "conflation" [10] of the $\leq_k$ order. $\triangle$ is Baaz's determinateness operator [1] (studied in **FDE** by Sano and Omori [19]). $\Box$ is the modality studied by Font and Rius [11]. $\circ$ is the classicality operator. The quarter turn operator $\circlearrowleft$ was introduced by Ruet [16]. The conditional $x \rightarrow_e y$ is defined as $\sim A \vee B$, while $\rightarrow_b$ is the Boolean conditional and $\rightarrow_l$ the Łukasiewicz conditional. See [14] for definitions and references.

Definition 3. The *matrix* for Dunn and Belnap's logic of First Degree Entailment is given by:

1. the set of *truth values* $V = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$,
2. the set $V^+ = \{\mathbf{t}, \mathbf{b}\} \subseteq V$ of *designated truth values*,
3. the truth functions for connectives $\sim$, $\neg_b$, $\wedge$, $\vee$, $\rightarrow_e$, $\rightarrow_b$, $\rightarrow_l$, $-$, $\otimes$, $\oplus$, $\circlearrowleft$, $\triangle$, $\Box$ and $\circ$, as given below;
4. the truth functions for quantifiers $\forall$, $\exists$, $\bigotimes$ and $\bigoplus$, as given below.

The set of *undesignated values* is $V^- = V \setminus V^+ = \{\mathbf{n}, \mathbf{f}\}$.

The truth functions for connectives $\sim$, $\neg_b$, $\wedge$, $\vee$, $\rightarrow_e$, $\rightarrow_b$, $\rightarrow_l$, $-$, $\otimes$, $\oplus$, $\circlearrowleft$, $\triangle$, $\Box$ and $\circ$ are defined by

$\widetilde{\sim}$	
t	**f**
b	**b**
n	**n**
f	**t**

$\widetilde{\neg_b}$	
t	**f**
b	**n**
n	**b**
f	**t**

$\widetilde{\wedge}$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$\widetilde{\vee}$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

$\widetilde{\rightarrow_e}$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**t**	**t**	**t**	**t**
f	**t**	**t**	**t**	**t**

$\widetilde{\rightarrow_b}$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**t**	**n**	**n**
n	**t**	**b**	**t**	**b**
f	**t**	**t**	**t**	**t**

$\widetilde{\rightarrow_l}$	**t**	**b**	**n**	**f**
t	**t**	**f**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**t**	**n**	**t**	**n**
f	**t**	**t**	**t**	**t**

$\widetilde{-}$	
t	**t**
b	**n**
n	**b**
f	**f**

$\widetilde{\otimes}$	**t**	**b**	**n**	**f**
t	**t**	**t**	**n**	**n**
b	**t**	**b**	**n**	**f**
n	**n**	**n**	**n**	**n**
f	**n**	**f**	**n**	**f**

$\widetilde{\oplus}$	**t**	**b**	**n**	**f**
t	**t**	**b**	**t**	**b**
b	**b**	**b**	**b**	**b**
n	**t**	**b**	**n**	**f**
f	**b**	**b**	**f**	**f**

$\widetilde{\circlearrowleft}$	
t	**b**
b	**f**
n	**t**
f	**n**

$\widetilde{\triangle}$	
t	**t**
b	**t**
n	**f**
f	**f**

$\widetilde{\Box}$	
t	**t**
b	**f**
n	**f**
f	**f**

$\widetilde{\circ}$	
t	**t**
b	**f**
n	**f**
f	**t**

The truth functions for quantifiers $\forall$, $\exists$, $\bigotimes$ and $\bigoplus$ are defined by

$\widetilde{\forall}$	
$\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{t},\mathbf{b},\mathbf{n}\}$	$\mathbf{f}$
$\{\mathbf{t},\mathbf{b},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{t},\mathbf{b}\}$	$\mathbf{b}$
$\{\mathbf{t},\mathbf{n},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{t},\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{t},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{t}\}$	$\mathbf{t}$
$\{\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{b},\mathbf{n}\}$	$\mathbf{f}$
$\{\mathbf{b},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{b}\}$	$\mathbf{b}$
$\{\mathbf{n},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{f}\}$	$\mathbf{f}$

$\widetilde{\exists}$	
$\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{b},\mathbf{n}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{b},\mathbf{f}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{b}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{n},\mathbf{f}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{n}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{f}\}$	$\mathbf{t}$
$\{\mathbf{t}\}$	$\mathbf{t}$
$\{\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{t}$
$\{\mathbf{b},\mathbf{n}\}$	$\mathbf{t}$
$\{\mathbf{b},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{b}\}$	$\mathbf{b}$
$\{\mathbf{n},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{f}\}$	$\mathbf{f}$

$\widetilde{\bigotimes}$	
$\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{t},\mathbf{b},\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{t},\mathbf{b},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{t},\mathbf{b}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{n},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{t},\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{t},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{t}\}$	$\mathbf{t}$
$\{\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{b},\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{b},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{b}\}$	$\mathbf{b}$
$\{\mathbf{n},\mathbf{f}\}$	$\mathbf{n}$
$\{\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{f}\}$	$\mathbf{f}$

$\widetilde{\bigoplus}$	
$\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{t},\mathbf{b},\mathbf{n}\}$	$\mathbf{b}$
$\{\mathbf{t},\mathbf{b},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{t},\mathbf{b}\}$	$\mathbf{b}$
$\{\mathbf{t},\mathbf{n},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{t},\mathbf{n}\}$	$\mathbf{t}$
$\{\mathbf{t},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{t}\}$	$\mathbf{t}$
$\{\mathbf{b},\mathbf{n},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{b},\mathbf{n}\}$	$\mathbf{b}$
$\{\mathbf{b},\mathbf{f}\}$	$\mathbf{b}$
$\{\mathbf{b}\}$	$\mathbf{b}$
$\{\mathbf{n},\mathbf{f}\}$	$\mathbf{f}$
$\{\mathbf{n}\}$	$\mathbf{n}$
$\{\mathbf{f}\}$	$\mathbf{f}$

Definition 4. A *structure* $\mathscr{M} = \langle D, \Phi_{\mathscr{M}} \rangle$ for a language $\mathscr{L}$ (an $\mathscr{L}$-structure) consists of the following:

1. A nonempty set D, called the *domain* (elements of D are called *individuals*).
2. A mapping $\Phi_{\mathscr{M}}$ satisfying the following:
   (a) Each free variable of $\mathscr{L}$ is mapped to an element of D.
   (b) Each n-ary function symbol f of $\mathscr{L}$ is mapped to a function $f_{\mathscr{M}} : D^n \to D$, or to an element of D if $n = 0$. Additionally, $\Phi_{\mathscr{M}}$ maps elements of D to themselves.

(c) Each n-ary predicate symbol P of $\mathscr{L}$ is mapped to a function $P_{\mathscr{M}}\colon D^n \to V$, or to a element of V if $n = 0$.

Definition 5. Let $\mathscr{M}$ be an $\mathscr{L}$-structure. An *assignment* s is a mapping from the free variables of $\mathscr{L}$ to individuals.

An *interpretation* $\mathfrak{I} = \langle \mathscr{M}, s\rangle$ is an $\mathscr{L}$-structure $\mathscr{M} = \langle D, \Phi_{\mathscr{M}}\rangle$ together with an assignment s. The mapping $\Phi_{\mathscr{M}}$ can be extended in the obvious way to a mapping $\Phi_{\mathfrak{I}}$ from terms to individuals:

1. If t is a free variable, then $\Phi_{\mathfrak{I}} := s(t)$.
2. If t is of the form $f(t_1, \ldots, t_k)$, where f is a k-ary function symbol and $t_1, \ldots, t_k$ are terms, then $\Phi_{\mathfrak{I}} := f_{\mathscr{M}}\big(\Phi_{\mathfrak{I}}(t_1), \ldots, \Phi_{\mathfrak{I}}(t_k)\big)$.

Definition 6. Given an interpretation $\mathfrak{I} = \langle \mathscr{M}, s\rangle$, we define the *valuation* $\mathrm{val}_{\mathfrak{I}}$ for formulas A to truth values as follows:

1. If A is atomic, $A = P(t_1, \ldots, t_n)$, where P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then let $\mathrm{val}_{\mathfrak{I}}(A) = P_{\mathscr{M}}\big(\Phi_{\mathfrak{I}}(t_1), \ldots, \Phi_{\mathfrak{I}}(t_n)\big)$.
2. If $A = \Box(A_1, \ldots, A_n)$, where $A_1, \ldots, A_n$ are formulas, and $\widetilde{\Box}$ is the associated truth function to $\Box$, then $\mathrm{val}_{\mathfrak{I}}(A) = \widetilde{\Box}\big(\mathrm{val}_{\mathfrak{I}}(A_1), \ldots, \mathrm{val}_{\mathfrak{I}}(A_n)\big)$.
3. If $A = (\mathsf{Q}x)(B(x))$, and $\widetilde{\mathsf{Q}}$ is the associated truth function to Q, then $\mathrm{val}_{\mathfrak{I}}(A) = \widetilde{\mathsf{Q}}\big\{\mathrm{val}_{\mathfrak{I}}(B(d)) | d \in D\big\}\big)$.

Definition 7. An interpretation $\mathfrak{I}$ *satisfies* a formula A, in symbols: $\mathfrak{I} \models A$, iff $\mathrm{val}_{\mathfrak{I}}(A) \in V^+$.

Definition 8. Δ *entails* A iff $\mathfrak{I} \models A$ for every interpretation $\mathfrak{I}$ such that $\mathfrak{I} \models B$ for all $B \in \Delta$. A is *valid* iff it is satisfied by every interpretation $\mathfrak{I}$.

3 Sequent calculus for Dunn and Belnap's logic of First Degree Entailment

Definition 9 (Syntax of Sequents)**.** A *sequent* Γ is a quadruple

$$\Gamma_{\mathbf{t}} \mid \Gamma_{\mathbf{b}} \mid \Gamma_{\mathbf{n}} \mid \Gamma_{\mathbf{f}}$$

of finite sequences Γ_v of formulas, where $v \in V$. The Γ_v are called the *components* of Γ.

For a sequence of formulas Δ, and $W \subseteq V$, let $[W\colon \Delta]$ denote the sequent whose component Γ_v is Δ if $v \in W$ and empty otherwise. For $[\{w_1, \ldots, w_k\}\colon \Delta]$ we also write $[w_1, \ldots, w_k\colon \Delta]$. If Γ and Γ' are sequents, then Γ, Γ' denotes the component-wise union, i.e., the v-component of Γ, Γ' is Γ_v, Γ'_v.

Definition 10. Let $\mathfrak{I}$ be an interpretation. $\mathfrak{I}$ *satisfies* a sequent Γ iff there is a $v \in V$ so that for some formula $A \in \Gamma_v$, $\mathrm{val}_{\mathfrak{I}}(F) = v$. $\mathfrak{I}$ is called a *model* of Γ, in symbols $\mathfrak{I} \models \Gamma$.

Γ is called *satisfiable* iff there is an interpretation $\mathfrak{I}$ so that $\mathfrak{I} \models \Gamma$ and *valid* iff for every interpretation $\mathfrak{I}$, $\mathfrak{I} \models \Gamma$.

Proposition 11. $\Delta \models A$ *iff the sequent* $[\mathbf{n}, \mathbf{f} \colon \Delta], [\mathbf{t}, \mathbf{b} \colon A]$ *is valid.*

Definition 12. The *sequent calculus* for Dunn and Belnap's logic of First Degree Entailment is given by:

1. axiom schemas of the form $[V \colon A]$,
2. weakening rules for every truth value v:

$$\frac{\Gamma}{\Gamma, [v \colon A]}\ \text{W:}v$$

3. exchange rules for every truth value v:

$$\frac{\Gamma, [v \colon A, B], \Delta}{\Gamma, [v \colon B, A], \Delta}\ \text{X:}v$$

4. contraction rules for every truth value v:

$$\frac{\Gamma, [v \colon A, A]}{\Gamma, [v \colon A]}\ \text{C:}v$$

5. cut rules for every two truth values $v \neq w$:

$$\frac{\Gamma, [v \colon A] \quad \Delta, [w \colon A]}{\Gamma, \Delta}\ \text{CUT:}vw$$

6. an introduction rule $\Box \colon v$ for every connective $\Box$ and every truth value v, as specified below,
7. an introduction rule $\mathsf{Q} \colon v$ for every quantifier Q and every truth value v, as specified below, where the free variables α occurring in the upper sequents satisfy the so-called *eigenvariable condition*: No α occurs in the lower sequent.

(2)–(5) are called *structural rules*. (6) and (7) are called *logical rules*.

The introduction rules for connective $\sim$ are given by

$$\frac{\Gamma, [\mathbf{f} \colon A]}{\Gamma, [\mathbf{t} \colon {\sim} A]}\ {\sim}\text{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b} \colon A]}{\Gamma, [\mathbf{b} \colon {\sim} A]}\ {\sim}\text{:}\mathbf{b} \qquad \frac{\Gamma, [\mathbf{n} \colon A]}{\Gamma, [\mathbf{n} \colon {\sim} A]}\ {\sim}\text{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{t} \colon A]}{\Gamma, [\mathbf{f} \colon {\sim} A]}\ {\sim}\text{:}\mathbf{f}$$

The introduction rules for connective $\neg_b$ are given by

$$\frac{\Gamma, [\mathbf{f}\colon A]}{\Gamma, [\mathbf{t}\colon \neg_b A]}\ \neg_b{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{n}\colon A]}{\Gamma, [\mathbf{b}\colon \neg_b A]}\ \neg_b{:}\mathbf{b} \qquad \frac{\Gamma, [\mathbf{b}\colon A]}{\Gamma, [\mathbf{n}\colon \neg_b A]}\ \neg_b{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{t}\colon A]}{\Gamma, [\mathbf{f}\colon \neg_b A]}\ \neg_b{:}\mathbf{f}$$

The introduction rules for connective $\wedge$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon B] \quad \Gamma, [\mathbf{t}\colon A]}{\Gamma, [\mathbf{t}\colon A \wedge B]}\ \wedge{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{t}, \mathbf{b}\colon B] \quad \Gamma, [\mathbf{b}\colon A, B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{b}\colon A \wedge B]}\ \wedge{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{t}, \mathbf{n}\colon B] \quad \Gamma, [\mathbf{n}\colon A, B] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A]}{\Gamma, [\mathbf{n}\colon A \wedge B]}\ \wedge{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon A, B] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A, B]}{\Gamma, [\mathbf{f}\colon A \wedge B]}\ \wedge{:}\mathbf{f}$$

The introduction rules for connective $\vee$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{b}\colon A, B] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A, B]}{\Gamma, [\mathbf{t}\colon A \vee B]}\ \vee{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{b}\colon A, B] \quad \Gamma, [\mathbf{b}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{b}\colon A \vee B]}\ \vee{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{n}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{n}\colon A, B] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{n}\colon A \vee B]}\ \vee{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{f}\colon B] \quad \Gamma, [\mathbf{f}\colon A]}{\Gamma, [\mathbf{f}\colon A \vee B]}\ \vee{:}\mathbf{f}$$

The introduction rules for connective $\to_e$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon B], [\mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{t}\colon A \to_e B]}\ \to_e{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b}\colon B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{b}\colon A \to_e B]}\ \to_e{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{n}\colon B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{n}\colon A \to_e B]}\ \to_e{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{f}\colon B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{f}\colon A \to_e B]}\ \to_e{:}\mathbf{f}$$

The introduction rules for connective $\to_b$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{n}\colon B], [\mathbf{b}, \mathbf{f}\colon A] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon B], [\mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{t}\colon A \to_b B]}\ \to_b{:}\mathbf{t}$$

$$\frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{b}\colon B], [\mathbf{n}\colon A] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A]}{\Gamma, [\mathbf{b}\colon A \to_b B]}\ \to_b{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{n}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{b}\colon A], [\mathbf{n}\colon B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{n}\colon A \to_b B]}\ \to_b{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{f}\colon B] \quad \Gamma, [\mathbf{t}\colon A]}{\Gamma, [\mathbf{f}\colon A \to_b B]}\ \to_b{:}\mathbf{f}$$

The introduction rules for connective $\to_l$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{n}\colon B], [\mathbf{f}\colon A] \quad \Gamma, [\mathbf{t}\colon B], [\mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{t}\colon A \to_l B]}\ \to_l{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b}\colon B] \quad \Gamma, [\mathbf{b}\colon A]}{\Gamma, [\mathbf{b}\colon A \to_l B]}\ \to_l{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{t}\colon A], [\mathbf{b}\colon A, B], [\mathbf{f}\colon B] \quad \Gamma, [\mathbf{n}\colon A, B]}{\Gamma, [\mathbf{n}\colon A \to_l B]}\ \to_l{:}\mathbf{n}$$

$$\frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{t}\colon A], [\mathbf{f}\colon B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{f}\colon A \to_l B]}\ \to_l{:}\mathbf{f}$$

The introduction rules for connective $-$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon A]}{\Gamma, [\mathbf{t}\colon -A]}\ -{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{n}\colon A]}{\Gamma, [\mathbf{b}\colon -A]}\ -{:}\mathbf{b} \qquad \frac{\Gamma, [\mathbf{b}\colon A]}{\Gamma, [\mathbf{n}\colon -A]}\ -{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{f}\colon A]}{\Gamma, [\mathbf{f}\colon -A]}\ -{:}\mathbf{f}$$

The introduction rules for connective $\otimes$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{b}\colon B] \quad \Gamma, [\mathbf{t}\colon A, B] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{t}\colon A \otimes B]}\ \otimes{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b}\colon B] \quad \Gamma, [\mathbf{b}\colon A]}{\Gamma, [\mathbf{b}\colon A \otimes B]}\ \otimes{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{t}, \mathbf{n}\colon A, B] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A, B]}{\Gamma, [\mathbf{n}\colon A \otimes B]}\ \otimes{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{f}\colon A, B] \quad \Gamma, [\mathbf{b}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{f}\colon A \otimes B]}\ \otimes{:}\mathbf{f}$$

The introduction rules for connective $\oplus$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{n}\colon B] \quad \Gamma, [\mathbf{t}\colon A, B] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A]}{\Gamma, [\mathbf{t}\colon A \oplus B]}\ \oplus{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{t}, \mathbf{b}\colon A, B] \quad \Gamma, [\mathbf{b}, \mathbf{f}\colon A, B]}{\Gamma, [\mathbf{b}\colon A \oplus B]}\ \oplus{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{n}\colon B] \quad \Gamma, [\mathbf{n}\colon A]}{\Gamma, [\mathbf{n}\colon A \oplus B]}\ \oplus{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{n}, \mathbf{f}\colon B] \quad \Gamma, [\mathbf{f}\colon A, B] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{f}\colon A \oplus B]}\ \oplus{:}\mathbf{f}$$

The introduction rules for connective $\circlearrowleft$ are given by

$$\frac{\Gamma, [\mathbf{n}\colon A]}{\Gamma, [\mathbf{t}\colon \circlearrowleft A]}\ \circlearrowleft{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{t}\colon A]}{\Gamma, [\mathbf{b}\colon \circlearrowleft A]}\ \circlearrowleft{:}\mathbf{b} \qquad \frac{\Gamma, [\mathbf{f}\colon A]}{\Gamma, [\mathbf{n}\colon \circlearrowleft A]}\ \circlearrowleft{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{b}\colon A]}{\Gamma, [\mathbf{f}\colon \circlearrowleft A]}\ \circlearrowleft{:}\mathbf{f}$$

The introduction rules for connective $\triangle$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{b}\colon A]}{\Gamma, [\mathbf{t}\colon \triangle A]}\ \triangle{:}\mathbf{t} \qquad \frac{\Gamma}{\Gamma, [\mathbf{b}\colon \triangle A]}\ \triangle{:}\mathbf{b} \qquad \frac{\Gamma}{\Gamma, [\mathbf{n}\colon \triangle A]}\ \triangle{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{f}\colon \triangle A]}\ \triangle{:}\mathbf{f}$$

The introduction rules for connective $\Box$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon A]}{\Gamma, [\mathbf{t}\colon \Box A]}\ \Box{:}\mathbf{t} \qquad \frac{\Gamma}{\Gamma, [\mathbf{b}\colon \Box A]}\ \Box{:}\mathbf{b} \qquad \frac{\Gamma}{\Gamma, [\mathbf{n}\colon \Box A]}\ \Box{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{n}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{f}\colon \Box A]}\ \Box{:}\mathbf{f}$$

The introduction rules for connective $\circ$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{f}\colon A]}{\Gamma, [\mathbf{t}\colon \circ A]}\ \circ{:}\mathbf{t} \qquad \frac{\Gamma}{\Gamma, [\mathbf{b}\colon \circ A]}\ \circ{:}\mathbf{b} \qquad \frac{\Gamma}{\Gamma, [\mathbf{n}\colon \circ A]}\ \circ{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{n}\colon A]}{\Gamma, [\mathbf{f}\colon \circ A]}\ \circ{:}\mathbf{f}$$

The introduction rules for quantifier $\forall$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon A(\alpha)]}{\Gamma, [\mathbf{t}\colon (\forall x)A(x)]}\ \forall{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b}\colon A(\tau)] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A(\alpha)]}{\Gamma, [\mathbf{b}\colon (\forall x)A(x)]}\ \forall{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{n}\colon A(\tau)] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A(\alpha)]}{\Gamma, [\mathbf{n}\colon (\forall x)A(x)]}\ \forall{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon A(\tau_1)] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A(\tau_2)]}{\Gamma, [\mathbf{f}\colon (\forall x)A(x)]}\ \forall{:}\mathbf{f}$$

The introduction rules for quantifier $\exists$ are given by

$$\frac{\Gamma, [\mathbf{t}, \mathbf{b}\colon A(\tau_1)] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A(\tau_2)]}{\Gamma, [\mathbf{t}\colon (\exists x)A(x)]}\ \exists{:}\mathbf{t} \qquad \frac{\Gamma, [\mathbf{b}\colon A(\tau)] \quad \Gamma, [\mathbf{b}, \mathbf{f}\colon A(\alpha)]}{\Gamma, [\mathbf{b}\colon (\exists x)A(x)]}\ \exists{:}\mathbf{b}$$

$$\frac{\Gamma, [\mathbf{n}\colon A(\tau)] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A(\alpha)]}{\Gamma, [\mathbf{n}\colon (\exists x)A(x)]}\ \exists{:}\mathbf{n} \qquad \frac{\Gamma, [\mathbf{f}\colon A(\alpha)]}{\Gamma, [\mathbf{f}\colon (\exists x)A(x)]}\ \exists{:}\mathbf{f}$$

The introduction rules for quantifier $\bigotimes$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon A(\tau)] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A(\alpha)]}{\Gamma, [\mathbf{t}\colon (\bigotimes x)A(x)]} \bigotimes\mathbf{:t} \qquad \frac{\Gamma, [\mathbf{b}\colon A(\alpha)]}{\Gamma, [\mathbf{b}\colon (\bigotimes x)A(x)]} \bigotimes\mathbf{:b}$$

$$\frac{\Gamma, [\mathbf{n}, \mathbf{f}\colon A(\tau_1)] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A(\tau_2)]}{\Gamma, [\mathbf{n}\colon (\bigotimes x)A(x)]} \bigotimes\mathbf{:n} \qquad \frac{\Gamma, [\mathbf{f}\colon A(\tau)] \quad \Gamma, [\mathbf{b}, \mathbf{f}\colon A(\alpha)]}{\Gamma, [\mathbf{f}\colon (\bigotimes x)A(x)]} \bigotimes\mathbf{:f}$$

The introduction rules for quantifier $\bigoplus$ are given by

$$\frac{\Gamma, [\mathbf{t}\colon A(\tau)] \quad \Gamma, [\mathbf{t}, \mathbf{n}\colon A(\alpha)]}{\Gamma, [\mathbf{t}\colon (\bigoplus x)A(x)]} \bigoplus\mathbf{:t} \qquad \frac{\Gamma, [\mathbf{b}, \mathbf{f}\colon A(\tau_1)] \quad \Gamma, [\mathbf{t}, \mathbf{b}\colon A(\tau_2)]}{\Gamma, [\mathbf{b}\colon (\bigoplus x)A(x)]} \bigoplus\mathbf{:b}$$

$$\frac{\Gamma, [\mathbf{n}\colon A(\alpha)]}{\Gamma, [\mathbf{n}\colon (\bigoplus x)A(x)]} \bigoplus\mathbf{:n} \qquad \frac{\Gamma, [\mathbf{f}\colon A(\tau)] \quad \Gamma, [\mathbf{n}, \mathbf{f}\colon A(\alpha)]}{\Gamma, [\mathbf{f}\colon (\bigoplus x)A(x)]} \bigoplus\mathbf{:f}$$

Definition 13. An upward tree of sequents is called a *proof* in the sequent calculus iff every leaf is an instance of an axiom, and all other sequents in it are obtained from the ones standing immediately above it by applying one of the rules. The sequent at the root of P is called its *end-sequent*. A sequent Γ is called *provable* iff it is the end-sequent of some proof.

Theorem 14 (Soundness and Completeness)**.** *A sequent is provable if and only if it is valid.*

Proof. See Theorems 3.1 and 3.2 of Baaz et al. [5] or Theorems 3.3.8 and 3.3.10 of Zach [25]. □

Corollary 15. *In Dunn and Belnap's logic of First Degree Entailment,* $\models A$ *iff* $[\mathbf{t}, \mathbf{b}\colon A]$ *has a sequent proof, and* $\Delta \models A$ *iff* $[\mathbf{n}, \mathbf{f}\colon \Delta], [\mathbf{t}, \mathbf{b}\colon A]$ *has a proof.*

Theorem 16 (Cut-elimination)**.** *The cut rule is eliminable in the sequent calculus for Dunn and Belnap's logic of First Degree Entailment.*

Proof. See Theorem 4.1 of Baaz et al. [5] or Theorem 3.5.3 of Zach [25]. □

Theorem 17 (Midsequent theorem)**.** *The midsequent theorem holds in the sequent calculus for Dunn and Belnap's logic of First Degree Entailment.*

Proof. See Theorem 6.1 of Baaz et al. [5] or Theorem 3.7.2 of Zach [25]. □

Theorem 18 (Maehara lemma)**.** *The Maehara lemma holds for the sequent calculus for Dunn and Belnap's logic of First Degree Entailment.*

Proof. See Theorem 3.8.1 of Zach [25]. □

4 Tableaux for Dunn and Belnap's logic of First Degree Entailment

Although the method of Surma [21] and Carnielli [8] for obtaining signed analytic tableaux systems applies to Dunn and Belnap's logic of First Degree Entailment, it has a drawback. As Hähnle [13] pointed out, to show that a formula is valid, it is required to provide as many closed tableaux as there are non-designated values. This is usually not desirable; the generalized approach by Hähnle [13] solves this problem. Below we give a tableau system for Dunn and Belnap's logic of First Degree Entailment using the sets of signs $V \setminus \{v\}$, i.e., the tableau system exactly dual to that of Carnielli (in the sense of [3]).

Definition 19. A *signed formula* is an expression of the form $v\colon A$ where $v \in V$ and A is a formula.

Definition 20. A *tableau* for a set of signed formulas Δ is a downward rooted tree of signed formulas where each one is either an element of Δ or results from a signed formula in the branch above it by a branch expansion rule. A tableau is *closed* if every branch contains, for some formula A, the signed formulas $v\colon A$ for all $v \in V$, or a signed formula $v\colon A$ with a branch expansion rule that explicitly closes the branch ($\otimes$).

The branch expansion rules for connective $\sim$ are given by

$$\frac{\mathbf{t}\colon {\sim}A}{\mathbf{f}\colon A} \qquad \frac{\mathbf{b}\colon {\sim}A}{\mathbf{b}\colon A} \qquad \frac{\mathbf{n}\colon {\sim}A}{\mathbf{n}\colon A} \qquad \frac{\mathbf{f}\colon {\sim}A}{\mathbf{t}\colon A}$$

The branch expansion rules for connective $\neg_b$ are given by

$$\frac{\mathbf{t}\colon \neg_b A}{\mathbf{f}\colon A} \qquad \frac{\mathbf{b}\colon \neg_b A}{\mathbf{n}\colon A} \qquad \frac{\mathbf{n}\colon \neg_b A}{\mathbf{b}\colon A} \qquad \frac{\mathbf{f}\colon \neg_b A}{\mathbf{t}\colon A}$$

The branch expansion rules for connective $\wedge$ are given by

$$\frac{\mathbf{t}\colon A \wedge B}{\begin{array}{cc} \mathbf{t}\colon B & \mathbf{t}\colon A \end{array}} \qquad \frac{\mathbf{b}\colon A \wedge B}{\begin{array}{ccc} \mathbf{t}\colon B & \mathbf{b}\colon A & \mathbf{t}\colon A \\ \mathbf{b}\colon B & \mathbf{b}\colon B & \mathbf{b}\colon A \end{array}} \qquad \frac{\mathbf{n}\colon A \wedge B}{\begin{array}{ccc} \mathbf{t}\colon B & \mathbf{n}\colon A & \mathbf{t}\colon A \\ \mathbf{n}\colon B & \mathbf{n}\colon B & \mathbf{n}\colon A \end{array}}$$

$$\frac{\mathbf{f}\colon A \wedge B}{\begin{array}{cc} \mathbf{b}\colon A & \mathbf{n}\colon A \\ \mathbf{f}\colon A & \mathbf{f}\colon A \\ \mathbf{b}\colon B & \mathbf{n}\colon B \\ \mathbf{f}\colon B & \mathbf{f}\colon B \end{array}}$$

The branch expansion rules for connective $\vee$ are given by

$$\frac{\mathbf{t}\colon A \vee B}{\begin{array}{cc} \mathbf{t}\colon A & \mathbf{t}\colon A \\ \mathbf{b}\colon A & \mathbf{n}\colon A \\ \mathbf{t}\colon B & \mathbf{t}\colon B \\ \mathbf{b}\colon B & \mathbf{n}\colon B \end{array}} \qquad \frac{\mathbf{b}\colon A \vee B}{\begin{array}{ccc} \mathbf{b}\colon B & \mathbf{b}\colon A & \mathbf{b}\colon A \\ \mathbf{f}\colon B & \mathbf{b}\colon B & \mathbf{f}\colon A \end{array}} \qquad \frac{\mathbf{n}\colon A \vee B}{\begin{array}{ccc} \mathbf{n}\colon B & \mathbf{n}\colon A & \mathbf{n}\colon A \\ \mathbf{f}\colon B & \mathbf{n}\colon B & \mathbf{f}\colon A \end{array}}$$

$$\frac{\mathbf{f}\colon A \vee B}{\mathbf{f}\colon B \qquad \mathbf{f}\colon A}$$

The branch expansion rules for connective $\rightarrow_e$ are given by

$$\frac{\mathbf{t}\colon A \rightarrow_e B}{\begin{array}{c}\mathbf{n}\colon A\\ \mathbf{f}\colon A\\ \mathbf{t}\colon B\end{array}} \qquad \frac{\mathbf{b}\colon A \rightarrow_e B}{\begin{array}{cc}\mathbf{b}\colon B & \mathbf{t}\colon A\\ & \mathbf{b}\colon A\end{array}} \qquad \frac{\mathbf{n}\colon A \rightarrow_e B}{\begin{array}{cc}\mathbf{n}\colon B & \mathbf{t}\colon A\\ & \mathbf{b}\colon A\end{array}} \qquad \frac{\mathbf{f}\colon A \rightarrow_e B}{\begin{array}{cc}\mathbf{f}\colon B & \mathbf{t}\colon A\\ & \mathbf{b}\colon A\end{array}}$$

The branch expansion rules for connective $\rightarrow_b$ are given by

$$\frac{\mathbf{t}\colon A \rightarrow_b B}{\begin{array}{cc}\mathbf{b}\colon A & \mathbf{n}\colon A\\ \mathbf{f}\colon A & \mathbf{f}\colon A\\ \mathbf{t}\colon B & \mathbf{t}\colon B\\ \mathbf{n}\colon B & \mathbf{b}\colon B\end{array}} \qquad \frac{\mathbf{b}\colon A \rightarrow_b B}{\begin{array}{ccc}\mathbf{b}\colon B & \mathbf{n}\colon A & \mathbf{t}\colon A\\ \mathbf{f}\colon B & \mathbf{b}\colon B & \mathbf{n}\colon A\end{array}} \qquad \frac{\mathbf{n}\colon A \rightarrow_b B}{\begin{array}{ccc}\mathbf{n}\colon B & \mathbf{b}\colon A & \mathbf{t}\colon A\\ \mathbf{f}\colon B & \mathbf{n}\colon B & \mathbf{b}\colon A\end{array}}$$

$$\frac{\mathbf{f}\colon A \rightarrow_b B}{\mathbf{f}\colon B \qquad \mathbf{t}\colon A}$$

The branch expansion rules for connective $\rightarrow_l$ are given by

$$\frac{\mathbf{t}\colon A \rightarrow_l B}{\begin{array}{cc}\mathbf{f}\colon A & \mathbf{n}\colon A\\ \mathbf{t}\colon B & \mathbf{f}\colon A\\ \mathbf{n}\colon B & \mathbf{t}\colon B\end{array}} \qquad \frac{\mathbf{b}\colon A \rightarrow_l B}{\mathbf{b}\colon B \qquad \mathbf{b}\colon A} \qquad \frac{\mathbf{n}\colon A \rightarrow_l B}{\begin{array}{cc}\mathbf{t}\colon A & \mathbf{n}\colon A\\ \mathbf{b}\colon A & \mathbf{n}\colon B\\ \mathbf{b}\colon B & \\ \mathbf{f}\colon B & \end{array}} \qquad \frac{\mathbf{f}\colon A \rightarrow_l B}{\begin{array}{ccc}\mathbf{b}\colon B & \mathbf{t}\colon A & \mathbf{t}\colon A\\ \mathbf{f}\colon B & \mathbf{f}\colon B & \mathbf{b}\colon A\end{array}}$$

The branch expansion rules for connective $-$ are given by

$$\frac{\mathbf{t}\colon -A}{\mathbf{t}\colon A} \qquad \frac{\mathbf{b}\colon -A}{\mathbf{n}\colon A} \qquad \frac{\mathbf{n}\colon -A}{\mathbf{b}\colon A} \qquad \frac{\mathbf{f}\colon -A}{\mathbf{f}\colon A}$$

The branch expansion rules for connective $\otimes$ are given by

$$\frac{\mathbf{t}\colon A \otimes B}{\begin{array}{ccc}\mathbf{t}\colon B & \mathbf{t}\colon A & \mathbf{t}\colon A\\ \mathbf{b}\colon B & \mathbf{t}\colon B & \mathbf{b}\colon A\end{array}} \qquad \frac{\mathbf{b}\colon A \otimes B}{\mathbf{b}\colon B \qquad \mathbf{b}\colon A} \qquad \frac{\mathbf{n}\colon A \otimes B}{\begin{array}{cc}\mathbf{t}\colon A & \mathbf{n}\colon A\\ \mathbf{n}\colon A & \mathbf{f}\colon A\\ \mathbf{t}\colon B & \mathbf{n}\colon B\\ \mathbf{n}\colon B & \mathbf{f}\colon B\end{array}}$$

$$\frac{\mathbf{f}\colon A \otimes B}{\begin{array}{ccc}\mathbf{b}\colon B & \mathbf{f}\colon A & \mathbf{b}\colon A\\ \mathbf{f}\colon B & \mathbf{f}\colon B & \mathbf{f}\colon A\end{array}}$$

The branch expansion rules for connective $\oplus$ are given by

$$\frac{\mathbf{t}\colon A \oplus B}{\begin{array}{ccc}\mathbf{t}\colon B & \mathbf{t}\colon A & \mathbf{t}\colon A\\ \mathbf{n}\colon B & \mathbf{t}\colon B & \mathbf{n}\colon A\end{array}} \qquad \frac{\mathbf{b}\colon A \oplus B}{\begin{array}{cc}\mathbf{t}\colon A & \mathbf{b}\colon A\\ \mathbf{b}\colon A & \mathbf{f}\colon A\\ \mathbf{t}\colon B & \mathbf{b}\colon B\\ \mathbf{b}\colon B & \mathbf{f}\colon B\end{array}} \qquad \frac{\mathbf{n}\colon A \oplus B}{\mathbf{n}\colon B \qquad \mathbf{n}\colon A}$$

$$\begin{array}{ccc} & \mathbf{f}\colon A \oplus B & \\ \hline \mathbf{n}\colon B & \mathbf{f}\colon A & \mathbf{n}\colon A \\ \mathbf{f}\colon B & \mathbf{f}\colon B & \mathbf{f}\colon A \end{array}$$

The branch expansion rules for connective $\circlearrowleft$ are given by

$$\frac{\mathbf{t}\colon \circlearrowleft A}{\mathbf{n}\colon A} \qquad \frac{\mathbf{b}\colon \circlearrowleft A}{\mathbf{t}\colon A} \qquad \frac{\mathbf{n}\colon \circlearrowleft A}{\mathbf{f}\colon A} \qquad \frac{\mathbf{f}\colon \circlearrowleft A}{\mathbf{b}\colon A}$$

The branch expansion rules for connective $\triangle$ are given by

$$\begin{array}{c} \mathbf{t}\colon \triangle A \\ \hline \mathbf{t}\colon A \\ \mathbf{b}\colon A \end{array} \qquad \begin{array}{c} \mathbf{f}\colon \triangle A \\ \hline \mathbf{n}\colon A \\ \mathbf{f}\colon A \end{array}$$

The branch expansion rules for connective $\Box$ are given by

$$\begin{array}{c} \mathbf{t}\colon \Box A \\ \hline \mathbf{t}\colon A \end{array} \qquad \begin{array}{c} \mathbf{f}\colon \Box A \\ \hline \mathbf{b}\colon A \\ \mathbf{n}\colon A \\ \mathbf{f}\colon A \end{array}$$

The branch expansion rules for connective $\circ$ are given by

$$\begin{array}{c} \mathbf{t}\colon \circ A \\ \hline \mathbf{t}\colon A \\ \mathbf{f}\colon A \end{array} \qquad \begin{array}{c} \mathbf{f}\colon \circ A \\ \hline \mathbf{b}\colon A \\ \mathbf{n}\colon A \end{array}$$

The branch expansion rules rules for quantifier $\forall$ are given by

$$\begin{array}{c} \mathbf{t}\colon (\forall x)A(x) \\ \hline \mathbf{t}\colon A(\alpha) \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\mathbf{b}\colon (\forall x)A(x)} \\ \hline \mathbf{b}\colon A(\tau) & \mathbf{b}\colon A(\alpha) \\ & \mathbf{t}\colon A(\alpha) \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\mathbf{n}\colon (\forall x)A(x)} \\ \hline \mathbf{n}\colon A(\tau) & \mathbf{n}\colon A(\alpha) \\ & \mathbf{t}\colon A(\alpha) \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{\mathbf{f}\colon (\forall x)A(x)} \\ \hline \mathbf{b}\colon A(\tau_1) & \mathbf{f}\colon A(\tau_2) \\ \mathbf{f}\colon A(\tau_1) & \mathbf{n}\colon A(\tau_2) \end{array}$$

The branch expansion rules rules for quantifier $\exists$ are given by

$$\begin{array}{cc} \multicolumn{2}{c}{\mathbf{t}\colon (\exists x)A(x)} \\ \hline \mathbf{b}\colon A(\tau_1) & \mathbf{n}\colon A(\tau_2) \\ \mathbf{t}\colon A(\tau_1) & \mathbf{t}\colon A(\tau_2) \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\mathbf{b}\colon (\exists x)A(x)} \\ \hline \mathbf{b}\colon A(\tau) & \mathbf{b}\colon A(\alpha) \\ & \mathbf{f}\colon A(\alpha) \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\mathbf{n}\colon (\exists x)A(x)} \\ \hline \mathbf{n}\colon A(\tau) & \mathbf{f}\colon A(\alpha) \\ & \mathbf{n}\colon A(\alpha) \end{array}$$

$$\frac{\mathbf{f}\colon (\exists x)A(x)}{\mathbf{f}\colon A(\alpha)}$$

The branch expansion rules rules for quantifier $\bigotimes$ are given by

$$\frac{\mathbf{t}\colon (\bigotimes x)A(x)}{\mathbf{t}\colon A(\tau) \quad \begin{array}{r}\mathbf{b}\colon A(\alpha)\\ \mathbf{t}\colon A(\alpha)\end{array}} \qquad \frac{\mathbf{b}\colon (\bigotimes x)A(x)}{\mathbf{b}\colon A(\alpha)} \qquad \frac{\mathbf{n}\colon (\bigotimes x)A(x)}{\begin{array}{r}\mathbf{f}\colon A(\tau_1)\\ \mathbf{n}\colon A(\tau_1)\end{array} \quad \begin{array}{r}\mathbf{n}\colon A(\tau_2)\\ \mathbf{t}\colon A(\tau_2)\end{array}}$$

$$\frac{\mathbf{f}\colon (\bigotimes x)A(x)}{\mathbf{f}\colon A(\tau) \quad \begin{array}{r}\mathbf{b}\colon A(\alpha)\\ \mathbf{f}\colon A(\alpha)\end{array}}$$

The branch expansion rules rules for quantifier $\bigoplus$ are given by

$$\frac{\mathbf{t}\colon (\bigoplus x)A(x)}{\mathbf{t}\colon A(\tau) \quad \begin{array}{r}\mathbf{n}\colon A(\alpha)\\ \mathbf{t}\colon A(\alpha)\end{array}} \qquad \frac{\mathbf{b}\colon (\bigoplus x)A(x)}{\begin{array}{r}\mathbf{b}\colon A(\tau_1)\\ \mathbf{f}\colon A(\tau_1)\end{array} \quad \begin{array}{r}\mathbf{b}\colon A(\tau_2)\\ \mathbf{t}\colon A(\tau_2)\end{array}} \qquad \frac{\mathbf{n}\colon (\bigoplus x)A(x)}{\mathbf{n}\colon A(\alpha)}$$

$$\frac{\mathbf{f}\colon (\bigoplus x)A(x)}{\mathbf{f}\colon A(\tau) \quad \begin{array}{r}\mathbf{f}\colon A(\alpha)\\ \mathbf{n}\colon A(\alpha)\end{array}}$$

Definition 21. An interpretation $\mathfrak{I}$ *satisfies* a signed formula $v\colon A$ iff $\mathrm{val}_{\mathfrak{I}}(A) \neq v$. A set of signed formulas is *satisfiable* if some interpretation $\mathfrak{I}$ satisfies all signed formulas in it.

Theorem 22. *A set of signed formulas is unsatisfiable iff it has a closed tableau.*

Proof. Apply Theorems 4.14 and 4.21 of Hähnle [13]; interpreting $v\colon A$ as $\mathsf{S}\ A$ where $\mathsf{S} = V \setminus \{v\}$. □

Corollary 23. *In Dunn and Belnap's logic of First Degree Entailment,* $\models A$ *iff* $\{v\colon A \mid v \in V^+\}$ *has a closed tableau.* $\Delta \models A$ *iff* $\{v\colon B \mid v \in V^-, B \in \Delta\} \cup \{v\colon A \mid v \in V^+\}$ *has a closed tableau.*

5 Natural deduction for Dunn and Belnap's logic of First Degree Entailment

Let Γ be a (set) sequent, $V^+ \subseteq V$ the set of *designated truth values*. The set of non-designated truth values is then $V^- = V \setminus V^+$. We divide the sequent Γ into its designated part Γ^+ and its non-designated part Γ^- in the obvious way:

$$\begin{array}{rcl}\Gamma^+ & := & \langle \Gamma_v \mid v \in V^+ \rangle \\ \Gamma^- & := & \langle \Gamma_v \mid v \in V^- \rangle\end{array}$$

Definition 24. The *natural deduction rules* for Dunn and Belnap's logic of First Degree Entailment are given by:

1. A weakening rule for all $v \in V^+$:

$$\frac{\Gamma^+}{\Gamma^+, [v\colon A]}\ \mathrm{w}{:}v$$

2. For every connective $\Box$ and every truth value v an introduction rule $\Box\text{I:}v$ (if $v \in V^+$) or an elimination rule $\Box\text{E:}v$ (if $v \in V^-$).

3. For every quantifier Q and every truth value v an introduction rule $\mathsf{Q}\text{I:}v$ (if $v \in V^+$) or an elimination rule $\mathsf{Q}\text{E:}v$ (if $v \in V^-$).

The introduction and elimination rules for connective $\sim$ are given by

$$\frac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{f}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{t}\colon {\sim}A]}\ {\sim}\text{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^-\\ \Gamma_1^+, [\mathbf{b}\colon A]\end{array}}{\Gamma_1^+, [\mathbf{b}\colon {\sim}A]}\ {\sim}\text{I}\colon \mathbf{b} \qquad \frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon {\sim}A]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon A]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon {\sim}A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ {\sim}\text{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon {\sim}A]\rceil & \Gamma_1^-\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon {\sim}A] & \Gamma_1^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ {\sim}\text{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\neg_b$ are given by

$$\frac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{f}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{t}\colon \neg_b A]}\ \neg_b\text{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{n}\colon A]\rceil\\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{b}\colon \neg_b A]}\ \neg_b\text{I}\colon \mathbf{b} \qquad \frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon \neg_b A]\rceil & \Gamma_1^-\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon \neg_b A] & \Gamma_1^+, [\mathbf{b}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_b\text{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon \neg_b A]\rceil & \Gamma_1^-\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon \neg_b A] & \Gamma_1^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \neg_b\text{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\wedge$ are given by

$$\frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-\\ \Gamma_1^+, [\mathbf{t}\colon B] & \Gamma_2^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon A \wedge B]}\ \wedge\text{I}\colon \mathbf{t} \qquad \frac{\begin{array}{ccc}\Gamma_1^- & \Gamma_2^- & \Gamma_3^-\\ \Gamma_1^+, [\mathbf{t}, \mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{b}\colon A, B] & \Gamma_3^+, [\mathbf{t}, \mathbf{b}\colon A]\end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{b}\colon A \wedge B]}\ \wedge\text{I}\colon \mathbf{b}$$

$$\frac{\begin{array}{cccc}\Gamma_0^-, \lceil[\mathbf{f}\colon A \wedge B]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{n}\colon A]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \wedge B] & \Gamma_1^+, [\mathbf{t}\colon B] & \Gamma_2^+ & \Gamma_3^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \wedge\text{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{n}\colon A \wedge B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A, B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}, \mathbf{f}\colon A, B]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \wedge B] & \Gamma_1^+, [\mathbf{b}\colon A, B] & \Gamma_2^+\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \wedge\text{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\vee$ are given by

$$\frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-, \lceil[\mathbf{n}\colon A, B]\rceil\\ \Gamma_1^+, [\mathbf{t}, \mathbf{b}\colon A, B] & \Gamma_2^+, [\mathbf{t}\colon A, B]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon A \vee B]}\ \vee\text{I}\colon \mathbf{t}$$

$$\dfrac{\begin{array}{ccc} \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^- & \Gamma_3^-, \lceil[\mathbf{f}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{b}\colon A, B] & \Gamma_3^+, [\mathbf{b}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{b}\colon A \vee B]}\ \vee\mathrm{I}\colon \mathbf{b}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{f}\colon A \vee B]\rceil & \Gamma_1^-, \lceil[\mathbf{n}, \mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{n}, \mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \vee B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+ \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \vee\mathrm{E}\colon \mathbf{n}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{n}\colon A \vee B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \vee B] & \Gamma_1^+ & \Gamma_2^+ \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \vee\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\rightarrow_e$ are given by

$$\dfrac{\begin{array}{c} \Gamma_1^-, \lceil[\mathbf{n}, \mathbf{f}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{t}\colon B] \end{array}}{\Gamma_1^+, [\mathbf{t}\colon A \rightarrow_e B]}\ \rightarrow_e\mathrm{I}\colon \mathbf{t} \qquad \dfrac{\begin{array}{cc} \Gamma_1^- & \Gamma_2^- \\ \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon A \rightarrow_e B]}\ \rightarrow_e\mathrm{I}\colon \mathbf{b}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{f}\colon A \rightarrow_e B]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon B]\rceil & \Gamma_2^- \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \rightarrow_e B] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon A] \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \rightarrow_e\mathrm{E}\colon \mathbf{n}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{n}\colon A \rightarrow_e B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^- \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \rightarrow_e B] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon A] \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \rightarrow_e\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\rightarrow_b$ are given by

$$\dfrac{\begin{array}{cc} \Gamma_1^-, \lceil[\mathbf{n}\colon B]\rceil, \lceil[\mathbf{f}\colon A]\rceil & \Gamma_2^-, \lceil[\mathbf{n}, \mathbf{f}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{t}\colon B], [\mathbf{b}\colon A] & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon B] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon A \rightarrow_b B]}\ \rightarrow_b\mathrm{I}\colon \mathbf{t}$$

$$\dfrac{\begin{array}{ccc} \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A]\rceil & \Gamma_3^-, \lceil[\mathbf{n}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{b}\colon B] & \Gamma_3^+, [\mathbf{t}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{b}\colon A \rightarrow_b B]}\ \rightarrow_b\mathrm{I}\colon \mathbf{b}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{f}\colon A \rightarrow_b B]\rceil & \Gamma_1^-, \lceil[\mathbf{n}, \mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon B]\rceil & \Gamma_3^- \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \rightarrow_b B] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{b}\colon A] & \Gamma_3^+, [\mathbf{t}, \mathbf{b}\colon A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \rightarrow_b\mathrm{E}\colon \mathbf{n}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{n}\colon A \rightarrow_b B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^- \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \rightarrow_b B] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{t}\colon A] \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \rightarrow_b\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\rightarrow_l$ are given by

$$\dfrac{\begin{array}{cc} \Gamma_1^-, \lceil[\mathbf{n}\colon B]\rceil, \lceil[\mathbf{f}\colon A]\rceil & \Gamma_2^-, \lceil[\mathbf{n}, \mathbf{f}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{t}\colon B] & \Gamma_2^+, [\mathbf{t}\colon B] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon A \to_l B]}\ {\to_l}\mathrm{I}\colon \mathbf{t} \qquad \dfrac{\begin{array}{cc} \Gamma_1^- & \Gamma_2^- \\ \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{b}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon A \to_l B]}\ {\to_l}\mathrm{I}\colon \mathbf{b}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{f}\colon A \to_l B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A, B]\rceil \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \to_l B] & \Gamma_1^+, [\mathbf{t}\colon A], [\mathbf{b}\colon A, B] & \Gamma_2^+ \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ {\to_l}\mathrm{E}\colon \mathbf{n}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{n}\colon A \to_l B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_3^- \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \to_l B] & \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{t}\colon A] & \Gamma_3^+, [\mathbf{t}, \mathbf{b}\colon A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ {\to_l}\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $-$ are given by

$$\dfrac{\begin{array}{c} \Gamma_1^- \\ \Gamma_1^+, [\mathbf{t}\colon A] \end{array}}{\Gamma_1^+, [\mathbf{t}\colon -A]}\ -\mathrm{I}\colon \mathbf{t} \qquad \dfrac{\begin{array}{c} \Gamma_1^-, \lceil[\mathbf{n}\colon A]\rceil \\ \Gamma_1^+ \end{array}}{\Gamma_1^+, [\mathbf{b}\colon -A]}\ -\mathrm{I}\colon \mathbf{b} \qquad \dfrac{\begin{array}{cc} \Gamma_0^-, \lceil[\mathbf{f}\colon -A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon -A] & \Gamma_1^+, [\mathbf{b}\colon A] \end{array}}{\Gamma_0^+, \Gamma_1^+}\ -\mathrm{E}\colon \mathbf{n}$$

$$\dfrac{\begin{array}{cc} \Gamma_0^-, \lceil[\mathbf{n}\colon -A]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon -A] & \Gamma_1^+ \end{array}}{\Gamma_0^+, \Gamma_1^+}\ -\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\otimes$ are given by

$$\dfrac{\begin{array}{ccc} \Gamma_1^- & \Gamma_2^- & \Gamma_3^- \\ \Gamma_1^+, [\mathbf{t}, \mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{t}\colon A, B] & \Gamma_3^+, [\mathbf{t}, \mathbf{b}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{t}\colon A \otimes B]}\ \otimes\mathrm{I}\colon \mathbf{t} \qquad \dfrac{\begin{array}{cc} \Gamma_1^- & \Gamma_2^- \\ \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+, [\mathbf{b}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon A \otimes B]}\ \otimes\mathrm{I}\colon \mathbf{b}$$

$$\dfrac{\begin{array}{ccc} \Gamma_0^-, \lceil[\mathbf{f}\colon A \otimes B]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon A, B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}, \mathbf{f}\colon A, B]\rceil \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \otimes B] & \Gamma_1^+, [\mathbf{t}\colon A, B] & \Gamma_2^+ \end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \otimes\mathrm{E}\colon \mathbf{n}$$

$$\dfrac{\begin{array}{cccc} \Gamma_0^-, \lceil[\mathbf{n}\colon A \otimes B]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{f}\colon A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon A \otimes B] & \Gamma_1^+, [\mathbf{b}\colon B] & \Gamma_2^+ & \Gamma_3^+, [\mathbf{b}\colon A] \end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \otimes\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\oplus$ are given by

$$\dfrac{\begin{array}{ccc} \Gamma_1^-, \lceil[\mathbf{n}\colon B]\rceil & \Gamma_2^- & \Gamma_3^-, \lceil[\mathbf{n}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{t}\colon B] & \Gamma_2^+, [\mathbf{t}\colon A, B] & \Gamma_3^+, [\mathbf{t}\colon A] \end{array}}{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+, [\mathbf{t}\colon A \oplus B]}\ \oplus\mathrm{I}\colon \mathbf{t}$$

$$\dfrac{\begin{array}{cc} \Gamma_1^- & \Gamma_2^-, \lceil[\mathbf{f}\colon A, B]\rceil \\ \Gamma_1^+, [\mathbf{t}, \mathbf{b}\colon A, B] & \Gamma_2^+, [\mathbf{b}\colon A, B] \end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon A \oplus B]}\ \oplus\mathrm{I}\colon \mathbf{b}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{f}\colon A\oplus B]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon A\oplus B] & \Gamma_1^+ & \Gamma_2^+\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \oplus\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cccc}\Gamma_0^-, \lceil[\mathbf{n}\colon A\oplus B]\rceil & \Gamma_1^-, \lceil[\mathbf{n},\mathbf{f}\colon B]\rceil & \Gamma_2^-, \lceil[\mathbf{f}\colon A, B]\rceil & \Gamma_3^-, \lceil[\mathbf{n},\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon A\oplus B] & \Gamma_1^+ & \Gamma_2^+ & \Gamma_3^+\end{array}}{\Gamma_0^+, \ldots, \Gamma_3^+}\ \oplus\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\circlearrowleft$ are given by

$$\frac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{n}\colon A]\rceil \\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{t}\colon \circlearrowleft A]}\ \circlearrowleft\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_1^+, [\mathbf{b}\colon \circlearrowleft A]}\ \circlearrowleft\mathrm{I}\colon \mathbf{b} \qquad \frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon \circlearrowleft A]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \circlearrowleft A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \circlearrowleft\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon \circlearrowleft A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \circlearrowleft A] & \Gamma_1^+, [\mathbf{b}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \circlearrowleft\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\triangle$ are given by

$$\frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\mathbf{t},\mathbf{b}\colon A]\end{array}}{\Gamma_1^+, [\mathbf{t}\colon \triangle A]}\ \triangle\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{b}\colon \triangle A]}\ \triangle\mathrm{I}\colon \mathbf{b} \qquad \frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon \triangle A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \triangle A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \triangle\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon \triangle A]\rceil & \Gamma_1^-, \lceil[\mathbf{n},\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \triangle A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \triangle\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\Box$ are given by

$$\frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_1^+, [\mathbf{t}\colon \Box A]}\ \Box\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{b}\colon \Box A]}\ \Box\mathrm{I}\colon \mathbf{b} \qquad \frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon \Box A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \Box A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \Box\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon \Box A]\rceil & \Gamma_1^-, \lceil[\mathbf{n},\mathbf{f}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \Box A] & \Gamma_1^+, [\mathbf{b}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \Box\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for connective $\circ$ are given by

$$\frac{\begin{array}{c}\Gamma_1^-, \lceil[\mathbf{f}\colon A]\rceil \\ \Gamma_1^+, [\mathbf{t}\colon A]\end{array}}{\Gamma_1^+, [\mathbf{t}\colon \circ A]}\ \circ\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^- \\ \Gamma_1^+\end{array}}{\Gamma_1^+, [\mathbf{b}\colon \circ A]}\ \circ\mathrm{I}\colon \mathbf{b} \qquad \frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon \circ A]\rceil & \Gamma_1^- \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \circ A] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \circ\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon \circ A]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon A]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon \circ A] & \Gamma_1^+, [\mathbf{b}\colon A]\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \circ\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for quantifier $\forall$ are given by

$$\frac{\begin{array}{c}\Gamma_1^-\\ \Gamma_1^+, [\mathbf{t}\colon A(\alpha)]\end{array}}{\Gamma_1^+, [\mathbf{t}\colon (\forall x)A(x)]}\ \forall\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-\\ \Gamma_1^+, [\mathbf{b}\colon A(\tau)] & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon A(\alpha)]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon (\forall x)A(x)]}\ \forall\mathrm{I}\colon \mathbf{b}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{f}\colon (\forall x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon A(\tau)]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A(\alpha)]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon (\forall x)A(x)] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{t}\colon A(\alpha)]\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \forall\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{n}\colon (\forall x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A(\tau_1)]\rceil & \Gamma_2^-, \lceil[\mathbf{n}, \mathbf{f}\colon A(\tau_2)]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon (\forall x)A(x)] & \Gamma_1^+, [\mathbf{b}\colon A(\tau_1)] & \Gamma_2^+\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \forall\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for quantifier $\exists$ are given by

$$\frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-, \lceil[\mathbf{n}\colon A(\tau_2)]\rceil\\ \Gamma_1^+, [\mathbf{t}, \mathbf{b}\colon A(\tau_1)] & \Gamma_2^+, [\mathbf{t}\colon A(\tau_2)]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon (\exists x)A(x)]}\ \exists\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-, \lceil[\mathbf{f}\colon A(\alpha)]\rceil\\ \Gamma_1^+, [\mathbf{b}\colon A(\tau)] & \Gamma_2^+, [\mathbf{b}\colon A(\alpha)]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon (\exists x)A(x)]}\ \exists\mathrm{I}\colon \mathbf{b}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{f}\colon (\exists x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon A(\tau)]\rceil & \Gamma_2^-, \lceil[\mathbf{n}, \mathbf{f}\colon A(\alpha)]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon (\exists x)A(x)] & \Gamma_1^+ & \Gamma_2^+\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \exists\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{n}\colon (\exists x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A(\alpha)]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon (\exists x)A(x)] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+}\ \exists\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for quantifier $\bigotimes$ are given by

$$\frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-\\ \Gamma_1^+, [\mathbf{t}\colon A(\tau)] & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon A(\alpha)]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon (\bigotimes x)A(x)]}\ \bigotimes\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{c}\Gamma_1^-\\ \Gamma_1^+, [\mathbf{b}\colon A(\alpha)]\end{array}}{\Gamma_1^+, [\mathbf{b}\colon (\bigotimes x)A(x)]}\ \bigotimes\mathrm{I}\colon \mathbf{b}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{f}\colon (\bigotimes x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{n}, \mathbf{f}\colon A(\tau_1)]\rceil & \Gamma_2^-, \lceil[\mathbf{n}\colon A(\tau_2)]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon (\bigotimes x)A(x)] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{t}\colon A(\tau_2)]\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \bigotimes\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{n}\colon (\bigotimes x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A(\tau)]\rceil & \Gamma_2^-, \lceil[\mathbf{f}\colon A(\alpha)]\rceil\\ \Gamma_0^+, [\mathbf{b}, \mathbf{t}\colon (\bigotimes x)A(x)] & \Gamma_1^+ & \Gamma_2^+, [\mathbf{b}\colon A(\alpha)]\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+}\ \bigotimes\mathrm{E}\colon \mathbf{f}$$

The introduction and elimination rules for quantifier $\bigoplus$ are given by

$$\frac{\begin{array}{cc}\Gamma_1^- & \Gamma_2^-, \lceil[\mathbf{n}\colon A(\alpha)]\rceil\\ \Gamma_1^+, [\mathbf{t}\colon A(\tau)] & \Gamma_2^+, [\mathbf{t}\colon A(\alpha)]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{t}\colon (\bigoplus x)A(x)]}\ \bigoplus\mathrm{I}\colon \mathbf{t} \qquad \frac{\begin{array}{cc}\Gamma_1^-, \lceil[\mathbf{f}\colon A(\tau_1)]\rceil & \Gamma_2^-\\ \Gamma_1^+, [\mathbf{b}\colon A(\tau_1)] & \Gamma_2^+, [\mathbf{t}, \mathbf{b}\colon A(\tau_2)]\end{array}}{\Gamma_1^+, \Gamma_2^+, [\mathbf{b}\colon (\bigoplus x)A(x)]}\ \bigoplus\mathrm{I}\colon \mathbf{b}$$

$$\frac{\begin{array}{cc}\Gamma_0^-, \lceil[\mathbf{f}\colon (\bigoplus x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{n}\colon A(\alpha)]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon (\bigoplus x)A(x)] & \Gamma_1^+\end{array}}{\Gamma_0^+, \Gamma_1^+} \bigoplus\mathrm{E}\colon \mathbf{n}$$

$$\frac{\begin{array}{ccc}\Gamma_0^-, \lceil[\mathbf{n}\colon (\bigoplus x)A(x)]\rceil & \Gamma_1^-, \lceil[\mathbf{f}\colon A(\tau)]\rceil & \Gamma_2^-, \lceil[\mathbf{n},\mathbf{f}\colon A(\alpha)]\rceil \\ \Gamma_0^+, [\mathbf{b},\mathbf{t}\colon (\bigoplus x)A(x)] & \Gamma_1^+ & \Gamma_2^+\end{array}}{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+} \bigoplus\mathrm{E}\colon \mathbf{f}$$

Definition 25. A *natural deduction derivation* is defined inductively as follows:

1. Let A be any formula. Then

$$\frac{[V^-\colon A]}{[V^+\colon A]}$$

   is a derivation of A from the assumption $[V^-\colon A]$ (an *initial derivation*).

2. If D_k are derivations of Γ_k^+, Δ_k^+ from the assumptions $\Gamma_k^-, \hat{\Delta}_k^-$, and

$$\frac{\left\langle \begin{array}{c}\Gamma_k^-, \lceil\Delta_k^-\rceil \\ \Gamma_k^+, \Delta_k^+\end{array}\right\rangle_{k\in K}}{\Pi^+}$$

   is an instance of a deduction rule with $\hat{\Delta}_k^-$ a subsequent of Δ_k^-, and all eigenvariable conditions are satisfied, then

$$\frac{\langle D_k\rangle_{k\in K}}{\Pi^+}$$

   is a derivation of Π^+ from the assumptions $\bigcup_{k\in K}\Gamma_k^-$. The formulas in $\hat{\Delta}_k^-$ which do not occur in $\bigcup_{k\in K}\Gamma_k^-$ are said to be *discharged* at this inference.

Theorem 26. *A partial sequent Γ^+ can be derived from the assumptions Γ^- in the natural deduction system for Dunn and Belnap's logic of First Degree Entailment iff, for every interpretation $\mathfrak{I}$, either some formula in Γ_v^- ($v \in V^-$) evaluates to the truth value v, or there is a $w \in V^+$ and a formula in Γ_w^+ that evaluates to w.*

Proof. See Theorems 4.7 and 5.4 of Baaz et al. [4] or Theorems 4.2.8 and 4.3.4 of Zach [25]. □

Corollary 27. $\Gamma \models A$ *iff there is a natural deduction derivation of* $[V^+\colon A]$ *from* $[V^-\colon \Gamma]$.

6 Resolution and clause formation rules for Dunn and Belnap's logic of First Degree Entailment

The many-valued resolution calculus of Baaz and Fermüller [2] applies to Dunn and Belnap's logic of First Degree Entailment. We present the framework here, as well as a language preserving clause translation system for Dunn and Belnap's logic of First Degree Entailment.

Definition 28 (Signed formula)**.** A *signed formula* is an expression of the form A^v, where A is a formula and $v \in V$. If A is atomic, A^v is a *signed atom*.

Definition 29 (Signed clause)**.** A (signed) *clause* $C = \{A_1^{v_1}, \ldots, A_n^{v_n}\}$ is a finite set of signed atoms (proper clause). The empty clause is denoted by $\Box$.

An *extended clause* is a finite set of signed formulas.

Definition 30 (Semantics of clause sets)**.** Let $\mathscr{M}$ be a structure. $\mathscr{M}$ *satisfies* a clause C iff for every assignment s, there is some signed formula $A^v \in C$, so that $\mathrm{val}_{\mathfrak{I}}(A) = v$ (where $\mathfrak{I} = \langle \mathscr{M}, s\rangle$). $\mathscr{M}$ satisfies a clause set $\mathscr{C}$ iff it satisfies every clause in $\mathscr{C}$. $\mathscr{C}$ is called *satisfiable* iff some structure satisfies it, and *unsatisfiable* otherwise.

The clause formation rules for connective $\sim$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\sim A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{f}}\}\}}\ \sim{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\sim A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}\}\}}\ \sim{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\sim A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{n}}\}\}}\ \sim{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\sim A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}\}\}}\ \sim{:}\mathbf{f}$$

The clause formation rules for connective $\neg_b$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_b A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{f}}\}\}}\ \neg_b{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_b A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{n}}\}\}}\ \neg_b{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\neg_b A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}\}\}}\ \neg_b{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\neg_b A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}\}\}}\ \neg_b{:}\mathbf{f}$$

The clause formation rules for connective $\wedge$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{t}}\},\ C \cup \{A^{\mathbf{t}}\}\}}\ \wedge{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{t}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{b}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}}\ \wedge{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{t}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{n}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{n}}\}\}}\ \wedge{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \wedge B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}, A^{\mathbf{f}}, B^{\mathbf{b}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}, B^{\mathbf{n}}, B^{\mathbf{f}}\}\}}\ \wedge{:}\mathbf{f}$$

The clause formation rules for connective $\vee$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}, B^{\mathbf{t}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{n}}, B^{\mathbf{t}}, B^{\mathbf{n}}\}\}}\ \vee{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{b}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{b}}, A^{\mathbf{f}}\}\}}\ \vee{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{n}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{n}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}\}\}}\ \vee{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \vee B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{f}}\}\}}\ \vee{:}\mathbf{f}$$

The clause formation rules for connective $\rightarrow_e$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_e B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}, B^{\mathbf{t}}\}\}}\ {\rightarrow_e}{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_e B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}}\ {\rightarrow_e}{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_e B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}}\ {\rightarrow_e}{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_e B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}}\ {\rightarrow_e}{:}\mathbf{f}$$

The clause formation rules for connective $\rightarrow_b$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_b B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}, A^{\mathbf{f}}, B^{\mathbf{t}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}, B^{\mathbf{t}}, B^{\mathbf{b}}\}\}}\ {\rightarrow_b}{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_b B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{n}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{n}}\}\}}\ {\rightarrow_b}{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_b B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{n}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{b}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}}\ {\rightarrow_b}{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_b B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{t}}\}\}}\ {\rightarrow_b}{:}\mathbf{f}$$

The clause formation rules for connective $\rightarrow_l$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_l B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{f}}, B^{\mathbf{t}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}, B^{\mathbf{t}}\}\}}\ {\rightarrow_l}{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_l B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{b}}\}\}}\ {\rightarrow_l}{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \rightarrow_l B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}, B^{\mathbf{b}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{n}}, B^{\mathbf{n}}\}\}}\ {\rightarrow_l}{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \to_l B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{t}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}} \ \to_l{:}\mathbf{f}$$

The clause formation rules for connective $-$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(-A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}\}\}} \ -{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(-A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{n}}\}\}} \ -{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(-A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}\}\}} \ -{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(-A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{f}}\}\}} \ -{:}\mathbf{f}$$

The clause formation rules for connective $\otimes$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \otimes B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{t}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{t}}, B^{\mathbf{t}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}} \ \otimes{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \otimes B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{b}}\}\}} \ \otimes{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \otimes B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}, A^{\mathbf{n}}, B^{\mathbf{t}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}, B^{\mathbf{n}}, B^{\mathbf{f}}\}\}} \ \otimes{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \otimes B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{b}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{f}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{b}}, A^{\mathbf{f}}\}\}} \ \otimes{:}\mathbf{f}$$

The clause formation rules for connective $\oplus$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(A \oplus B)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{t}}, B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{t}}, B^{\mathbf{t}}\},\ C \cup \{A^{\mathbf{t}}, A^{\mathbf{n}}\}\}} \ \oplus{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \oplus B)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}, B^{\mathbf{t}}, B^{\mathbf{b}}\},\ C \cup \{A^{\mathbf{b}}, A^{\mathbf{f}}, B^{\mathbf{b}}, B^{\mathbf{f}}\}\}} \ \oplus{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \oplus B)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{n}}\},\ C \cup \{A^{\mathbf{n}}\}\}} \ \oplus{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(A \oplus B)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{B^{\mathbf{n}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{f}}, B^{\mathbf{f}}\},\ C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}\}\}} \ \oplus{:}\mathbf{f}$$

The clause formation rules for connective $\circlearrowleft$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\circlearrowleft A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{n}}\}\}} \ \circlearrowleft{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\circlearrowleft A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}\}\}} \ \circlearrowleft{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\circlearrowleft A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{f}}\}\}} \ \circlearrowleft{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\circlearrowleft A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}\}\}} \ \circlearrowleft{:}\mathbf{f}$$

The clause formation rules for connective $\triangle$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\triangle A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}, A^{\mathbf{b}}\}\}}\ \triangle{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\triangle A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C\}}\ \triangle{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\triangle A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C\}}\ \triangle{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\triangle A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{n}}, A^{\mathbf{f}}\}\}}\ \triangle{:}\mathbf{f}$$

The clause formation rules for connective $\Box$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\Box A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}\}\}}\ \Box{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\Box A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C\}}\ \Box{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\Box A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C\}}\ \Box{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\Box A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}, A^{\mathbf{n}}, A^{\mathbf{f}}\}\}}\ \Box{:}\mathbf{f}$$

The clause formation rules for connective $\circ$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{(\circ A)^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{t}}, A^{\mathbf{f}}\}\}}\ \circ{:}\mathbf{t} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\circ A)^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C\}}\ \circ{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{(\circ A)^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C\}}\ \circ{:}\mathbf{n} \qquad \frac{\mathscr{C} \cup \{C \cup \{(\circ A)^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A^{\mathbf{b}}, A^{\mathbf{n}}\}\}}\ \circ{:}\mathbf{f}$$

The clause formation rules for quantifier $\forall$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{((\forall x)A(x))^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A(b)^{\mathbf{t}}\}\}}\ \forall{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\forall x)A(x))^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{b}}\},\ C \cup \{A(b)^{\mathbf{b}}, A(b)^{\mathbf{t}}\}\}}\ \forall{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\forall x)A(x))^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{n}}\},\ C \cup \{A(b)^{\mathbf{n}}, A(b)^{\mathbf{t}}\}\}}\ \forall{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\forall x)A(x))^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f_1(\vec{a}))^{\mathbf{b}}, A(f_1(\vec{a}))^{\mathbf{f}}\},\ C \cup \{A(f_2(\vec{a}))^{\mathbf{f}}, A(f_2(\vec{a}))^{\mathbf{n}}\}\}}\ \forall{:}\mathbf{f}$$

The clause formation rules for quantifier $\exists$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{((\exists x)A(x))^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f_1(\vec{a}))^{\mathbf{b}}, A(f_1(\vec{a}))^{\mathbf{t}}\},\ C \cup \{A(f_2(\vec{a}))^{\mathbf{n}}, A(f_2(\vec{a}))^{\mathbf{t}}\}\}}\ \exists{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\exists x)A(x))^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{b}}\},\ C \cup \{A(b)^{\mathbf{b}}, A(b)^{\mathbf{f}}\}\}}\ \exists{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\exists x)A(x))^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{n}}\},\ C \cup \{A(b)^{\mathbf{f}}, A(b)^{\mathbf{n}}\}\}}\ \exists{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\exists x)A(x))^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A(b)^{\mathbf{f}}\}\}}\ \exists{:}\mathbf{f}$$

The clause formation rules for quantifier $\bigotimes$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigotimes x)A(x))^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{t}}\},\ C \cup \{A(b)^{\mathbf{b}}, A(b)^{\mathbf{t}}\}\}}\ \bigotimes{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigotimes x)A(x))^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A(b)^{\mathbf{b}}\}\}}\ \bigotimes{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigotimes x)A(x))^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f_1(\vec{a}))^{\mathbf{f}}, A(f_1(\vec{a}))^{\mathbf{n}}\},\ C \cup \{A(f_2(\vec{a}))^{\mathbf{n}}, A(f_2(\vec{a}))^{\mathbf{t}}\}\}}\ \bigotimes{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigotimes x)A(x))^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{f}}\},\ C \cup \{A(b)^{\mathbf{b}}, A(b)^{\mathbf{f}}\}\}}\ \bigotimes{:}\mathbf{f}$$

The clause formation rules for quantifier $\bigoplus$ are given by

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigoplus x)A(x))^{\mathbf{t}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{t}}\},\ C \cup \{A(b)^{\mathbf{n}}, A(b)^{\mathbf{t}}\}\}}\ \bigoplus{:}\mathbf{t}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigoplus x)A(x))^{\mathbf{b}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f_1(\vec{a}))^{\mathbf{b}}, A(f_1(\vec{a}))^{\mathbf{f}}\},\ C \cup \{A(f_2(\vec{a}))^{\mathbf{b}}, A(f_2(\vec{a}))^{\mathbf{t}}\}\}}\ \bigoplus{:}\mathbf{b}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigoplus x)A(x))^{\mathbf{n}}\}\}}{\mathscr{C} \cup \{C \cup \{A(b)^{\mathbf{n}}\}\}}\ \bigoplus{:}\mathbf{n}$$

$$\frac{\mathscr{C} \cup \{C \cup \{((\bigoplus x)A(x))^{\mathbf{f}}\}\}}{\mathscr{C} \cup \{C \cup \{A(f(\vec{a}))^{\mathbf{f}}\},\ C \cup \{A(b)^{\mathbf{f}}, A(b)^{\mathbf{n}}\}\}}\ \bigoplus{:}\mathbf{f}$$

In the translation rules for quantifiers, the indicated free variables b are new free variables that do not already occur in the premise, and terms $f(\vec{a})$ are formed using a new function symbol f and $\vec{a}$ all the free variables of the corresponding clause in the premise.

Theorem 31. *Let $T(\mathscr{C})$ be the result of exhaustively applying the translation rules to a clause set $\mathscr{C}$. Then $T(\mathscr{C})$ is a set of proper clauses, i.e., $T(\mathscr{C})$ contains only signed atoms (all connectives and quantifiers are eliminated). Furthermore, $T(\mathscr{C})$ is satisfiable iff $\mathscr{C}$ is.*

Proposition 32. *Let A be a sentence and Δ be a set of sentences. Then*

1. $\models A$ *iff* $\{\{A^v \mid v \in V^-\}\}$ *is unsatisfiable.*
2. $\Delta \models A$ *iff*

$$\{\{B^w \mid w \in V^+\} \mid B \in \Delta\} \cup \{\{A^v \mid v \in V^-\}\}$$

   is unsatisfiable.

Definition 33. A clause R is a *resolvent* of clauses C_1, C_2 if $R = (C_1 \setminus \{A_1^{v_1}\})\sigma \cup (C_2 \setminus \{A_2^{v_2}\})\sigma$ where

1. C_1 and C_2 have no free variables in common,
2. A_1 and A_2 are unifiable with most general unifier σ,
3. $v_1 \neq v_2$.

If C_1 and C_2 have free variables in common, we say that R is a resolvent of C_1 and C_2 if it is a resolvent of variable-disjoint renamings C_1' and C_2' of C_1 and C_2, respectively.

Definition 34. A *resolution refutation* of a clause set $\mathscr{C}$ is a sequence of clauses $C_1, \ldots, C_n$ so that for every i, $C_i \in \mathscr{C}$ or C_i is a resolvent of C_j, C_k with $j, k < i$, and $C_n = \emptyset$.

Theorem 35. *A clause set $\mathscr{C}$ is unsatisfiable iff it has a resolution refutation.*

Proof. See Theorems 3.14 and 3.19 of Baaz and Fermüller [2] or Theorems 2.5.5 and 2.5.8 of Zach [25]. □

Corollary 36. $\Delta \models A$ *iff*

$$T(\{\{B^w \mid w \in V^+\} \mid B \in \Delta\} \cup \{\{A^v \mid v \in V^-\}\})$$

has a resolution refutation.

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A fde.lgc – specification of Dunn and Belnap's logic of First Degree Entailment

```
logic "First Degree Entailment".
truth_values { t , b , n , f }.
designated_truth_values { b , t }.
ordering(truth,"f < {n, b} < t").
operator(neg/1, mapping {
          (t) : f,
          (b) : b,
          (n) : n,
          (f) : t
         }
).
operator(bneg/1, mapping {
          (t) : f,
          (b) : n,
          (n) : b,
          (f) : t
         }
).
operator(and/2, inf(truth)).
operator(or/2, sup(truth)).
```

```
operator(impe/2, table [
              t, b, n, f,
          t,  t, b, n, f,
          b,  t, b, n, f,
          n,  t, t, t, t,
          f,  t, t, t, t
        ]
).
operator(impb/2, table [
              t, b, n, f,
          t,  t, b, n, f,
          b,  t, t, n, n,
          n,  t, b, t, b,
          f,  t, t, t, t
        ]
).
operator(impl/2, table [
              t, b, n, f,
          t,  t, f, n, f,
          b,  t, b, n, f,
          n,  t, n, t, n,
          f,  t, t, t, t
        ]
).
ordering(inform,"n < {t, f} < b").
operator(confl/1, mapping {
         (t) : t,
         (b) : n,
         (n) : b,
         (f) : f
        }
).
operator(times/2, inf(inform)).
operator(plus/2, sup(inform)).
operator(ruet/1, mapping {
         (t) : b,
         (b) : f,
         (n) : t,
         (f) : n
        }
).
operator(delta/1, mapping {
         (t) : t,
         (b) : t,
         (n) : f,
         (f) : f
        }
).
operator(box/1, mapping {
         (t) : t,
         (b) : f,
         (n) : f,
         (f) : f
        }
).
operator(class/1, mapping {
         (t) : t,
```

```
          (b) : f,
          (n) : f,
          (f) : t
         }
).
quantifier(forall, induced_by inf(truth)).
quantifier(exists, induced_by sup(truth)).
quantifier(bigtimes, induced_by inf(inform)).
quantifier(bigplus, induced_by sup(inform)).
```

B fde.cfg – LaTeX translations

```
texName(b,"\\mathbf{b}").
texName(t,"\\mathbf{t}").
texName(n,"\\mathbf{n}").
texName(f,"\\mathbf{f}").
texName(forall, \\forall).
texName(exists, \\exists).
texName(bigtimes, \\bigotimes).
texName(bigplus, \\bigoplus).
texName(and,    \\land  ).
texName(or,     \\lor   ).
texName(times,  \\otimes  ).
texName(plus,   \\oplus   ).
texName(neg,    "\\mathord{\\sim}"  ).
texName(bneg,   "\\lnot_b"  ).
texName(delta,  \\triangle).
texName(box,    "\\Box").
texName(class,  \\circ).
texName(confl,  "-").
texName(ruet,  \\circlearrowleft).
texName(impe,    "\\to_e").
texName(impl,    "\\to_l").
texName(impb,    "\\to_b").

texInfix(and).
texInfix(or).
texInfix(times).
texInfix(plus).
texInfix(impe).
texInfix(impb).
texInfix(impl).
texPrefix(neg).
texPrefix(bneg).
texPrefix(confl).
texPrefix(box).
texPrefix(delta).
texPrefix(class).
texPrefix(confl).
texPrefix(ruet).


texExtra("FullNameOfLogic","Dunn and Belnap's logic of First Degree
     Entailment").
texExtra("FDE","\\ensuremath{\\mathbf{FDE}}").
```

```
texExtra("ShortName", "\\FDE").
texExtra("Intro","\\FDE{} was introduced as a ‘‘useful four-valued
    logic’’ in \\cite{Belnap1975} and \\cite{Dunn1976}.  The
    operators we consider are (most of) those discussed in Omori
    and Wansing’s survey~\\cite{OmoriWansing2017}, whose notation
    we also follow. The sequent systems presented below are
    extensions of those in \\cite{BaazFermullerZach1994}, where \\
    FDE{} was used as the running example (the $\\bigotimes$
    quantifier was denoted \\textsf{U} there). The rules also
    improve on those of Ruet’s \\cite{Ruet1996} sequent calculus.
    4-sided sequent systems for extensions of FDE similar to ours
    were also given by Wintein and Muskens~\\cite{Wintein2012,
    Wintein2016}.").
texExtra("Semantics","The set of truth values of \\FDE{} forms a so
    -called bilattice (see \\cite{Ginsberg1988}) with two orderings
    ~$\\le_t$ (truth) and $\\le_k$ (information). They are defined
    as $\\mathbf{f} <_t \\mathbf{b}, \\mathbf{n} <_t \\mathbf{t}$
    and $\\mathbf{n} <_k \\mathbf{t}, \\mathbf{f} <_t \\mathbf{b}$.
     The operators $\\land$ and $\\lor$ are the inf and sup in the
    $\\le_t$ ordering; the operators $\\otimes$ and $\\oplus$ the
    inf and sup in the~$\\le_k$ ordering.  The quantifiers $\\
    forall$, $\\exists$, $\\bigotimes$, and $\\bigoplus$,
    respectively, are the quantifiers induced by these connectives
    (i.e., the inf and sup of the truth-value distributions of the
    matrix $A(x)$ of $(\\quantor x)A(x)$). The operator $\\sim$ is
    the usual negation of \\FDE, while $\\lnot_b$ is the Boolean
    negation, and $-$ is the ‘‘conflation’’ \\cite{Fitting2006} of
    the $\\le_k$ order. $\\triangle$ is Baaz’s determinateness
    operator \\cite{Baaz1996} (studied in \\FDE{} by Sano and Omori
     \\cite{SanoOmori2014}). $\\Box$ is the modality studied by
    Font and Rius~\\cite{FontRius2000}. $\\circ$ is the
    classicality operator. The quarter turn operator $\\
    circlearrowleft$ was introduced by Ruet \\cite{Ruet1996}. The
    conditional $x \\to_e y$ is defined as $\\sim A \\lor B$, while
     $\\to_b$ is the Boolean conditional and $\\to_l$ the \\L
    ukasiewicz conditional. See \\cite{OmoriWansing2017} for
    definitions and references.").
texExtra("Link", "https://logic.at/multlog/fde.pdf").
```