On vagueness and granularity

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The endless debate on vagueness among philosophers should somehow be addressed.

To begin with: A personal issue.

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Why is it actually easier to read KANT
than contemporary philosophers (of vagueness)?

The personal issue, contd.

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- not to "solve" the paradox –
  - I just note that
    A change of an object observed when standing next to it might not be observable when standing far from it
  - I choose a model of the situation,
    e.g. using Łukasiewicz logic (HÁJEK, NOVÁK)
  - and I am satisfied.

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  - I choose a model of the situation,
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  - and I am satisfied.
- but to see why philosophers do not accept this "solution".

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For this occasion, let me complain about philosophers as well as my own community.

It seems that

- philosophers rely a lot on paradigms not expected in modern philosophy;
- mathematicians rely a lot on symbolism without asking what it is about.

Implicit assumptions: the philosophical side

Shapiro in his monograph *Vagueness in Context* asks about the "source" of vagueness:

Is it a purely linguistic matter,
concerned with how we represent the world via language,
or is there a sense in which the world itself is vague?

Implicit assumptions: the philosophical side

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or is there a sense in which the world itself is vague?

I dare to conclude:

- An observer-independent reality is assumed:
  things are there, if we observe them or not.
- We look at the "world" and have to find the correct words
  to express what's going on.

A sharply contrasting view

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- For short: We deal with well-defined sequences of **observations** and their associated probabilities.

- The assumption of an observer-independent world and of a "single history" is not supported.

Implicit assumptions: the mathematical side

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In mathematics, what comes first: the structures or the axioms?

- In contemporary mathematics, symbolism is overemphasised.

  Symbolic logic is the method, not the content.
- The origin of the structures we reason about is not really considered as being of interest.

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A progress would require flexible views
both in mathematics and in philosophy.

In particular:

A better explanation of the role of mathematics is needed
to clarify possible roles of formal methods for accounts of vagueness.

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- Mathematical structures are modelled upon forms of perception.

Structures and perception

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We will called it the "fine model" for the concept.

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- In the infinite case:
  A structure does not reflect perceptions,
  but a way how perceptions are imaginable.

  It is closed under a specific way to imagine objects.

  Statements about the structure cannot be
  mapped back one-to-one to perceptions.

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To take into account vagueness means to take into account actual perceptions.

To this end, the refinement process in the construction needs to be described.

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Ad (ii). Also usual mathematics, but with a “coarser” relation.

Ad (iii): To be fixed as our problem.

Vagueness and granularity

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A possibility:

- to use (as usual) the fine model;
- to define a parametrised set of relations: finer and coarser ones.

  We endow the fine model with a
  metric or a **similarity relation**,
  corresponding to the different levels of granularity.

A formal approach: the idea

(We restrict to the propositional level.)

We consider graded implications

$$\alpha \xrightarrow{t} \beta$$

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$$\alpha \xrightarrow{t} \beta$$

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and

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Similarity-based reasoning
(RUSPINI)

Let S be a set of distinguished situations.

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Let σ: $S \times S \to [0,1]$ be a similarity relation.

Let $A, B \subseteq S$.

We consider the *implication measures*

$$I(A,B) \; = \; \inf_{a \in A} \sup_{b \in B} \sigma(a,b)$$

and

$$J(A,B) \; = \; \sup_{a \in A} \inf_{b \in B} \sigma(a,b).$$

Approach based on multi-modal logic

(Esteva, Godo, Garcia, Rodríguez, Dubois, Prade)

Consider the logic extending CPL by modal operators $\Diamond_t$, $t \in [0,1]_{\mathbb{Q}}$.

Interpret $\Diamond$ according to:

$$\Diamond_t A \;=\; \{x \in S\colon\; \sigma(x,a) \geq t \text{ for some } a \in A\}$$

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Consider the logic extending CPL by modal operators $\diamondsuit_t$, $t \in [0,1]_{\mathbb{Q}}$.

Interpret $\diamondsuit$ according to:

$$\diamondsuit_t A \;=\; \{x \in S\colon\; \sigma(x,a) \geq t \text{ for some } a \in A\}$$

We can express "$\alpha \xrightarrow{t} \beta$" by

$$\alpha \to \diamondsuit_t \beta$$

and "$\alpha \overset{t}{\Rightarrow} \beta$" by

$$\diamondsuit_t \alpha \to \beta.$$

Approaches based on graded implications only

(Godo, Rodríguez)

Consider the logic extending CPL by graded implications $\xrightarrow{t}$ or $\overset{t}{\Rightarrow}$.

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Consider the logic extending CPL by graded implications $\stackrel{t}{\longrightarrow}$ or $\stackrel{t}{\Longrightarrow}$.

Axiomatisation (of the first case) by R.O.Rodríguez.

A modification: Logic of graded tolerance

Implications only at the outermost level. No m.e.c.

$$\frac{\Gamma, \alpha, \beta \xrightarrow{t} \delta}{\Gamma, \alpha \wedge \beta \xrightarrow{t} \delta} \qquad \frac{\Gamma \xrightarrow{t} \delta}{\Gamma, \alpha \xrightarrow{t} \delta}$$

$$\frac{\Gamma, \alpha \xrightarrow{t} \gamma \quad \Gamma, \beta \xrightarrow{t} \gamma}{\Gamma, \alpha \vee \beta \xrightarrow{t} \gamma} \qquad \frac{\Gamma \xrightarrow{t} \alpha}{\Gamma \xrightarrow{t} \alpha \vee \beta}$$

$$\frac{\Gamma \xrightarrow{s} \alpha \quad \alpha \xrightarrow{t} \beta}{\Gamma \overset{s \odot t}{\longrightarrow} \beta}$$

$$\frac{\Gamma \xrightarrow{s} \alpha}{\Gamma \xrightarrow{t} \alpha}, \text{ where } s \geq t \qquad \alpha \xrightarrow{1} \beta \text{ if } \alpha \to \beta \text{ in CPL}$$

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Completeness: w.r.t. not necessarily symmetric similarity relations.

Logic of graded safety

Like the previous logic, but based on $\Rightarrow$.

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Completeness: w.r.t. metric spaces with a certain property.

Modelling vague properties

The formalism needs to be brought to the first-order level.

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"Sorites" then works in principle as follows:

Let $\mathbb{N}\backslash\{0\}$ be our fine model, endowed with the metric d:

$$d(m,n) \;=\; \max\{\frac{m}{n},\frac{n}{m}\} - 1,$$

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Let H be the property "being a heap".

We have

$$H(n) \overset{\frac{1}{n-1}}{\Rightarrow} H(n-1)$$

and

$$H(10000) \overset{9999}{\longrightarrow} H(1).$$

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Unfortunately, it seems that the proposed approach to vagueness is not related to any of the accounts of vagueness in the philosophical literature.

However, the so-called "Logic of graded safety" is conceptually closely related to Williamson's Logic of Clarity, which in turn is based on an account of vagueness diametrically opposed to ours.