A sample theory
Consider a language with the binary predicate symbol ‘≡’ (written infix) as its only non-logical symbol.
We want to study (and count) models of \( \text{Eq} = \text{Re} \land \text{Sy} \land \text{Tr} \):

\[
\begin{align*}
\text{Re} & : \forall x \ x \equiv x \\
\text{Sy} & : \forall x \forall y \ x \equiv y \rightarrow y \equiv x \\
\text{Tr} & : \forall x \forall y \forall z \ (x \equiv y \land y \equiv z) \rightarrow x \equiv z
\end{align*}
\]

[As a ‘warm up’: describe two non-isomorphic 2-element models of \( \text{Eq} \)]

Equivalence relations
The models of \( \text{Eq} \) are equivalence relations (over some set \( X \)). Formally: Structures \((X, \sim)\), where for all \( a, b, c \in X \):

(E1) Reflexivity: \( a \sim a \),
(E2) Symmetry: If \( a \sim b \) then also \( b \sim a \),
(E3) Transitivity: If \( a \sim b \) as well as \( b \sim c \), then \( a \sim c \).

Equivalence relations can be obtained from partitions.
DEF: A partition \( \Pi \) of \( X \) is a set of nonempty subsets of \( X \), s.t.

(P1) Disjointness: \( A, B \in \Pi \) implies either \( A = B \) or \( A \cap B = \emptyset \),
(P2) Exhaustiveness: Every \( a \in X \) belongs to some \( A \in \Pi \).

IMPORTANT: We consider only denumerable models \( M \) of \( \text{Eq} \).
DEF:
The signature of \( M \) is a mapping \( \sigma_M : \omega \rightarrow \omega \cup \{ \infty \} \), where \( \sigma_M(n) \) is the number of equivalence classes of size \( n \), \( n \geq 1 \), and \( \sigma_M(0) \) is the number of equivalence classes of infinite size.

We denote signatures as infinite vectors
\( (\sigma_M(0), \sigma_M(1), \sigma_M(2), \sigma_M(3), \ldots) \)

Example: (A promiscuous model)
Let \( M \) be a model of \( \text{Eq} \) and also of \( \text{E} \text{prom} \):

\( \forall x \forall y \ x \equiv y. \)

Then \( a \equiv^M b \) holds for all \( a, b \in |M| \).
Therefore there is only one equivalence class (which, by definition, is of infinite size): i.e., \( \sigma_M = (1, 0, 0, 0, \ldots) \). All models of \( \text{Eq} \land \forall x \forall y \ x \equiv y \) are isomorphic to \( M \); i.e., there is only one isomorphism type.
Example: (An eremitic model)
Add the following sentence to $Eq$:

$$E_{erem}: \forall x \forall y \ x \equiv y \leftrightarrow x \equiv y.$$  

All models $M$ have signature $\sigma_M = (0, \infty, 0, 0, \ldots)$.  

Again, there is only one isomorphism type.

Example: (Two isomorphism types)
$Eq \land (E_{prom} \lor E_{erem})$ is made true only by the eremitic and the promiscuous model.  

Hence there are exactly two isomorphism types.

Example: (An uxorious model)

$$E_{ux}: \forall x \exists y (x \neq y \land x \equiv y \land \forall z (z \equiv x \rightarrow (z = x \lor z = y)))$$

is made true only by models $M$ with only 2-element equivalence classes; i.e. $\sigma_M = (0, 0, \infty, 0, \ldots)$.

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Example: (n isomorphism types)

Analogously to $E_{ux}$ we can enforce $n$-element equivalence classes for every $n \geq 1$.

For every $k > 0$, exactly $k$ isomorphism types can be enforced by disjunction over corresponding sentences.

Example: (denumerably many isomorphism types)

$$E_{den}: \forall x \forall y (\exists u (u \neq x \land u \equiv x \land \exists v (v \neq y \land v \equiv y)) \rightarrow x \equiv y)$$

has models, where all elements that are not ‘isolated’ are in a single equivalence class (which can be of any size).  

The following signatures arise:

$$(0, \infty, 0, 0, \ldots) \quad (1, 0, 0, 0, \ldots)$$
$$(1, \infty, 0, 0, \ldots) \quad (1, 1, 0, 0, \ldots)$$
$$(0, \infty, 1, 0, \ldots) \quad (1, 2, 0, 0, \ldots)$$
$$(0, \infty, 0, 1, \ldots) \quad (1, 3, 0, 0, \ldots)$$
$$(0, \infty, 0, 0, \ldots) \quad \ldots \quad \ldots$$

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The Löwenheim-Skolem theorem

Remember: We can enforce models of all finite sizes, but also models of infinite size.

Can we enforce non-enumerable models? (Think, e.g., of $\mathbb{R}$.)

Obviously yes, if we allow non-enumerably many constants and non-enumerably many sentences.

However: We forbid non-enumerable languages (but don’t restrict sets of sentences otherwise).

The answer then is no:

Theorem (downward Löwenheim-Skolem theorem)

Every satisfiable set of sentences has an enumerable model.

[Before proving this we explore its consequences]
Consequences of the Löwenheim-Skolem theorem 1
An immediate application of the Löwenheim-Skolem theorem is the existence of certain non-standard models.

Corollary: (Skolem’s paradox)
The first-order theory of $\mathbb{R}$, i.e., the set all of sentences (over some enumerable language) that are made true by $\mathbb{R}$, has enumerable models.

Consequently:
The reals cannot be characterized canonically by any set of sentences: There are always models $\mathcal{R}$, non-isomorphic to $\mathbb{R}$.

Similar remarks hold for first-order set theory.

For cognoscenti:
‘Powers sets’ need not be interpreted to ‘contain really’ all subsets, but only those that can be ‘described’ within the theory.

Consequences of the Löwenheim-Skolem theorem 2

Corollary: (Canonical-domains lemma 1)
Let $\Gamma$ be a sentence or a set of sentences.
If $\Gamma$ is satisfiable, then it has a model with domain either $\{0, 1, \ldots, n-1\}$ or $\omega$.

Corollary: (Canonical-domains lemma 2)
Let $\Delta$ be a sentence or a set of sentences without identity and without function symbols.
If $\Delta$ is satisfiable, then it has a model with domain $\omega$.

Proof:
In addition to Löwenheim-Skolem one needs the following observation:
Without identity and function symbols we can map all ‘too large’ integers into a particular element of a finite model. (See [BBJ])

Consequences of the Löwenheim-Skolem theorem 3
$\Gamma$ ... set of sentences

DEF:
$\Gamma$ is (implicationally) complete if for every sentence $A$ of its language either $\Gamma \models A$ or $\Gamma \models \neg A$.

DEF:
$\Gamma$ is denumerably categorical if all denumerable models of $\Gamma$ are isomorphic.

Corollary: (Vaught’s test).
If $\Gamma$ is denumerably categorical but not finitely satisfiable, then $\Gamma$ is complete.

[[ Proof on blackboard ]]