Interpolation

Motivation: Consider
\[ A_1, \ldots, A_n \models C, \]
where \( C \) is some mathematical theorem and the \( A_i \) are specific axioms, definitions, lemmas, or other theorems.

(\text{Let} \( A = A_1 \land \cdots \land A_n \).

\( A \) may contain symbols that do not occur in \( C \), and vice versa.

Can we find a sentence \( B \), such that
\[ A \models B \quad \text{and} \quad B \models C \]
where \( B \) contains only symbols occurring in both: \( A \) and \( C \)?

As stated above, the answer is \textbf{no}.

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Proof of Craig's Interpolation Theorem

\text{Degenerate case:} \( A \) and \( C \) are valid or unsatisfiable.

Let \( A \) be \( \exists x F(x) \land \forall x \neg F(x) \) and let \( C \) be \( \exists u P(u) \).

The \( \exists x \neq x \) may serve as interpolant.

\[ \Rightarrow \quad \text{We may have to use} \quad \ldots \quad \text{in the interpolant} \quad B, \]

\[ \text{even if it neither occurs in} \quad A \text{\ or nor in} \quad C: \]

Other ways deal to with this case:

\[ \diamond \quad \text{Exclude unsatisfiable sentences} \quad A \quad \text{and valid sentences} \quad C. \]

\[ \diamond \quad \text{Include the truth constants} \quad \bot, \top \text{ as (0-ary) connectives.} \]

For the rest of the proof we may assume that \( A \) is satisfiable and that \( C \) is not valid.

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Example: (Failure of interpolation)

\text{Let} \( A \) be \( \exists x F(x) \land \forall x \neg F(x) \)\text{ and let } \( C \) be \( \exists u \exists v u \neq v \).

\( A \models C \), but there are no common symbols (except \( \exists \) and \( \neg \)).

Focussing on more modest requirements we can state:

\text{Proposition:} (Interpolation w.r.t. constants)

If \( A \models C \) then there is a \( B \) such that \( A \models B \) and \( B \models C \), where \( B \) contains only constants that occur in \( A \) as well as in \( C \).

\text{Proof:} Take \( \exists v_1 \ldots \exists v_n A^* \) for \( B \), where \( A^* \) results from \( A \) by replacing the constants \( c_1, \ldots, c_n \), that do not occur in \( C \) by the new variables \( v_1, \ldots, v_n \). It is easy to check \( \text{q.e.d.} \)

More interestingly, let \( L(F) \) denote all non-logical symbols in \( F \):

\text{Theorem:} (Craig's Interpolation Theorem)

If \( A \models C \) then there is a \( B \) such that \( A \models B \) and \( B \models C \), where \( L(B) \subseteq L(A) \cap L(C) \).

\( B \) is called \textbf{interpolant} from \( A \) to \( C \).

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Note:

- \( A \models C \quad \text{iff} \quad \{ A, \neg C \} \) is unsatisfiable.
- \( B \) is an interpolant of \( A \) to \( C \) \text{iff}
\[ A \models B, \quad \neg C \models \neg B, \quad \text{and} \quad L(B) \subseteq L(A) \cap L(C). \]

We proceed indirectly and show:

If there is no interpolant of \( A \) to \( C \), then \( \{ A, \neg C \} \) is satisfiable.

We use the model existence lemma, \text{i.e.: appeal to (S0)--(S8)}.

\text{Some useful terminology:}

\[ \diamond \quad B \text{ is said to separate } \Gamma_L \text{ from } \Gamma_R \text{ iff } \Gamma_L \models B, \Gamma_R \models \neg B, \text{ and } L(B) \subseteq L(\Gamma_L) \cap L(\Gamma_R). \]

\[ \diamond \quad \Gamma \text{ is called divisible (without separation) if it can be written as } \Gamma = \Gamma_L \cup \Gamma_R, \text{ where } \Gamma_L \text{ and } \Gamma_R \text{ are satisfiable, and no sentence separates } \Gamma_L \text{ from } \Gamma_R. \]

\( B \) is an interpolant of \( A \) to \( C \) \text{ iff } \( B \) separates \( \{ A \} \) from \( \{ \neg C \} \).
Proof without identity and function symbols

Let $S$ be the set of divisible $L$-sets.
It remains to establish (S0)–(S6) for $S$.

(S0) trivial [why ?]

(S1): Suppose $\Gamma = \Gamma_L \cup \Gamma_R$ is divisible, and $D, \neg D \in \Gamma$.
Since $\Gamma_L$ and $\Gamma_R$ are satisfiable we conclude w.l.o.g.: $D \in \Gamma_L, \neg D \in \Gamma_R$. But this means that $D$ separates $\Gamma_L$ from $\Gamma_R$, in contradiction to the assumption of divisibility.

(S2): Suppose $\Gamma = \Gamma_L \cup \Gamma_R$ is divisible, and $\neg \neg D \in \Gamma$.
W.l.o.g., $\neg D \in \Gamma_L$ and hence $\Gamma_L \models D$. This implies that also $(\Gamma_L \cup \{D\}) \cup \Gamma_R$ divisible, and thus that $\Gamma \cup \{D\} \in S$.

(S4)–(S6): Analogous to (S2). [[ Blackboard, if needed ]]

Adding identity and function symbols

The case with identity (but without function symbols) is reduced to the case without identity by replacing $=$ with a non-logical predicate symbol $\equiv$.

Terminology:
Let $E_A [E_C]$ be the conjunction of the equivalence axioms and the congruence axioms for the predicates in $A [C]$ (using $\equiv$ for the congruence relation).
$F^* \ldots F$, where $=$ is replaced by $\equiv$.

Fact:
Any interpolant $B^*$ from $E_A \land A^* \to E_C \to C^*$ can be re-translated into an interpolant $B$ from $A$ to $C$.
Similarly, appropriate ‘definitions’ of functions by predicates allow to reduce the case with function symbols to the case without function symbols. [[ Details in [BBJ] ]]

Combining theories — joint consistency

DEF:
A theory $T$ (in $L$) is a set of sentences (over $L$) that is closed w.r.t. logical consequence: $T \models F$ implies $F \in T$.
$F$ is also called a theorem of $T$.

Combining theories is an important and practically relevant topic in software verification.
[ Can you explain why? ]

Note:
For satisfiable theories $T_1$, $T_2$, in general:
$\diamond T_1 \cup T_2$ is not a theory
$\diamond$ the theory $\{ F : T_1 \cup T_2 \models F \}$ is not satisfiable
even if the languages $L(T_1)$ and $L(T_2)$ are disjoint.
Lemma:
Let \( T_1, T_2 \) be theories. \( T_1 \cup T_2 \) is satisfiable iff there is no sentence \( A \in T_1 \), where \( \neg A \in T_2 \).

Note:
\( \diamond \) The lemma expresses the following:
joint satisfiability = joint consistency
(Note that this is a kind of completeness statement!)

\( \diamond \) The lemma is wrong for many non-classical logics.

Proof: The 'only if' part is trivial.
The 'if' part follows from compactness and interpolation:
Suppose \( T_1 \cup T_2 \) is unsatisfiable, then already some finite \( S_0 \subseteq T_1 \cup T_2 \) is unsatisfiable.
We will show that there is an \( A \in T_1 \), where \( \neg A \in T_2 \).

Joint consistency (ctd.)
DEF: Theory \( T' \) is a conservative extension of theory \( T \) if \( T \subseteq T' \) and every \( F \in T' \) over \( L(T) \) is already in \( T \).

Theorem: (Joint conservative extensions theorem)
For \( i = 0, 1, 2 \) let \( T_i \) be a theory over \( L_i \), where \( L_0 = L_1 \cap L_2 \).
Let \( T_3 \) consist of the consequences of \( T_1 \cup T_2 \) over \( L_1 \cup L_2 \).
If \( T_1 \) and \( T_2 \) are conservative extensions of \( T_0 \), then so is \( T_3 \).

Proof: We have to show: \( B \in T_3 \) for \( B \) in \( T_0 \).
\( B \in T_3 \) implies \( T_1 \cup T_2 \cup \{ \neg B \} \) is unsatisfiable.
By the above Lemma, for some \( D \in T_1 \): \( \neg D \in T_2 \cup \{ \neg B \} \), where \( D \) is in \( L_0 \). Hence also \( \neg B \rightarrow \neg D \) is in \( L_0 \).
\( T_2 \cup \{ \neg B \} \models \neg D \). \( \neg B \rightarrow \neg D \in T_2 \). Since \( T_2 \) is a conservative extensions of \( T_0 \), we conclude \( \neg B \rightarrow \neg D \in T_0 \). Since \( T_1 \) is a conservative extensions of \( T_0 \), we have \( D \in T_0 \). But \( \{ D, \neg B \rightarrow \neg D \} \models B \), and therefore \( B \in T_0 \). Q.e.d..

Corollary: (Robinson’s joint consistency theorem)
For \( i = 0, 1, 2 \) let \( T_i \) be a theory over \( L_i \), where \( L_0 = L_1 \cap L_2 \).
If \( T_0 \) is complete, and \( T_1 \supseteq T_0 \) as well as \( T_2 \supseteq T_0 \) are satisfiable, then \( T_1 \cup T_2 \) is satisfiable.

Proof:
Any satisfiable extension of a complete theory is conservative.
Any conservative extension of a satisfiable theory is satisfiable.
Thus if the \( T_i \) are as specified, then we may apply the joint conservative extensions theorem to conclude that the theory consisting of all consequences of \( T_1 \cup T_2 \) is satisfiable.
Therefore \( T_1 \cup T_2 \) itself is satisfiable. Q.e.d..
Note:
We have proved Robinson’s joint consistency theorem using Craig’s interpolation theorem. Also the converse is possible. (See [BBJ], page 265.)

In any case: the (term) model existence lemma is essentially involved in proving all of the following:

- Compactness theorem
- Löwenheim-Skolem theorem
- Completeness (in different versions)
- Craig’s interpolation theorem
- Robinson’s joint consistency theorem
- Beth’s definability theorem