Probability, Fuzzy Logic and bets: foundational issues and some mathematics.

Logical Models of Reasoning with Vague Information

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§1. The concept of probability.

The concept of probability of an event is related both to vagueness and to many-valued logic. Indeed, first of all, it is hard to find a convincing mathematical definition of the probability of an event (hence, probability is a vague concept).

Second, even though there is a clear distinction between fuzzy logic and probability logic, it is possible to interpret probability logic into (modal) fuzzy logic: according to Hàjek, the probability of an event ϕ is the truth value of the sentence 'ϕ is probable'. But clearly, classical logic is not sufficient for such an interpretation, as such truth degrees are in general numbers between 0 and 1.

But we want to start from the definition of probability of an event ϕ. There are several different proposals:

(1) The frequentist approach: the probability of an event ϕ is the relative frequency of ϕ over a large number of experiments.

Example. Suppose that a roulette player observes that over $37 \cdot 10^6$ experiments the number 13 occurred 10^6 times. Then, a reasonable guess is that the probability of the number 13 is $\frac{1}{37}$.

Criticism: this approach makes sense only under the assumption that it is concretely possible to repeat the experiment a big number of times and that its probability never changes. Both these assumptions are not always verified.

A negative example. The Italian Prime Minister, Silvio Berlusconi, wants to build a bridge over the sea connecting the cities of Reggio Calabria and Messina. A concrete problem is: what is the probability that the bridge resists for, say, 200 years at least?

The frequentist answer would be: build a huge number of bridges between Reggio Calabria and Messina, wait for 200 years and compute the ratio between the number of bridges which resisted for 200 years and the total number of bridges. It is clear that this method does not work.

(2) The principle of indifference: Suppose that an experiment has N possible outcomes and that, for any two possible outcomes, there is no logical reason to believe that one of them is more likely than the other one.

Then, it is reasonable to assume that all the outcomes have the same probability.

Example: we toss a coin. Since we have no logical reason to believe that heads is more probable than tail or viceversa, it is safe to assume that tail and head have the same probability.

Criticism. We have to be careful with our interpretation of the sentence: there is no logical reason to believe that one outcome is more likely than another one. In the case of coin tossing, there is some symmetry between the outcomes, which suggests that the outcomes have the same probability.

But if we do not have any idea about the probability of the outcomes, then it makes no sense to assume that they have all the same probability: in this way, from no information at all, we infer the relevant information that all the outcomes have the same probability. This inference is completely unjustified.

Moreover, according to a rough interpretation of this principle, we do not distinguish between an event for which we have strong reasons to believe that it has probability $\frac{1}{2}$ and an event to which we assign probability $\frac{1}{2}$ just because we have no idea about its probability.

A negative example. Suppose we are discussing about the existence of some form of life in a remote galaxy. Consider the events E_1: there is life in at least one planet of that galaxy and E_2: there is life in at least two planets of that galaxy. Since we have no idea about the probability of E_1 and of E_2, a rough application of the principle of indifference would lead to the conclusion that both E_1 and E_2 have probability $\frac{1}{2}$. But this would imply that the event E_3: there is life in exactly one planet of that galaxy has probability 0, which is completely counterintuitive.

(3) The subjective approach by de Finetti. Bruno de Finetti, [DF], suggested a new approach to probability: expressed in a modern language, the probability of an event ϕ (whose outcome is unknown now and will be known later) is the amount of money α that a rational and reversible bookmaker (called Ada in the sequel) would propose for the following bet: a bettor (called Blaise in the sequel) bets a real number λ and pays $\lambda\alpha$ to Ada now. If ϕ will be true, then he will get back λ from Ada, and if ϕ will be false, then he will get nothing from her.

Remarks. (a) In this approach, α represents the amount of money that Blaise has to pay to get 1 when ϕ is true. In the modern betting games, the bookmaker declares the inverse $\frac{1}{\alpha}$ of α, meaning that, if the bettor pays λ and ϕ will be true, then he will get $\frac{\lambda}{\alpha}$. But these formulations are equivalent, (in both cases, if the bettor pays $\lambda\alpha$ he will get $\frac{\lambda\alpha}{\alpha} = \lambda$) although the second one looks more attractive for the bettor.

(b) According to de Finetti, the probability of an event ϕ is not absolute, but subjective. Different rational bookmaker may have different opinions about the betting odd α.

(c) Ada is a reversible bookmaker. That is, if Blaise believes that Ada's betting odd α is unfair, he may bet a negative amount λ of money (we agree that paying $\lambda < 0$ is the same as receiving $-\lambda$. Hence, betting a negative amount of money is the same as reversing the roles of bookmaker and bettor).

But: the symmetry between Ada and Blaise is not complete. The choice of the bets is up to Blaise, and the interchange of the roles is only possible after Blaise has made his choices.

It remains to explain what we mean by rational bookmaker. To this purpose, assume that Ada accepts bets on different events $\phi_1, ..., \phi_n$.

In this case, she chooses a finite set $\Gamma = (\phi_1, \alpha_1), ..., (\phi_n, \alpha_n)$, where for $i = 1, ..., n$, ϕ_i is an event and α_i is Ada's betting odd, that is, the amount of money that Ada chooses for a bet on ϕ_i of the form described above. Such a finite set Γ is called a book.

If for $i = 1, ..., n$, Blaise bets λ_i (where the λ_i may be negative) on ϕ_i, and the truth value of ϕ_i is $v(\phi_i)$, then Ada's payoff will be

$$\sum_{i=1}^{n} \lambda_i(\alpha_i - v(\phi_i)).$$

A rationality criterion suggested by De Finetti is the following:

A book $\Delta = (\phi_1, \alpha_1), ..., (\phi_n, \alpha_n)$ is said to be rational or coherent if there is no winning strategy or Dutch Book for Blaise, that is, there is no system of bets $\lambda_1, ..., \lambda_n$ on $\phi_1, ..., \phi_n$ such that Blaise's payoff

$$\sum_{i=1}^{n} \lambda_i(v(\phi_i) - \alpha_i)$$

is strictly positive independently of the truth values $v(\phi_1), ..., v(\phi_n)$.

De Finetti proved the following:

Theorem 0. A book $\Delta = (\phi_1, \alpha_1), ..., (\phi_n, \alpha_n)$ is coherent iff there is a probability distribution P such that $P(\phi_i) = \alpha_i$, $i = 1, ..., n$.

Hence, a book is rational if it agrees with the following laws of probability:

(1) The probability of an event is a number between 0 and 1.

(2) The probability of the certain event is 1.

(3) If two events are incompatible (i.e., they never occur simultaneously) the probability of their union is the sum of their probabilities.

Criticism. Although very convincing from a theoretical point of view, de Finetti's approach looks a little bit too liberal: for instance, consider the event L:

L: Next year, I will win one million dollars at the lottery.

Then, if Ada chooses the betting odd 1 for this event, Blaise has no winning strategy. (But I might have a sure win if I bet -1,000,000 dollars on this bet: either I win 1,000,000 dollars at the lottery and I loose nothing on this bet, or I win nothing at the lottery and I win 1,000,000 with this bet).

Hence, from de Finetti's pont of view it is coherent to assume that the event L has probability 1. I am happy with this, but... it's a dream!

De Finetti's point of view is that the beliefs of a rational agent should be respected even if they look strange: e.g., in the example of the lottery, the agent might know for sure that the lottery is not fair.

§2. **Variants of de Finetti's rationality criterion: many-valued events**. In the real life, we have to take decisions on the ground of non crisp events, like There will be traffic on the highway, or The market will be stable in the next days.

Does it make sense to speak of probability of a such many-valued (fuzzy) event?

In the frequentist approach, the relative frequency should be replaced by a sort of mean value of the truth values of the event. Such mean value is represented by a state.

Formally, a state on the algebra of the events is a map s from the set of all events into $[0, 1]$ such that:

(1) If two events ϕ and ψ are provably equivalent, then $s(\phi) = s(\psi)$.

(2) If ϕ is a Łukasiewicz tautology, the $s(\phi) = 1$.

(3) If ϕ and ψ are incompatible, that is, $\phi \to \neg\psi$ is provable, then $s(\phi \oplus \psi) = s(\phi) + s(\psi)$, where $\oplus$ is the Łukasiewicz multiplicative disjunction with truth table $v(\phi \oplus \psi) = \min\{1, v(\phi) + v(\psi)\}$.

States can be interpreted as mean values. The idea is the following.

An event, regarded as a formula ϕ of Łukasiewicz logic ϕ, may be identified with the function ϕ^* which associates to every valuation v into the MV-algebra on [0,1] the value $v(\phi) \in [0, 1]$.

Then, it can be proved that a state is the mean value $\int_V \phi^* d\mu$ of ϕ^* with respect to a Borel probability measure μ on the set V of all valuations.

As regards to de Finetti's approach, the simplest idea is to define the probability of a fuzzy event ϕ as the betting odd α of a rational and reversible bookmaker for the bet defined as in the classical case, with the only difference that $v(\phi)$ need not be 0 or 1, but is in general a number in $[0, 1]$.

Hence, if Blaise bets λ (possibly, $\lambda < 0$) on an event ϕ with betting odd α, his payoff will be $\lambda(v(\phi) - \alpha)$ as in the classical case. The same for books on a finite number of events.

The rationality criterion is as in the classical case: a book is rational or coherent if there is no strategy of bets for Blaise which leads him to a sure win, independently of the evaluations of the events. Daniele Mundici [Mu] proved the following:

Theorem 1. A book $(\phi_1, \alpha_1), ..., (\phi_n, \alpha_n)$ (where $\phi_1, ..., \phi_n$ are fuzzy events and $\alpha_1, ..., \alpha_n$ are the corresponding betting odds) is coherent in the above sense iff there is a state s on the algebra of formulas of Łukasiewicz logic such that for $i = 1, .., n$, $s(\phi_i) = \alpha_i$.

§**3. Another variant of de Finetti's criterion: conditional probability on fuzzy events**. It is hard to imagine a frequentist approach to conditional probability on fuzzy events. But de Finetti's criterion has a very easy and natural generalization.

We consider the classical case first. Let ϕ and ψ be crisp events.

The probability of ϕ given ψ is the betting odd α that a rational and reversible bookmaker would choose for the following bet:

The bettor chooses a number λ (possibly, $\lambda < 0$) and pays $\lambda\alpha$ to the bookmaker (hence, he receives $-\lambda\alpha$ if $\lambda < 0$).

If ψ and ϕ will be true, then the bettor receives λ (if $\lambda < 0$, then he pays $-\lambda$).

If ψ will be true and ϕ false, then the bettor receives nothing.

If ψ will be false, the bet is invalidated, and the bettor receives back the amount he payed before, that is, $\lambda\alpha$ (if $\lambda < 0$, he pays $-\lambda\alpha$).

The bettor's payoff is described by the simple formula

$$\lambda v(\psi)(v(\phi) - \alpha).$$

What happens if not only ϕ, but also the conditioning event ψ is fuzzy? There are several choices:

One might stipulate that the bet is not valid if ψ is not completely true.

Or else, one may stipulate that betting λ on ϕ given ψ with betting odd α is the same as betting λ on ϕ with the condition that only a part of the bet proportional to the truth value of ψ is valid. In this second case, the bettor's payoff is

$\lambda v(\psi)(v(\phi) - \alpha)$ as in the classical case.

Example. (The Messi example). Suppose we bet that the Barcelona team wins the final of the Champions League provided that Messi plays in that final. Suppose that Barcelona wins (as it happened in the final with the Manchester United) and that Messi plays all the game except for the last three minutes (as it happened in that final).

Then, the conditioning event Messi plays is not completely true. Should we invalidate the bet? If you think so, then perhaps you are competent about probability, but I don't think you are competent about soccer.

I think that the previous example has convinced at leat the Barcelona people that this is the right definition of conditional probability over fuzzy events. Hence, the rationality criterion should be the following:

Let $\Delta = (\phi_1|\psi_1, \alpha_1)..., (\phi_n|\psi_n, \alpha_n)$ be a book on the conditional events $\phi_1|\psi_1, ..., \phi_n|\psi_n$. We say that Δ is rational, or coherent iff there is no system of bets λ_i on $\phi_i|\psi_i$. $i = 1, ..., n$, such that that Blaise's payoff

$$\sum_{i=1}^{n} \lambda_i v(\psi_i)(v(\phi_i) - \alpha_i)$$

is strictly positive.

Is there any mathematical characterization of rationality in terms of states? The answer is YES.

Theorem 2. (FM). Let $\psi \cdot \phi$ be the formula whose truth value is the product of the truth values of ψ and of ϕ. Let $\Delta = (\phi_1|\psi_1, \alpha_1)..., (\phi_n|\psi_n, \alpha_n)$ be a book on the conditional events $\phi_1|\psi_1, ..., \phi_n|\psi_n$.

Then Δ is rational iff there is a state s on the algebra of formulas of Łukasiewicz logic with product such that $s(\phi_i \cdot \psi_i) = \alpha_i s(\psi_i)$, $i = 1, ..., n$. Thus, rationality corresponds to Kroupa's formula $P(\phi|\psi) \cdot P(\psi) = P(\phi \cdot \psi)$.

§**4. Yet another variant of de Finetti's criterion: imprecise probabilities over fuzzy events**. There are several reasons which suggest the use of imprecise probabilities and of non-reversible betting games (i.e., games in which negative bets are not allowed), cf [Hal], [Wa], [Wi], [AL]).

First of all, a real bookmaker might argue that the game is not completely balanced, as the choice of the bets is up to Blaise. Hence, she may want to protect herself from the risk of a big loss. If negative bets are forbidden and if the truth value of ϕ is very hard to predict, a safe strategy for Ada would be to choose a betting odd α for ϕ and a betting odd β for $\neg\phi$ where β is more than $1-\alpha$.

Second, if the bookmaker has no precise idea about the probability of an event ϕ, she might consult some experts. These experts may suggest different betting odds for ϕ, and a safe strategy is to choose the maximum of these betting odds and then to allow only positive bets.

Example. (Halpern example). A box contains 100 balls of the same weight and of the same shape. 30 of them are red, and 70 are either yellow or blue, but even the bookmaker does not know how many balls are blue and how many balls are yellow.

A ball will be chosen at random, and the bookmaker accepts bets on the events: R: the chosen ball will be red, B: the chosen ball will be blue and Y: the chosen ball will be yellow.

Then, a safe strategy for the bookmaker is to choose a betting odd $\frac{30}{100}$ for R, $\frac{70}{100}$ for B and $\frac{70}{100}$ for Y.

But in this case, the bookmaker should not accept negative bets, otherwise the bettor would clearly have a winning strategy.

Hence, we will consider betting games in which only positive bets are allowed. These games will be called non-reversible betting games in the sequel.

Note that if the book, besides a betting odd α for ϕ contains a betting odd $1-\alpha$ for $\neg\phi$, then the game becomes equivalent to a reversible game: betting $\lambda < 0$ on ϕ in a reversible betting game is equivalent (in the sense that gives to Blaise the same payoff) as betting $-\lambda$ on $\neg\phi$.

We are looking for a rationality criterion for non-reversible games. To this purpose, we will examine first some rationality criteria that are equivalent in the case of reversible games, and we will discuss their extensions to non reversible games.

Let Δ be a book. A winning strategy (a loosing strategy respectively) for Blaise (based on Δ) consists of a system of bets such that for every valuation v the corresponding payoff of Blaise is strictly positive (strictly negative respectively).
A bad bet is a bet for which there is an alternative strategy which ensures to Blaise a strictly better payoff independently of the truth values of the formulas involved.

The following result is straightforward:

Theorem 3. Let Δ be a book in a reversible game. The following are equivalent:
(1) There is no winning strategy for Blaise.
(2) There is no loosing strategy for Blaise.
(3) There is no bad bet for Blaise.

Proof. (1) $\Leftrightarrow$ (2). Betting $\lambda_1, ..., \lambda_n$ on $\phi_1, ..., \phi_n$ respectively is a winning strategy iff betting $-\lambda_1, ..., -\lambda_n$ on the same events is a loosing strategy.

(2) $\Leftrightarrow$ (3). If betting $\lambda_1, ..., \lambda_n$ on $\phi_1, ..., \phi_n$ respectively is a loosing strategy for Blaise, then betting λ_1 on ϕ_1 is a bad bet: a better strategy is betting $-\lambda_i$ on ϕ_i for $i = 2, ..., n$.

Conversely, if betting λ on ϕ is a bad bet and betting λ_i on ϕ_i, $i = 1, ..., n$, is a strategy which ensures a strictly better payoff, then betting λ on ϕ and $-\lambda_i$ on ϕ_i, $i = 1, ..., n$ is a loosing strategy.

It follows that in a reversible game rationality prevents not only the bookmaker Ada, but also the bettor Blaise from a sure loss. Moreover, if a book is rational, then one can never find a strategy for Blaise which is strictly better than another one, independently of their valuation.

The situation changes in the case of non reversible games.

(1) In the Halpern example, there is no winning strategy for the bettor and there is no bad bet, but there is a loosing strategy, namely, betting 1 on both B and Y: the bettor pays $\lambda \frac{140}{100}$ and in the best case he gets back λ.

(2) Consider the book $\Gamma = (\phi, \frac{1}{3}), (\psi, \frac{1}{3}), (\phi \vee \psi, 1)$, where ϕ and ψ are propositional variables, and hence they can assume any truth value in $[0, 1]$. Then, there is no winning strategy for the bettor, but there is a bad bet, namely, betting 1 on $\phi \vee \psi$: a better strategy would be to bet 1 on both ϕ and ψ.

In our opinion, this second book is not rational: in a non-reversible betting game, the bookmaker should not only choose a safe book, (this can be always obtained if all betting odds are equal to 1), but also an attractive book, otherwise no bettor will bet on it. In example (2) Ada might choose her betting odd for $\phi \vee \psi$ to be $\frac{2}{3}$ instead of 1. In this way, she might make her book more attractive for Blaise without loosing money when he plays his best strategy.

This argument shows that only the non existence of a bad bet may be considered a good rationality criterion in a non reversible game.

Problem: is there an axiomatization of upper probabilities which models this rationality criterion?

The answer is YES, provided that we work in divisible Łukasiewicz logic, i.e., in Łukasiewicz logic with division operators $\frac{\phi}{n}$.

For every valuation v and natural number n, $v(\frac{\phi}{n})$ is defined to be $\frac{v(\phi)}{n}$.

If $m \leq n$, then $\frac{\phi}{n} \oplus ... \oplus \frac{\phi}{n}$ (m times) is abbreviated by $\frac{m}{n}\phi$.

Hence, we have a scalar multiplication by any rational number in $[0, 1]$. When ϕ is a tautology of divisible Łukasiewicz logic, we write just q for $q\phi$.

In this language, the axioms of an upper probability U are:

(U0) $0 \leq U(\phi) \leq 1$, and $U(\phi) = 1$ for every Łukasiewicz tautology ϕ..

(U1) if $\phi \rightarrow \psi$ is provable in divisible Łukasiewicz logic, then $U(\phi) \leq U(\psi)$.

(U2) $U(\phi \oplus \psi) \leq U(\phi) + U(\psi)$.

(U3) If q is a rational in $[0, 1]$, then $U(q\phi) = qU(\phi)$.

(U4) If $q \rightarrow \neg\phi$ is provable in divisible Łukasiewicz logic, (q a rational in $[0, 1]$), then $U(\phi \oplus q) = U(\phi) + q$.

It turns out that (U0),...,(U4) give a complete description of the rationality criterion consisting of the non existence of a bad bet, in a sense that will be explained in the next part.

§**6. Some mathematics**. This part is mainly due to Klaus Keimel and Walter Roth, who generalized a previous result due to Martina Fedel and to myself. All the material will appear in a joint paper by the above authors. This part is only for experts of MV-algebras and of functional analysis, hence not for me.

An infinitesimal of an MV-algebra is an element ε such that for every natural number n, $n\varepsilon \leq \neg\varepsilon$. Since infinitesimals have probability 0, we can safely work in MV-algebras without non-zero infinitesimals. Such MV-algebras are called semisimple . For example, the algebra of divisible Łukasiewicz logic modulo provable equivalence is semisimple.

The elements of a semisimple MV-algebra $\mathbf{A}$ can be regarded as continuous functions from a compact Hausdorff space X into $[0, 1]$, with operations defined pointwise. Let $C(X)$ denote the MV-algebra of all continuous maps from X into $[0, 1]$. We may consider it as a subspace of $[0, 1]^X$ with the product topology. Then, $C(X)$ is compact and $\mathbf{A}$ becomes a uniformly dense sublagebra of $C(X)$.

Now consider the set $P(\mathbf{A})$ of all states on $\mathbf{A}$ with the weak* topology, i.e., the least topology for which the evaluation maps Φ_f from $P(\mathbf{A})$ into $[0, 1]$ defined, for every state s, by $\Phi_f(s) = s(f)$, are continuous.

Then we can state the main result of our joint paper.

Theorem 4. Let $(\phi_1, \alpha_1), ..., (\phi_n, \alpha_n)$ be a book in a non reversible game. The following are equivalent:

(1) There is no bad bet based on this book.

(2) There is a function U from the algebra of events of divisible Łukasiewicz logic into $[0, 1]$ satisfying (U0),...,(U4) such that $U(\phi_i) = \alpha_i$ for $i = 1, ..., n$.

(3) There is a closed (wrt the weak* topology) and convex subset K of $P(\mathbf{A})$ such that, for $i = 1, ..., n$, $\alpha_i = \max\{s(\phi_i) : s \in K\}$ (this last condition says that $U(\phi_i)$ is the maximum of all probabilities $s(\phi_i)$ when s ranges over K and then it models the situation with many experts which suggest different betting odds).

Remarks. (1) If we admit only negative bets, then the rationality criterion that there should not be a bad bet for Blaise corresponds to the fact that the book may be extended to a lower probability (infimum of a set of states).

(2) Tomás Kroupa has investigated a special case of upper and lower probabilities, namely, the belief functions see [Kr2].

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