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# Reasoning About Uncertainty of Fuzzy Events: An Overview

TOMMASO FLAMINIO, LLUÍS GODÓ, AND ENRICO MARCHIONI<sup>1</sup>

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## 1 Introduction and motivations

Similarly to many areas of Artificial Intelligence, Logic as well has approached the definition of inferential systems that take into account elements from real-life situations. In particular, logical treatments have been trying to deal with the phenomena of vagueness and uncertainty. While a degree-based computational model of vagueness has been investigated through fuzzy set theory [88] and fuzzy logics, the study of uncertainty has been dealt with from the measure-theoretic point of view, which has also served as a basis to define logics of uncertainty (see e.g. [57]).

Fuzzy logics rely on the idea that truth comes in degrees. The inherent vagueness in many real-life declarative statements makes it impossible to predicate their full truth or full falsity. For this reason, propositions are taken as statements that can be regarded as partially true.

Measures of uncertainty aim at formalizing the strength of our beliefs in the occurrence of some events by assigning to those events a degree of belief concerning their occurrence. From the mathematical point of view, a measure of uncertainty is a function that assigns to each event (understood here as a formula in a specific logical language  $\mathcal{L}^{\mathcal{C}}$ ) a value from a given scale, usually the real unit interval  $[0, 1]$ , under some suitable constraints. A well-known example is given by probability measures which try to capture our degree of confidence in the occurrence of events by additive  $[0, 1]$ -valued assignments.

Both fuzzy set theory and measures of uncertainty are linked by the need of intermediate values in their semantics, but they are essentially different. In particular, in the field of logics, a significant difference between fuzzy and probabilistic logic regards the fact that, while intermediate degrees of truth in fuzzy logic are compositional (i.e. the truth degree of a compound formula  $\varphi \circ \psi$  only depends on the truth degrees of the simpler formulas  $\varphi$  and  $\psi$ ), degrees of belief are not. In fact, for instance, the probability of a conjunction  $\varphi \wedge \psi$  is not always a function of the probability of  $\varphi$  and the probability of  $\psi$ . Therefore, while fuzzy logics behave as (truth-functional) many-valued logics, probabilistic logics can be rather regarded as a kind of modal logics (cf. [50, 51] for instance).

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The conclusion arising from the above mentioned differences is that the degree of truth of a formula cannot be understood, in general, as the degree of belief of the *same* formula. Still, we can interpret the degree of belief of a formula  $\varphi$  as the degree of truth of the modal formula  $P\varphi$  that states that  $\varphi$  is *plausible* or *likely*.

This approach was first suggested by Hájek and Harmancová in [55], and later followed by Hájek, Esteva and Godo in [54, 52, 44, 45] where reasoning under uncertainty (modelled by probabilities, necessity and possibility measures, or even Dempster-Shafer belief functions) with classical propositions was captured in the framework of t-norm based logics. Indeed, given an assertion as “The proposition  $\varphi$  is plausible (probable, believable)”, its degree of truth can be interpreted as the degree of uncertainty of the proposition  $\varphi$ . In fact, the higher our degree of confidence in  $\varphi$  is, the higher the degree of truth of the above sentence will be. In a certain sense, the predicate “is plausible (believable, probable)” can be regarded as a *fuzzy* modal operator over the proposition  $\varphi$ . Then, given a class of uncertainty measures, one can define modal many-valued formulas  $\mathcal{M}\varphi$ , whose interpretations are given by real numbers corresponding to the degree of uncertainty assigned to  $\varphi$  under measures  $\mu$  of the given class. Furthermore, one can translate the specific postulates governing the behavior of particular classes of uncertainty measures into axioms on the modal formulas over a certain t-norm based logic, depending on the operations we need to represent.<sup>2</sup>

This logical approach to reason about uncertainty was also adopted to treat conditional probability in [71, 46, 47]; (generalized) conditional possibility and necessity in [67, 68]; and simple and conditional non-standard probability in [39]. A generalized treatment for both simple and conditional measures of uncertainty over Boolean events that covers most of the above ones was given by Marchioni in [69, 70].

Our aim, in this overview paper, is to give a comprehensive logical treatment of several generalizations of main classes of measures of uncertainty over fuzzy events. In particular, we will show how it is possible to represent and logically formalize reasoning about classes of measures such as probabilities, plausibility, possibilities and necessities over several classes of many-valued events. Fuzzy logics provide a powerful framework to handle and combine fuzziness and uncertainty. Indeed, in such logics the operations associated to the evaluation of the connectives are functions defined over the real unit interval  $[0, 1]$ , that correspond, directly or up to some combinations, to operations used to compute degrees of uncertainty. Then, such algebraic operations can be embedded in the connectives of the many-valued logical framework, resulting in clear and elegant formalizations.

This article is organized as follows. In Section 2, we provide the necessary logical background for the different fuzzy logics we will use throughout the paper. In Section 3, we introduce the basic concepts regarding some classes of measures over non-classical events. In Section 4, we deal with several modal expansions of particular fuzzy logics to treat classes of measures over fuzzy events. In Section 5, we study how to expand the language of those modal fuzzy logics by adding truth constants from the rational unit interval  $[0, 1]$ . In Section 6, we rely on those modal expansions to characterize, in purely logical terms, the problem of extending a partial uncertainty assignment over

<sup>2</sup>Needless to say, there are logics that are better suited than others to express the axioms of specific uncertainty measures, since some logics are not rich enough to capture the particular behavior of certain measures.

fuzzy events to a measure over the whole algebra they generate. We conclude with Section 7, where we discuss further and complementary readings about the topic of uncertainty measures over non-classical many-valued events.

## 2 Logical background

In this section, we introduce the background notions concerning the logic MTL [29], its extensions and expansions.

### 2.1 Core and $\Delta$ -core fuzzy logics

The language of MTL consists of a countable set  $V = \{p_1, p_2, \dots\}$  of propositional variables and a set of connectives  $\mathcal{LC} = (\&, \rightarrow, \wedge, \perp)$  of type  $(2, 2, 2, 0)$ . The set  $Fm_V$  of formulas defined from the variables in  $V$  and the above connectives is built with the usual inductive clauses.

MTL has the following axiomatization:

- |       |  |       |  |
|-------|--|-------|--|
| (A1)  | $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$                    | (A2)  | $\varphi \& \psi \rightarrow \varphi$  |
| (A3)  | $\varphi \& \psi \rightarrow \psi \& \varphi$  | (A4)  | $\varphi \wedge \psi \rightarrow \varphi$  |
| (A5)  | $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$  | (A6)  | $\varphi \& (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$                        |
| (A7a) | $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$                               | (A7b) | $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$ |
| (A8)  | $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$ | (A9)  | $\perp \rightarrow \varphi.$   |

The only inference rule of MTL is modus ponens: from  $\varphi$  and  $\varphi \rightarrow \psi$ , infer  $\psi$ . A *proof* in MTL is a sequence  $\varphi_1, \dots, \varphi_n$  of formulas such that each  $\varphi_i$  either is an axiom of MTL, or follows from some preceding  $\varphi_j, \varphi_k$  ( $j, k < i$ ) by modus ponens. As usual, a set of formulas is called a *theory*. We say that a formula  $\varphi$  can be derived from a theory  $\Gamma$ , denoted as  $\Gamma \vdash_{\text{MTL}} \varphi$ , if there is a proof of  $\varphi$  from a set  $\Gamma' \subseteq \Gamma$ . A theory  $\Gamma$  is said to be consistent if  $\Gamma \not\vdash_{\text{MTL}} \perp$ .

Other definable connectives are the following:

- |                                |    |  |
|--------------------------------|----|--|
| $\varphi \vee \psi$            | is | $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi),$ |
| $\varphi \leftrightarrow \psi$ | is | $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi),$  |
| $\neg \varphi$                 | is | $\varphi \rightarrow \perp,$   |
| $\top$                         | is | $\neg \perp.$  |

We also use the following abbreviation: for all  $n \in \mathbb{N}$ , and for every  $\varphi \in Fm_V$ ,  $\varphi^n$  stands for  $\varphi \& \dots \& \varphi$  ( $n$ -times).

**DEFINITION 1** (1) *Let  $\varphi(p_1, \dots, p_k)$  be a formula in  $Fm_V$ . Then the axiom schema defined by  $\varphi$  is the set of all those formulas in  $Fm_V$  that can be defined from  $\varphi$  by substituting every propositional variable  $p_i$  occurring in  $\varphi$ , by a formula  $\psi_i \in Fm_V$ .*

(2) *A logic in the language  $\mathcal{LC}$  is said to be a schematic extension of MTL if its axioms are those of MTL plus additional axiom schemas, with modus ponens as the unique inference rule.*

(3) *Consider a language  $\mathcal{LC}' \supset \mathcal{LC}$ . A logic axiomatized in the language  $\mathcal{LC}'$  containing all the axioms and rules of MTL is said to be an expansion of MTL.*

An important expansion of MTL is the one obtained by expanding the language  $\mathcal{LC}$  with the unary connective  $\Delta$  (known in the literature as *Baaz's delta*, [6]), and adding the following axiom schemas:

$$\begin{array}{ll} (\Delta 1) & \Delta\varphi \vee \neg\Delta\varphi \\ (\Delta 2) & \Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi) \\ (\Delta 3) & \Delta\varphi \rightarrow \varphi \\ (\Delta 4) & \Delta\varphi \rightarrow \Delta\Delta\varphi \\ (\Delta 5) & \Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi) \end{array}$$

along with the deduction rule of  $\Delta$ -necessitation: from  $\varphi$ , deduce  $\Delta\varphi$ . The above logical system is called  $\text{MTL}_\Delta$ .

**THEOREM 2** *Let  $\mathcal{L} \in \{\text{MTL}, \text{MTL}_\Delta\}$ . Consider the following properties for every set of formulas  $\Gamma \cup \{\varphi, \psi, \chi\}$  in the language of  $\mathcal{L}$ :*

$$\begin{array}{ll} (ldt) & \Gamma, \varphi \vdash_{\mathcal{L}} \psi \text{ iff there is an } n \in \mathbb{N} \text{ such that } \Gamma \vdash_{\mathcal{L}} \varphi^n \rightarrow \psi. \\ (\Delta dt) & \Gamma, \varphi \vdash_{\mathcal{L}} \psi \text{ iff } \Gamma \vdash_{\mathcal{L}} \Delta\varphi \rightarrow \psi. \\ (cong) & \varphi \leftrightarrow \psi \vdash_{\mathcal{L}} \chi(\varphi) \leftrightarrow \chi(\psi). \end{array}$$

Then, MTL satisfies (ldt) and (cong) while  $\text{MTL}_\Delta$  satisfies ( $\Delta dt$ ) and (cong).

Following [14], we say that a logic  $\mathcal{L}$  is a *core fuzzy logic* if  $\mathcal{L}$  expands MTL and satisfies (ldt) and (cong). A logic  $\mathcal{L}$  is a  $\Delta$ -*core fuzzy logic* if  $\mathcal{L}$  expands  $\text{MTL}_\Delta$  and satisfies ( $\Delta dt$ ) and (cong).

An MTL-algebra is a structure  $A = (A, \odot, \Rightarrow, \wedge, \vee, 0_A, 1_A)$  of type  $(2, 2, 2, 2, 0, 0)$  such that:

- (1) The reduct  $(A, \wedge, \vee, 0_A, 1_A)$  is a bounded lattice,
- (2) The reduct  $(A, \odot, 1_A)$  is a commutative monoid,
- (3) The operations  $\odot$  and  $\Rightarrow$  form an adjoint pair:

$$\text{for all } x, y, z \in A, x \odot y \leq z \text{ iff } x \leq y \Rightarrow z.$$

- (4) The prelinearity condition is satisfied:

$$\text{for all } x, y \in A, (x \Rightarrow y) \vee (y \Rightarrow x) = 1_A.$$

Since MTL and all ( $\Delta$ -)core fuzzy logics are algebraizable in the sense of Blok and Pigozzi [7], we simply say that for any ( $\Delta$ -)core fuzzy logic  $\mathcal{L}$ , the class (variety) of  $\mathcal{L}$ -algebras coincides with the equivalent algebraic semantics for  $\mathcal{L}$ . We refer to [14] for a more complete treatment.

Basic examples of MTL-algebras are obtained by equipping the real unit interval  $[0, 1]$  with a left continuous t-norm  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  (cf. [52, 59]), its residuum  $\Rightarrow_*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  and the usual lattice order of the reals. The main three examples of continuous t-norms are the Łukasiewicz t-norm ( $x * y = \max(x + y - 1, 0)$ ), the product t-norm ( $x * y = x \cdot y$ ) and the Gödel t-norm ( $x * y = \min(x, y)$ ). These structures

$([0, 1], *, \Rightarrow_*, \min, \max, 0, 1)$  are called *real MTL-chains*,<sup>3</sup> and they will play a crucial role in the rest of this paper. Of course, whenever we deal with particular expansions of MTL, we must take care of the standard interpretation of the symbols that expand the MTL-language. Recall that the standard interpretation of Baaz's delta is the following: for all  $x \in [0, 1]$ ,  $\Delta(x) = 1$  if  $x = 1$ , and  $\Delta(x) = 0$  otherwise.

An evaluation of  $Fm_V$  into a real MTL-chain is a map  $e$  from the propositional variables in  $V$  into  $[0, 1]$  that extends to formulas by truth-functionality. An evaluation  $e$  is a *model* for a formula  $\varphi$  if  $e(\varphi) = 1$ . An evaluation  $e$  is a model for a theory  $\Gamma$ , if  $e(\psi) = 1$ , for every  $\psi \in \Gamma$ .

Let now  $\mathcal{L}$  denote any  $(\Delta)$ -core fuzzy logic. Then we say that  $\mathcal{L}$  enjoys:

- *Real completeness* (RC) if for every formula  $\varphi$ ,  $\vdash_{\mathcal{L}} \varphi$  iff for every evaluation  $e$  into a real  $\mathcal{L}$ -chain,  $e(\varphi) = 1$ .
- *Finite strong real completeness* (FSRC) if for every finite theory  $\Gamma \cup \{\varphi\}$ ,  $\Gamma \vdash_{\mathcal{L}} \varphi$  iff for every evaluation  $e$  into a real  $\mathcal{L}$ -chain that is a model for  $\Gamma$ ,  $e(\varphi) = 1$ .
- *Strong real completeness* (SRC) if for every theory  $\Gamma \cup \{\varphi\}$ ,  $\Gamma \vdash_{\mathcal{L}} \varphi$  iff for every evaluation  $e$  into a real  $\mathcal{L}$ -chain that is a model for  $\Gamma$ ,  $e(\varphi) = 1$ .
- *Strong hyperreal completeness* (SR\*C) if for every theory  $\Gamma \cup \{\varphi\}$ ,  $\Gamma \vdash_{\mathcal{L}} \varphi$  iff for every evaluation  $e$  into a ultraproduct of real  $\mathcal{L}$ -chains that is a model for  $\Gamma$ ,  $e(\varphi) = 1$ .

Jenei and Montagna proved in [58] that MTL enjoys SRC. We refer to [14] for a complete and in-depth study of such different notions of completeness for all the most prominent  $(\Delta)$ -core fuzzy logics.

**DEFINITION 3** A  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  is said to be *locally finite* iff for every finite set  $V_0$  of propositional variables, the Lindenbaum-Tarski algebra<sup>4</sup>  $Fm_{V_0}$  of  $\mathcal{L}$  generated by the variables in  $V_0$  is a finite algebra.

## 2.2 Expansions with an involutive negation

As we pointed out above, in any  $(\Delta)$ -core fuzzy logic, we can define a negation connective  $\neg$ , as  $\neg\varphi := \varphi \rightarrow \perp$ . This negation, in its standard interpretation, behaves quite differently depending on the chosen left-continuous t-norm and, in general, is not an involution, i.e. it does not satisfy the equation  $\neg\neg x = x$ .

A relevant expansion of a  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  is obtained by adding an involutive negation  $\sim$  that does not depend on the chosen left-continuous t-norm [15, 30, 38]. In particular, we recall that  $MTL_{\sim}$  is the logic obtained by expanding  $MTL_{\Delta}$  with the unary symbol  $\sim$ , together with the following axioms:

- ( $\sim$ 1)  $\sim\sim\varphi \leftrightarrow \varphi$
- ( $\sim$ 2)  $\Delta(\varphi \rightarrow \psi) \rightarrow (\sim\psi \rightarrow \sim\varphi)$ .

<sup>3</sup>In the literature of mathematical fuzzy logic, algebras over the real unit  $[0, 1]$  are also called *standard*.

<sup>4</sup>We remind the reader that, whenever we fix a language  $\mathcal{LC}$ , a set of variables  $V$ , and a logic  $\mathcal{L}$  together with its consequence relation  $\vdash_{\mathcal{L}}$ , the Lindenbaum-Tarski algebra  $Fm_V$  is the quotient algebra of formulas modulo the equi-derivability relation. We invite the reader to consult [12] for further details.

MTL $\sim$ -algebras, the algebraic counterpart of MTL $\sim$ , are structures

$$(A, \odot, \Rightarrow, \wedge, \vee, \sim, \Delta, 0_A, 1_A)$$

of type  $(2, 2, 2, 2, 1, 1, 0, 0)$  and are obviously defined. It is worth noticing that, as proved in [38], extensions of MTL $\sim$  preserve (finite) strong standard completeness.

MTL $\sim$  extensions are particularly interesting because, in each of their standard algebras, any operation  $\oplus$  defined as:  $x \oplus y := \sim(\sim x \odot \sim y)$  is interpreted as a t-conorm, thus making the system  $(\odot, \oplus, \sim)$  a De Morgan triple [40]. We will see later that these structures allow to define a basic representation of possibility and necessity measures of fuzzy events (see Section 3.2.1).

### 2.3 Expansions with rational truth constants

Other notable expansions of a  $\Delta$ -core fuzzy logic  $\mathcal{L}$  are obtained by expanding its language with a set  $C$  of truth constants from  $[0, 1]$ . More precisely, let  $*$  be any left continuous t-norm and  $[0, 1]_*$  be the corresponding real algebra  $([0, 1], *, \Rightarrow_*, \min, \max, 0, 1)$ . Denote by  $\mathcal{L}_*$  the algebraizable (in the sense of [7]) core fuzzy logic whose equivalent algebraic semantics is constituted by the variety  $HSP([0, 1]_*)$ , i.e. the variety generated by the standard algebra  $[0, 1]_*$ . The logic  $\mathcal{L}_{*,\Delta}$  denotes as usual the expansion of  $\mathcal{L}_*$  by the Baaz connective  $\Delta$ .

Let  $C$  be a countable subset of  $[0, 1]_*$ . Then the logic  $\mathcal{L}_{*,\Delta}(C)$  is the expansion of  $\mathcal{L}_{*,\Delta}$  obtained by adding to its language the elements of  $C$  as constants and the following book-keeping axioms, where, for every  $c \in C$ , we denote its associated constant by  $\bar{c}$  (notice that we still denote the top and bottom elements as 0 and 1):

$$\begin{aligned} (R1) \quad & \bar{c}_1 \& \bar{c}_2 \leftrightarrow \overline{c_1 * c_2}, \\ (R2) \quad & \bar{c}_1 \rightarrow \bar{c}_2 \leftrightarrow \overline{c_1 \Rightarrow_* c_2}, \\ (R3) \quad & \Delta \bar{c} \leftrightarrow \overline{\Delta c}. \end{aligned}$$

For the logic  $\mathcal{L}_{*,\Delta}(C)$ , a different version of completeness has been introduced to interpret canonically the constant symbols [28]. In particular, we say that  $\mathcal{L}_{*,\Delta}(C)$  has the *canonical* (finite) strong real-chain completeness iff  $\mathcal{L}_{*,\Delta}(C)$  is (finitely) strong complete w.r.t. the real algebra  $([0, 1], *, \Rightarrow_*, \min, \max, \{c\}_{c \in C})$ , so that evaluations interpret every symbol  $\bar{c}$  by the real number  $c$  (for all  $c \in C$ ). Then, we have:

**THEOREM 4** ([28, 31]) *Let  $*$  be **CONT-fin**<sup>5</sup>  $\cup$  **WNM-fin**<sup>6</sup>, and let  $C \subset [0, 1]_*$  be a suitable countable subalgebra. Then:*

- (1)  $\mathcal{L}_{*,\Delta}(C)$  has the canonical finite strong real completeness.
- (2)  $\mathcal{L}_{*,\Delta}(C)$  has the canonical strong real completeness iff  $*$   $\in$  **WNM-fin**.

<sup>5</sup>By the Mostert-Shields theorem [52, Theorem 2.1.16] every continuous t-norms  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is an *ordinal sum* of the three basic t-norms: Gödel, product, and Łukasiewicz. **CONT-fin** denotes the class of all those continuous t-norms that are ordinal sums with finitely many components.

<sup>6</sup>Every nilpotent minimum t-norm  $*$  (cf. [29]) is uniquely characterized by its associated weak negation  $n_*$ :  $[0, 1] \rightarrow [0, 1]$ . The t-norm  $*$  is said to have a *finite partition* if its associated weak negation  $n_*$  is constant over finitely many intervals. **WNM-fin** denotes the class of all those weak nilpotent minimum t-norms having a finite partition. Notice that Gödel t-norm also belongs to this class.

For a given left-continuous t-norm  $*$  and an involutive negation  $n: [0,1] \rightarrow [0,1]$  closed over the rational unit interval  $[0,1]_{\mathbb{Q}}$ , let  $\mathcal{L}_{*,n}([0,1]_{\mathbb{Q}})$  be the axiomatic extension of  $\text{MTL}_{\sim}$  that is complete with respect to the variety of  $\text{MTL}_{\sim}$ -algebras generated by the real algebra  $([0,1], *, \Rightarrow, n, \Delta, 0, 1)$ . Then,  $\mathcal{L}_{*,n}([0,1]_{\mathbb{Q}})$  is the expansion of  $\mathcal{L}_{*,n}$  with truth-constants from the rational unit interval, together with the book-keeping axioms (R1)–(R3) plus the following one for the involutive negation:

$$(R4) \quad \sim \bar{c} \leftrightarrow \overline{n(c)}.$$

Adopting the same techniques used in [38, Theorem 5.6, Theorem 5.13], it is not hard to show that the same conclusions of Theorem 4 can also be obtained for any  $\mathcal{L}_{*,n}([0,1]_{\mathbb{Q}})$ , whenever  $* \in \mathbf{CONT}\text{-fin} \cup \mathbf{WNM}\text{-fin}$ .

## 2.4 Łukasiewicz logics

Łukasiewicz logic  $\mathbb{L}$  was introduced in [66], and has been widely studied by many authors both from the syntactical and algebraic point of view (cf. [8, 13, 52]). As a core fuzzy logic,  $\mathbb{L}$  is obtained from MTL, by adding the following axioms:

$$\begin{aligned} (\text{div}) \quad & \varphi \wedge \psi \rightarrow (\varphi \& (\varphi \rightarrow \psi)), \\ (\text{inv}) \quad & \neg\neg\varphi \rightarrow \varphi. \end{aligned}$$

Due to axiom (inv), the defined negation of Łukasiewicz logic is involutive. This allows us to define a connective of strong disjunction as follows:  $\varphi \oplus \psi$  is  $\neg(\neg\varphi \& \neg\psi)$ .

For each  $n \in \mathbb{N}$ , the  $n$ -valued Łukasiewicz logic  $\mathbb{L}_n$  is the schematic extension of  $\mathbb{L}$  with the axiom schemas:

$$(\mathbb{L}_n1) \quad (n-1)\varphi \leftrightarrow n\varphi, \quad (\mathbb{L}_n2) \quad (k\varphi^{k-1})^n \leftrightarrow n\varphi^k,$$

for each integer  $k = 2, \dots, n-2$  that does not divide  $n-1$ , and where  $n\varphi$  is an abbreviation for  $\varphi \oplus \dots \oplus \varphi$  ( $n$  times).

The algebraic counterpart for Łukasiewicz logic is the class of MV-algebras [11, 13]. These structures were introduced by Chang [11] using a different presentation that is equivalent to the one given as extensions of MTL-algebras. In its original language, an MV-algebra is a structure  $\mathcal{A} = \langle A, \oplus, \neg, 0_A \rangle$  satisfying the following equations:

$$\begin{aligned} (\text{MV1}) \quad & x \oplus (y \oplus z) = (x \oplus y) \oplus z, & (\text{MV2}) \quad & x \oplus y = y \oplus x, \\ (\text{MV3}) \quad & x \oplus 0_A = x, & (\text{MV4}) \quad & \neg\neg x = x, \\ (\text{MV5}) \quad & x \oplus \neg 0_A = \neg 0_A, & (\text{MV6}) \quad & \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x. \end{aligned}$$

As we stated, MV-algebras are MTL-algebras satisfying (see [13, 52]):

$$\begin{aligned} x \wedge y &= x \odot (x \Rightarrow y); & (\text{Divisibility}) \\ (x \Rightarrow 0_A) \Rightarrow 0_A &= x. & (\text{Involution}) \end{aligned}$$

Indeed, in the signature  $\langle \oplus, \neg, 0_A \rangle$ , the monoidal operation  $\odot$  can be defined as  $x \odot y := \neg(\neg x \oplus \neg y)$ , while the residuum of  $\odot$  is definable as  $x \Rightarrow y := \neg x \oplus y$ . The top element is defined as  $1_A := \neg 0_A$ , and the order relation is obtained by defining  $x \leq y$  iff  $x \Rightarrow y = 1_A$ , while the lattice operations are given by  $x \wedge y := x \odot (\neg x \oplus y)$  and  $x \vee y := (x \odot \neg y) \oplus y$ . Moreover, we define the following useful abbreviation: for every natural  $n$  and  $x \in A$ ,  $n x$  will denote  $x \oplus \dots \oplus x$ , and  $x^n$  will denote  $x \odot \dots \odot x$ .

For each  $n \in \mathbb{N}$ , an  $MV_n$ -algebra is an  $MV$ -algebra that satisfies the equations:

$$(MV7) \quad (n-1)x = nx \qquad (MV8) \quad (kx^{k-1})^n = nx^k$$

for each integer  $k = 2, \dots, n-2$  not dividing  $n-1$ .

The class of  $MV$ -algebras ( $MV_n$ ) forms a variety  $\mathbb{MV}$  ( $\mathbb{MV}_n$ ) that clearly is the equivalent algebraic semantics for  $\mathbb{L}$  ( $\mathbb{L}_n$ ), in the sense of Blok and Pigozzi [7].  $\mathbb{MV}$  is generated as a quasivariety by the standard  $MV$ -algebra  $[0, 1]_{MV}$ , i.e. the  $MV$ -algebra over the real unit interval  $[0, 1]$  with  $x \oplus y = \min(x+y, 1)$  and  $\neg x = 1-x$ .<sup>7</sup> Each  $\mathbb{MV}_n$  is generated by the linearly ordered  $MV$ -algebra over the set  $S_n = \{0, 1/n, \dots, (n-1)/n, 1\}$  and whose operations are those of the  $MV$ -algebra over  $[0, 1]$ , restricted to  $S_n$ .

Interesting examples of  $MV$ -algebras are the so-called Łukasiewicz *clans* of functions. Given a non-empty set  $X$ , consider the set of functions  $[0, 1]^X$  endowed with the pointwise extensions of the operations of the standard  $MV$ -algebra  $[0, 1]_{MV}$ . Then a (Łukasiewicz) *clan* over  $X$  is any subalgebra  $\mathcal{C} \subseteq [0, 1]^X$ , i.e. a set such that

- (1) if  $f, g \in \mathcal{C}$  then  $f \oplus g \in \mathcal{C}$ ,
- (2) if  $f \in \mathcal{C}$  then  $\neg f \in \mathcal{C}$ ,
- (3)  $\bar{0} \in \mathcal{C}$ ,

where  $\bar{0}$  denotes the function constantly equal to 0. A clan  $\mathcal{T}$  over  $X$  is called a (Łukasiewicz) *tribe* when it is closed with respect to a countable (pointwise) application of the  $\oplus$  operation, i.e. if the following condition

$$\text{if } \{f_n \mid n \in \mathbb{N}\} \subseteq \mathcal{T} \text{ then } \bigoplus_{n \in \mathbb{N}} f_n \in \mathcal{T}$$

holds. Similarly, one can define an  $\mathbb{L}_n$ -clan of functions over some set  $X$  to be any subalgebra  $\mathcal{C} \subseteq (S_n)^X$ .

The fact that  $\mathbb{MV}$  is the equivalent algebraic semantics for Łukasiewicz logic  $\mathbb{L}$  and is generated as a quasivariety by the standard  $MV$ -algebra  $[0, 1]_{MV}$  implies that Łukasiewicz logic enjoys FSRC. However  $\mathbb{L}$  does not have SRC (cf. [14, 52]). On the other hand, for every  $n \in \mathbb{N}$ , the logic  $\mathbb{L}_n$  is strongly complete with respect to the  $MV$ -algebra  $S_n$  (cf. [34]).

Rational Łukasiewicz logic  $R\mathbb{L}$  is a conservative expansion of Łukasiewicz logic introduced by Gerla in [42, 43], obtained by adding the set of unary connectives  $\delta_n$ , one for each  $n \in \mathbb{N}$ , together with the following axioms:

$$(D1) \quad n\delta_n\varphi \leftrightarrow \varphi, \qquad (D2) \quad \neg\delta_n\varphi \oplus \neg(n-1)\delta_n\varphi.$$

The algebraic semantics for  $R\mathbb{L}$  is given by the variety of  $DMV$ -algebras (divisible  $MV$ -algebras), i.e. structures  $\mathcal{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n \in \mathbb{N}}, 0_A \rangle$  such that  $\langle A, \oplus, \neg, 0_A \rangle$  is an  $MV$ -algebra and the following equations hold for all  $x \in A$  and  $n \in \mathbb{N}$ :

$$(\delta_n 1) \quad n(\delta_n x) = x, \qquad (\delta_n 2) \quad \delta_n x \odot (n-1)(\delta_n x) = 0_A.$$

<sup>7</sup>Notice that there exist uncountably many  $MV$ -algebras whose universe is the real unit interval  $[0, 1]$ , but they are all isomorphic to each other, and, in particular, to the standard  $MV$ -algebra.

An evaluation  $e$  of RŁ formulas into the real unit interval is just a Łukasiewicz logic evaluation extended for the connectives  $\delta_n$  as follows:  $e(\delta_n \varphi) = e(\varphi)/n$ .

Notice that in RŁ all rationals in  $[0, 1]$  are definable (as truth constants) as:

- $\overline{1/n}$  is definable as  $\delta_n \overline{1}$ , and
- $\overline{m/n}$  is definable as  $m(\delta_n \overline{1})$

since for any evaluation  $e$ :  $e(\delta_n \overline{1}) = 1/n$  and  $e(m(\delta_n \overline{1})) = (1/n) \oplus \dots \oplus (1/n) = m/n$ .

As shown in [43], the variety of DMV-algebras is generated as a quasivariety by the standard DMV-algebra  $[0, 1]_{\text{DMV}}$  (i.e. the expansion of  $[0, 1]_{\text{MV}}$  with the  $\delta_n$  operations), and consequently RŁ enjoys FSRC. However, since it is a conservative expansion of Ł, RŁ does not have SRC.

We also introduce here a logic simpler than RŁ that we will make use of later in the paper. For every  $n \in \mathbb{N}$ , we denote by  $\mathbb{L}_n^+$  the expansion of  $\mathbb{L}_n$  obtained by expanding its language with the truth constant  $\overline{1/n}$  together with the axioms:

- (n1)  $n(\overline{1/n})$ ,
- (n2)  $\neg(\overline{1/n} \ \& \ (n-1)\overline{1/n})$ .

It is not difficult to see that the logic  $\mathbb{L}_n^+$  is strongly complete with respect to its related algebraic semantics, i.e. the MV-algebra over  $S_n$  expanded with a truth constant  $\overline{1/n}$  satisfying the two equations corresponding to axioms (n1) and (n2).

**THEOREM 5 ([13])** *The logics Ł and RŁ are not locally finite. For every  $n \in \mathbb{N}$ , the logics  $\mathbb{L}_n$  and  $\mathbb{L}_n^+$  are locally finite.*

### 3 Uncertainty measures over non-classical events

In this section we introduce the basic concepts regarding uncertainty measures over non-classical events. We start by introducing the definition of those uncertainty measures over Boolean algebras that we will later generalize for weaker structures. The modal logics we will present in Section 4 will be based on these generalizations.

#### 3.1 The classical case

Classical representations of uncertainty are based on a set of *possible situations* (or worlds), sometimes called a *sample space* or a *frame of discernment*, which represents all the possible outcomes. A typical example is the toss of a die. In this case, the sample space is given by six different situations, each of them corresponding to a certain outcome. An event can be simply regarded as a subset of the sample space corresponding to the set of those situations in which the event is true. In the case of the toss of a die, for instance, the event “the outcome will be an even number” corresponds to the set given by  $\{2, 4, 6\}$ . Complex events can be seen as Boolean combinations of subsets of the sample space. For instance, the event “the outcome will be an even number and it will be strictly greater than 4” is nothing but the intersection of the sets  $\{2, 4, 6\}$  and  $\{5, 6\}$ . Measures of uncertainty are classically defined over the Boolean algebra generated by subsets of a given sample space.

An event can be also identified with the proposition whose meaning is the set of situations that make it true. From a logical point of view, we can associate to a proposition the set of classical evaluations in which the proposition is true. Each of those evaluations, in fact, corresponds to a possible situation.

In what follows we will use the words “event” and “proposition” with the same meaning, and they will refer to a set of situations, or equivalently to a set of classical evaluations. Given that measures are defined over the Boolean algebra of subsets of a sample space, we can consider measures as defined over the Boolean algebra of provably equivalent classical propositions.

In general, measures of uncertainty aim at formalizing our degree of confidence in the occurrence of an event by assigning a value from a partially ordered bounded scale. In its more general sense, this is encoded by the concept of plausibility measure introduced by Halpern (see [57]).<sup>8</sup> Given a partially ordered set  $\langle L, \leq, 0, 1 \rangle$ , an *L-valued plausibility measure* on a Boolean algebra  $\mathcal{B} = (B, \wedge, \vee, \neg, 0_B, 1_B)$  of events is a mapping  $\rho: B \rightarrow L$  satisfying the following properties:

- i.  $\rho(0_B) = 0$ , and  $\rho(1_B) = 1$ ,
- ii. for every  $x, y \in B$  with  $x \leq y$ ,  $\rho(x) \leq \rho(y)$ , where  $x \leq y$  denotes the order relation between elements of  $B$ .

The first two conditions mean that the certain event  $1_B$  and the impossible event  $0_B$  have measure 1 and 0, respectively.<sup>9</sup> Indeed, the certain event is satisfied in every possible situation, while the impossible event never occurs. The third condition corresponds to monotonicity, i.e. if the situations in which an event can occur are included in those that support another event, then the degree of uncertainty of the former is smaller than the degree of uncertainty of the latter.

Uncertainty measures are usually defined as real valued functions where the partially ordered scale is identified with the real unit interval  $[0, 1]$ . Plausibility measures of this kind are also known as *fuzzy measures*, and were first introduced by Sugeno in [87]. Thus, (classical) fuzzy measures are in fact plausibility measures assigning values from  $[0, 1]$  to elements of the Boolean algebra of events.

Besides such common properties, each class of fuzzy measures basically differs from the others in how the measure of compound propositions or events is related to the measure of their components. In other words, what specifies the behavior of a fuzzy measure is how from assessments of uncertainty concerning different events we can determine the measure of (some of) their combinations. In a certain sense, we can say that classes of fuzzy measures are characterized by the satisfaction of some compositional properties. However, it is well-known that a proper fuzzy measure  $\mu$  cannot be fully compositional.<sup>10</sup>

<sup>8</sup>We want to warn the reader not to confuse *plausibility measures* in the sense of [57] with *plausibility functions* in the sense of Dempster-Shafer theory, cf. [84].

<sup>9</sup>From now on, when no danger of confusion is possible, we will omit the subscripts of the bottom and top elements of the Boolean algebra  $0_B$  and  $1_B$  respectively, and we will simply write 0 and 1.

<sup>10</sup>In the sense that there do not exist functions  $f_\wedge, f_\vee: B \times B \rightarrow L$  and  $f_\neg: B \rightarrow L$  such that, for every  $x, y \in B$ ,  $\mu(x \wedge y) = f_\wedge(\mu(x), \mu(y))$ ,  $\mu(x \vee y) = f_\vee(\mu(x), \mu(y))$ ,  $\mu(\neg x) = f_\neg(\mu(x))$ .

**THEOREM 6 ([25])** *Let  $\mu : B \rightarrow L$  be any  $L$ -valued fuzzy measure. If  $\mu$  is fully compositional then it collapses into a two-valued function, i.e. for all  $x \in B$ ,  $\mu(x) \in \{0, 1\}$ .*

Typical examples of classes of fuzzy measures are probability measures, and possibility and necessity measures.<sup>11</sup>

(Finitely additive) probability measures, first introduced from a measure-theoretic perspective by Kolmogorov in [60], are fuzzy measures defined over a Boolean algebra  $\mathcal{B}$  that satisfy the law of finite additivity:

$$\text{for every } x, y \in B \text{ such that } x \wedge y = 0_B, \mu(x \vee y) = \mu(x) + \mu(y).$$

Any probability measure  $\mu$  over a finite Boolean algebra  $\mathcal{B}$  is uniquely determined by a corresponding probability distribution  $p$  on the (finite) set of atoms  $\{a_i\}_{i \in I}$  of  $\mathcal{B}$ : by defining  $p(a_i) = \mu(\{a_i\})$ , so that  $\sum_{i \in I} p(a_i) = 1$ , it holds that, for any  $x \in \mathcal{B}$ ,  $\mu(x) = \sum_{a_j \leq x} p(a_j)$ .

Possibility measures (introduced by Zadeh in [90], and deeply studied by Dubois and Prade [22, 24]) are a class of fuzzy measures satisfying the following law of composition w.r.t. the maximum t-conorm:

$$\mu(x \vee y) = \max(\mu(x), \mu(y)).$$

Similarly, necessity measures [22] are fuzzy measures satisfying the following law of composition w.r.t. the minimum t-norm:

$$\mu(x \wedge y) = \min(\mu(x), \mu(y)).$$

Possibility and necessity measures are dual in the sense that, given a possibility measure  $\Pi$  (a necessity measure  $N$ ), one can derive its dual necessity measure as follows:

$$N(x) = 1 - \Pi(\neg x) \quad [\Pi(x) = 1 - N(\neg x)].$$

Similarly to probability measures, any possibility measure  $\Pi$  over a finite Boolean algebra  $\mathcal{B}$  is uniquely determined by a possibility distribution  $\pi$  on the set of atoms  $\{a_i\}_{i \in I}$  of  $\mathcal{B}$ . Indeed, by defining  $\pi(a_i) = \Pi(\{a_i\})$ , one has  $\sup_{i \in I} \pi(a_i) = 1$ , and  $\Pi(u) = \sup_{a_j \leq u} \pi(a_j)$  for any  $u \in \mathcal{B}$ . As for the dual necessity measure, we have  $N(u) = \inf_{a_i \leq u} 1 - \pi(a_i)$ .

### 3.2 Non-classical events

In the literature, there seems not to be a general definition of the notion of a fuzzy measure defined over structures weaker than Boolean algebras. Generalized treatments have just covered specific cases, as we will see below, such as probability and necessity / possibility measures. Since those treatments study measures over particular subclasses of MTL-algebras, it seems natural to give a definition for those kinds of structures.

**DEFINITION 7** *Given an MTL-algebra  $\mathcal{A}$ , a generalized fuzzy measure on  $\mathcal{A}$  is a mapping  $\mu : A \rightarrow [0, 1]$  such that  $\mu(0_A) = 0$ ,  $\mu(1_A) = 1$ , and for  $x, y \in A$ ,  $\mu(x) \leq \mu(y)$  whenever  $x \leq y$ .*

<sup>11</sup>Notice that we do not discuss here the appropriateness of a class of measures w.r.t. uncertainty phenomena and we do not compare them to each other. For such an analysis the reader is referred e.g. to papers by Smets [85, 86], Halpern's book [57] and the references therein.

In what follows, we are going to study particular classes of generalized fuzzy measures that are extensions of those introduced for the Boolean case.

### 3.2.1 Possibility and necessity measures

In this section we give a definition of generalized possibility and necessity measures over MTL-algebras (although we only make use of the underlying lattice structure). Notice that, even if the real unit interval  $[0, 1]$  is the most usual scale for all kinds of uncertainty measures, any bounded totally ordered set can be actually used (possibly equipped with suitable operations), especially in the case of non-additive measures of a more qualitative nature like possibility and necessity measures.

**DEFINITION 8** *Let  $\mathcal{A}$  be an MTL-algebra and let  $\mu : A \rightarrow [0, 1]$  be a generalized fuzzy measure over  $\mathcal{A}$ . Then:*

- $\mu$  is called a basic possibility measure when for all  $x, y \in A$

$$\mu(x \vee y) = \max(\mu(x), \mu(y)),$$

- $\mu$  is called a basic necessity measure when for all  $x, y \in A$

$$\mu(x \wedge y) = \min(\mu(x), \mu(y)).$$

For the case of  $\mathcal{A}$  being a lattice of  $[0, 1]$ -valued functions on a set  $X$  (i.e. a lattice of fuzzy sets), say  $A = [0, 1]^X$ , several extensions of the notions of possibility and necessity measures for fuzzy sets have been proposed in relation to different logical systems extending the well-known Dubois-Lang-Prade's Possibilistic logic to fuzzy events, see e.g. [22, 21, 48, 4, 3, 5]. Actually, the different proposals in the literature arise from two observations. First of all, contrary to the classical case,  $[0, 1]$ -valued basic possibility and necessity measures  $\Pi, N : [0, 1]^X \rightarrow [0, 1]$  are not univocally determined by a possibility distribution  $\pi$  on the set  $X$ . The second observation is that, in the classical case, the expressions of possibility and necessity measures of subsets of  $X$  in terms of a possibility distribution on  $X$  can be equivalently rewritten as

$$\Pi(f) = \sup_{x \in X} \min(\pi(x), f(x)), \quad N(f) = \inf_{x \in X} \max(1 - \pi(x), f(x))$$

where  $f : X \rightarrow \{0, 1\}$  is two-valued function, which can be obviously identified with a subset of  $X$ . Therefore, natural generalizations of these expressions when  $f : X \rightarrow [0, 1]$  is a fuzzy subset of  $X$  are

$$\Pi(f) = \sup_{x \in X} \pi(x) \odot f(x), \quad N(f) = \inf_{x \in X} \pi(x) \Rightarrow f(x) \quad (*)$$

where  $\odot$  is a t-norm and  $\Rightarrow$  is some suitable fuzzy implication function.<sup>12</sup> In particular, the following implication functions have been discussed in the literature as instantiations of the  $\Rightarrow$  operation in (\*):

<sup>12</sup>The minimal properties required for a binary operation  $\Rightarrow : [0, 1] \times [0, 1] \rightarrow [0, 1]$  to be considered as a fuzzy counterpart of the classical  $\{0, 1\}$ -valued implication truth-function are:  $0 \Rightarrow 0 = 1$ ,  $1 \Rightarrow 0 = 0$ ,  $\Rightarrow$  is non-increasing in the first variable and non-decreasing in the second variable.

$$(1) u \Rightarrow_{\text{KD}} v = \max(1 - u, v) \quad (\text{Kleene-Dienes implication})$$

$$(2) u \Rightarrow_{\text{RG}} v = \begin{cases} 1, & \text{if } u \leq v \\ 1 - u, & \text{otherwise} \end{cases} \quad (\text{reciprocal of Gödel implication})$$

$$(3) u \Rightarrow_{\text{Ł}} v = \min(1, 1 - u + v). \quad (\text{Łukasiewicz implication})$$

All these functions actually lead to proper extensions of the above definition of necessity over classical sets or events in the sense that if  $f$  describes a crisp subset of  $X$ , i.e.  $f$  is a function  $f: X \rightarrow \{0, 1\}$ , then (\*) gives  $N(f) = \inf_{x: f(x)=0} 1 - \pi(x)$ .

Moreover, if  $\Pi$  and  $N$  are required to be dual with respect to the standard negation, i.e.  $\Pi(f) = 1 - N(1 - f)$ , then one is led to consider the fuzzy implication  $\Rightarrow$  defined as  $x \Rightarrow y = (1 - x) \uplus y$  where  $\uplus$  is the t-conorm dual of  $\odot$ . These kinds of fuzzy implications are commonly known as *strong implications*. Notice that  $\Rightarrow_{\text{KD}}$  is the strong implication for  $\odot = \min$  and  $\Rightarrow_{\text{Ł}}$  is the strong implication for the Łukasiewicz t-norm.

Interestingly enough, these two notions of generalized possibilistic measures can be understood as a special kind of *fuzzy integrals*, called (generalized) *Sugeno integrals* [87]. Indeed, given a fuzzy measure  $\mu: 2^X \rightarrow [0, 1]$ , the *Sugeno integral* of a function  $f: X \rightarrow [0, 1]$  with respect to  $\mu$  is defined as

$$\oint_S f d\mu = \max_{i=1, \dots, n} \min(f(x_{\sigma(i)}), \mu(A_{\sigma(i)}))$$

where  $\sigma$  is a permutation of the indices such that  $f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \dots \geq f(x_{\sigma(n)})$ , and  $A_{\sigma(i)} = \{x_{\sigma(1)}, \dots, x_{\sigma(i)}\}$ . When  $\mu$  is a (classical) possibility measure on  $2^X$  induced by a (normalized) possibility distribution  $\pi: X \rightarrow [0, 1]$ , i.e.  $\mu(A) = \max\{\pi(x) \mid x \in A\}$  for every  $A \subseteq X$ , then the above expression of the Sugeno integral becomes (see e.g. [10])

$$\oint_S f d\pi = \max_{x \in X} \min(\pi(x), f(x)).$$

When the above minimum operation is replaced by an arbitrary t-norm  $\odot$ , we obtain the so-called generalized Sugeno integral [87]

$$\oint_{S, \odot} f d\mu = \max_{i=1, \dots, n} f(x_{\sigma(i)}) \odot \mu(A_{\sigma(i)}),$$

which, in the case of  $\mu$  being the possibility measure on  $2^X$  defined by a possibility distribution  $\pi$ , becomes

$$\oint_{S, \odot} f d\pi = \max_{x \in X} \pi(x) \odot f(x).$$

The next theorem offers an axiomatic characterization of those measures for which there exists a possibility distribution that allows a representation in terms of a generalized Sugeno integral. The formulation we provide here is very general and makes only use of the structure of De Morgan triples<sup>13</sup> over the real unit interval.

<sup>13</sup>A De Morgan triple (see e.g. [40]) is a structure on the real unit interval  $(\odot, \uplus, \neg)$  where  $\odot$  is a t-norm,  $\uplus$  a t-conorm,  $\neg$  a strong negation function such that  $x \uplus y = \neg(-x \odot -y)$  for all  $x, y \in [0, 1]$ .

**THEOREM 9** *Let  $X$  be a finite set, let  $(\odot, \uplus, 1 - x)$  be a De Morgan triple, and let  $N, \Pi: [0, 1]^X \rightarrow [0, 1]$  be a pair of dual basic necessity and possibility measures. Then,  $N$  satisfies the following property for all  $r \in [0, 1]$*

$$N(\bar{r} \uplus f) = r \uplus N(f)$$

(or equivalently,  $\Pi(\bar{r} \odot f) = r \odot \Pi(f)$ )

*if, and only if, there exists  $\pi: X \rightarrow [0, 1]$  such that  $\Pi(f) = \max_{x \in X} \pi(x) \odot f(x)$  and  $N(f) = 1 - \Pi(1 - f) = \min_{x \in X} (1 - \pi(x)) \uplus f(x)$ .*

**Proof** Suppose  $N$  is such that  $N(\bar{r} \uplus f) = \bar{r} \uplus_L N(f)$  for every  $f \in [0, 1]^X$  and  $r \in [0, 1]$ . It is easy to check that every  $f \in [0, 1]^X$  can be written as

$$f = \bigwedge_{x \in X} \mathbf{x}^c \uplus \overline{f(x)},$$

where  $\mathbf{x}^c: X \rightarrow [0, 1]$  is the characteristic function of the complement of the singleton  $\{x\}$ , i.e.  $\mathbf{x}^c(y) = 1$  if  $y \neq x$  and  $\mathbf{x}^c(x) = 0$ , and  $\overline{f(x)}$  stands for the constant function of value  $f(x)$ .

Now, by applying the axioms of a basic necessity measure and the assumption that  $N(\bar{r} \uplus f) = \bar{r} \uplus N(f)$ , we obtain that

$$N(f) = N\left(\bigwedge_{x \in X} \mathbf{x}^c \uplus \overline{f(x)}\right) = \min_{x \in X} N\left(\mathbf{x}^c \uplus \overline{f(x)}\right) = \min_{x \in X} N(\mathbf{x}^c) \uplus f(x).$$

Finally, by putting  $\pi(x) = 1 - N(\mathbf{x}^c)$ , we finally get

$$N(f) = \min_{x \in X} (1 - \pi(x)) \uplus f(x),$$

which, of course, by duality implies that

$$\Pi(f) = \max_{x \in X} \pi(x) \odot f(x)$$

The converse is easy. □

This type of integral representation can be easily generalized when we replace the real unit interval  $[0, 1]$  as the scale for the measures by more general algebraic structures, for instance by residuated lattices with involution. The details are out of the scope of this paper.

### 3.2.2 Finitely additive measures

The classical notion of (finitely additive) probability measure on Boolean algebras was generalized in [74] by the notion of *state on MV-algebras*.

**DEFINITION 10** ([74]) *By a state on an MV-algebra  $\mathcal{A} = \langle A, \oplus, \neg, 0_A \rangle$  we mean a function  $s: A \rightarrow [0, 1]$  satisfying:*

- (i)  $s(1_A) = 1$ ,
- (ii) if  $u \odot v = 0_A$ , then  $s(u \oplus v) = s(u) + s(v)$ .

The following proposition collects some properties that states enjoy.

**PROPOSITION 11** ([74]) *Let  $s$  be a state on an MV-algebra  $A$ . Then the following hold properties hold:*

- (iii)  $s(\neg u) = 1 - s(u)$ ,
- (iv) if  $u \leq v$ , then  $s(u) \leq s(v)$ ,
- (v)  $s(u \oplus v) = s(u) + s(v) - s(u \odot v)$ .

Moreover, a map  $s: A \rightarrow [0, 1]$  is a state iff (i) and (v) hold.

In [89], Zadeh introduced the following notion of probability on fuzzy sets. A fuzzy subset of a (finite) set  $X$  can be considered just as a function  $f \in [0, 1]^X$ . Then, given a probability distribution  $p: X \rightarrow [0, 1]$  on  $X$ , the *probability* of  $f$  is defined as

$$p^*(f) = \sum_{x \in X} f(x) \cdot p(x),$$

where we have written  $p(x)$  for  $p(\{x\})$ . Indeed,  $p^*$  is an example of state over the tribe  $[0, 1]^X$ . The restriction of  $p^*$  over the  $S_n$ -valued fuzzy sets is also an example of state over  $(S_n)^X$ .

The notion of state on a clan can be applied to define what a state on formulas is. Let  $W$  and  $W_n$  be the set of  $[0, 1]_{\text{MV}}$ -evaluations and  $S_n$ -evaluations respectively over the set of formulas  $Fm(V)$  in the language of Łukasiewicz logic built from a set of propositional variables  $V$ . For each  $X \subseteq W$ , and each  $\varphi \in Fm(V)$ , let

$$\varphi_X^*: X \rightarrow [0, 1]$$

be defined by  $\varphi_X^*(w) = w(\varphi)$ , where  $w$  is any  $[0, 1]_{\text{MV}}$ -evaluation in  $X$ . Analogously, for any  $Y \subseteq W_n$ , define

$$\varphi_{Y,n}^*: Y \rightarrow S_n.$$

Then, both  $Fm_X = \{\varphi_X^* \mid \varphi \in Fm(V)\}$  and  $Fm_Y = \{\varphi_{Y,n}^* \mid \varphi \in Fm(V)\}$  are clans over  $W$  and  $W_n$  respectively. Then any state  $s$  on  $Fm_X$  (resp. on  $Fm_Y$ ) induces a state on formulas  $s': Fm(V) \rightarrow [0, 1]$  by putting  $s'(\varphi) = s(\varphi_X^*)$  (resp.  $= s'(\varphi_{Y,n}^*)$ ). Notice that  $s'(\varphi) = s'(\psi)$  whenever  $\varphi \leftrightarrow \psi$  is provable in  $\mathbb{L}$  or in  $\mathbb{L}_n$  respectively.

Paris proved in [79, Appendix 2] that every state  $s$  on a finitely generated  $Fm_Y$  can be represented as an integral:

**THEOREM 12** (Paris, [79]) *Let  $V_0$  be a finite set of propositional variable, and let  $Y$  be the subset of  $W_n$  of all the evaluations of  $V_0$  into  $S_n$ . Then for every state  $s$  on  $Fm_Y$ , there is a probability distribution  $p$  on  $Y$  such that, for every  $\varphi_{Y,n}^* \in Fm_Y$ ,*

$$s(\varphi_{Y,n}^*) = \sum_{w \in Y} p(w) \cdot w(\varphi).$$

More general and sophisticated integral representation for states on MV-algebras were independently proved by Kroupa [63], and Panti [77]: for every MV-algebra  $\mathcal{A}$ , the set of all states on  $\mathcal{A}$  is in one-to-one correspondence with the class of regular Borel probability measure on a compact Hausdorff space  $\mathbb{X}$ . In particular for every state  $s$  on  $\mathcal{A}$  there is a regular Borel probability measure  $p$  on  $\mathbb{X}$  such that  $s$  is the integral with respect to  $p$ . A discussion about this topic is beyond the scope of this paper (see [63, 77] for a detailed treatment).

#### 4 Fuzzy modal logics for some classes of generalized plausibility measures

As seen in the previous section, generalized fuzzy measures assign to non-classical events values from the real unit interval  $[0, 1]$ . As also mentioned in the introduction, the underlying idea of the fuzzy logic-based treatment of uncertainty is to introduce in a given fuzzy logic a modal operator  $\mathcal{M}$ , so that  $\mathcal{M}\varphi$  denotes that  $\varphi$  is *likely*, (*plausible*, *probable*, *possible*, *etc.*), where  $\varphi$  is a proposition denoting an event (classical or non-classical). Then, taking advantage of the real semantics of  $(\Delta)$ -core fuzzy logics over the unit real interval  $[0, 1]$ , particular truth-functions over  $[0, 1]$  can be used to express specific compositional properties of different classes of measures.

For instance, consider the class of generalized plausibility measures over an  $\mathcal{L}$ -algebra of events, for some  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$ . Recall that this class is characterized by the normalization axioms,  $\rho(1) = 1$  and  $\rho(0) = 0$ , and monotonicity: whenever  $x \leq y$ ,  $\rho(x) \leq \rho(y)$ . These properties can be easily captured within  $\mathcal{L}$  itself over a language expanded by a modal operator  $Pl$  by considering the axioms  $Pl\top$  and  $\neg Pl\perp$ , together with the inference rule: from  $\varphi \rightarrow \psi$  infer  $Pl\varphi \rightarrow Pl\psi$ . Indeed, for any evaluation  $e$  over any real  $\mathcal{L}$ -algebra,  $e(Pl\varphi \rightarrow Pl\psi) = 1$  iff  $e(Pl\varphi) \leq e(Pl\psi)$ . Therefore, we can say that any  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  is *adequate* for the class of generalized fuzzy measures over  $\mathcal{L}$ -algebras of events.

However, if we then want to rely on a certain logic to represent a particular subclass of fuzzy measures, we need to take into account whether the operations needed in the definition of the subclass can be defined in that logic.

To be more specific, consider a  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  which is complete with respect to a class  $\mathcal{C}$  of real  $\mathcal{L}$ -algebras. Then any formula  $\varphi(p_1, \dots, p_n)$  over propositional variables  $p_1, \dots, p_n$  in the language of  $\mathcal{L}$  defines a function  $t_\varphi^A: [0, 1]^n \rightarrow [0, 1]$  for every real algebra  $\mathcal{A} \in \mathcal{C}$ , by stipulating  $t_\varphi^A(a_1, \dots, a_n) = e(\varphi)$ , where  $e$  is the  $\mathcal{L}$ -interpretation such that  $e(p_1) = a_1, \dots, e(p_n) = a_n$ . Then we say that a certain function  $f: [0, 1]^n \rightarrow [0, 1]$  is *definable* in a  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  if:

- (1) there exists a class  $\mathcal{C}$  of real algebras for which  $\mathcal{L}$  is complete, and
- (2) there exists an  $\mathcal{L}$ -formula  $\varphi(p_1, \dots, p_n)$  such that, for all  $\mathcal{A} \in \mathcal{C}$ ,  $t_\varphi^A(a_1, \dots, a_n) = f(a_1, \dots, a_n)$  for all  $a_1, \dots, a_n \in A$ .

For instance, the formulas  $p_1 \wedge p_2$  and  $p_1 \vee p_2$  define over any class of real MTL-chains the min and max functions respectively.

Informally speaking, we say that a  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  is *compatible* with a given subclass of fuzzy measures if the algebraic operations or relations playing a role in the axiomatic postulates of the given class of measures can be expressed by means of functions definable in  $\mathcal{L}$ .

We give an example to clarify this notion of compatibility.

**EXAMPLE 13** Consider the class of (finitely additive) probability measures on, say, classical events. In this case not every  $(\Delta)$ -core fuzzy logic  $\mathcal{L}$  is suitable to axiomatize a logic to reason about probabilities. In fact, the operation of (bounded) sum is necessary

to express the law of finite additivity, and this operation is not present in all real algebras of all logics, but it is present, for instance, in the standard algebra  $[0, 1]_{MV}$  of Łukasiewicz logic  $\mathbf{L}$ , and in the standard algebra of some of its expansions like Rational Łukasiewicz logic  $\mathbf{RL}$ . These logics, therefore, allow to axiomatize a modal logic to reason about probability (also remember that  $\mathbf{L}$  has an involutive negation), by allowing to express the additivity with the connective  $\oplus$ , whose standard interpretation is the truncated sum (recall Section 2.4, and see Section 4.4):  $P(\varphi \vee \psi) \leftrightarrow P\varphi \oplus P\psi$ , in case  $\vdash \neg(\varphi \wedge \psi)$  over Classical Logic. In contrast, it is easy to observe that, for instance, a probability logic cannot be axiomatized over Gödel logic since the (truncated) addition cannot be expressed by means of Gödel logic truth-functions.

In the rest of this section, we consider different fuzzy modal logics (in a restricted sense that will be clarified in the following definitions) axiomatizing reasoning about several classes of fuzzy measures. We introduce the fundamental syntactical and semantical frameworks that we will specifically enrich in the following subsections to deal with the distinguished classes of measures we have already recalled.

Unless stated otherwise, for the rest of this section we always consider  $\mathcal{L}_1$  to be a ( $\Delta$ -)core fuzzy logic used to represent events, and  $\mathcal{L}_2$  to be a ( $\Delta$ -)core fuzzy logic compatible with the specific class of measures we are going to reason about. As a matter of notation, let us denote by  $\mathcal{M}$  any class of fuzzy measures as those we axiomatized in the first part of this section. We introduce the basic framework to formalize reasoning about fuzzy measures in  $\mathcal{M}$ . The syntactical apparatus built over  $\mathcal{L}_1$  and  $\mathcal{L}_2$  is denoted by  $\text{FM}(\mathcal{L}_1, \mathcal{L}_2)$ .

**Syntax.** The syntax of  $\text{FM}(\mathcal{L}_1, \mathcal{L}_2)$  comprises a countable set of propositional variables  $V = \{x_1, x_2, \dots\}$ , connectives from  $\mathcal{L}_1$  and  $\mathcal{L}_2$ <sup>14</sup> and the unary modality  $\mathcal{M}$ . Formulas belong to two classes:

**EF:** The class of formulas from  $\mathcal{L}_1$ . They are inductively defined as in Section 2, and will be used to denote events. The class of those formulas will be denoted by  $\mathcal{E}$ .

**MF:** The class of modal formulas is defined inductively: for every formula  $\varphi \in \mathcal{E}$ ,  $\mathcal{M}\varphi$  is an atomic modal formula, all truth-constants of  $\mathcal{L}_2$  are also atomic modal formulas, and, moreover, compound formulas are defined from the atomic ones and using the connectives of  $\mathcal{L}_2$ . We will denote by  $\mathcal{MF}$  the class of modal formulas.

Note that connectives appearing in the scope of the modal operator  $\mathcal{M}$  are from  $\mathcal{L}_1$ , while those outside are from  $\mathcal{L}_2$ .

**Semantics.**<sup>15</sup> Let  $\mathcal{C}_1$  be a class of  $\mathcal{L}_1$ -chains over a same universe  $U_1$  for which  $\mathcal{L}_1$  is complete, and let  $\mathcal{A}_2$  be real  $\mathcal{L}_2$ -chain and such that it is compatible with  $\mathcal{M}$ . A semantics with respect to  $\mathcal{C}_1$  and  $\mathcal{A}_2$  for the language  $\text{FM}(\mathcal{L}_1, \mathcal{L}_2)$  is defined as follows: a real  $\{\mathcal{C}_1, \mathcal{A}_2\}$ - $\mathcal{M}$  model is a triple  $\langle W, e, \rho \rangle$  where:

- $W$  is a non-empty set whose elements are called *nodes* or *possible words*.

<sup>14</sup>We will not distinguish the connective symbols of both logics since it will become clear from the context.

<sup>15</sup>The semantical framework we adopt here is inspired by the approach of [82] in the general setting of two-layered fuzzy modal logics. We thank Petr Cintula for bringing this work to our knowledge.

- $e: \mathcal{E} \times W \rightarrow U_1$ , where  $U_1$  is the common universe of the chains in  $\mathcal{C}_1$ , is a map such that, for every fixed  $w \in W$ , the map  $e(\cdot, w): \mathcal{E} \rightarrow U_1$  is an evaluation of non-modal formulas over a particular algebra  $\mathcal{A}_w \in \mathcal{C}_1$
- $\rho: Fm_W(V) \rightarrow [0, 1]$  is an  $\mathcal{M}$ -fuzzy measure, where  $Fm_W(V)$  is defined as follows. For every formula  $\varphi \in \mathcal{E}$ , define the map  $f_\varphi: W \rightarrow U_1$  such that, for every  $w \in W$ ,  $f_\varphi(w) = e(\varphi, w)$ . Then  $Fm_W(V)$  is the  $\mathcal{L}_1$ -algebra of all the functions defined in this way, with the pointwise application of the operations in the  $\mathcal{A}_w$ 's.

Let  $M = \langle W, e, \rho \rangle$  be a  $\{\mathcal{C}_1, \mathcal{A}_2\}$ - $\mathcal{M}$  model, let  $w$  be a fixed node in  $W$ , and let  $\phi$  be a formula of  $F\mathcal{M}(\mathcal{L}_1, \mathcal{L}_2)$ . Then, the truth value of  $\phi$  in  $M$  at the node  $w$  (we will denote this value by  $\|\phi\|_{M,w} \in [0, 1]$ ) is inductively defined as follows:

- If  $\phi$  is a formula in  $\mathcal{E}$ , then  $\|\phi\|_{M,w} = e(\phi, w)$ .
- If  $\phi$  is an atomic modal formula of the form  $\mathcal{M}\psi$ , then  $\|\mathcal{M}\psi\|_{M,w} = \rho(f_\psi)$ .
- If  $\phi$  is a compound modal formula, then  $\|\phi\|_{M,w}$  is computed by truth functionality and using the operations of  $\mathcal{A}_2$ .

Notice that when  $\phi$  is modal, its truth value  $\|\phi\|_{M,w}$  does not depend on the chosen world  $w$ , hence in these cases we will simplify the notation by dropping the subscript  $w$ , and we will write  $\|\phi\|_M$ .  $M$  will be called a *model* for  $\phi$  when  $\|\phi\|_M = 1$ , and will be called a model for a modal theory  $\Gamma$  (i.e.  $\Gamma \subseteq \mathcal{M}\mathcal{F}$ ) when it is a model for each formula in  $\Gamma$ .

In the remaining part of this section, it will be useful to consider  $\{\mathcal{C}_1, \mathcal{A}_2\}$ - $\mathcal{M}$  models  $\langle W, e, \rho \rangle$ , where the measure  $\rho$  takes values in an  $\mathcal{L}_2$ -chain  $\mathcal{A}_2$  whose domain coincides with a non trivial ultrapower  $*[0, 1]$  of  $[0, 1]$ . Those models will be called *hyperreal*. Evaluations into a hyperreal  $\{\mathcal{C}_1, \mathcal{A}_2\}$ - $\mathcal{M}$  model are defined accordingly.

**REMARK 14** (1) *To simplify the reading, and without danger of confusion, we will henceforth avoid mentioning the class of chains  $\mathcal{C}_1$  and the algebra  $\mathcal{A}_2$  when referring to the models introduced above. We will simply say that a triple  $\langle W, e, \rho \rangle$  is a (real or hyperreal)  $\mathcal{M}$ -model. The class  $\mathcal{C}_1$  and the algebra  $\mathcal{A}_2$  will be always clear by the context.*

(2) *In the following subsections, we will axiomatize particular classes of fuzzy measures. Case by case we will adopt a notation consistent with the class of measures we will deal with. Therefore, we will denote by  $\mathcal{PL}$  the class of generalized plausibility measures, by  $\Pi$  the class of possibility measures, and so forth. For example, we will denote by  $F\mathcal{PL}(\mathcal{L}_1, \mathcal{L}_2)$  the logic for generalized plausibility and, also referring to what we stressed in (1), we will call its models the plausibilistic models. Clearly the same notation (mutatis mutandis) will be also adopted for all the particular classes of fuzzy measures we are going to treat.*

#### 4.1 A modal logic for generalized plausibility measures

In this section we take  $\mathcal{M}$  to be the class of *generalized plausibility measures*, denoted as  $\mathcal{PL}$ , and let  $\mathcal{L}_1, \mathcal{L}_2$  be two core fuzzy logics. Recall that any core fuzzy logic is compatible with  $\mathcal{PL}$ . The logic that allows to reason about generalized plausibility measures

of over  $\mathcal{L}_1$ -events over the logic  $\mathcal{L}_2$  will be called  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ , and its axioms and rules are the following:

**Ax1** All axioms and rules of  $\mathcal{L}_1$  restricted to formulas in  $\mathcal{E}$ .

**Ax2** All axioms and rules of  $\mathcal{L}_2$  restricted to modal formulas.

**Ax3** Axiom for the modality  $Pl$ :

$$\mathbf{PI} \quad \neg Pl(\perp),$$

**M** The rule of *monotonicity* for  $Pl$ : from  $\varphi \rightarrow \psi$ , deduce  $Pl(\varphi) \rightarrow Pl(\psi)$ .  
(where  $\varphi, \psi \in \mathcal{E}$ )

**N** The rule of *necessitation* for  $Pl$ : from  $\varphi$ , deduce  $Pl(\varphi)$ .  
(where  $\varphi \in \mathcal{E}$ )

Notice that nested modalities are not allowed, nor are formulas which contain modal formulas but also non-modal formulas that are not under the scope of any modality. That is to say that, for example, if  $\varphi, \psi \in \mathcal{E}$ , then neither  $Pl(Pl(\varphi))$  nor  $\psi \rightarrow Pl(\varphi)$  is a well-founded formula in our language.

The notion of *proof* in  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  is defined as usual, and we denote by  $\vdash_{\text{F}\mathcal{P}\mathcal{L}}$  the relation of logical consequence. A *theory* is a set of formulas, and a *modal theory* is a set of modal formulas. For any theory  $\Gamma$ , and for every formula  $\phi$ , we write  $\Gamma \vdash_{\text{F}\mathcal{P}\mathcal{L}} \phi$  to denote that  $\phi$  follows from  $\Gamma$  in  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ .

**PROPOSITION 15** *The logic  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  proves the following:*

- (1) *The modality  $Pl$  is normalized, that is  $\vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\perp) \leftrightarrow \perp$ , and  $\vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\top) \leftrightarrow \top$  (where, as usual,  $\top = \neg\perp$ ).*
- (2) *The rule of substitution of equivalents:  $\tau \leftrightarrow \gamma \vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\tau) \leftrightarrow Pl(\gamma)$ .*

**Proof** (1) Since in  $\mathcal{L}_2$  the negation can be defined as  $\neg\phi = \phi \rightarrow \perp$ , the axiom **PI** actually states that  $Pl(\perp) \rightarrow \perp$ . Moreover,  $\perp \rightarrow Pl(\perp)$  trivially holds, and therefore  $\vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\perp) \leftrightarrow \perp$ . Finally, since  $\vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\top)$ , then  $\vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\top) \leftrightarrow \top$ .

(2) As usual  $\tau \leftrightarrow \gamma$  can be split in  $\tau \rightarrow \gamma$  and  $\gamma \rightarrow \tau$ . Now, from  $\tau \rightarrow \gamma$ , and using (1),  $\tau \rightarrow \gamma \vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\tau) \rightarrow Pl(\gamma)$ . Similarly  $\gamma \rightarrow \tau \vdash_{\text{F}\mathcal{P}\mathcal{L}} Pl(\gamma) \rightarrow Pl(\tau)$ .  $\square$

As for the semantics, given a class  $\mathcal{C}_1$  of real  $\mathcal{L}_1$ -algebras for which  $\mathcal{L}_1$  is complete and a real  $\mathcal{L}_2$ -algebra  $\mathcal{A}_2$  compatible with a generalized plausibility measure  $\rho$ , a  $\{\mathcal{C}_1, \mathcal{A}_2\}$ - $\mathcal{P}\mathcal{L}$  model, for short a *plausibilistic model*, will be a triple  $M = \langle W, e, \rho \rangle$  with the same definition and notation used above for the general case (see Remark 14).

**REMARK 16** *The compatibility assumption of the algebra  $\mathcal{A}_2$  with respect to the measure  $\rho$  is what guarantees that the logic  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ , and in particular its genuine modal axiom(s) and rule(s), is sound with respect to the class of plausibilistic models. The same observation applies to the other modal logics we will consider in the next subsections.*

**DEFINITION 17** *Let  $\Gamma \cup \{\Phi\}$  be a modal theory of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ . Then we say that the logic  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  is:*

- *Finitely strongly complete with respect to real plausibilistic models (real-FSC) if whenever  $\Gamma$  is finite, and  $\Gamma \not\vdash_{\text{FP}\mathcal{L}} \Phi$ , there is a real plausibilistic model  $M$  for  $\Gamma$  such that  $\|\Phi\|_M < 1$ .*
- *Strongly complete with respect to real plausibilistic models (real-SC) if for every  $\Gamma$  such that  $\Gamma \not\vdash_{\text{FP}\mathcal{L}} \Phi$ , there is a real plausibilistic model  $M$  for  $\Gamma$  such that  $\|\Phi\|_M < 1$ .*
- *Strongly complete with respect to hyperreal plausibilistic models (hyperreal-SC) if for every  $\Gamma$  such that  $\Gamma \not\vdash_{\text{FP}\mathcal{L}} \Phi$ , there is a hyperreal plausibilistic model  $M$  for  $\Gamma$  such that  $\|\Phi\|_M < 1$ .*

Now we introduce a general way to prove (finite, strong) completeness for  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  with respect to the class of real and hyperreal plausibility models. The same methods will be then applied in the following sections when we will study those extensions of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  that allow to deal with more specific uncertainty measures.

First, we define a translation mapping from the modal language of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  into the propositional language of  $\mathcal{L}_2$ . This translation works as follows: for every atomic modal formula  $Pl(\varphi)$ , we introduce a new variable  $p_\varphi$  in the language of  $\mathcal{L}_2$ . Then, we inductively define the translation  $\bullet$  as follows:

- $(Pl(\varphi))^\bullet = p_\varphi$ .
- $\perp^\bullet = \perp$ .
- $(\star(\Phi_1, \dots, \Phi_n))^\bullet = \star((\Phi_1)^\bullet, \dots, (\Phi_n)^\bullet)$  for every  $n$ -ary connective  $\star$  of  $\mathcal{L}_2$ .

For any modal theory  $\Gamma$  of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ , in accordance with  $\bullet$ , we define

$$\Gamma^\bullet = \{\Psi^\bullet \mid \Psi \in \Gamma\}$$

$$\text{FP}\mathcal{L}^\bullet = \{\Theta^\bullet \mid \Theta \text{ is an instance of PI}\} \cup \{p_\varphi \mid \vdash_{\mathcal{L}_1} \varphi\} \cup \{p_\varphi \rightarrow p_\psi \mid \vdash_{\mathcal{L}_1} \varphi \rightarrow \psi\}.$$

**LEMMA 18** *Let  $\Gamma \cup \{\Phi\}$  be a modal theory of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ . Then*

$$\Gamma \vdash_{\text{FP}\mathcal{L}} \Phi \text{ iff } \Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet.$$

**Proof** ( $\Rightarrow$ ) An  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ -proof  $\Psi_1, \dots, \Psi_n$  of  $\Phi$  in  $\Gamma$  is made into an  $\mathcal{L}_2$ -proof of  $\Phi^\bullet$  in  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet$  by deleting all  $\mathcal{L}_1$ -formulas and taking, for each modal formula  $\Psi_i$ , the  $\mathcal{L}_2$  formula  $\Psi_i^\bullet$ .

( $\Leftarrow$ ) Conversely, each  $\mathcal{L}_2$ -proof of  $\Phi^\bullet$  has the form  $\Psi_1^\bullet, \dots, \Psi_n^\bullet$ , where  $\Psi_i$  are modal formulas. Therefore the previous proof is converted into an  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ -proof of  $\Phi$  in  $\Gamma$ , by adding for each  $\Psi_i$  of the form  $p_\varphi$  ( $\varphi$  being an  $\mathcal{L}_1$ -theorem) a proof in  $\mathcal{L}_1$  of  $\varphi$ , and then applying a step of necessitation (**N**) in order to get  $Pl(\varphi)$ , and for each  $\Psi_j$  of the form  $p_\varphi \rightarrow p_\psi$  a proof in  $\mathcal{L}_1$  of  $\varphi \rightarrow \psi$ , and then applying a step of the monotonicity rule (**M**) in order to get  $Pl(\varphi) \rightarrow Pl(\psi)$ .  $\square$

Now, assume  $\Gamma \cup \{\Phi\}$  to be a finite modal theory over  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ , and let  $V_0$  be the following set of propositional variables:

$$V_0 = \{v_i \mid v_i \text{ occurs in some } \varphi, P(\varphi) \text{ is a subformula of } \Psi, \Psi \in \Gamma \cup \{\Phi\}\},$$

i.e.  $V_0$  is the set of all the propositional variables occurring in all the  $\mathcal{L}_1$ -formulas occurring in some modal formula of  $\Gamma \cup \{\Phi\}$ . Clearly  $V_0$  is finite.

We can identify  $\mathcal{F}_{\mathcal{L}_1}(V_0)$  with the Lindenbaum-Tarski algebra of  $\mathcal{L}_1$  of formulas generated in the restricted language having  $V_0$  as set of variables. Therefore, for every  $[\varphi] \in \mathcal{F}_{\mathcal{L}_1}(V_0)$  we choose a representative of the class  $[\varphi]$ , that we will denote by  $\varphi^\square$ . Then, consider the following further translation map:

- For every modal formula  $\Phi$ , let  $\Phi^\square$  be the formula resulting from the substitution of each propositional variable  $p_\varphi$  occurring in  $\Phi^\bullet$  by  $p_{\varphi^\square}$ ,
- $(\star(\Phi_1, \dots, \Phi_n))^\square = \star((\Phi_1)^\square, \dots, (\Phi_n)^\square)$  for every  $n$ -ary connective  $\star$  of  $\mathcal{L}_2$ .

In accordance with that translation, we define  $\Gamma^\square$  and  $\text{FP}\mathcal{L}^\square$  as:

$$\begin{aligned} \Gamma^\square &= \{\Psi^\square \mid \Psi^\bullet \in \Gamma^\bullet\} \\ \text{FP}\mathcal{L}^\square &= \{\Upsilon^\square \mid \Upsilon^\bullet \in \text{FP}\mathcal{L}^\bullet\}. \end{aligned}$$

LEMMA 19  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet$  iff  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square \vdash_{\mathcal{L}_2} \Phi^\square$ .

**Proof** ( $\Leftarrow$ ) Let  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square \vdash_{\mathcal{L}_2} \Phi^\square$ . Then, in order to prove the claim we have to show that  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet$  for each  $\Phi$  such that its  $\square$ -translation is  $\Phi^\square$ . For instance, if  $\Phi = Pl(\psi)$  then  $\Phi^\square = p_{\psi^\square} = p_{\gamma^\square}$  for each  $\gamma \in [\psi]$ , therefore, if  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square \vdash_{\mathcal{L}_2} p_{\varphi^\square}$  we have to show that  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} p_\gamma$  for each  $\gamma \in [\varphi]$ .

First, we notice that the following fact immediately follows from Proposition 15(2):

**Claim 1** Let  $\varphi, \psi$  be  $\mathcal{L}_1$ -formulas. If  $\vdash_{\mathcal{L}_1} \varphi \leftrightarrow \psi$ , then  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2) \vdash Pl(\varphi) \leftrightarrow Pl(\psi)$  (and in particular  $\text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} p_\varphi \leftrightarrow p_\psi$ ).

Let us now turn back to the proof of Lemma 19. Let  $\Phi$  be a modal formula of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  and let  $Pl(\varphi_1), \dots, Pl(\varphi_k)$  be all the atomic modal formulas occurring in  $\Phi$ . If  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square \vdash_{\mathcal{L}_2} \Phi^\square$ , then, it easily follows from the above claim that  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet$  where  $\Phi^\bullet$  is any  $\mathcal{L}_2$ -formula obtained by replacing each occurrence of a propositional variable  $p_{\varphi_i}$  with another  $p_{\psi_i}$  such that  $\psi_i \in [\varphi_i]$ . In fact, if  $\psi_i \in [\varphi_i]$ , then  $\vdash_{\mathcal{L}_1} \psi_i \leftrightarrow \varphi_i$  and therefore, from Claim 1,  $\text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} p_{\psi_i} \leftrightarrow p_{\varphi_i}$ . Thus  $p_{\varphi_i}$  can be substituted with  $p_{\psi_i}$  without loss of generality in the proof. Therefore, in particular  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet$  and this direction is complete.

( $\Rightarrow$ ) In order to prove the other direction let us assume  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet$  and let  $\Psi_1^\bullet, \dots, \Psi_k^\bullet$  be an  $\mathcal{L}_2$ -proof of  $\Phi^\bullet$  in  $\Gamma^\bullet \cup \text{FP}\mathcal{L}^\bullet$ . For each  $1 \leq j \leq k$  replace  $\Psi_j^\bullet$  with  $\Psi_j^\square$ , the representative of its equivalence class in  $\mathcal{F}_{\mathcal{L}_1}(V_0)$ . Clearly  $\Psi_1^\square, \dots, \Psi_k^\square$  is an  $\mathcal{L}_2$ -proof of (a formula logically equivalent to)  $\Phi^\square$ . In fact, if  $\Psi_k^\bullet = \Phi^\bullet$ , then  $\Psi_k^\square \leftrightarrow \Phi^\square$ . Moreover, for each  $1 \leq i < k$  one of the following holds:

- (i)  $\Psi_i^\square$  is (logically equivalent to) an axiom of  $\mathcal{L}_2$ ,
- (ii)  $\Psi_i^\square \in \Gamma^\square \cup \text{FP}\mathcal{L}^\square$ ,

- (iii) If  $\Psi_t^\bullet$  is obtained by modus ponens from  $\Psi_s^\bullet \rightarrow \Psi_t^\bullet$  and  $\Psi_s^\bullet$ , then we claim that  $\Psi_t^\square$  is obtained by modus ponens from  $\Psi_s^\square \rightarrow \Psi_t^\square$  and  $\Psi_s^\square$ . In fact we have just to note that  $(\Psi_s \rightarrow \Psi_t)^\square = \Psi_s^\square \rightarrow \Psi_t^\square$  and thus the claim easily follows.

Moreover, since modus ponens is the only inference rule of  $\mathcal{L}_2$  we have nothing to add, and our claim is settled.  $\square$

Now, we are ready to state and prove our completeness theorem.

**THEOREM 20** *Let  $\mathcal{L}_1$  be a logic for events, and let  $\mathcal{L}_2$  be a logic compatible with plausibility measures. Then the following hold:*

- (1) *If  $\mathcal{L}_1$  is locally finite, and  $\mathcal{L}_2$  enjoys the FSRC, then  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$  is real-FSC.*
- (2) *If  $\mathcal{L}_2$  has the SRC, then  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$  is real-SC.*
- (3) *If  $\mathcal{L}_2$  has the FSRC, then  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$  is hyperreal-SC.*

**Proof** (1) Assume  $\mathcal{L}_1$  to be locally finite and complete with respect to a class  $\mathcal{C}_1 = \{L_i\}_{i \in I}$  of  $\mathcal{L}_1$ -chains over a same universe  $U_1$ . Let  $\Gamma \cup \{\Phi\}$  be a modal theory of  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$  such that  $\Gamma \not\vdash_{\text{FPL}} \Phi$ . Then, by Definition 3, and by definition of  $\square$ , it follows that  $\Gamma^\square \cup \text{FPL}^\square$  is a finite theory of  $\mathcal{L}_2$ . Moreover, by Lemma 19,  $\Gamma \not\vdash_{\text{FPL}} \Phi$  iff  $\Gamma^\square \cup \text{FPL}^\square \not\vdash_{\mathcal{L}_2} \Phi^\square$ . Since  $\mathcal{L}_2$  enjoys FSRC, there is an evaluation  $v$  into a real  $\mathcal{L}_2$ -algebra  $\mathcal{A}_2$  which is a model for  $\Gamma^\square \cup \text{FPL}^\square$ , but  $v(\Phi^\square) < 1$ .

Now consider the model  $M = \langle W, e, \rho \rangle$  (cf. [82]), where:

- $W = \cup_{i \in I} W_i$  where  $W_i$  is the set of all evaluations on the algebra  $L_i$ .
- $e: V \times W \rightarrow U_1$  is defined as follows: for every  $w \in W_i$ , and every  $p \in V$ ,

$$e(p, w) = \begin{cases} w(p) & \text{if } p \in V_0, \\ 0 & \text{otherwise.} \end{cases}$$

- $\rho: \text{Fm}_W(V_0) \rightarrow [0, 1]$  is defined as: for all  $f_\varphi \in \text{Fm}_W(V_0)$ ,

$$\rho(f_\varphi) = v(\text{Pl}(\varphi)^\square).$$

**Claim 2** *The model  $M = \langle W, e, \rho \rangle$  is a plausibilistic model.*

**Proof** (of the Claim 2) We only need to prove that  $\rho$  is a plausibility measure. Then, recalling that  $\perp^\square = \perp$ , we have  $\top^\square = \top$ , and so  $\rho(f_\perp) = v(\perp^\square) = v(\perp) = 0$ . Analogously  $\rho(f_\top) = 1$ . To prove monotonicity, assume that  $f_\varphi \leq f_\psi$  in  $\text{Fm}_W(V_0)$ . Now,  $f_\varphi \leq f_\psi$  means that for every chain  $L_i$  and every evaluation  $w$  on  $L_i$ ,  $w(\varphi) \leq w(\psi)$ , and by completeness of  $\mathcal{L}_1$  with respect to  $\mathcal{C}_1$ ,  $\vdash_{\mathcal{L}_1} \varphi \rightarrow \psi$ . By the monotonicity rule **M**,  $\vdash_{\text{FPL}} \text{Pl} \varphi \rightarrow \text{Pl} \psi$ . Hence  $\top = (\text{Pl} \varphi \rightarrow \text{Pl} \psi)^\square = (\text{Pl} \varphi)^\square \rightarrow (\text{Pl} \psi)^\square \in \text{FPL}^\square$ . Since  $v$  is a model of  $\text{FPL}^\square$ , we have  $v((\text{Pl} \varphi)^\square \rightarrow (\text{Pl} \psi)^\square) = 1$ . But  $v((\text{Pl} \varphi)^\square \rightarrow (\text{Pl} \psi)^\square) = 1$  iff  $v((\text{Pl} \varphi)^\square) \leq v((\text{Pl} \psi)^\square)$  iff  $\rho(f_\varphi) \leq \rho(f_\psi)$ . Therefore  $M$  is a plausibilistic model as required.  $\square$

Let  $\Psi$  be any modal formula of  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$ . By induction on  $\Psi$ , it is now easy to show that  $\|\Psi\|_M = v(\Psi^\square)$ , hence  $M$  is a plausibilistic model that satisfies every formula of  $\Gamma$ , and such that  $\|\Phi\|_M < 1$  as required.

(2) Let now  $\Gamma \cup \{\Phi\}$  be any arbitrary modal theory of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ , and in particular assume  $\Gamma$  to be infinite. Therefore, independently from the fact that  $\mathcal{L}_1$  is locally finite or not, the  $\mathcal{L}_2$  propositional theory  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square$  is infinite. Assume  $\Gamma \not\vdash_{\text{FP}\mathcal{L}} \Phi$ : from Lemma 19,  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square \not\vdash_{\mathcal{L}_2} \Phi^\square$ . Since  $\mathcal{L}_2$ , by hypothesis, has strong real completeness, there exists, again, an evaluation  $v$  into a real  $\mathcal{L}_2$ -algebra such that  $v$  is a model of  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square$ , and  $v(\Phi^\square) < 1$ .

Then, the same plausibilistic model  $M$  we defined in the proof of (1) is appropriate for our purposes. Then, (2) is proved as well.<sup>16</sup>

(3) Assume now  $\Gamma$  to be any arbitrary modal theory of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ . Assume that  $\Gamma \not\vdash_{\text{FP}\mathcal{L}} \Phi$ : so  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square \not\vdash_{\mathcal{L}_2} \Phi^\square$  by Lemma 19. By Definition 3,  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square$  is not a finite theory of  $\mathcal{L}_2$ , but since  $\mathcal{L}_2$  has FSRC, then by [34, Theorem 3.2],  $\mathcal{L}_2$  has SR\*C. Consequently, there is an evaluation  $v$  into a non-trivial ultraproduct of real  $\mathcal{L}_2$ -chains satisfying all the formulas in  $\Gamma^\square \cup \text{FP}\mathcal{L}^\square$ , and  $v(\Phi^\square) < 1$ .

Again, the same strategy used in the proof of the claims (1) and (2) shows that the model  $M = \langle W, e, \rho \rangle$ , defined as in the proof of (1), evaluates into 1 all the modal formulas of  $\Gamma$ , and  $\|\Phi\|_M < 1$ . Notice that in this peculiar case, for every  $f_\varphi \in \text{Fm}_W(V_0)$ ,  $\rho(f_\varphi) = v(\text{Pl}(\varphi)^\square) \in {}^*[0, 1]$ , and  $M$  is in fact a hyperreal plausibilistic model.  $\square$

## 4.2 Logics for generalized possibility and necessity

As we discussed in Section 3.2.1, possibility and necessity measures can be generalized to be defined on any lattice-ordered structure. Now, we show the logical counterpart of these measure-theoretical approaches introducing schematic extensions of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  so as to capture these more peculiar mappings.

Since the formalisms we introduce are intended to deal with necessity and possibility measures, we are going to consider as  $\mathcal{L}_1$ , and  $\mathcal{L}_2$  only those  $\Delta$ -core fuzzy logics that are extensions of  $\text{MTL}_\sim$ . This will allow us to treat not only necessity but possibility measures as well, since they are definable as  $\Pi(\varphi) := \sim N(\sim \varphi)$ . With an abuse of notation, we denote by  $N$  (necessity) the modal operator of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$ .

The logic  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  is the schematic extension of  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  given by the basic axiom schema

$$\text{FN } N(\varphi \wedge \psi) \leftrightarrow N(\varphi) \wedge N(\psi).$$

Necessity models for  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  are particular plausibilistic models. Indeed, they are triples of the form  $\langle W, e, N \rangle$ , where  $W$  and  $e$  are defined as in the case of plausibilistic models, and where  $N: \text{Fm}(\mathcal{L}_1)_W \rightarrow [0, 1]$  is a necessity measure. Whenever  $N$  ranges over a non-trivial ultrapower  ${}^*[0, 1]$  of the unit interval  $[0, 1]$  we speak about *hyperreal necessity model*.

**THEOREM 21** *Let  $\mathcal{L}_1$  be a logic for events, and let  $\mathcal{L}_2$  be a logic compatible with necessity measures. Then the following hold:*

- (1) *If  $\mathcal{L}_1$  is locally finite, and  $\mathcal{L}_2$  has the FSRC, then  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  is real-FSC.*

<sup>16</sup>In fact, in this case where  $\mathcal{L}_2$  is assumed to have the SRC, the same result could have been obtained directly from the first translation  $\bullet$ , i.e. without the further second translation  $\square$ .

(2) If  $\mathcal{L}_2$  has the SRC, then  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  is real-SC.

(3) If  $\mathcal{L}_2$  has the FSRC, then  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  is hyperreal-SC.

**Proof** The claims can be easily proved by following the same lines of Lemmas 18 and 19, and Theorem 20. Indeed, using easy adaptations of Lemmas 18 and 19, one has to show that, given a modal theory  $\Gamma$  and a modal formula  $\Phi$ ,  $\Gamma \vdash_{\text{FN}} \Phi$  iff  $\Gamma^\bullet \cup \text{FN}^\bullet \vdash_{\mathcal{L}_2} \Phi^\bullet$  iff  $\Gamma^\square \cup \text{FN}^\square \vdash_{\mathcal{L}_2} \Phi^\square$ . The only point here is that when building the theory  $\text{FN}^\bullet$  one has to additionally consider countably many instances of the axiom **FN**. Then one has to show that the plausibilistic model  $M = \langle W, e, N \rangle$  arising from the adaptation of the proof of Theorem 20, is indeed a necessity model. Adopting the same notation of the proof of Theorem 20, call  $\nu$  the  $\mathcal{L}_2$ -model of  $\Gamma^\square \cup \text{FN}^\square$ , and call  $M = \langle W, e, N \rangle$  the plausibilistic model, where for every  $f_\varphi \in \text{Fm}_W(V_0)$ , we define  $N(f_\varphi) = \nu(N(\varphi)^\square)$ . Then, since  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  is the basic schematic extension of  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$  by the schema **FN**, for every  $f_\varphi, f_\psi \in \text{Fm}_W(V_0)$ ,

$$N(f_\varphi \wedge f_\psi) = N(f_{\varphi \wedge \psi}) = \nu(N(\varphi \wedge \psi)^\square) \text{ and } \nu((N(\varphi)^\square \wedge N(\psi)^\square) \leftrightarrow N(\varphi \wedge \psi)^\square) = 1$$

because  $(N(\varphi)^\square \wedge N(\psi)^\square) \leftrightarrow N(\varphi \wedge \psi)^\square \in \text{FN}^\square$  and  $\nu$  is a model of  $\text{FN}^\square$ , and hence  $N(f_\varphi \wedge f_\psi) = N(f_\varphi) \wedge N(f_\psi)$ . Therefore  $N$  is a necessity and the claim is settled.  $\square$

### 4.3 Logics for representable generalized possibility and necessity

For every t-norm  $*$  in  $\text{CONT-fin} \cup \text{WNM-fin}$ , let  $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_*([0, 1]_{\mathbb{Q}})$ , as defined in Section 2.2. The logic  $\text{FN}^{\mathbb{Q}}(\mathcal{L}_1, \mathcal{L}_2)$  is the basic schematic extension of  $\text{FN}(\mathcal{L}_1, \mathcal{L}_2)$  given by the axiom schema

$$\text{QN} \quad N(\bar{r} \uplus \varphi) \leftrightarrow \bar{r} \uplus N(\varphi) \text{ for every } r \in [0, 1] \cap \mathbb{Q},$$

and where  $\varphi \uplus \psi$  stands for  $\sim(\sim\varphi \& \sim\psi)$  in  $\mathcal{L}_*([0, 1]_{\mathbb{Q}})$ .

Notice that the logic  $\text{FPI}^{\mathbb{Q}}(\mathcal{L}_1, \mathcal{L}_2)$ , where necessity measures are replaced by possibility measures, is in fact the same as  $\text{FN}^{\mathbb{Q}}(\mathcal{L}_1, \mathcal{L}_2)$ , since the involutive negations of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  allow the definition of possibility from necessity by duality. Therefore, we only focus on  $\text{FN}^{\mathbb{Q}}(\mathcal{L}_1, \mathcal{L}_2)$ .

*Homogeneous necessity models* are necessity models  $\langle W, e, N^{\mathbb{Q}} \rangle$  where

$$N^{\mathbb{Q}}: \text{Fm}_W(V) \rightarrow [0, 1]$$

further satisfies:  $N^{\mathbb{Q}}(\bar{r} \uplus \varphi) = \bar{r} \uplus N^{\mathbb{Q}}(\varphi)$ . Whenever the homogeneous necessity measure takes values in a non-trivial ultrapower  ${}^*[0, 1]$  of the real unit interval, we speak, as usual, of *hyperreal homogeneous necessity models*. Unlike all the previously studied cases, it is now possible to introduce a stronger class of models. This is the class of *strong necessity models* of the form  $M^{\mathbb{Q}} = \langle W, e, \pi \rangle$  where  $W$  and  $e$  are defined as above, and where  $\pi: W \rightarrow [0, 1]$  is a normalized possibility distribution, i.e.  $\sup_{w \in W} \pi(w) = 1$ . Evaluations in a strong necessity model are defined as usual, except for atomic modal formulas  $N(\psi)$  that are now evaluated as follows:

$$\|N(\psi)\|_{M^{\mathbb{Q}}} = \inf_{w \in W} \left( \|\psi\|_{M^{\mathbb{Q}}, w} \uplus \pi(w) \right).$$

**THEOREM 22** *Let  $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_*([0, 1]_{\mathbb{Q}})$  for a  $t$ -norm  $*$   $\in$  **CONT-fin**  $\cup$  **WNM-fin**. Then the following hold:*

- (1) *If  $\mathcal{L}_1$  is locally finite and  $\mathcal{L}_2$  has FSRC, then the logic  $\text{FN}^{\mathbb{Q}}(\mathcal{L}_1, \mathcal{L}_2)$  is real-FSC with respect to the class of homogeneous necessity models, and the class of strong necessity models.*
- (2) *If  $\mathcal{L}_2$  has FSRC, then the logic  $\text{FN}^{\mathbb{Q}}(\mathcal{L}_1, \mathcal{L}_2)$  is hyperreal-SC with respect to the class of homogeneous necessity models.*

**Proof** An inspection of the proof of Theorem 20 and a similar technique used in the proof of Theorem 21, applied to **QN**, shows the first part of (1) and (2).

Take, now, a finite modal theory  $\Gamma \cup \{\Phi\}$  such that  $\Gamma \not\vdash_{\text{FN}^{\mathbb{Q}}} \Phi$ , and let  $M = \langle W, e, N^{\mathbb{Q}} \rangle$  be the homogeneous necessity model satisfying all the formulas in  $\Gamma$ , and  $\|\Phi\|_M < 1$ .

$N^{\mathbb{Q}}$  is a homogeneous necessity measure on  $\text{Fm}_W(V_0)$  and  $W$  coincides with the class of all  $\mathcal{L}_1$ -evaluations. Moreover, both  $\langle W, *, \uplus, \sim \rangle$  and  $\langle [0, 1], *, \uplus, \sim \rangle$  are De Morgan triples, and, being  $M$  a model for **QN**, we have that  $N^{\mathbb{Q}}(\bar{r} \uplus f) \leftrightarrow \bar{r} \uplus N^{\mathbb{Q}}(f)$  for every  $r \in [0, 1] \cap \mathbb{Q}$  and every  $f \in \text{Fm}_W(V_0)$ . Then, Theorem 9(1) ensures the existence of a normalized possibility distribution  $\pi$  on  $W$  such that, for every  $f_{\varphi} \in \text{Fm}_W(V_0)$ ,

$$N^{\mathbb{Q}}(f_{\varphi}) = \bigwedge_{w \in W} \sim \pi(w) \uplus e(w, \varphi).$$

Thus  $M^{\mathbb{Q}} = \langle W, e, \pi \rangle$  is a strong necessity model that satisfies  $\Gamma$ , but  $\|\Phi\|_{M^{\mathbb{Q}}} < 1$ .  $\square$

**REMARK 23** *An alternative modal-style treatment of (representable) possibility and necessity measure on many-valued events can be found in [20], where the authors rely on  $G_{\Delta}(\mathbb{Q})$  (i.e. Gödel logic with  $\Delta$  and truth constants from the rationals in  $[0, 1]$ ) as a logic for modal formulas. In fact, the only necessary ingredients to correctly axiomatize representable necessity and possibility modal formulas are the rational truth constants and the lattice operations. These requirements are fulfilled by  $G_{\Delta}(\mathbb{Q})$  (i.e. in the present notation  $G_{\Delta}(\mathbb{Q})$  is compatible with necessity and possibility over many-valued events).*

#### 4.4 Logics for generalized probability

Now, we describe a logical treatment of probability measures. To keep the notation uniform, we denote by  $P$  the modal operator that interprets probability measures on fuzzy-events.

In what follows  $\mathcal{L}_1$  stands for either  $\mathbb{L}_k$ , or  $\mathbb{L}$ , and  $\mathcal{L}_2$  is any expansion of Łukasiewicz logic  $\mathbb{L}$ . The logic  $\text{FP}(\mathcal{L}_1, \mathcal{L}_2)$  is the schematic extension of  $\text{FPL}(\mathcal{L}_1, \mathcal{L}_2)$  obtained by the following axioms:

**P1**  $P(\neg\varphi) \leftrightarrow \neg P(\varphi)$ .

**P2**  $P(\varphi \oplus \psi) \leftrightarrow [(P(\varphi) \rightarrow P(\varphi \& \psi)) \rightarrow P(\psi)]$ .

The notion of proof in  $\text{FP}(\mathcal{L}_1, \mathcal{L}_2)$  will be denoted by  $\vdash_{\text{FP}}$ . Obviously the properties of normalization, and monotonicity we proved in Proposition 15, still hold for  $\text{FP}(\mathcal{L}_1, \mathcal{L}_2)$ . In addition  $\text{FP}(\mathcal{L}_1, \mathcal{L}_2)$  satisfies the following:

**PROPOSITION 24** *The modality  $P$  is finitely additive, that is, for every  $\tau, \gamma \in \mathcal{E}$  it holds:  $\tau \& \gamma \rightarrow \perp \vdash_{\text{FP}} P(\tau \oplus \gamma) \leftrightarrow (P(\tau) \oplus P(\gamma))$ .*

**Proof** Recall from Proposition 15, that  $P(\perp) \leftrightarrow \perp$  holds in  $\text{FP}(\mathcal{L}_1, \mathcal{L}_2)$ . Now, since  $\tau \& \gamma \rightarrow \perp$ , we have  $\tau \& \gamma \leftrightarrow \perp$ , and by the rule of substitution of the equivalents (Proposition 15(2)),  $P(\tau \& \gamma) \leftrightarrow \perp$ . Therefore by **P2**, we get  $\tau \& \gamma \rightarrow \perp \vdash_{\text{FP}} P(\tau \oplus \gamma) \leftrightarrow [(P(\tau) \rightarrow \perp) \rightarrow P(\psi)]$ , and so  $\tau \& \gamma \rightarrow \perp \vdash_{\text{FP}} P(\tau \oplus \gamma) \leftrightarrow (\neg P(\tau) \rightarrow P(\psi))$ .  $\square$

Models for  $\text{FP}(\mathcal{L}_1, \mathcal{L}_2)$  are special cases of plausibilistic models: a (weak) *probabilistic model* is a triple  $M = \langle W, e, s \rangle$ , where  $W$  and  $e$  are defined as in the case of plausibilistic models, and  $s: \text{Fm}(\mathcal{L})_W \rightarrow [0, 1]$  is a state. The evaluation of a formulas into a model  $M$  is defined as in the previous cases.

A probabilistic model is a *hyperreal probabilistic model*, whenever the measure  $s$  takes values from a non-trivial ultrapower  ${}^*[0, 1]$  of the unit interval  $[0, 1]$ .

In analogy to the case of representable necessity and possibility measures, also for the case of probability, we can introduce the notion of *strong probabilistic model*. Indeed, strong probabilistic models are a triples  $\langle W, e, p \rangle$  where  $W$  and  $e$  are as in the case of weak probabilistic models, and  $p: W \rightarrow [0, 1]$  is such that  $W_0 = \{w \in W \mid p(w) > 0\}$  is countable, and  $\sum_{w \in W_0} p(w) = 1$ . Evaluations of (modal) formulas are defined as usual, with the exception of atomic modal formulas that are defined as follows: for every  $P(\psi) \in \mathcal{MF}$ ,

$$\|P(\psi)\|_M = \sum_{w \in W} p(w) \cdot \|\psi\|_{M,w}.$$

The following is, again, a direct consequence of Theorem 20.

**THEOREM 25** *For every  $k \in \mathbb{N}$ , the logic  $\text{FP}(\mathbb{L}_k, \mathbb{L})$  is real-FSC with respect to both the class of probabilistic models and the class of strong probabilistic models. Moreover, the logic  $\text{FP}(\mathbb{L}, \mathbb{L})$  is hyperreal-SC.*

**Proof** Again, one starts by adapting Lemmas 18 and 19, by showing that, given a modal theory  $\Gamma$  and a modal formula  $\Phi$ ,  $\Gamma \vdash_{\text{FP}} \Phi$  iff  $\Gamma^\bullet \cup \text{FP}^\bullet \vdash_{\mathbb{L}} \Phi^\bullet$  iff  $\Gamma^\square \cup \text{FP}^\square \vdash_{\mathbb{L}} \Phi^\square$ , taking into account now that when building the theory  $\text{FP}^\bullet$  one has to additionally consider instances of the axiom **P1** and **P2**. Then, the only necessary modification with respect to the proof of Theorem 20 regards the fact that we have to ensure that the measure  $s: \text{Fm}_W(V_0) \rightarrow [0, 1]$  of  $M = \langle W, e, s \rangle$ , defined as  $s(f_\varphi) = v(P(\varphi)^\square) = v(p_{\varphi^\square})$ , is a state. Following similar proofs in [52, Th. 8.4.9] and [35, Th. 4.2], it is easy to check that

$$s(f_\varphi \oplus f_\psi) = s(f_\varphi) + s(f_\psi) - s(f_\varphi \& f_\psi).$$

Therefore  $s$  is a state from Proposition 11.

To conclude our proof consider a finite modal theory  $\Gamma \cup \{\Phi\}$  and assume  $\Gamma \not\vdash_{\text{FP}} \Phi$ . From what we proved above, there is a probabilistic model  $M = \langle W, e, s \rangle$  that is a model for  $\Gamma$ , and  $\|\Phi\|_M < 1$ . Adopting the same notation of Theorem 12, call  $Y$  the (finite) set of all the evaluations from  $V_0$  into  $S_k$ . Then the state  $s$  is defined on the MV-algebra  $\text{Fm}_Y$ , hence, from Theorem 12, there exists a probability distribution  $p$  on  $Y$  such that for every  $f_\varphi \in \text{Fm}_Y$ ,  $s(f_\varphi) = \sum_{w \in Y} p(w) \cdot w(\varphi)$ .

Now, we define  $M' = \langle W_k, e, \hat{p} \rangle$  where  $W_k$  is the set of all the evaluations of variables in  $V$  into  $S_k$ , for every  $w \in W_n$  and every variable  $q$ ,  $e(q, w) = w(q)$  and  $\hat{p}: W_n \rightarrow [0, 1]$  satisfies:

$$\hat{p}(w) = \begin{cases} p(w) & \text{if } w \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $M'$  is a strong probabilistic Łukasiewicz model, and it can be easily proved that for every modal formula  $\Psi$  of  $\text{F}\mathcal{P}\mathcal{L}(\mathbb{L}_k, \mathbb{L})$ ,  $\|\Psi\|_{M'} = \|\Psi\|_M$ . Therefore  $M'$  is a model of  $\Gamma$ , and  $\|\Phi\|_{M'} < 1$  as required.  $\square$

## 5 Expansions with rational truth constants

In this section, we rely on basic schematic extensions of  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$ . Notice that the class  $\mathcal{M}\mathcal{F}$  of modal formulas of  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  is taken as closed under the operators  $\delta_n$ , for every  $n \in \mathbb{N}$ , and therefore, for every modal formula  $\Phi$ ,  $\delta_n\Phi$  is modal as well. We stress this fact because we adopt now the same notation we introduced in Section 2.4, and therefore, for every rational number  $r = m/n \in [0, 1]$  with  $n, m$  being natural numbers, we write  $\bar{r}$  or even  $m/n$  instead of  $m\delta_n(\top)$ .

We are going to study here plausibility measures from the general point of view, and so we consider only the modality  $Pl$ . The other cases involving (representable) necessity and possibility, and probability measures are similar and hence omitted. A complete treatment for those classes of measures can be found in [36, 35].

The logic  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  is significantly more expressive than a logic  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathcal{L}_2)$  where  $\mathcal{L}_2$  does not allow to define rational values. In fact it is now possible to deal with formulas like, for instance,  $Pl(\varphi) \leftrightarrow \frac{1}{2}$  and  $Pl(\psi) \rightarrow \frac{1}{3}$  whose intended interpretation is that *the plausibility of  $\varphi$  is  $\frac{1}{2}$  and the plausibility of  $\psi$  is at most  $\frac{1}{3}$* , respectively.

Using Theorem 20, it is not difficult to prove that  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  is sound and (finitely) strongly complete with respect to the class of plausibilistic models. In fact  $\mathbb{R}\mathbb{L}$  has finite strong real completeness (see [43]). On the other hand, when we expand a logic by means of rational truth values, it is possible to define the notions of *provability degree* and *truth degree* of a formula  $\psi$  over an arbitrary theory  $\Gamma$ . For  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  they are defined as follows:

**DEFINITION 26** *Let  $\Gamma$  be an  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  modal theory and let  $\Phi$  be a modal formula. Then, the provability degree of  $\Phi$  over  $\Gamma$  is defined as*

$$|\Phi|_{\Gamma} = \sup\{r \in [0, 1] \cap \mathbb{Q} : \Gamma \vdash_{\text{F}\mathcal{P}\mathcal{L}} \bar{r} \rightarrow \Phi\},$$

*and the truth degree of  $\Phi$  over  $\Gamma$  is defined as*

$$\|\Phi\|_{\Gamma} = \inf\{\|\Phi\|_M \mid M \text{ is a plausibilistic model of } \Gamma\}.$$

*We say that  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  is Pavelka-style complete, or that  $\text{F}\mathcal{P}\mathcal{L}(\mathcal{L}_1, \mathbb{R}\mathbb{L})$  enjoys the Pavelka-style completeness theorem iff for every modal theory  $\Gamma \cup \{\Phi\}$ ,*

$$|\Phi|_{\Gamma} = \|\Phi\|_{\Gamma}.$$

Now we are going to show that  $\text{FP}\mathcal{L}(\mathcal{L}_1, \mathbf{RL})$  is Pavelka-style complete. Just as a remark notice that, with respect to this kind of completeness, we are allowed to relax the hypothesis about the cardinality of the modal theory we are working with. In fact  $\Gamma$  is assumed to be an arbitrary (countable) theory, not necessarily finite. This is due to the fact that  $\mathbf{RL}$  is indeed *strongly* Pavelka-style complete (cf. [43, Theorem 5.2.10]).

**THEOREM 27** *Let  $\Gamma$  be a modal theory of  $\text{FP}\mathcal{L}(\mathcal{L}, \mathbf{RL})$ , and let  $\phi$  be a modal formula of  $\text{FP}\mathcal{L}(\mathcal{L}, \mathbf{RL})$ . Then, the truth degree of  $\phi$  in  $\Gamma$  equals the provability degree of  $\phi$  in  $\Gamma$ :*

$$\|\phi\|_{\Gamma} = |\phi|_{\Gamma}.$$

**Proof** We are simply going to sketch the proof of Pavelka-style completeness for  $\text{FP}\mathcal{L}(\mathcal{L}, \mathbf{RL})$ . The argument used is, in fact, routine, and more details can be found in [52, Theorem 8.4.9] for the case of Boolean events, and probability measure (but the same argument easily holds for our more general case).

Let  $\Gamma \cup \{\Phi\}$  be an arbitrary modal theory of  $\text{FP}\mathcal{L}(\mathcal{L}, \mathbf{RL})$ . Adopting the same notation of the above section, from Lemma 18, and Lemma 19,  $\Gamma \vdash_{\text{FP}\mathcal{L}} \Phi$  iff  $\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square} \vdash_{\mathbf{RL}} \Phi^{\square}$ . Moreover, since the connectives of  $\mathbf{RL}$  are all continuous, it is easy to show that

$$(1) \quad |\Phi|_{\Gamma} = |\Phi^{\square}|_{\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square}}.$$

We know from [43, Theorem 5.2.10], that  $\mathbf{RL}$  is Pavelka-style complete, hence

$$(2) \quad |\Phi^{\square}|_{\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square}} = \|\Phi^{\square}\|_{\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square}}.$$

A routine verification (see for instance the proof of Theorem 20) shows that from the map  $\|\cdot\|_{\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square}}$  evaluating the truth degree of formulas of the form  $\varphi^{\square}$  into  $[0, 1]$ , one can easily define a plausibilistic model capturing the same truth values of  $\|\cdot\|_{\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square}}$ . Therefore

$$(3) \quad \|\Phi^{\square}\|_{\Gamma^{\square} \cup \text{FP}\mathcal{L}^{\square}} = \|\Phi\|_{\Gamma}.$$

Consequently, from (1), (2), and (3), we obtain  $|\Phi|_{\Gamma} = \|\Phi\|_{\Gamma}$ . □

## 6 On the coherence problem

Take a finite set of events  $\phi_1, \dots, \phi_k \in \mathcal{E}$ , and a map  $\mathbf{a}: \phi_i \mapsto \alpha_i \in [0, 1]$ .

*Can the map  $\mathbf{a}$  be extended to an uncertainly measure on the algebra generated by the formulas  $\phi_1, \dots, \phi_k$ ?*

This problem generalizes a well-known and deeply-studied classical one. In fact, if we ask the above question in terms of classical events, and probability measures, then the above problem is known in the literature as *de Finetti coherence problem* [17, 18, 19].

We are now going to introduce a way to treat and characterize the above coherence criterion to deal with many-valued (and in general non-Boolean) events, and measures different from the additive ones.<sup>17</sup>

<sup>17</sup>De Finetti's coherence criterion has been recently studied for states and MV-algebras in [65, 75].

**DEFINITION 28** Let  $\phi_1, \dots, \phi_k$  be formulas in the language of  $\mathcal{L}$  and let  $\mathcal{M}$  be a class of generalized plausibility measures. Then a map  $\mathbf{a}: \{\phi_1, \dots, \phi_k\} \rightarrow [0, 1]$  is said to be:

- (i) A rational assignment, provided that for every  $i = 1, \dots, k$ ,  $\mathbf{a}(\phi_i)$  is a rational number.
- (ii)  $\mathcal{M}$ -Coherent if there is an uncertainty measure  $\mu \in \mathcal{M}$  on the Lindenbaum-Tarski algebra  $Fm_V$  generated by the variables occurring in  $\phi_1, \dots, \phi_k$ , such that, for all  $i = 1, \dots, k$ ,  $\mathbf{a}(\phi_i) = \mu([\phi_i])$ .

Consider a finite set of  $\mathcal{L}$ -formulas  $\phi_1, \dots, \phi_k$ , and a rational assignment

$$\mathbf{a}: \phi \mapsto \frac{n_i}{m_i}, \text{ (for } i = 1, \dots, k),$$

where  $n_i$  and  $m_i$  are co-prime positive integers and such that  $n_i \leq m_i$ . Then, e.g. the following formulas are definable in the language of  $\text{FM}(\mathcal{L}, \mathbf{RL})$ :

$$(4) \quad \mathcal{M}(\phi_i) \leftrightarrow \overline{n_i/m_i}.$$

The following theorem characterizes  $\mathcal{M}$ -coherent rational assignments in terms of consistency of the formulas defined in (4). Since the proof of the following theorem is similar for every class  $\mathcal{M}$  of measures, we will concentrate on generalized plausibility measures, and we will omit the other cases (like necessity and probability).

**THEOREM 29** Let  $\phi_1, \dots, \phi_k$  be formulas in  $\mathcal{L}$ , and let

$$\mathbf{a}: \phi_i \mapsto \frac{n_i}{m_i}$$

be a rational assignment. Then the following are equivalent:

- (i)  $\mathbf{a}$  is  $\mathcal{PL}$ -coherent,
- (ii) the modal theory  $\Gamma = \{Pl(\phi_i) \leftrightarrow \overline{n_i/m_i} \mid i = 1, \dots, k\}$  is consistent in  $\text{FPL}(\mathcal{L}_1, \mathbf{RL})$  (i.e.  $\Gamma \not\vdash_{\text{FPL}} \perp$ ).

**Proof** (i)  $\Rightarrow$  (ii). Let  $\mathbf{a}$  be  $\mathcal{PL}$ -coherent, and let  $\rho: \mathcal{F}_{\mathcal{L}_1}(V_0) \rightarrow [0, 1]$  be a plausibility measure on the Lindenbaum-Tarski algebra of  $\mathcal{L}_1$  defined from the set of variables  $V_0$  occurring in  $\phi_1, \dots, \phi_k$ , extending  $\mathbf{a}$ . Then, let  $W$  be defined as in the proof of Theorem 20 and consider the model  $M = \langle W, e, \hat{\rho} \rangle$  where for every variable  $p$  and every  $w \in W$ ,  $e(p, w) = w(p)$ , and where  $\hat{\rho}: Fm_W(V) \rightarrow [0, 1]$  is the plausibility measure such that for all  $f_\phi \in Fm_W(V)$ ,  $\hat{\rho}(f_\phi) = \rho([\phi])$ . Then  $M$  is a plausibilistic model for  $\Gamma$ . In fact, for every  $i = 1, \dots, k$ ,

$$\begin{aligned} \|Pl(\phi_i) \leftrightarrow \overline{n_i/m_i}\|_M &= 1 && \text{iff} \\ \|Pl(\phi_i)\|_M \leftrightarrow \|\overline{n_i/m_i}\|_M &= 1 && \text{iff} \\ \|Pl(\phi_i)\|_M &= n_i/m_i && \text{iff} \\ \hat{\rho}(f_{\phi_i}) = \rho([\phi_i]) &= n_i/m_i. \end{aligned}$$

Therefore  $\Gamma$  has a model, and so  $\Gamma \not\vdash_{\text{FPL}} \perp$ .

(ii)  $\Rightarrow$  (i). Assume, conversely, that  $\Gamma \not\vdash_{\text{FPL}} \perp$ . Then, there exists a plausibilistic model  $M = \langle W, e, \rho \rangle$  such that  $\|\phi\|_M = 1$  for each  $\phi \in \Gamma$ . Consider the map  $\hat{\rho} : \text{Fm}_W(V) \rightarrow [0, 1]$  defined as follows: for every  $[\psi] \in \mathcal{F}_{\mathcal{L}_1}(V)$ ,

$$\hat{\rho}([\psi]) = \|Pl(\psi)\|_M = \rho(f_\psi).$$

Then  $\hat{\rho}$  is a generalized plausibility measure. In fact:

(i)  $\hat{\rho}([\top]) = \|Pl(\top)\|_M = \|\top\|_M = 1$ , and analogously  $\hat{\rho}([\perp]) = 0$ .

(ii) Assume that  $[\phi] \leq [\psi]$ . Then  $[\phi \rightarrow \psi] = [\top]$ , and hence, by the monotonicity rule one has  $\|Pl(\phi) \rightarrow Pl(\psi)\|_M = 1$  as well. But, this is equivalent to  $\|Pl(\phi)\|_M \leq \|Pl(\psi)\|_M$ , i.e.  $\hat{\rho}([\phi]) \leq \hat{\rho}([\psi])$ . Then  $\hat{\rho}$  is monotone.

Moreover, for every  $i = 1, \dots, k$ ,  $\hat{\rho}([\phi_i]) = \|Pl(\phi_i)\|_M = n_i/m_i$ . In fact, by definition of  $\Gamma$ ,  $Pl(\phi_i) \leftrightarrow n_i/m_i \in \Gamma$ , hence  $\|Pl(\phi_i) \leftrightarrow n_i/m_i\|_M = 1$ , i.e.  $\|Pl(\phi_i)\|_M = n_i/m_i$ . Consequently,  $\hat{\rho}$  is a plausibility measure on  $\mathcal{F}_{\mathcal{L}_1}(V)$  that extends  $\mathbf{a}$ . Therefore, the claim is proved.  $\square$

## 7 Conclusions and further readings

The monographs [57, 78] are standard references for a wide overview on classical uncertainty measures and reasoning under uncertainty. It is also worth mentioning the book [76] (consisting of two volumes) that offers a survey on measure theory with its many different branches, from the classical one to additive and non-additive measures on many-valued and quantum structures, along with many other related topics.

Normalized and additive maps on MV-algebras have been introduced by Kôpka and Chovanec in [61], and then by Mundici under the name of MV-algebraic states (or simply states) in [74]. More specifically, the notion of a state on MV-algebras is intimately connected with that of a state on an Abelian  $\ell$ -group that can be found in Goodearl [49]. We also refer to the paper [26] for a comprehensive survey on the topic of states on MV-algebras and applications.

States have been also studied in a different framework than that of MV-algebras. The literature about this general approach includes several papers. In particular, we mention the work by Aguzzoli, Gerla and Marra [2] where they studied states on Gödel algebras, the paper [1] by Aguzzoli and Gerla where states were studied in the more general setting of Nilpotent Minimum algebras (cf. [29]). Dvurečenskij and Rachůnek studied in [27] probabilistic-style measures in bounded commutative and residuated  $\ell$ -monoids. We also mention the work by Riečan on probability on BL-algebras, and IF-events [80, 81], and the paper by Mertanen and Turunen [72] dealing with states on semi-divisible residuated lattices.

Extensions of de Finetti's coherence criterion to deal with states on MV-algebras are studied in [79] for the case of events being (equivalence classes of) formulas of finitely valued Łukasiewicz logic. A first approach to the case of infinite valued Łukasiewicz logic was made by Gerla in [41], and subsequently characterized completely by Mundici [75]. In [65], Kühr and Mundici solved the problem of extending de Finetti's criterion to deal with formulas of any  $[0, 1]$ -valued algebraic logic having connectives whose interpretation is given by continuous functions.

The problem of checking the coherence (in the sense of de Finetti) of a partial probabilistic assignment was shown to be NP-complete in [78]. This result was applied in [56] by Hájek and Tulipani to show that the satisfiability problem for a modal probabilistic logic for classical events is still NP-complete. The computational complexity of de Finetti's criterion for Łukasiewicz finitely valued events was studied by Hájek in [53], and a final NP-completeness result for the coherence problem of infinitely-valued Łukasiewicz events was proved by Bova and Flaminio in [9].

To conclude, we recall some fundamental papers on the topic of generalized measure on fuzzy events. In [73], Montagna studied de Finetti coherence criterion for conditional events in the sense of conditional states introduced by Kroupa in [62]. In [32], Fedel, Kreimel, Montagna and Roth characterized a coherent rationality criterion for non-reversible games on (divisible) MV-algebras by means of upper and lower probabilities. A multimodal based logical approach to upper and lower probability on MV-algebras was introduced in [33]. In [64, 37], the authors have begun a study of belief functions on particular classes of semisimple MV-algebras.

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Tommaso Flaminio, Lluís Godo, and Enrico Marchioni  
IIIA, Artificial Intelligence Research Institute  
CSIC, Spanish Scientific Research Council  
Campus de la Universitat Autònoma de Barcelona s/n  
08193 Bellaterra, Spain  
Emails: {tommaso, godo, enrico}@iiia.csic.es