# Fuzzy Logic and Higher-Order Vagueness

Nicholas J.J. Smith

Department of Philosophy, The University of Sydney
www.personal.usyd.edu.au/~njjsmith/

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# The Problem: Higher-Order Vagueness/Artificial Precision

> [Fuzzy logic] imposes artificial precision. . . [T]hough one is not obliged to require that a predicate either definitely applies or definitely does not apply, one is obliged to require that a predicate definitely applies to such-and-such, rather than to such-and-such other, degree (e.g. that a man 5 ft 10 in tall belongs to tall to degree 0.6 rather than 0.5) (Haack 1979)

One immediate objection which presents itself to [the fuzzy] line of approach is the extremely artificial nature of the attaching of precise numerical values to sentences like ‘73 is a large number’ or ‘Picasso’s Guernica is beautiful’. In fact, it seems plausible to say that the nature of vague predicates precludes attaching precise numerical values just as much as it precludes attaching precise classical truth values. (Urquhart 1986)

[T]he degree theorist's assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. In so far as a degree theory avoids determinacy over whether *a* is *F*, the objection here is that it does so by enforcing determinacy over the degree to which *a* is *F*. All predications of "is red" will receive a unique, exact value, but it seems inappropriate to associate our vague predicate "red" with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321? (Keefe 1998)

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Also: Copeland 1997, Goguen 1968–9, Lakoff 1973, Machina 1976, Rolf 1984, Schwartz 1990, Tye 1995, Williamson 1994...

(NB includes both proponents and opponents of degrees of truth)

# The problem in a nutshell

It is artificial/implausible/inappropriate to associate each vague predicate in natural language with a function which assigns one particular fuzzy truth value (real number between 0 and 1) with each object (the object's degree of possession of that property).

It is artificial/implausible/inappropriate to associate each sentence in natural language which predicates a vague property of an object with one particular fuzzy truth value (the sentence's degree of truth).

# Proposed Solutions

1. Fuzzy epistemicism
2. Fuzzy metalanguage
3. Blurry sets
4. Fuzzy plurivaluationism

# Fuzzy epistemicism

Statements such as 'Bob is tall' do indeed have unique fuzzy truth values (e.g. 0.4).

However in general we cannot know what these values are.

That is why it seems (falsely) to us as though these statements do not have unique fuzzy truth values.

Cf. Machina 1976, Copeland 1997, Keefe 1998...

# Fuzzy metalanguage

If a vague language requires a continuum-valued semantics, that should apply in particular to a vague meta-language. The vague meta-language will in turn have a vague meta-meta-language, with a continuum-valued semantics, and so on all the way up the hierarchy of meta-languages. (Williamson 1994)

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Cf. also Cook 2002, Edgington 1997, Field 1974, Horgan 1994, Keefe 2000, McGee and McLaughlin 1995, Rolf 1984, Sainsbury 1990, Tye 1990, 1994, 1995, 1996, Varzi 2001, Williamson 1994, 2003...

The idea is this:

1. Present a semantics for vague language which assigns vague sentences real numbers as truth values,
2. then say that the metalanguage in which these assignments were made is itself subject to a semantics of the same sort.

On this view, statements of the form The degree of truth of 'Bob is tall' is 0.4 need not be simply True or False: they may themselves have intermediate degrees of truth.

So rather than exactly one sentence of the form The degree of truth of 'Bob is tall' is $x$ being True and the others False, many of them might be true to various degrees.

Thus there is a sense in which sentences in natural language which predicate vague properties of objects are <u>not</u> each assigned just one particular fuzzy truth value.

# Blurry Sets

Smith 2004 'Vagueness and Blurry Sets' (JPL 33, pp.165–235).

The truth values of this system are *DF*'s (degree functions).

Each *DF* is a function $f : [0, 1]^* \rightarrow [0, 1]$

$[0, 1]^*$ is the set of words on the alphabet [0,1]

(i.e. the set of all finite sequences of elements of [0,1], including the empty sequence $\langle\rangle$).

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If $f(\langle 0.3, 0.4 \rangle) = 0.5$, then it is 0.5 true that it is 0.4 true that Bob is tall to degree 0.3. The assignments to all sequences of length 2 together constitute a third level of approximation to Bob's degree of tallness/the degree of truth of (B).

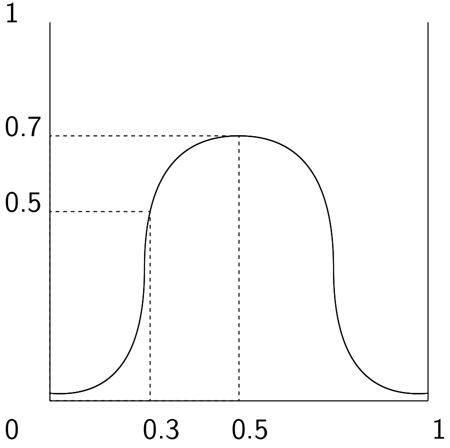
$\vdots$

In addition:

The assignments made by $f$ to sequences of length 1 determine a function $f_{\langle\rangle} : [0,1] \rightarrow [0,1]$.

This can be seen as encoding a density function.

We require that its centre of mass is $f(\langle\rangle)$.




Bob's degree of tallness: second approximation

Likewise:

The assignments made by $f$ to sequences $\langle a, x \rangle$ of length 2 whose first member is $a$ determine a function $f_{\langle a \rangle} : [0, 1] \rightarrow [0, 1]$.

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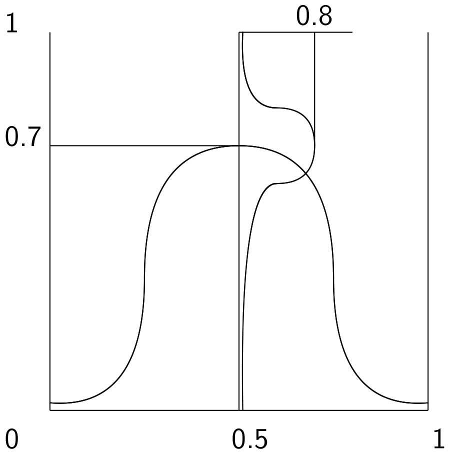
We require that its centre of mass is $f(\langle a \rangle)$.




Bob's degree of tallness: third approximation (part view)

And so on. . .

Image: We can picture a degree as a region of varying shades of grey spread between 0 and 1 on the real line.

If you focus on any point in this region, you see that what appeared to be a point of a particular shade of grey is in fact just the centre of a further such grey region.

The same thing happens if you focus on a point in this further region, and so on.

The region is blurry all the way down: no matter how much you increase the magnification, it will not come into sharp focus.

On this view, as on the fuzzy metalanguage view, statements of the form The degree of truth of 'Bob is tall' is 0.4 need not be simply True or False: they may themselves have intermediate degrees of truth.

So rather than exactly one sentence of the form The degree of truth of 'Bob is tall' is $x$ being True and the others False, many of them might be true to various degrees.

Thus there is a sense in which sentences in natural language which predicate vague properties of objects are not each assigned just one particular fuzzy truth value.

NB On both views, we have a hierarchy of statements, none of which tells us the full and final story of the degree of truth of 'Bob is tall'.

However there is a crucial difference between the two views:

Fuzzy metalanguage: A hierarchy of assignments of simple truth values.

Blurry sets: A single assignment of a complex truth value which has an internal hierarchical structure. Each vague sentence is assigned a unique degree function as its truth value. These assignments can be described in a classical metalanguage.

# Classical plurivaluationism

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> ~~supervaluationism~~ plurivaluationism tells us two things. The first is that the semantics of our language is not fully determinate, and that statements in this language are open to a variety of interpretations each of which is compatible with our ordinary linguistic practices. The second thing is that when the multiplicity of interpretations turns out to be irrelevant, we should ignore it. If what we say is true under all the admissible interpretations of our words, then there is no need to bother being more precise. (Varzi 2003)

Classical plurivaluationism trades only in classical models.

But instead of supposing that each discourse has one intended model, it allows that a discourse may have multiple acceptable models.

## Orthodox classical (monovaluationist) picture

When I utter 'Bob is tall', I say something, I make a claim:

that this guy (the referent of 'Bob' in the intended model) has that property (the extension of 'is tall' in the intended model).

What I say is true on some models and false on others.

It is true (simpliciter) if it is true on the intended model.

## Classical plurivaluationist picture

When I utter 'Bob is tall', I say many things at once: one claim for each acceptable model.

I mean each of these things equally: there is not a unique meaning.

Semantic indeterminacy — or equally, semantic plurality.

But if all the claims I make are true/false, we can pretend (talk as if) I made only one claim, which is true/false.

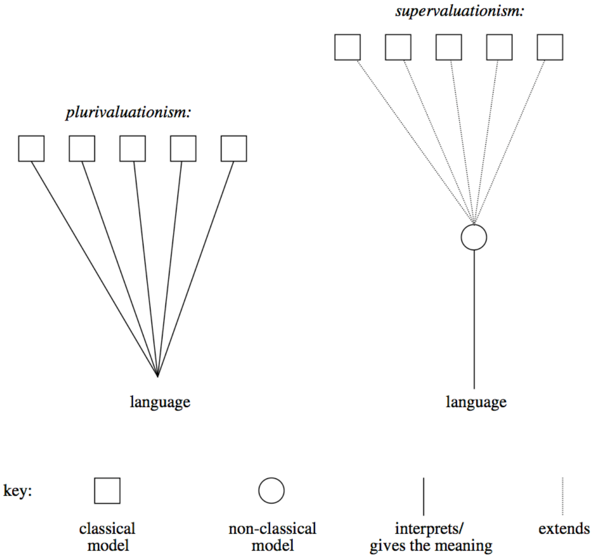
Image: If all the shotgun pellets go through the bullseye, we can talk as if there was just one bullet, which went through.

## NB This is not supervaluationism (properly so-called)

Supervaluationism involves one intended non-classical model, and its classical extensions.

A proposition is true/false in the intended non-classical model iff it is true/false in every classical extension thereof. The function which assigns truth values to sentences in the non-classical model on this basis is the supervaluation.

Here the classical models are not equally-good interpretations of the discourse. There is only one interpretation: the non-classical model. Its extensions are simply used to calculate truth values of sentences in this interpretation.


supervaluationism:
plurivaluationism:
language
language
key:
classical model
non-classical model
interprets/ gives the meaning
extends

# Fuzzy plurivaluationism

Smith 2008 Vagueness and Degrees of Truth (OUP).

Just like classical plurivaluationism except that the models are fuzzy, not classical.

Upshot: there is not one uniquely correct assignment of truth value to 'Bob is tall'. There are multiple, equally-correct assignments: one in each acceptable model.

# Choosing a Solution

Four solutions on the table: which is the right one?

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We can rule out the fuzzy metalanguage view on methodological grounds.

> Our models are typical purely exact constructions, and we use ordinary exact logic and set theory freely in their development. . . It is hard to see how we can study our subject at all rigorously without such assumptions. (Goguen 1968–9)

We understand fuzzy model theory as standard mathematics, presented in the usual precise language of mathematics.

If you say at the end of presenting fuzzy model theory that the language in which you made your presentation was governed by the very semantics you just presented, then we do not really understand your presentation after all.

Formal semantic treatments of vague languages—many-valued logics, supervaluations and the like—are characteristically framed in a meta-language that is conceived as precise. Thus one cannot say in the precise meta-language what utterances in the vague object-language say, for to do so one must speak vaguely; one can only make precise remarks about those vague utterances. Since the expressive limitations of such a meta-language render it incapable of giving the meanings of object-language utterances, it can hardly be regarded as adequate for a genuine semantic treatment of the object-language. . . . the formality of the semantics [comes] at the cost of giving up the central task of genuine semantics: saying what utterances of the object language mean. (Williamson 1994)

What does it mean to give the semantics/meaning of ‘Bob is tall’?

1. Make a (different) claim which has the same content as ‘Bob is tall’
2. Give an account of the semantic relations between parts of the sentence ‘Bob is tall’ and parts of the world, and of how these combine to determine the truth status of the whole sentence.

Contra Williamson, 2 is the task of genuine semantics. (1 is the task of translation.)

And 2 requires a precise metalanguage.

That still leaves three solutions on the table:

1. Fuzzy epistemicism
2. Blurry sets
3. Fuzzy plurivaluationism

How do we decide which is the right one?

We need a clearer idea of the true nature and source of the problem.

Haack: no diagnosis.

Urquhart: the nature of vague predicates precludes attaching precise numerical values.

Keefe: what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321?

# Surface characterizations vs fundamental definitions

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Fundamental definition of $P$: the fundamental underlying nature or essence of the $P$'s, which <u>explains</u> why they have such-and-such surface characteristics.

Example: water.

Surface characterization: clear, tasteless, potable liquid which falls as rain.

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Fundamental definition: $H_2O$

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- its extension has a blurred boundary
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Before we can say whether something — e.g. assignment of a unique fuzzy truth value to each vague sentence — conflicts with the nature of vagueness, we need a fundamental definition of vagueness.

# Candidate Definitions

1) Borderline cases

Not fundamental. E.g. ‘is schort’ has borderline cases, but is not vague:

1. If $x$ is less than four feet in height, then ‘$x$ is schort’ is true.
2. If $x$ is more than six feet in height, then ‘$x$ is schort’ is false. (The end)

2) Blurred boundaries

Too vague for a fundamental definition.

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Too vague for a fundamental definition.

3) Sorites susceptibility

Not fundamental. Missing explanation: vague predicates generate Sorites paradoxes <u>because</u> they are vague.

4) Semantic indeterminacy (of the sort involved in plurivaluationism)

Not fundamental. E.g. ‘gavagai’ (Quine) and ‘mass’ (Field): even assuming they exhibit semantic indeterminacy (rabbits vs undetached rabbit parts; rest mass vs relativistic mass), they are not vague.

5) Tolerance. A predicate $F$ is tolerant with respect to $\phi$ if there is some positive degree of change in respect of $\phi$ that things may undergo, which is "insufficient ever to affect the justice with which $F$ is applied to a particular case" (Wright 1975).

Given a sorites series for $F$, $F$ cannot be tolerant, on pain of contradiction.

So if tolerance is the essence of vagueness, we must either accept true contradictions, or else deny that there are any vague predicates.

# Vagueness as Closeness

$F$ is vague iff it satisfies the following condition, for any objects $a$ and $b$:

Closeness If $a$ and $b$ are very close (similar) in $F$-relevant respects, then '$Fa$' and '$Fb$' are very close in respect of truth.

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Think of this as a weakening of:

**Tolerance** If $a$ and $b$ are very close in $F$-relevant respects, then '$Fa$' and '$Fb$' are identical in respect of truth.

Related to, but not the same as, the idea that vague predicates are those whose extensions (thought of as functions from objects to truth values) are continuous.

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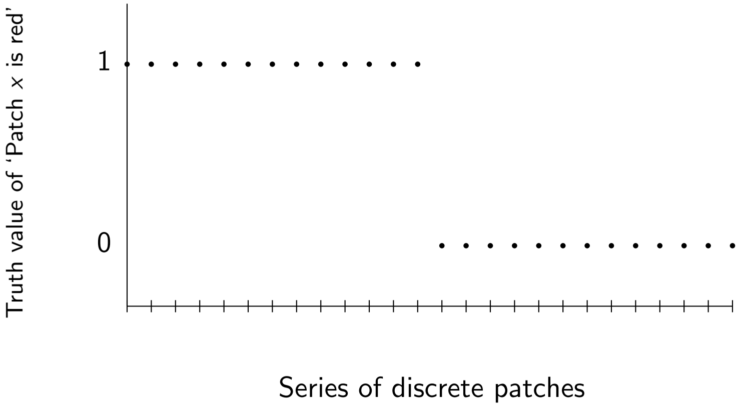
The informal statement of continuity employs such a notion: a small change in the input produces at most a small change in the output.

But this drops out of the formal definition: for any positive sized target area in the codomain (whether or not we would ordinarily regard it as 'small') we can find a positive sized launch area in the domain (which again need not be 'small' in the ordinary absolute sense) such that everything sent by the function from that launch area lands in that target area.

# Redd: continuous but not vague



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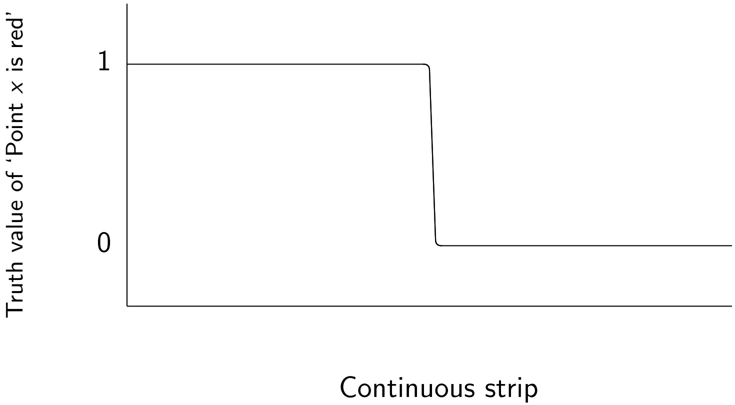

Advantages of this definition:

1. Accommodates tolerance intuitions, without contradiction.
2. Yields an explanation of why vague predicates have the three characteristic surface features.
3. Accommodates intuitions about higher-order vagueness, in the definition of vagueness itself.

For details see Smith 2005 'Vagueness as Closeness' (AJP 83:157–83) and Smith 2008.

# The Problem of the Intended Interpretation

1. Facts of type $T$ do not determine a unique intended interpretation of discourse $D$.
2. No facts of any type <u>other</u> than $T$ are relevant to determining the intended interpretation of $D$.
3. From 1 and 2: All the facts together do not determine a unique intended interpretation of $D$.
4. It cannot be a <u>primitive</u> fact—i.e. a fact not determined by other facts—that some interpretation $\mathfrak{M}$ is the unique intended interpretation of $D$.
5. From 3 and 4: It is not a fact at all that $D$ has a unique intended interpretation.

# Examples

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  Type $T$ includes all and only the publicly accessible facts concerning what people say in what circumstances.

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  Type $T$ includes all and only the publicly accessible facts concerning what people say in what circumstances.
- Kripkenstein's sceptical puzzle:

  Type $T$ also includes dispositional facts, and private mental facts.

# Vagueness

Type $T$ includes:

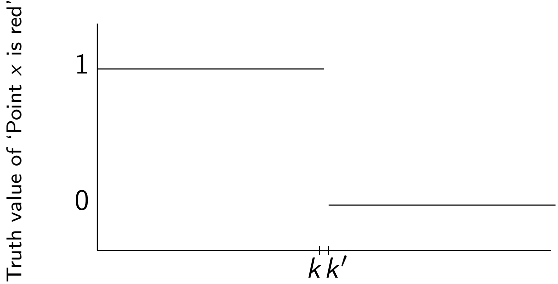
- All the facts as to what speakers of $D$ actually say and write, including the circumstances in which these things are said and written, and any causal relations obtaining between speakers and their environment.
- All the facts as to what speakers of $D$ are disposed to say and write in all kinds of possible circumstances.
- All the facts concerning the eligibility as referents of objects and sets.
- All the facts concerning the simplicity or complexity of the candidate interpretations.

Any theory of vagueness must cohere with this picture of how meaning is determined.

If the theory says that a vague predicate has a meaning of such-and-such a kind (e.g. a function from objects to classical truth values, or a function from objects to fuzzy truth values), we should be able to satisfy ourselves that the type $T$ facts could indeed determine such a meaning for actual vague predicates.

# The classical view

Does it conflict with the nature of vagueness? Yes!




$k$ and $k'$ are very close in respects relevant to the application of 'is red', but '$k$ is red' and '$k'$ is red' are not close in respect of truth.

Does it encounter the problem of the intended interpretation? Yes!

It seems that the type $T$ facts do not suffice to pick out a particular height dividing the tall from the non-tall, etc.

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It seems that the type $T$ facts do not suffice to pick out a particular height dividing the tall from the non-tall, etc.

Williamson: the classical view is not (logically) incompatible with the view that use determines meaning.

True: but we have no idea how use could determine a unique meaning.

So the classical view faces two problems:

1. The existence of a sharp drop-off from True to False in a sorites series: conflicts with the nature of vagueness.

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1. The existence of a sharp drop-off from True to False in a sorites series: conflicts with the nature of vagueness.

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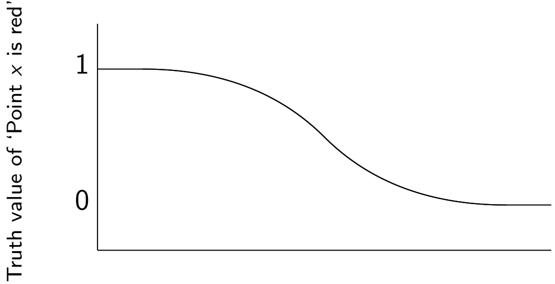
2. The location of the drop-off: conflicts with our best views about how meaning is determined.

   I call this the location problem.

# The basic fuzzy view

Does it conflict with the nature of vagueness? No!




No jolt problem.

Does it encounter the problem of the intended interpretation? Yes!

It seems that the type $T$ facts do not suffice to pick out a particular function from objects to fuzzy truth values representing the extension of ‘is tall’, etc.

So there is a location problem.

This is the higher-order vagueness/artificial precision problem.

# Return to the proposed solutions

Keeping in mind:

- The basic fuzzy view already solves the jolt problem. Assigning a unique degree of truth to each vague sentence does not conflict with the nature of vagueness.

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Keeping in mind:

- The basic fuzzy view already solves the jolt problem. Assigning a unique degree of truth to each vague sentence does not conflict with the nature of vagueness.

But:

- The basic fuzzy view does face the location problem. Assigning a unique degree of truth to each vague sentence does conflict with our best views about the determination of meaning.

# Fuzzy epistemicism

Does not solve the location problem.

The problem is how there could be a unique function which is the extension of 'is tall', given that our usage (etc.) does not suffice to pick out a unique such function.

Saying that we do not know which function it is just misses the point of the problem.

# Blurry sets

Does not solve the location problem.

The type $T$ facts do not suffice to pick out a unique fuzzy set (function from objects to fuzzy truth values) as the extension of 'is tall'.

Likewise, the type $T$ facts do not suffice to pick out a unique blurry set (function from objects to degree functions) as the extension of 'is tall'.

# Fuzzy plurivaluationism

Solves the location problem.

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The problem (artificial precision): The type $T$ facts do not suffice to pick out a unique fuzzy set as the extension of ‘is tall’, etc — in general, a unique intended interpretation of vague discourse.

The solution (fuzzy plurivaluationism):

An acceptable interpretation is one which is not ruled out as incorrect by the type $T$ facts.

In light of the problem, there is not a unique acceptable interpretation of vague discourse — i.e. one intended interpretation.

A fortiori, fuzzy plurivaluationism is correct: for it is precisely the view that there is no unique intended interpretation of vague discourse.

Instead, there are many equally correct interpretations — the acceptable ones.

Thus 'Bob is tall' does not have a uniquely correct degree of truth. It is assigned many different degrees of truth—one on each acceptable interpretation—and none of these is more correct than any of the others.

This is the desired result. That this was not the case on the original fuzzy view was precisely the problem with which we started.