Tentative overview:

0 What is meant by ‘nonclassical logics’?
   What are these logics good for?

1 A (possible) classification —
   Relations between various logics

2 A reminder on classical logic

3 Modal logics
   ▶ What are modal logics?
   ▶ General theory: syntax, semantics, proof systems, expressibility, relations between important modal logics, ‘correspondence theory’, multi-modal logics, . . .
   ▶ Epistemic logic(s) (for modelling multi-agent systems)
   ▶ Hints at other families of modal logics: temporal logic, deontic logic, dynamic logic, provability logic, . . .
Tentative overview (ctd.):

4 Intuitionistic logic (constructive logic)
▶ general motivation: ‘From Brouwer to program extraction’
▶ different semantics: Kripke/Beth style semantics, Brouwer-Heyting-Kolmogorov interpretation, topological semantics
▶ different proof systems: Hilbert type, sequent system(s), ‘natural deduction’, …
▶ dialogue games characterizing constructive (and other) logics

5 Many-valued logics, fuzzy logics:
▶ finite valued logics: syntax, semantics, important examples of 3- and 4-valued logics (Kleene, Belnap) and families of logics (Gödel, Łukasiewicz, …)
▶ internal and external proof systems
▶ short appetizer \( t \)-norm based fuzzy logics
  (specialized course “Fuzzy Logic” by P. Cintula ony 2017/18)

6 Overview of further topics:
▶ e.g., substructural logics, logic in games and games in logic, logical dynamics, paraconsistent logics, relevance logics, …

Let me know your own interests!
Do we need nonclassical logics? Why?

A possible reason for inadequateness of classical logic (CL) as a model for correct formal reasoning:

- CL-semantics is too coarse: e.g., lacking ‘truth value gaps’ or degrees of truth; concerns about ‘constructivity’; concerns about ‘paradoxes of material implication’

Exercise 1:
Explain (in your own words!) the paradoxes of material implication. Make your sources explicit!

Exercise 2:
Consider the following statement $S$, that we will assume to be true:

*If god doesn’t exist then it is not the case that if I pray to god she will answer my prayers.*

This has the logical form $\neg G \supset \neg (P \supset A)$. I don’t pray to god means that $P$ is false. But if $P$ is false then $P \supset A$ is true and therefore $\neg (P \supset A)$ is false. Since we assume $S$ to be true, $\neg G$ has also to be false and thus $G$ (‘God exists’) is true.

Explain what exactly is wrong here?
Do we need nonclassical logics? Why? (ctd.)

Another possible reason:

- insufficient expressibility: e.g., concerning temporal dependencies or the knowledge/belief/truth distinction

An important remark:
First-order classical logics can be considered universal. But it is often computationally & conceptionally more adequate to use, e.g., propositional modal logic instead of a classical first-order theory.
Note:
All nonclassical logics refer in some specific way to classical logic:
(Not necessarily exclusively) either
– to propose a fundamental alternative to it, or
– to extend/augment it in a specific way, or
– to generalize some fundamental feature of it, or
– to refine it from a certain perspective.

[ picture developed on blackboard → handout ]

As a consequence,
we need to have a clear understanding of the main concepts of classical logic and of logics in general:

▶ Syntax (‘classical’ connectives and quantifiers)
▶ Semantics (‘Tarski-style’)
▶ Proof systems / algorithms (‘Gentzen-style’, ‘Frege-Hilbert-style’, tableaux, resolution, etc.)
Basic concepts of CL — A quick reminder

Note:
Logic — not only classical logic! — comes in levels/orders:

- propositional logic
- first-order logic
- second-order logic
- ...(higher orders)

Main characteristics of CL at the different levels:

- propositional logic: decidable (validity coNP-complete)
- first-order logic: undecidable, but still axiomatizable
  (‘r.e.’/‘c.e.’, validity $\Pi^0_1$-complete = ‘semi-decidable’)
- second and higher orders: not (recursively) axiomatizable
  (validity not ‘arithmetical’, $\Pi^1_1$-complete)
Classical propositional logics — the logic of ‘bivalence’

Syntax: inductive definition of formulas $FORM^0$:
$A, B, \ldots, F, G, \ldots$

- propositional variables (atoms) $PV$:
  $p, q, r, \ldots$
- logical connectives (operators):
  $\supset$ (implication), $\lor$ (disjunction),
  $\land$ (conjunction), $\neg$ (negation), $\bot$ (‘falsum’)
- additional symbols: parenthesis

Note: We make liberal use of usual priority rules, associativity and commutativity of $\land$ and $\lor$ etc.

Semantics: truth tables — e.g., for implication

<table>
<thead>
<tr>
<th>$A \supset B$</th>
<th>$B : 1$</th>
<th>$B : 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A : 1$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$A : 0$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where $1, 0$ are the truth values ‘true’, ‘false’
Semantics (ctd):
An interpretation \( I \) is an
- assignment: \( PV \mapsto \{1, 0\} \).

Every interpretation \( I \) induces — via the truth tables — an
- evaluation function \( v_I : FORM^0 \mapsto \{1, 0\} \).

\( F \) is satisfiable \( \ldots \) \( F \) evaluates to 1 (‘is true’) under some interpretation
\( F \) is valid \( \ldots \) \( F \) evaluates to 1 (‘is true’) under all interpretations

**Exercise 3:**
Clarify the following notions for propositional CL:
tautology, model, interpretation, (un)satisfiability, consequence.

**Exercise 4\*:**
Show that the satisfiability problem for the fragment of proposi-
tional CL without \( \neg \) and \( \bot \) (‘positive fragment’) is trivial, whereas the validity problem is still coNP-complete.
Classical predicate logic (first-order CL)

Syntax: (set of formulas $FORM^1$)

Atomic formulas (atoms) are analyzed as structured

- predicate symbols $PS_n$ of arity $n$ ($P$, $Q$, $R$, ...)
  (‘0-ary’ predicate symbol = propositional variable)
- (object) terms ($s$, $t$, $t_i$, ...) are inductively built up from
  - (object) variables $V$ ($x$, $y$, $z$, $u$, ...)
  - constant symbols $KS$ ($a$, $b$, $c$, ...)
  - function symbols $FS_n$ ($f$, $g$, $h$, ...) of arity $n$

resulting in atoms $P(f(a, x), g(f(x, y)), c)$ etc.

- general formulas ($A$, $B$, ..., $F$, $G$, ...) build up from connectives as well as quantifiers $\forall$, $\exists$

Exercise 5:
Present a formal definition of $free(F)$ (set of free variables occurring in the formula $F$), based on a formal definition of $FORM^1$. 
First-order CL (ctd.)

Semantics:
An interpretation is a tuple \( I = (D, \Phi, d) \)

(1) \( D \) is a non-empty set — the domain of \( I \)
(2) \( \Phi \) is a signature mapping:
   (2.1) \( \Phi(c) \in D \) for all \( c \in KS \).
   (2.2) \( \Phi(f) \) is a function of type \( D^n \mapsto D \) for \( f \in FS_n \).
   (2.3) \( \Phi(P) \) is a relation, represented as function of type \( D^n \mapsto \{0, 1\} \) for \( P \in PS_n \).
(3) \( d \) is a variable assignment, i.e., a function of type \( V \to D \)

Note:
Sometimes all such structures \( I \) are called ‘models’.
(As in: ‘model theory’).
First-order CL — semantics (ctd.)

Given an interpretation \( I = (D, \Phi, d) \) the value \( v_I(t) \) of a term \( t \) in \( I \) is defined inductively as follows:

- if \( t \in V \) then \( v_I(t) = d(t) \)
- if \( t \in KS \) then \( v_I(t) = \Phi(t) \)
- otherwise \( t \) is of the form \( f(t_1, \ldots, t_n) \) and \( v_I(t) = \Phi(f)(v_I(t_1), \ldots, v_I(t_n)) \)

**Exercise 6:**

Present a formal definition of the evaluation function \( v_I \), that assigns a truth value in \( I \) to every formula \( F \).

**Exercise 7:**

Let \( G = (P(x) \supset P(f(x, y))) \). Find models and counter-models for the following formulas: \( \exists x \exists y \ G, \forall x \exists y \ G, \exists x \forall y \ G, \exists y \forall x \ G \).

**Note:**

Proof systems don't refer to models, but are purely syntactical.
# Syntax vs. semantics

<table>
<thead>
<tr>
<th>Semantics</th>
<th>Syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>F</em> is valid</td>
<td><em>F</em> is provable</td>
</tr>
<tr>
<td>(true in every interpretation)</td>
<td>(in a calculus, i.e.) is a theorem</td>
</tr>
<tr>
<td><em>F</em> is unsatisfiable</td>
<td><em>F</em> is refutable</td>
</tr>
<tr>
<td>(false in every interpretation)</td>
<td>(¬<em>F</em> is a theorem)</td>
</tr>
<tr>
<td><em>F</em> follows from/is a consequence of set $\Gamma \supseteq {G_1, \ldots, G_n}$</td>
<td><em>F</em> is derivable from $G_1, \ldots, G_n$</td>
</tr>
<tr>
<td><em>F</em> and <em>G</em> are logically equivalent</td>
<td>$(F \supseteq G) \land (G \supseteq F)$ is provable</td>
</tr>
<tr>
<td><em>F</em> and <em>G</em> are true/false in the same interpretations</td>
<td>(abbrev.: $F \equiv G$ or $F \leftrightarrow G$)</td>
</tr>
</tbody>
</table>

**Exercise 8:**
Explain how soundness and completeness relates syntax and semantics (in particular in case of ‘follows’ vs. ‘derivable’).
Types of calculi (proof systems)

- **(Frege-)Hilbert type:** following the scheme of axiomatic theories: few, simple derivation rules (e.g. only *modus ponens*)
  - well suited for presenting/comparing a wide range of logics
  - bad for proof search

- **Gentzen type:** rule-oriented, analytic, close to semantics
  - sequent calculi (e.g., *LK*)
  - (analytic) tableaux
  - matrix systems, connection graphs
  - hypersequent calculi
  - display calculi, . . .

**Exercise 9:**
Describe at least one more type of proof systems. Explain what *soundness* and *completeness* means for each system. What could the distinction between ‘strong’ and ‘weak/ordinary’ completeness refer to?