Comments on *Vagueness, Tolerance and Non-Transitive Entailment* by Robert van Rooij

ONDREJ MAJER

Robert van Rooij presents an original framework dealing with the formalization of vague predicates. His solution is based on a non-classical entailment, which he calls tolerant entailment. The author thoroughly discusses the relation of his solution to the major approaches to vagueness (supervaluationism, contextualism, epistemic approaches, ...) and gives an analysis of higher order vagueness.

The paper discusses several topics related to the analysis of vagueness. We shall not comment on all of them but instead concentrate on the notion of tolerant truth and its connections to modal logic.

1 Tolerance

In the beginning of the paper the author introduces the notion of tolerance, which according to him is a property constitutive to the meaning of vague predicates. Formally the tolerance principle is expressed as:

(P) For any $x, y \in D$: $(Px \wedge x \sim_P y) \rightarrow Py$.

The tolerance principle is responsible for the most famous problem related to vague predicates—the Sorites paradox.

The author argues that an adequate formalization of vague predicates should allow (at least in some weak sense) for a truth of the tolerance principle (as it is constitutive), but that on the other hand it should block the Sorites paradox.

The author starts his analysis with a discussion of the notion of gradable predicates (like bald, tall, ...) and introduces a relation which is supposed to order individuals in the given domain with respect to the degree in which they obey the property P (the author calls it "P-er than" and denotes $<_P$). The author assumes it is natural $<_P$ to be a strict weak order (irreflexive, transitive and almost connected) and shows that the corresponding indifference relation is an equivalence and so is potentially open to the Sorites series.

It might be worth discussing in more detail in which sense the properties of the relation $<_P$ are natural (e.g. why not something weaker like partial order) as the author himself does not use it. He instead introduces a relation of semiorder and argues that it is the central notion in his approach to vagueness.

Van Rooij mentions an interesting characterization of semi-orders in the terms of a function f_P assigning to an individual a real number which can be understood as a degree in which the individual possesses the property P:

$x <_P y$ is true	iff	there is a real-valued function f_P and some fixed (small) real number ε (the margin of error) such that $f_P(x) > f_P(y) + \varepsilon$.

The corresponding indifference relation is not equivalence (it is not transitive) and can be interpreted in the terms of a margin of error:

$$x \sim_P y \quad \text{iff} \quad |f_P(x) - f_P(y)| \leq \varepsilon.$$

Though the author does not use this characterization in the rest of the article, it might be interesting to reformulate his system in terms of the "measurement function" f_P and to see if there can be found some resemblances to the degree-theoretic approach (although van Rooij shares some objections against it).

2 Non-classical entailment

Robert van Rooij proposes a framework dealing with vague expressions based on a non-classical notion of entailment which is non-transitive. He starts with a classical first order language with unary vague predicates. The corresponding models are however non-classical—each model is equipped with a relation $<_P$ (and consequently with a similarity relation $\sim_P$) for each vague predicate P. This relation strongly resembles the accessibility relation in the standard Kripkean semantics for modal logics and allows the definition of a non-classical notion of satisfaction. While formulae without a vague predicate (if any) are defined in a standard way, an atomic formula Pa is *tolerantly true* in M iff there is an individual d indistinguishable from a, such that Pd.

$M \models^t Pa$ iff $M \models Pd$ for some individual d such that $\mathbf{a} \sim_P \mathbf{d}$.
($\mathbf{a}$ is the individual corresponding to the name a)

The author defines two notion of entailment based on the notion of tolerant satisfaction:

(tt) $\phi \models^{tt} \psi$ iff $M \models_t \phi$ implies $M \models_t \psi$ for all models M

(ct) $\phi \models^{ct} \psi$ iff $M \models \phi$ implies $M \models_t \psi$ for all models M.

Let us note that strictly speaking the satisfaction relation should be indexed by the class of models in question as "all models" in the definition cannot mean all the first order models, but the first order models equipped by a relation of similarity for each vague predicate contained in the language.

Then the author provides an alternative definition of (possibility like) tolerant truth and the corresponding dual notion of (necessity like) strict truth. Tolerant truth in a model is defined like before, except for the clause for negation:

$M \models^t Pa$ iff $M \models Pd$ for some individual d such that $\mathbf{a} \sim_P \mathbf{d}$

$M \models^t \neg\phi$ iff $M \not\models^s \phi$.

Where the strict truth is defined as:

$M \models^s Pa$ iff $M \models Pd$ for all individuals d such that $\mathbf{a} \sim_P \mathbf{d}$

$M \models^s \neg\phi$ iff $M \not\models^t \phi$.

Finally the similarity relation $\sim_P$ is added to the language (with the classical interpretation under all notions of truth), allowing the Tolerance principle to be explicitly expressed in the language. It turns out that it is tolerance valid, i.e.

$$M \models^t \forall x, y((Px \wedge x \sim_P y) \rightarrow Py) \text{ for all models } M.$$

It also nicely deals with the problem of cutoff points (points a, b such that $a \sim_P b, Pa$, but $\neg Pb$) as

$$M \not\models^s (Pa \wedge \neg Pa), \text{ for all } M, \text{ but } M \models^t (Pa \wedge \neg Pa) \text{ for some } M.$$

The author then compares his solution with the main approaches to vagueness and argues that some sort of non-classical entailment is implicitly or explicitly used in the majority of them. The solution is certainly impressive and seems to solve the majority of problems connected to the formal approach to vagueness. There are still, however, some additional questions to be asked.

Which one of the tolerant consequences shall we use? The author prefers (ct) consequence, but admits also (sc). Are there any reasons to choose one rather then the other? Or any non-transitive consequence would do? Shall we use classical consequence for crisp predicates and the tolerant for the vague ones?

3 Modal reformulation

As mentioned above, the notions of strict and tolerant truth are modal by nature (at least for a logician) and seem to correspond straightforwardly to necessity and possibility. The author discusses a 'modal' reformulation of his approach, but the modalities are just shortcuts for the notions defined before. It might seem trivial, but we think it is worth trying to take the modal approach literally and check to see how straightforward the correspondence is. The attractiveness of this approach is that we keep a standard notion of consequence in a modal frame and the modal attribution of vague properties to objects is made explicit.

Let us assume we have a standard modal frame consisting of a universe W, of points $a, b, \ldots \in W$ and an accessibility relation $\sim_P$ defined on $W \times W$. Our language consists only of one propositional symbol P which (maybe a bit unusually) represents some vague property. The fact that an individual a has the property P is then represented as $a \models P$ (we shall abbreviate it as a_p, if convenient). The tolerant (strict) truth of a formula Pa in a first order model with a similarity relation corresponds to the truth of a modal formula in a frame:

(t) $a \models \diamond P$ iff there is an $b, a \sim_P b$ such that $b \models P$

(s) $a \models \Box P$ iff for all $b, a \sim_P b$ it holds that $b \models P$.

In the modal formalization the tolerance principle is not expressible in the object language, but only on the meta-level:

(P) for any $a, b \in W$ if $a \models P$ and $a \sim_P b$ then $b \models P$.

This condition says that truth is preserved along the accessibility relation (persistence) and is just valid for a certain class of frames. So in this form the tolerance principle does not hold in general. On the other hand, their modal versions, which as the author mentions correspond to Williamson's epistemic approach and Shapiro's contextualist approach, follow from the standard definition of necessity/possibility and are valid in all modal frames:

(PW) for any $a, b \in W$ if $a \models \Box P$ and $a \sim_P b$ then $b \models P$

(PS) for any $a, b \in W$ if $a \models P$ and $a \sim_P b$ then $b \models \diamond P$.

These versions of the principle hold independently of the properties of the similarity relation. If we rewrite van Rooij's tolerant version of the tolerance principle, we obtain:

(PR) for any $a, b \in W$ if $a \models \Box P$ and $a \sim_P b$ then $b \models \diamond P$.

Which is equivalent to: for any $a, b \in W, a \sim_P b$ it holds that $a \not\models \Box P$ or $b \models \diamond P$. This is valid for all frames, where the accessibility relation corresponds to semi-order:

– either $a \models P$ and then $b \models \diamond P$ (from the symmetry of $\sim_P$)
– or $a \not\models P$ hence $a \models \diamond\neg P$ (from the reflexivity of $\sim_P$).

In fact (PR) holds for all frames with reflexive and symmetric accessibility relation.

Tolerant and strict entailment in the modal setup

Tolerant entailment in van Rooij's system is defined on a class of classical first order models equipped with a similarity relation satisfying some properties; namely, those following from the fact that it is based on a semiorder. Its modal translation will be defined on a class of modal frames, again with a certain kind of accessibility relation.

Tolerant entailment (tt) says, that ϕ tolerantly entails ψ (in a class $\mathscr{F}$ of modal frames) iff $F, a \models \diamond\phi$, implies $F, a \models \diamond\psi$ for all $a \in F$ and all $F \in \mathscr{F}$ which we can rewrite as

(tt) $\phi \models^{tt}_{\mathscr{F}} \psi$ iff $F \models \diamond\phi \to \diamond\psi$ for all $F \in \mathscr{F}$.

Similarly, ϕ (ct)-entails ψ (in a class of modal frames $\mathscr{F}$) iff $F, a \models \phi$, implies $F, a \models \diamond\phi$ for all $a \in F$ and all $F \in \mathscr{F}$ which we can rewrite as

(ct) $\phi \models^{ct}_{\mathscr{F}} \psi$ iff $F \models \phi \to \diamond\psi$ for all $F \in \mathscr{F}$.

The last entailment author discusses is (sc), which is equivalent to $F, a \models \Box\phi$, implies $F, a \models \psi$ for all $a \in F$ and all $F \in \mathscr{F}$ which we can rewrite as

(sc) $\phi \models^{sc}_{\mathscr{F}} \psi$ iff $F \models \Box\phi \to \psi$ for all $F \in \mathscr{F}$.

There is no room here to develop this idea further, let us just suggest some (possible) advantages of this approach:

- the non-standard notion of tolerant truth/entailment is replaced by the standard notion of truth/validity in a modal frame about which there are good intuitions and which is well studied

- it might be interesting to observe if different sorts of non-standard entailment can be forced by different conditions on accessibility relations in modal frames, in particular if there is any other relation than the similarity based on a semi-order satisfying the conditions of tolerant entailment
- higher-order vagueness would be represented as a standard embedding of modalities and does not need a special treatment

There are obviously some disadvantages that should be mentioned. Unlike in the original article the tolerance principle is not formulated in the object language but only in the metalanguage. This, however, should not be a big problem as the principle itself is about reasoning with vague predicates. For some it might also seem unintuitive or at least not straightforward to model vague properties as propositions rather than predicates.

The presented proposals do not suggest that the framework introduced in the original article should be completely reformulated. Rather, they represent an attempt to see this framework from a slightly different point of view which might show some interesting connections with standard, well developed logical frameworks.

Ondrej Majer
Institute of Philosophy
Academy of Sciences of the Czech Republic
Jilská 1
110 00 Prague 1, Czech Republic
Email: majer@flu.cas.cz