Nonclassical logics
(Nichtklassische Logiken)
VU 185.249 (lecture + exercises)

http://www.logic.at/lvas/ncl/

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Lecture 1
Tentative overview:

0 What is meant by ‘nonclassical logics’?
   What are these logics good for?

1 A (possible) classification —
   Relations between various logics

2 A reminder on classical logic

3 Modal logics
   ▶ What are modal logics?
   ▶ General theory: syntax, semantics, proof systems,
     expressibility, relations between important modal logics,
     ‘correspondence theory’, multi-modal logics, . . .
   ▶ Epistemic logic(s) (for modelling multi-agent systems)
   ▶ Hints at other families of modal logics: temporal logic,
     deontic logic, dynamic logic, provability logic, . . .
Tentative overview (ctd.):

4 Intuitionistic logic (constructive logic)
   - general motivation: ‘From Brouwer to program extraction’
   - different semantics: Kripke/Beth style semantics, Brouwer-Heyting-Kolmogorov interpretation, topological semantics
   - different proof systems: Hilbert type, sequent system(s), ‘natural deduction’, . . .
   - dialogue games characterizing constructive (and other) logics

5 Many-valued logics, fuzzy logics:
   - finite valued logics: syntax, semantics, important examples of 3- and 4-valued logics (Kleene, Belnap) and families of logics (Gödel, Łukasiewicz, . . .)
   - internal and external proof systems
   - short appetizer $t$-norm based fuzzy logics
     (specialized course “Fuzzy Logic” by P. Cintula next term(?))

6 Overview of further topics:
   - e.g., substructural logics, logic in games and games in logic, logical dynamics, paraconsistent logics, relevance logics, . . .

Let me know your own interests!
Do we need nonclassical logics? Why?

A (related) possible reason for inadequateness of classical logic (CL) as a model for correct formal reasoning:

- **CL-semantics** is **too coarse**: e.g., lacking ‘truth value gaps’ or degrees of truth; concerns about ‘constructivity’; concerns about ‘paradoxes of material implication’

**Exercise 1:**
Explain (in your own words!) the paradoxes of material implication. Make your sources explicit!

**Exercise 2:**
Consider the following statement $S$, that we will assume to be true:

*If god doesn’t exist then it is not the case that if I pray to god then she will answer my prayers.*

This has the logical form $\neg G \supset \neg (P \supset A)$. I don’t pray to god means that $P$ is false. But if $P$ is false then $P \supset A$ is true and therefore $\neg (P \supset A)$ is false. Since we assume $S$ to be true, $\neg G$ has also to be false and thus $G$ (‘God exists’) is true.

Explain what exactly is wrong here?
Do we need nonclassical logics? Why? (ctd.)

Another possible reason:

- **insufficient expressibility**: e.g., concerning temporal dependencies or the knowledge/belief/truth distinction

An important remark: (motivation for NCL)
First-order classical logics can be considered universal. But it is often **computationally & conceptionally more adequate** to use, e.g., propositional modal logic instead of a classical first-order theory.
Note:
All nonclassical logics refer in some specific way to classical logic: (Not necessarily exclusively) either
– to propose a fundamental alternative to it, or
– to extend/augment it in a specific way, or
– to generalize some fundamental feature of it, or
– to refine it from a certain perspective.

[ picture developed on blackboard → handout ]

As a consequence,
we need to have a clear understanding of the main concepts of classical logic and of logics in general:

▶ Syntax (‘classical’ connectives and quantifiers )
▶ Semantics (‘Tarski-style’)
▶ Proof systems / algorithms (‘Gentzen-style’, ‘Frege-Hilbert-style’, tableaux, resolution, etc.)
Basic concepts of CL
— A quick reminder

Note:
Logic — not only classical logic! — comes in levels/orders:

▶ propositional logic
▶ first-order logic
▶ second-order logic
▶ ...(higher orders)

Main characteristics of CL at the different levels:

▶ propositional logic: decidable (validity coNP-complete)
▶ first-order logic: undecidable, but still axiomatizable
  (‘r.e.’/‘c.e.’, validity $\Pi^0_1$-complete = ‘semi-decidable’)
▶ second and higher orders: not (recursively) axiomatizable
  (validity not ‘arithmetical’, $\Pi^1_1$-complete)
Classical propositional logics — the logic of ‘bivalence’

Syntax: inductive definition of formulas $FORM^0$:
$A, B, \ldots, F, G, \ldots$

▶ propositional variables (atoms) $PV$:
$p, q, r, \ldots$

▶ logical connectives (operators):
$\supset$ (implication), $\lor$ (disjunction),
$\land$ (conjunction), $\neg$ (negation), $\perp$ (‘falsum’)

▶ additional symbols: parenthesis

Note: We make liberal use of usual priority rules,
associativity and commutativity of $\land$ and $\lor$ etc.

Semantics: truth tables — e.g., for implication

<table>
<thead>
<tr>
<th>$A \supset B$</th>
<th>$B: 1$</th>
<th>$B: 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A: 1$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$A: 0$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where 1, 0 are the truth values ‘true’, ‘false’
Semantics (ctd):
An interpretation $I$ is an

- assignment: $PV \mapsto \{1, 0\}$.

Every interpretation $I$ induces — via the truth tables — an

- evaluation function $v_I : FORM^0 \mapsto \{1, 0\}$.

$F$ is satisfiable \ldots $F$ evaluates to 1 (‘is true’) under some interpretation

$F$ is valid \ldots $F$ evaluates to 1 (‘is true’) under all interpretations

<table>
<thead>
<tr>
<th>Exercise 3:</th>
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<tbody>
<tr>
<td>Clarify the following notions for propositional CL: tautology, model, interpretation, (un)satisfiability, consequence. (Be aware of different terminolgy – specify your source(s)!)</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>Exercise 4*:</th>
</tr>
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<tbody>
<tr>
<td>Show that the satisfiability problem for the fragment of propositional CL without $\neg$ and $\bot$ (‘positive fragment’) is trivial, whereas the validity problem is still coNP-complete.</td>
</tr>
</tbody>
</table>
Classical predicate logic (first-order CL)

Syntax: (set of formulas $FORM^1$)

Atomic formulas (atoms) are analyzed as structured

- predicate symbols $PS_n$ of arity $n$ ($P$, $Q$, $R$, …) ('0-ary' predicate symbol = propositional variable)
- (object) terms ($s$, $t$, $t_i$, …) are inductively built up from
  - (object) variables $V$ ($x$, $y$, $z$, $u$, …)
  - constant symbols $KS$ ($a$, $b$, $c$, …)
  - function symbols $FS_n$ ($f$, $g$, $h$, …) of arity $n$
- resulting in atoms $P(f(a, x), g(f(x, y)), c)$ etc.
- general formulas ($A$, $B$, …, $F$, $G$, …) build up from connectives as well as quantifiers $\forall$, $\exists$

**Exercise 5:**
Present a formal definition of $FORM^1$ and define free($F$) (set of free variables occurring in the formula $F$), accordingly.
First-order CL (ctd.)

Semantics: (be aware of variations in the literature!)
An interpretation is a tuple $I = (D, \Phi, d)$

1. $D$ is a non-empty set — the domain of $I$
2. $\Phi$ is a signature mapping:
   2.1 $\Phi(c) \in D$ for all $c \in KS$.
   2.2 $\Phi(f)$ is a function of type $D^n \mapsto D$ for $f \in FS_n$.
   2.3 $\Phi(P)$ is a relation, represented as function of type $D^n \mapsto \{0, 1\}$ for $P \in PS_n$.
3. $d$ is a variable assignment, i.e., a function of type $V \rightarrow D$

Note:
Sometimes all such structures $I$ are called ‘models’.
(As in: ‘model theory’).
First-order CL — semantics (ctd.)

Given an interpretation $I = (D, \Phi, d)$ the value $v_I(t)$ of a term $t$ in $I$ is defined inductively as follows:

- if $t \in V$ then $v_I(t) = d(t)$
- if $t \in KS$ then $v_I(t) = \Phi(t)$
- otherwise $t$ is of the form $f(t_1, \ldots, t_n)$ and $v_I(t) = \Phi(f)(v_I(t_1), \ldots, v_I(t_n))$

**Exercise 6:**

Present a formal definition of the evaluation function $v_I$, that assigns a truth value in $I$ to every formula $F$.

**Exercise 7:**

Let $G = (P(x) \supset P(f(x, y)))$. Find models and counter-models for the following formulas: $\exists x \exists y \ G$, $\forall x \exists y \ G$, $\exists x \forall y \ G$, $\exists y \forall x \ G$.

**Note:**

Proof systems don’t refer to models, but are purely syntactical.
Syntax vs. semantics

<table>
<thead>
<tr>
<th>Semantics</th>
<th>Syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>F</em> is valid</td>
<td><em>F</em> is provable</td>
</tr>
<tr>
<td>(true in every interpretation)</td>
<td>(in a calculus, i.e.) is a theorem</td>
</tr>
<tr>
<td><em>F</em> is unsatisfiable</td>
<td><em>F</em> is refutable</td>
</tr>
<tr>
<td>(false in every interpretation)</td>
<td>(∼<em>F</em> is a theorem)</td>
</tr>
<tr>
<td><em>F</em> follows from/is a consequence of set Γ ⊇ {G₁, . . . , Gₙ}</td>
<td><em>F</em> is derivable from <em>G₁, . . . , Gₙ</em></td>
</tr>
<tr>
<td><em>F</em> and <em>G</em> are logically equivalent</td>
<td>(<em>F ⊃ G</em>) ∧ (<em>G ⊃ F</em>) is provable (abbrev.: <em>F</em> ≡ <em>G</em> or <em>F</em> ↔ <em>G</em>)</td>
</tr>
<tr>
<td><em>F</em> and <em>G</em> are true/false in the same interpretations</td>
<td></td>
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**Exercise 8:**

Explain how soundness and completeness relates syntax and semantics (in particular in case of ‘follows’ vs. ‘derivable’).