VIENNA SUMMER OF LOGIC 2014

Abstract Booklet

Logic Colloquium

Logic, Algebra and Truth Degrees

Vienna Summer of Logic, July 9–24, 2014

organized by the kurt gödel society

Preface

This booklet contains the abstracts of the talks given at the *ASL Logic Colloquium 2014* and at *Logic, Algebra and Truth Degrees (LATD) 2014*. These two conferences are both held as part of the Vienna Summer of Logic (VSL), the largest scientific event in the history of logic.

The VSL is organized by the Kurt Gödel Society and combines 12 large conferences and numerous workshops in the three research areas *Mathematical Logic*, *Logic in Computer Science*, and *Logic in Artificial Intelligence*. It welcomes the keynote speakers F. Baader, E. Clarke, Ch. Papadimitriou, and A. Wilkie and – as can be seen in this booklet – many other eminent scientists. We hope that holding these conferences jointly in Vienna will lead to new collaborations in this area.

We would like to thank all members of the organizing committees, in particular S. Eberhard, B. Mallinger, and J. Tapolczai for their invaluable help in the preparation of these two conferences.

Matthias Baaz
Agata Ciabattoni
Stefan Hetzl

Contents

1 Logic Colloquium 2014

1.1 Tutorials

▸ KRZYSZTOF R. APT, *A Tutorial on Strategic and Extensive Games.*
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Abstract. The aim of this tutorial is to introduce the most fundamental concepts and results concerning strategic and extensive games. No prior knowledge of the subject is assumed.

Strategic games deal with the analysis of interaction between rational players, where rationality is understood as payoff maximization. In strategic games the players take their actions simultaneously and the payoff for each player depends on the resulting joint action.

We shall begin by introducing the fundamental notions of a Nash equilibrium and of mixed strategies. Then we shall discuss the fundamental result of Nash stating that every finite game has a Nash equilibrium in mixed strategies and compare it with an earlier result of Von Neumann concerning equilibria in zero-sum games.

Subsequently we shall discuss various ways of elimination of strategies, in particular iterated elimination of strictly and of weakly strategies, and the concept of rationalizability due to Bernheim and Pearce.

The final part of the tutorial will deal with extensive games. These are games in which the players take their actions in turn. We shall discuss the so-called Zermelo result about the game of chess. Finally, we shall introduce the notion of a subgame perfect equilibrium due to Selten and relate it to the procedure of backward induction.

A short guide to the literature. The first book on game theory was [15] that profoundly influenced the subsequent developments. There are by now several excellent books on strategic and extensive games. Most of them are written from the perspective of applications to Economics and cover also other topics.

[9] is a broad in its scope, undergraduate level textbook, while [10] is probably the best book on the market for the graduate level. Undeservedly less known is the short and lucid [14]. An elementary, short introduction, focusing on the concepts, is [12]. In turn, [11] is a comprehensive book on strategic games and and extensive games. Finally, [4] is an insightful and occasionally entertaining introduction to game theory.

Several textbooks on microeconomics include introductory chapters on game theory, notably strategic and extensive games. Two good examples are [6] and [5]. In turn, [8] is a collection of surveys and introductions to the computational aspects of game theory, with a number of articles concerned with strategic games.

Finally, [7] is a most recent, very comprehensive account of the main areas of game theory, while [13] is an elegant introduction to the subject. We conclude by mentioning three references to our work that we shall rely upon: [3], [1] and [2].

References.

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[15] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior.* Princeton University Press, 1944.

► ALEXANDRE MIQUEL, *A tutorial on classical realizability and forcing.*
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The theory of classical realizability was introduced by Krivine [4] in the middle of the 90's to analyze the computational contents of classical proofs, following the connection between classical reasoning and control operators discovered by Griffin [2]. More than an extension of Kleene's intuitionistic realizability [3], classical realizability is a complete reformulation of the principles of realizability, with strong connections with Cohen forcing [1, 5, 7, 6].

The aim of this tutorial is to present the basics of classical realizability as well as some of its connections with Cohen forcing. For that, I will first present the theory in the framework of second-order arithmetic (PA2), focusing on its computational aspects and on classical program extraction. Then I will show how to combine classical realizability with Cohen forcing (in PAω) and give a computational interpretation of this combination. Finally, I will present some research directions, explaining why classical realizability can be seen as a noncommutative form of forcing.

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[4] J.-L. Krivine. *Realizability in classical logic.* ***Interactive models of computation and program behaviour***, *Panoramas et synthèses*, vol. 27, Société Mathématique de France, 2009, pp. 197–229.

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[6] ——— *Realizability algebras II: new models of ZF + DC.* ***Logical Methods in Computer Science***, vol. 8 (1:10), 2012, pp. 1–28.

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1.2 Plenary talks

▸ MATTHIAS ASCHENBRENNER, *Logic meets number theory in o-minimality.*
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In the past, applications of logic to number theory have mostly come through the model theory of certain *algebraic* structures (such as the field of p-adic numbers, or fields equipped with a derivation). The work of the Karp Prize winners Peterzil, Pila, Starchenko, and Wilkie harnesses the power of model-theoretic structures which have a more *analytic* flavor but are seemingly far removed from arithmetical considerations: o-minimal expansions of the field of real numbers. This leads to novel applications to number theory. A high point of these developments to date is the proof of certain special cases of the André-Oort Conjecture by Pila. Indispensable ingredients in this proof are a counting theorem by Pila-Wilkie as well as definability results due to Peterzil-Starchenko. I plan to survey this circle of ideas, with as few extra-logical prerequisites as possible.

▸ ANDREJ BAUER, *Reductions in computability theory from a constructive point of view.*
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In constructive mathematics we often consider implications between non-constructive reasoning principles. For instance, it is well known that the Limited principle of omniscience implies that equality of real numbers is decidable. Most such reductions proceed by reducing an instance of the consequent to an instance of the antecedent. We may therefore define a notion of *instance reducibility*, which turns out to have a very rich structure. Even better, under Kleene's function realizability interpretation instance reducibility corresponds to Weihrauch reducibility, while Kleene's number realizability relates it to truth-table reducibility. We may also ask about a constructive treatment of other reducibilities in computability theory. I shall discuss how one can tackle Turing reducibility constructively via Kleene's number realizability. One can then ask whether the constructive formulation of Turing degrees relates them to standard mathematical concepts.

▶ PATRICIA BLANCHETTE, *The birth of semantic entailment.*
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The relation of semantic entailment, i.e. of a conclusion's being true on every model of its premises, currently plays a central role in logic, and is arguably the canonical entailment-relation in most contexts. But it wasn't always this way; the relation doesn't come into its own until shortly before its starring role in the completeness theorem for first-order logic. This talk investigates the development of the notion of *model* from the mid-19th century to the early 20th century, and the parallel emergence of logic's concern with the relation of semantic entailment. We will be especially interested in clarifying some of the ways in which the emergence of the modern conceptions of model and of entailment are tied to a changing view of the nature of axiomatic foundations.

▶ ANDRÉS CORDÓN–FRANCO, *On local induction schemes.*
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First–order Peano arithmetic PA is axiomatized over a finite algebraic base theory by the full induction scheme

$$\varphi(0, v) \wedge \forall x\, (\varphi(x, v) \rightarrow \varphi(x+1, v)) \rightarrow \forall x\, \varphi(x, v),$$

where $\varphi(x, v)$ ranges over all formulas in the language of arithmetic $\{0, 1, +, \cdot, <\}$. Fragments of arithmetic are obtained by restricting, in one way or another, the induction scheme axiomatizing PA. Classical examples include the theories of Σ_n and Π_n induction and their parameter free counterparts.

In this talk we present a new kind of restriction on the induction scheme, giving rise to new subsystems of arithmetic that we collectively call *local induction* theories. Roughly speaking, local indiction axioms have the form

$$\varphi(0, v) \wedge \forall x\, (\varphi(x, v) \rightarrow \varphi(x+1, v)) \rightarrow \forall x \in \mathcal{D}\, \varphi(x, v).$$

That is to say, we restrict the x's for which the axiom claims $\varphi(x, v)$ to hold to the elements of a prescribed subclass $\mathcal{D}$ of the universe. Natural choices for $\mathcal{D}$ are the sets of the Σ_n–definable elements of the universe as well as other related substructures of definable elements.

We will study the basic properties of the local induction theories obtained in this way and derive a number of applications to the study of 'classical' fragments of PA. Remarkably, we show that the hierarchy of local reflection principles can be reexpressed in terms of our local induction theories, thus filling a gap in our understanding of the equivalence between reflection and induction in arithmetic.

($\star$) This is joint work with F. Félix Lara–Martín (University of Seville).

($\star\star$) Partially supported by grant MTM2011–26840, Ministerio de Ciencia e Innovación, Spanish Government.

▶ KIRSTEN EISENTRÄGER, *Generalizations of Hilbert's Tenth Problem.*
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Hilbert's Tenth Problem in its original form was to find an algorithm to decide, given a multivariate polynomial equation with integer coefficients, whether it has a solution over the integers. In 1970 Matiyasevich, building on work by Davis, Putnam and Robinson, proved that no such algorithm exists, i.e. Hilbert's Tenth Problem is undecidable. Since then, analogues of this problem have been studied by asking the same question for polynomial equations with coefficients and solutions in other commutative rings. The biggest open problem in the area is Hilbert's Tenth Problem over the rational numbers. In this talk we will construct some subrings R of the rationals that have the property that Hilbert's Tenth Problem for R is Turing equivalent to Hilbert's Tenth Problem over the rationals. We will also discuss some recent undecidability results for function fields of positive characteristic.

▸ VERA FISCHER, *Cardinal invariants and template iterations.*
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The cardinal invariants of the continuum arise from combinatorial, topological and measure theoretic properties of the reals, and are often defined to be the minimum size of a family of reals satisfying a certain property.

An example of such an invariant is the minimum size of a subgroup of S_∞, all of whose non-identity elements have only finitely many fixed points and which is maximal (with respect to this property) under inclusion. This cardinal invariant is denoted $\mathfrak{a}_g$. Another well-known invariant, denoted $\mathrm{non}(\mathcal{M})$, is the minimum size of a set of reals which is not meager. It is a ZFC theorem that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_g$. A third invariant, denoted $\mathfrak{d}$, is the minimum size of a family $\mathcal{F}$ of functions in ${}^\omega\omega$ which has the property that every function in ${}^\omega\omega$ is eventually dominated by an element of $\mathcal{F}$. In contrast to the situation between $\mathfrak{a}_g$ and $\mathrm{non}(\mathcal{M})$, ZFC cannot prove either of the inequalities $\mathfrak{a}_g \leq \mathfrak{d}$ or $\mathfrak{d} \leq \mathfrak{a}_g$. The classical forcing techniques seem, however, to be inadequate in addressing the consistency of $\mathfrak{d} < \mathfrak{a}_g$ which was obtained only after a ground-breaking work by Shelah and the appearance of his "template iteration" forcing techniques.

We further develop these techniques to show that $\mathfrak{a}_g$, as well as some of its relatives, can be of countable cofinality. In addition we will discuss other recent developments of the technique and conclude with open questions and directions for further research.

▸ MATT FOREMAN, *The Singular Cardinals Problem after 130 years or so.*.
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We trace the history of singular cardinals problem from its inception to the remarkable work of Shelah and Gitik, culminating in the PCF theory and the PCF conjecture.

▸ NOAM GREENBERG, *Applications of admissible computability.*
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Admissible computability is an extension of traditional computability theory to ordinals beyond the finite ones. I will discuss two manifestations of admissible computability in the study of effective randomness and in the study of effective properties of uncountable structures.

▶ JULIA F. KNIGHT, *Computable structure theory and formulas of special forms.*
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In computable structure theory, we ask questions about complexity of structures and classes of structures. For a particular countable structure $\mathcal{M}$, how hard is it to build a copy? Can we do it effectively? How hard is it to describe $\mathcal{M}$, up to isomorphism, distinguishing it from other countable structures? For a class K, how hard is it to characterize the class, distinguishing members from non-members? How hard is it to classify the elements of K, up to isomorphism. In the lecture, I will describe some results on these questions, obtained by combining ideas from computability, model theory, and descriptive set theory. Of special importance are formulas of special forms.

▶ LESZEK KOŁODZIEJCZYK, *The problem of a model without collection and without exponentiation.*
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$\mathrm{I}\Delta_0$ is the fragment of first-order arithmetic obtained by restricting the induction scheme to bounded formulas. $\mathrm{B}\Sigma_1$ extends $\mathrm{I}\Delta_0$ by the collection scheme for bounded formulas, that is by the axioms

$$\forall x\!<\!v\,\exists y\,\psi(x,y) \Rightarrow \exists w\,\forall x\!<\!v\,\exists y\!<\!w\,\psi(x,y),$$

where ψ is bounded (and may contain additional parameters).

It has been known since the seminal work of Parsons and of Paris and Kirby in the 1970s that $\mathrm{B}\Sigma_1$ does not follow from $I\Delta_0$, even though it is Π^0_2-conservative over $\mathrm{I}\Delta_0$. However, all constructions of a model of $\mathrm{I}\Delta_0$ not satisfying $\mathrm{B}\Sigma_1$ make use of the axiom Exp, which asserts that 2^x is a total function. From the perspective of $\mathrm{I}\Delta_0$, which does not prove the totality of any function of superpolynomial growth, the totality of exponentiation is a very strong unprovable statement. This led Wilkie and Paris [1] to ask whether $\mathrm{I}\Delta_0 + \neg\mathrm{Exp}$ proves $\mathrm{B}\Sigma_1$.

It is generally believed that the answer to Wilkie and Paris's question is negative, and there are various statements from computational complexity theory, in some cases mutually contradictory, known to imply a negative answer. However, an unconditional proof of a negative answer remains elusive.

I plan to survey some facts related to Wilkie and Paris's question, focusing on two recent groups of theorems:

(i) the results of the paper [2], which seem to suggest that we are a "small step" away from building a model of $\mathrm{I}\Delta_0 + \neg\mathrm{Exp}$ without collection,

(ii) some new results suggesting that the "small step" will be very hard to take, because there is a complexity-theoretic statement, almost certainly false but possibly not disprovable using present-day methods, which implies that $B\Sigma_1$ does follow from $\neg\mathrm{Exp}$.

[1] A. Wilkie and J. Paris, *On the existence of end extensions of models of bounded induction*, ***Logic, Methodology, and Philosophy of Science VIII*** (Moscow 1987), (J.E. Fenstad, I.T. Frolov, and R. Hilpinen, editors), North-Holland, 1989, pp. 143–162.

[2] Z. Adamowicz, L. A. Kołodziejczyk, and J. Paris, *Truth definitions without exponentiation and the Σ_1 collection scheme*, ***Journal of Symbolic Logic***, vol. 77 (2012), no. 2, pp. 649–655.

▶ BENJAMIN D. MILLER, *Definable cardinals just beyond* $\mathbb{R}/\mathbb{Q}$.
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Over the last few decades, a definable refinement of the usual notion of cardinality has been employed to great effect in shedding new light on many classification problems throughout mathematics. In order to best understand such applications, one must investigate the abstract nature of the definable cardinal hierarchy.

It is well known that the initial segment of the hierarchy below $\mathbb{R}/\mathbb{Q}$ looks quite similar to the usual cardinal hierarchy. On the other hand, if one moves sufficiently far beyond $\mathbb{R}/\mathbb{Q}$, the two notions diverge wildly.

After reviewing these results, we will discuss recent joint work with Clinton Conley, seeking to explain the difficulty in understanding definable cardinality beyond $\mathbb{R}/\mathbb{Q}$ by showing that the aforementioned wild behavior occurs immediately thereafter.

▶ MARK REYNOLDS, *Synthesis for monadic logic over the reals.*
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We say that a first-order monadic logic of order (FOMLO) sentence is satisfiable over the reals if there is some valuation for the monadic predicates which makes the formula true. Burgess and Gurevich showed that satisfiability for this logic is decidable. They built on pioneering work by Lauchli and Leonard who, in showing a similar result for linear orders in general, had presented some basic operations for the compositional building of monadic linear structures.

We look at some recent work in using these basic operations to give a synthesis result. That is, we present an algorithm which given a FOMLO sentence which is satisfiable over the reals, outputs a specific finite description of a model.

▶ MARIYA I. SOSKOVA, *Definability, automorphisms and enumeration degrees.*
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The enumeration degrees are an upper semi-lattice with a least element and jump operation. They are based on a positive reducibility between sets of natural numbers, enumeration reducibility, introduced by Friedberg and Rogers in 1959. The Turing degrees have a natural isomorphic copy in the structure of the enumeration degrees, namely the substructure of the total enumeration degrees. A long-standing question of Rogers [5] is whether the substructure of the total enumeration degrees has a natural first order definition. The first advancement towards an answer to this question was made by Kalimullin [4]. He discovered the existence of a special class of pairs of enumeration degrees, $\mathcal{K}$-pairs, and showed that this class has a natural first order definition in $\mathcal{D}_e$. Building on this result, he proved the first order definability of the enumeration jump operator and consequently obtained a first order definition of the total enumeration degrees above $\mathbf{0}'_e$. Ganchev and Soskova [3] showed that when we restrict ourselves to the local structure of the enumeration degrees bounded by $\mathbf{0}'_e$, the class of $\mathcal{K}$-pairs is still first order definable. In [2] they investigated maximal $\mathcal{K}$-pairs and showed that within the local structure the total enumeration degrees are first order definable as the least upper bounds of maximal $\mathcal{K}$-pairs.

The question of the global definability of the total enumeration degrees is finally solved by Cai, Ganchev, Lempp, Miller and Soskova [1]. They show that Ganchev and Soskova's local definition of total enumeration degrees is valid globally. Then using this fact, they show that the relation "c.e. in", restricted to total enumeration degrees is also first order definable. We will discuss these results and certain consequences, regarding the automorphism problem in both degree structures.

This research was supported by a BNSF grant No. DMU 03/07/12.12.2011, by a Sofia University SF grant and by a Marie Curie international outgoing fellowship STRIDE (298471) within the 7th European Community Framework Programme.

[1] M. Cai, H. A. Ganchev, S. Lempp, J. S. Miller and M. I. Soskova, *Defining totality in the enumeration degrees*, submitted.

[2] H. A. Ganchev and M. I. Soskova, *Definability via Kalimullin pairs in the structure of the enumeration degrees*, To appear in ***Transactions of the AMS***.

[3] H. A. Ganchev and M. I. Soskova, *Cupping and definability in the local structure of the enumeration degrees*, ***J. Symbolic Logic*** vol. 77 (2012), no. 1, pp. 133–158.

[4] I. Sh. Kalimullin, *Definability of the jump operator in the enumeration degrees*, ***Journal of Mathematical Logic***, vol. 3 (2003), pp. 257–267.

[5] H. Rogers Jr., *Theory of recursive functions and effective computability*, ***McGraw-Hill Book Company***, New York, 1967.

▸ ALBERT VISSER, *On a Theorem of McAloon*.
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A theory is *restricted* if there is a fixed bound on the complexity of its axioms. In his classical paper [1], Kenneth McAloon proves that every restricted arithmetical theory that is consistent with Peano Arithmetic has a model in which the standard natural numbers are definable. In slogan, one could say that McAloon shows that one needs the full language to exclude the standard numbers in principle.

In this talk we discuss the idea of generalizing McAloon's result to the class of consistent restricted sequential theories. We only obtain a weaker statement for the more general case. Whether the stronger statement holds remains open.

Sequential theories are, as a first approximation, theories with sufficient coding machinery for the construction of partial satisfaction predicates of a certain sort. Specifically, we have satisfaction for classes of formulas with complexity below n for a complexity measure like *depth of quantifier alternations*. Sequential theories were introduced by Pavel Pudlák in [2]. There are several salient general results concerning sequential theories. For example the degrees of interpretability of sequential theories have many good properties. Examples of sequential theories are PA^-, S^1_2, $\mathrm{I}\Sigma_1$, PA, ACA_0, ZF, GB.

To any sequential model $\mathcal{M}$ we can uniquely assign an arithmetical model $\mathcal{J}_{\mathcal{M}}$. This is, roughly, the intersection of all definable cuts of an internal model $\mathcal{N}$ of a weak arithmetic like S^1_2. We can show that $\mathcal{J}_{\mathcal{M}}$ is independent of the specific choice of $\mathcal{N}$. Our theorem says that any consistent restricted sequential theory U has a model $\mathcal{M}$ such that $\mathcal{J}_{\mathcal{M}}$ is isomorphic to the standard model.

In the talk, we will briefly indicate how McAloon's proof works and discuss some immediate generalizations. Then, we will outline the basic ideas behind the proof of the result concerning consistent restricted sequential theories.

[1] K. McAloon, *Completeness theorems, incompleteness theorems and models of arithmetic*, ***Transactions of the American Mathematical Society***, vol. 239 (1978), no. X, pp. 253–277.

[2] P. Pudlák, *Some prime elements in the lattice of interpretability types*, ***Transactions of the American Mathematical Society***, vol. 280 (1983), pp. 255–275, .

1.3 Special sessions

Logic of Games and Rational Choice

Organizer: R. Ramanujam.

► JOHAN VAN BENTHEM, *The DNA of logic and games.*
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Logic and games are entangled in delicate ways. Logics **of** games are used to analyze how players reason and act in a game. I will discuss dynamic-epistemic logics that analyze various phases of play in this mode. But one can also study logic **as** games, casting major logical notions as game-theoretic concepts. The two perspectives create a circle, or double helix if you will, of contacts all around. I will address this entanglement, and the issues to which it gives rise ([1]).

[1] van Benthem, ***Logic in games***, MIT Press, Cambridge MA, 2014.

► ROHIT PARIKH, *Elections and Knowledge.*
City University of New York.
E-mail: `rparikh@gc.cuny.edu`. There are (at least) two ways in which knowledge can enter into elections. 1. When a voter strategizes, i.e. votes for someone who is not her first preference, then she needs to know something about how the others are voting. Perhaps they want to know how she is voting. There are various possible scenarios here. 2. When a politician campaigns, he wants to influence the voters' beliefs. What should he say in order to appeal to them in the best way? We will make use of previous work by ourselves, Samir Chopra, Hans van Ditmarsch, Walter Dean and Eric Pacuit, as well as suggest some new ideas.

[1] Samir Chopra, Eric Pacuit and Rohit Parikh, *Knowledge theoretic properties of strategic voting*, ***Ninth European Conference on Logics in Artificial Intelligence*** (Lisbon, Portugal), 2004

[2] Walter Dean and Rohit Parikh, *The Logic of Campaigning*, ***Indian Conference in Logic and Applications*** (Delhi, India), (M. Banerjee and A. Seth, editors), vol. X, Springer 2011, pp. 38-39

[3] Hans van Ditmarsch, Jerome Lang and Abdallah Safidine, *Strategic Voting and the Logic of Knowledge*, ***Theoretical Aspects of Rationality and Knowledge*** (Chennai, India), (Burkhard C. Schipper, editor), Association for Computing Machinery 2013, pp. 61-70

▸ GABRIEL SANDU, *Nash equilibrium semantics for Independence-Friendly logic.*
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Henkin (1961) enriched first-order logic with so-called branching or Henkin quantifiers such as $\begin{pmatrix} \forall x \\ \exists y \end{pmatrix}$ and $\begin{pmatrix} \forall x & \exists y \\ \forall z & \exists w \end{pmatrix}$. The former is intended to express the fact that the existential qua ntifier $\exists y$ is independent of the universal quantifier $\forall x$. The latter is more easily introduced in terms of the idea of dependence: the existential quantifier $\exists y$ depends only on the universal quantifier $\forall x$, and the existential quantifier $\exists w$ depends only on the universal quantifier $\forall z$. The notions of independence and dependence are codified in terms of the existence of certain (Skolem) functions. It turns out that prefixing first-order formulas with branching quantifiers results in a logic which is strictly stronger than ordinary first-order logic.

In the first part of my presentation I will quickly review various formalisms which develop Henkin's ideas. One of them is Independence-Friendly logic introduced by Hintikka and Sandu (1989). The first branching quantifier is expressed in IF logic by $\forall x(\exists y/\{x\})$ ("for all x there is a y which is independent of x"). Similarly, the second branching quantifier is expressed by $\forall x \exists y \forall z(\exists w/\{x, y\})$ ("for all x there is a y and for all z there is a w which is independent from both x and y"). The notion of independence is spelled out in game-theoretical terms. With each IF formula φ, model $\mathbb{M}$, and partial assignment s whose domain is restricted to the free variables of φ, we associate an extensive win-lose game of imperfect information $G(\mathbb{M}, \varphi, s)$. When φ is the sentence $\forall x(\exists y/\{x\})x = y$, and s is the empty assignment, in a play of the game $G(\mathbb{M}, \varphi, s)$ $\forall$ chooses an individual $a \in M$ to be the value of x after which $\exists$ chooses an individual $b \in M$ to be the value of y without knowing the choice made earlier by $\forall$. Player $\exists$ wins the play if a is the same individual as b. Otherwise player $\forall$ wins. We stipulate that φ is true (false) in $\mathbb{M}$ if there is a winning strategy for player $\exists$ ($\forall$). The notion of strategy is the standard notion of choice function in classical game theory. In games of imperfect information such a function is required to be uniform. A comprehensive presentation of the model-theoretical properties of IF logic may be found in Mann, Sandu, and Sevenster (2011). Hintikka (1996) explores the significance of IF logic for the foundations of mathematics.

As expected, games of imperfect information may be indeterminate. For instance, on models with at least two elements, the IF sentence $\forall x(\exists y/\{x\})x = y$ is neither true nor false. Blass and Gurevich (1986) follow a suggestion by Aitaj and resolve the indeterminacy of this sentence by applying von Neumann's Minimax theorem: $\forall x(\exists y/\{x\})x = y$ gets the probabilistic value $\frac{1}{n}$ on any finite model with n elements. This value is the expected utility returned to the existential player by any mixed strategy equilibrium in the underlying game. This idea has been explored systematically for the first time in Sevenster (2006), and then further developed in Sevenster and Sandu (2010), and in Mann, Sandu and Sevenster (2011). In the second part of my talk I will review some of the recent results on probabilistic IF logic.

Finally I will address the question: What kind of probabilistic logic is probabilistic IF logic? Here I shall draw some comparisons to other probabilistic semantics (Leblanc's probabilistic semantics, Bacchus' and Halpern's probabilistic interpretations of first-order logic.)

[1] A. Blass and Y. Gurevich, *Henkin quantifiers and complete problems*, ***Annals of pure and Applied Logic***, vol. 32 (1986), no. 1, pp. 1–16.

[2] L. Henkin, *Some remarks on infinitely long formulas*, ***Intuitionistic Methods: Proceedings of the Symposium on Foundations of Mathematics*** (P. Bernays, editor), Pergamon Press, Oxford, 1959, pp. 169–183.

[3] J. Hintikka and G. Sandu, *Informational Independence as a Semantic Phenomenon*, ***Logic, Methodology and Philosophy of Science*** (J. E. Fenstead et al, editors), Elsevier, Amsterdam, 1989, pp. 571–589.

[4] A. I. Mann, G. Sandu and M. Sevenster, ***Independence-Friendly Logic: A Game-Theoretic Approach***, Cambridge University Press, 2011.

[5] M. Sevenster, ***Branches of Imperfect Information: Logic, Games, and Computation***, PhD Thesis, University of Amsterdam, 2006.

[6] M. Sevenster and G. Sandu, *Equilibrium Semantics of Languages of Imperfect Information*, ***Annals of Pure and Applied Logic***, vol. 161 (2010), pp. 618–631.

▸ JOUKO VÄÄNÄNEN, *Dependence and independence—a logical approach.*
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I will give an overview of dependence logic [1], the goal of which is to establish a basic logical theory of dependence and independence underlying seemingly unrelated subjects such as game theory, causality, random variables, database theory, experimental science, the theory of social choice, Mendelian genetics, etc. There is an abundance of new results in this field demonstrating a remarkable convergence. The concepts of (in)dependence in the different fields of humanities and sciences have surprisingly much in common and a common logic is starting to emerge.

[1] Jouko Väänänen, ***Dependence logic***, volume 70 of London Mathematical Society Student Texts, Cambridge University Press, 2007.

Model Theory

Organizer: Z. Chatzidakis.

▸ PANTELIS E. ELEFTHERIOU,
Department of Mathematics and Statistics, University of Konstanz, Zukunftskolleg, Box 216, 78457 Konstanz, Germany. *Pregeometries and definable groups.*
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We will describe a program for analyzing groups and sets definable in certain pairs $\langle \mathcal{R}, \mathcal{P} \rangle$. Examples include:

1. $\mathcal{R}$ is an o-minimal ordered group and P is a real closed field with bounded domain (joint work with Peterzil).
2. $\mathcal{R}$ is an o-minimal structure and P is a dense elementary substructure of $\mathcal{R}$ (work

in progress with Hieronymi).

In each of these cases, a relevant notion of a pregeometry and genericity is used.

▸ MEERI KESÄLÄ, *Quasiminimal structures and excellence.*

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A structure M is quasiminimal if every definable subset of M is either countable or co-countable. The field of complex numbers is a strongly minimal structure and hence quasiminimal, but if we add the natural exponential function, the quasiminimality of the structure becomes an open problem. Boris Zilber defined the non-elementary framework of *quasiminimal excellent classes* in 2005 in order to show that his class of *pseudoexponential fields* is uncountably categorical. He conjectured that the unique pseudoexponential field of cardinality $2^{\aleph_0}$ fitting into this framework is isomorphic to the complex numbers with exponentiation. A key property for the categoricity of quasiminimal excellent classes was the technical axiom of excellence, which was adopted from Shelah's work for excellent sentences in $L_{\omega_1\omega}$. However, the original proof of the categoricity of pseudoexponential fields turned out to have a gap and the problem lay in sh! owing that the excellence axiom holds.

In the paper *Quasiminimal structures and excellence*[1] we fill the gap in the proof with a surprising result: the excellence axiom is actually redundant in the framework of quasiminimal excellent classes. This result elegantly combines methods from classification theory that were generalized to different non-elementary frameworks by a group of people. These methods have a combinatorial core idea that is independent of the compactness of first order logic. We also study whether other quasiminimal structures fit into this uncountably categorical framework.

The paper strengthens the belief that non-elementary methods can provide effective tools to analyse structures that are out of reach for traditional model-theoretic methods. Different frameworks have been suggested and the methods refined and there are many interesting paths in the ongoing research. The paper is joint work of Martin Bays, Bradd Hart, Tapani Hyttinen, MK and Jonathan Kirby.

[1] Martin Bays, Bradd Hart, Tapani Hyttinen, Meeri Kesälä and Jonathan Kirby, *Quasiminimal structures and excellence*, ***Bulletin of the London Mathematical Society***, vol. 46 (2014), no. 1, pp. 155–163.

▸ JOCHEN KOENIGSMANN, *Definable valuation rings.*
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The question which valuation rings on a field are first-order definable in the language of rings and if so by what kind of formula and in what kind of uniformity in families naturally arises in model theory of valued fields, but also, for example, in the context of Hilbert's 10th Problem or of motivic integration. It has gained momentum in recent years. We shall report on the latest developments and discuss some open problems.

▸ DANIEL PALACÍN, *The Fitting subgroup of a supersimple group.*
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The Fitting subgroup of a given group G is the subgroup generated by all nilpotent normal subgroups of G. While it is always normal, it may not be nilpotent. Wagner proved that the Fitting subgroup of a stable is always nilpotent. However, this is not known for the wider class of groups with a simple theory.

A certain amount of model-theoretic ideas for groups in the stable context can be adapted to the more general framework of simple theories. For instance, stabilizers and generic types exist. In this talk we present some of the main tools and notions of groups in simple theories, and focus on those which have ordinal Lascar rank. Our aim is to prove that the Fitting subgroup of a type-definable supersimple group is again nilpotent. This generalizes a proof of Milliet in the finite rank case.

Perspectives on Induction (joint with CSL-LICS)

Organizers: M. Baaz, S. Hetzl.

▸ ALAN BUNDY, *Automating inductive proof.*
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The automation of inductive proof plays a pivotal role in the formal development of ICT systems: both software and hardware. It is required to reason about all forms of *repetition*, which arises in: recursive and iterative programs; parameterised hardware; traces of program runs; program invariants; etc. Since formal proof is a highly skilled and time-consuming activity, industry requires as much automation as possible to enable formal methods to be used cost effectively.

Unfortunately, inductive reasoning is much harder to automate than, for instance, first-order reasoning. Negative results from mathematical logic underpin these difficulties. These results include incompleteness, the undecidability of termination and the absence of cut elimination. Of these, the absence of cut elimination creates the most practical problems. The proofs of even some very simple and obviously true conjectures require the injection of cut formulae. These formulae typically take the form of intermediate lemmas, generalisations of the conjecture or non-standard induction rules. Cut rule steps are generally assumed to require human intervention with an interactive prover to provide an appropriate cut formula.

We have developed a proof technique called *rippling* [Bundy *et al*, 2005] that guides the manipulation of the induction conclusion until the induction hypothesis can be used in its proof. In fact, rippling can be used in any situation where a given embeds in a goal. It rewrites the goal while preserving and re-grouping the embedding until an instance of the given appears as a sub-expression of the goal.

The main contribution of rippling, however, is not its guidance of the step case, but the way it informs the application of the cut rule. It provides a strong expectation of the direction of the proof, but is not always successful. When it fails, an analysis of the failure suggests an appropriate application of cut: the form of a missing lemma, a generalisation or a non-standard induction rule [Ireland & Bundy, 1996]. This increases the scope of inductive-proof automation, which has economic implications for the use of formal methods in the ICT industry.

[Bundy *et al*, 2005]Bundy, A., Basin, D., Hutter, D. and Ireland, A. (2005). *Rippling: Meta-level Guidance for Mathematical Reasoning*, volume 56 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press.

[Ireland & Bundy, 1996]Ireland, A. and Bundy, A. (1996). Productive use of failure

in inductive proof. *Journal of Automated Reasoning*, 16(1–2):79–111.

▸ MICHAEL DETLEFSEN, *Inductive proofs and the knowledge they give.*
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Proofs by mathematical induction require certain interdependencies between the instances of the generalizations they prove. The character of these interdependencies and the conditions under which they obtain will be the principal concerns of this talk.

▸ GEORG GOTTLOB, *Decidable languages for knowledge representation and inductive definitions: From Datalog to Datalog+/-.*
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Datalog is a language based on function-free Horn clauses used to inductively define new relations from finite relational structures. Datalog has many nice computational and logical properties. For example, Datalog captures PTIME on ordered structures, which means that evaluating fixed Datalog programs (i.e. rule sets) over finite structures is in PTIME and, moreover, every PTIME-property on ordered structures can be expressed as a datalog program (see [1] for a survey). After giving a short overview of Datalog we argue that Datalog has certain shortcomings and is not ideal for knowledge representation, in paticular, for inductive ontological knowledge representation and reasoning. We consequently introduce Datalog+/- which is a new framework for tractable ontology querying, and for a variety of other applications. Datalog+/- extends plain Datalog by features such as existentially quantified rule heads, negative constraints, and equalities in rule heads, and, at the same time, restricts the rule syntax so as to achieve decidability and tractability. In particular, we discuss three paradigms ensuring decidability: chase termination, guardedness, and stickiness, which were introduced and studied in [2, 3, 4, 5].

[1] Dantsin, E.; Eiter, T.; Georg, G.; and Voronkov, A. 2001. Complexity and expressive power of logic programming. *ACM Comput. Surv.* 33(3):374–425.

[2] Calì, A.; Gottlob, G.; and Kifer, M. 2013. Taming the infinite chase: Query answering under expressive relational constraints. *Journal of Artificial Intelligence Research* 48:115–174.

[3] Calì, A.; Gottlob, G.; and Lukasiewicz, T. 2012. A general Datalog-based framework for tractable query answering over ontologies. *Journal of Web Semantics* 14:57–83.

[4] Calì, A.; Gottlob, G.; and Pieris, A. 2012. Towards more expressive ontology languages: The query answering problem. *Artificial Intelligence* 193:87–128.

[5] Gottlob, G.; Manna, M.; and Pieris, A. 2013. Combining decidability paradigms for existential rules. *Theory and Practice of Logic Programming* 13(4-5):877–892.

▸ GERHARD JÄGER, *Weak well orders and related inductions.*
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It is an interesting program to investigate the relationship between the proof theory of second order arithmetic and more general second order systems (e.g. theories of sets and classes such as von Neumann-Bernays-Gödel set theory and Morse-Kelley set theory). Which proof-theoretic results can be lifted from second order arithmetic to theories of sets and classes, for which is this not the case, and what are the reasons? What is specific of second order number theory and what additional insights can we gain?

One of the crucial questions is how to distinguish between "small" and "large" in the various contexts. In second order arithmetic, the small objects are the natural numbers whereas the large objects are the infinite sets of natural numbers. Hence it seems natural to identify the small objects in sets and classes with sets and the large objects with proper classes.

As long as only comparatively weak systems are concerned, the moving up from second order arithmetic to sets and classes seems to be a matter of routine. However, as soon as well orderings enter the picture, the situation becomes interesting. In second order arithmetic, every Π^1_1 statement is equivalent to the question whether a specific arithmetic relation is well ordered; on the other hand, in set theory a simple elementary formulas expresses the well foundedness of a given relation.

We propose studying the (new) notion of *weak well order* in sets and classes as the proof-theoretically adequate analogue of well order in second order arithmetic. To support this claim several results about inductions and recursions in connection with weak well orders will be presented. This is joint work with D. Flumini.

[1] D. Flumini, ***Weak Well Orders***, PhD Thesis, University of Bern, 2013.

[2] G. Jäger, *Operations, sets and classes*, ***Logic, Methodology and Philosophy of Science, Proceedings of the Thirteenth International Congress*** (Beijing), (C. Glymour, W. Wei, D. Westerståhl, editors), College Publications, 2009, pp. 74–96.

[3] G. Jäger and J. Krähenbühl, Σ^1_1 *choice in a theory of sets and classes*, ***Ways of Proof Theory*** (R. Schindler, editor), Ontos Verlag, 2010, pp. 283–313.

Philosophy of Mathematics

Organizer: Ø. Linnebo.

▸ PATRICIA BLANCHETTE, *Frege on mathematical progress.*
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Progress in mathematics has often involved a good deal of conceptual clarification, including increasingly precise characterizations of concepts (e.g. those of infinity, of continuity, perhaps of set, etc.) that were less clearly understood by earlier theorists. But the sometimes-vast difference between the earlier and later concepts that go by the same name raises the possibility that such conceptual refinement really brings with it a whole new subject-matter for the branch of mathematics in question, rather than a clarified understanding of the concepts used by earlier generations. This talk investigates Gottlob Frege's approach to understanding this kind of conceptual progress, and assesses the plausibility of his view that a given subject-matter can survive essentially unscathed despite fairly radical changes in the surrounding theory.

▸ LEON HORSTEN, *Reflection, Trust, Entitlement.*
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It has been argued by Feferman and others that when we accept a mathematical theory, we implicitly commit ourselves to reflection principles for this theory. When we reflect on this implicit commitment, we come to explicitly believe certain reflection principles. In my talk I will discuss our epistemic warrant for this resulting explicit belief in reflection principles.

▸ LUCA INCURVATI AND BENEDIKT LÖWE, *Restrictiveness relative to notions of interpretation.*
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In [4], Maddy gives a semi-formal account of *restrictiveness* by defining a corresponding formal notion based on a class of interpretations. In [2] and [3], Maddy's notion of restrictiveness was discussed and the theory ZF + 'Every uncountable cardinal is singular' was presented as a potential witness to the restrictiveness of ZFC. More recently, Hamkins has given more examples and pointed out some structural issues with Maddy's definition [1]. We look at Maddy's definitions from the point of view of an abstract *interpretation* relation. We consider various candidates for this interpretation relation, including one that is close to Maddy's original notion, but fixes the issues raised in [1]. Our work brings to light additional structural issues that we also discuss.

[1] Joel David Hamkins, *A multiverse perspective on the axiom of constructibility*, ***Infinity and Truth, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore*** (Singapore), (Chitat Chong, Qi Feng, Theodore A Slaman, and W Hugh Woodin, editors), vol. 25, World Scientific, 2013, pp. 25–45.

[2] Benedikt Löwe, *A first glance at non-restrictiveness*, ***Philosophia Mathematica***, vol. 9 (2001), no. 3, pp. 347–354.

[3] ——— *A second glance at non-restrictiveness*, ***Philosophia Mathematica***, vol. 11 (2003), no. 3, pp. 323–331.

[4] Penelope Maddy, ***Naturalism in Mathematics***, Clarendon Press, 1997.

▸ GABRIEL UZQUIANO, *On Bernays' Generalization of Cantor's Theorem.*
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Cantor's theorem states that there is no one-to-one correspondence between the members of a set a and the subsets of a. In [1], Paul Bernays showed how to encode the claim that there is no one-to-one correspondence between the members of a class A and the subclasses of A by means of a sentence of the language of class theory. Moreover, he proved his generalization of Cantor's theorem by means of a diagonal argument: given a one-to-one assignment of subclasses of A to members of A, he defined a subclass of A, which, on pain of contradiction, is not assigned to any member of A. It follows from Bernays' observation that if one assigns a member of A to every subclass of A, then the assignment is not one-one. Unfortunately, familiar arguments for this claim

fail to provide an explicit characterization of two different subclasses of A to which one and the same member of A is assigned by the assignment. George Boolos tackled a related problem in [2], where he showed how to specify explicit counterexamples to the claim that a function from the power set of a set a into the set a is one-one. Similar constructions turn out to be available in the case of classes, but they are sensitive to the presence of global choice and impredicative class comprehension. We explore some ramifications of this observation for traditional philosophical puzzles raised by the likes of Russell's paradox of propositions in Appendix B of [4] and Kaplan's paradox in [3].

[1] Paul Bernays, *A System of Axiomatic Set Theory: Part IV*, ***The Journal of Symbolic Logic***, vol. 7 (1942), no. 4, pp. 133–145.

[2] George Boolos, *Constructing Cantorian Counterexamples*, ***Journal of Philosophical Logic***, vol. 26 (1997), no. 3, pp. 237–239.

[3] David Kaplan, *A Problem in Possible-World Semantics*, ***Modality, Morality and Belief: Essays in Honor of Ruth Barcan Marcus*** (Walter Sinnott-Armstrong and Diana Raffman and Nicholas Asher, editors), Cambridge University Press, Cambridge, 1995, pp. 41–52.

[4] Bertrand Russell, ***The Principles of Mathematics***, W. W. Norton & Company, 1903.

Recursion Theory

Organizers: E. Fokina, D. Turetsky.

▸ JOHANNA N.Y. FRANKLIN, *UD-randomness and the Turing degrees.*
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The roots of UD-randomness are firmly analytic: Avigad defined it in 2013 using concepts from a 1916 theorem of Weyl concerning uniform distribution. Avigad showed in his original paper that UD-randomness is very weak. While every Schnorr random real is UD-random, the class of UD-random reals is incomparable with the class of the Kurtz random reals. In this talk, I will present some subsequent work on the Turing degrees of the UD-random reals and the relationships between UD-randomness and other randomness notions.

This work is joint with Wesley Calvert.

▸ RUPERT HÖLZL, *Randomness in the Weihrauch degrees.*
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(Based on joint work with Vasco Brattka and Guido Gherardi, and on joint work with Paul Shafer.) It is a recurring theme of theoretical computer science how access to sources of random information can enable the computation of certain mathematical objects. While this is particularly evident in the context of complexity theory, the question can also be studied in more general settings. Many different versions have been studied in the field of algorithmic randomness. It can be argued that this approach better represents the original question of what can be computed with access to randomness than, for example, the complexity theoretic approach, as in this setting

space or time bounds are not considered, meaning we are getting a better idea of the computational *content* of random objects — as opposed to a gauge of their ability to *speed up* a computation until it can be performed within polynomial time. For this reason, the results from algorithmic randomness and computability theory are of high importance.

In this talk we will look at the question from yet another angle, and give arguments why we think that this is the correct way to formalize the question of what random information can be used for. In fact, the field of algorithmic randomness does already provide two answers to the question: First there is the Kučera-Gács-Theorem, which, informally stated, says that everything can be computed from some random object. We will argue that despite the high importance of this theorem it does not provide an answer to the initial informal question, when we formalize it in a way that actually captures the intention behind it. Secondly, there is Sacks' theorem, which states that no non-trivial information can be generated from a set of oracles of positive measure. Again we will argue that this is not the answer we are looking for: Sacks' theorem only applies if we want to compute a single set A, as the proof relies essentially on a majority vote argument.

But there are many very valid settings where we do not want to compute a single set: Often we are given a mathematical problem and want to find a solution to it, and we want to know whether randomness can help us to find such a solution. For a given instance of such a task there may be many legal solutions; each of these solutions may have low probability of being produced by a Turing machine, so that a majority vote mechanic would fail.

To overcome this limitation we therefore need a different framework, without abolishing completely the ideas of computability theory. This new framework is provided by the Weihrauch degrees, the lattice induced by Weihrauch reducibility. In the talk we will introduce the framework and give arguments for why we think it is the correct way to approach the initial question.

We will study computation from sets of oracles of positive measure in this framework. Among other results, we will in particular identify two natural models of randomized computation: One is computation with access to Martin-Löf random oracles. The other is computation with what we call a Las Vegas algorithm, a Weihrauch degree version of Babai's similarly named notion from complexity theory. This second model of randomized computation can be naturally identified with Weak Weak Kőnig's Lemma.

We will then see that these two models of randomized computation can be separated in the Weihrauch degrees. This contrasts with results in the related field of reverse maths, where they are known to coincide. We will discuss what the origin of this different behavior is.

To conclude, we will briefly discuss some other ways in which algorithmic randomness and related notions show up in the Weihrauch lattice, to illustrate that the study of algorithmic randomness with Weihrauch tools is a fruitful topic with many open questions to explore.

▸ ISKANDER KALIMULLIN, *Uniform and non-uniform reducibilities of algebraic structures.*
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The talk will be devoted to various versions of algorithmic reducibility notion between algebraic structures. In particular, the reducibilities under Turing operetors, enumeration operators, and under Σ-formulas will be considered. Several constructions of jump inversion where these reducibilities do not coincide. Furthermore, the Σ-reducibility between the direct sums of cyclic p-groups will be studied in detail.

Set Theory

Organizers: J. Kellner, M. Goldstern.

▸ DAISUKE IKEGAMI, *Large cardinals, forcing axioms, and the theory of subsets of ω_1.*
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The goal of this research is to rule out "natural" independence phenomena in Set Theory by maximizing your theory in terms of large cardinals and forcing axioms. Using large cardinals in ZFC, by the results of Woodin [1], we have a clear understanding of the 1st order theory of sets of natural numbers and what it should be.

In this talk, we try to extend this understanding to the 1st order theory of subsets of ω_1 by using large cardinals, forcing axioms, and some hypothesis from inner model theory in ZFC. This is joint work with Matteo Viale.

[1] W. Hugh Woodin, ***The axiom of determinacy, forcing axioms, and the nonstationary ideal***, de Gruyter Series in Logic and its Applications, volume 1, Walter de Gruyter GmbH & Co. KG, Berlin, 2010.

▸ PHILIPP LÜCKE, *Locally definable well-orders.*
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A classical theorem of Mansfield shows that there exists a well-ordering of the set ${}^{\omega}\omega$ of all functions from ω to ω that is definable over the collection $\mathrm{H}(\omega_1)$ of all hereditarily countable sets by a Π_1-formula without parameters if and only if every such function is contained in Gödel's constructible universe L. In particular, the existence of such a well-ordering implies that the continuum hypothesis holds.

We consider the question whether this implication generalizes to higher cardinalities: does the existence of a well-ordering of the set ${}^{\omega_1}\omega_1$ of all functions from ω_1 to ω_1 that is definable over $\mathrm{H}(\omega_2)$ by a Π_1-formula without parameters imply that the GCH holds at ω_1?

This is joint work with Peter Holy (Bristol).

▸ DIEGO A. MEJÍA, *Matrix iterations and Cichoń's diagram.*
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Using matrix iterations of ccc posets we prove the consistency, with ZFC, of some constellations of Cichoń's diagram where the cardinals on the right hand side assume three different values. We also discuss the influence of the constructed models on other classical cardinal invariants of the continuum.

[1] D.A. Mejía, *Matrix iterations and Cichon's diagram*, ***Archive for Mathematical Logic***, vol. 52 (2013), no. 3–4, pp. 261–278.

[2] ——— *Models of some cardinal invariants with large continuum*, ***Kyōto daigaku sūrikaiseki kenkyūsho kōkyūroku***, (2013), pp. 36–48.

▸ KONSTANTIN SLUTSKY, *Regular cross-sections of Borel flows.*
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When working with measurable flows, it is sometimes convenient to choose a countable cross-section and to reduce a problem of interest to a similar question for the action induced by the flow on this cross-section. In some cases, one wants to impose additional restrictions on the cross-section, usually by restricting possible distances between points within each orbit.

Historically, cross-sections of flows were studied mainly in the context of ergodic theory. One of the most important results here is a theorem of D. Rudolph [3], which states that any free measure preserving flow, when restricted to an invariant subset of full measure, admits a cross-section with only two possible distances between adjacent points.

Borel dynamics deals with actions of groups on standard Borel spaces, when the latter is not equipped with any measure. In this more abstract context, one needs to construct cross-sections that are regular on all orbits without exceptions, and methods of ergodic theory, which tend to produce cross-sections only almost everywhere, are therefore frequently insufficient. In this regard, M. G. Nadkarni [2] posed a question whether the analog of Rudolph's Theorem holds true in the Borel setting: Does every free Borel flow admit a cross-section with only two different distances between adjacent points?

The talk will provide an overview of these and other results concerning the existence of regular cross-sections, and a positive answer to Nadkarni's question will be given. As an application of our methods, we give a classification of free Borel flows up to Lebesgue Orbit Equivalence, by which we understand orbit equivalence preserving Lebesgue measure on each orbit. This classification is an analog of the classification of hyperfinite equivalence relations obtained by R. Daugherty, S. Jackson, and A. S. Kechris [1].

[1] Randall Dougherty, Steve Jackson, Alexander S. Kechris, *The structure of hyperfinite Borel equivalence relations*, ***Transactions of the American Mathematical Society***, vol. 341 (1994), no. 1, pp. 193–225.

[2] Mahendra G. Nadkarni, ***Basic ergodic theory***, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, 1998.

[3] Daniel Rudolph, *A two-valued step coding for ergodic flows*, ***Mathematische Zeitschrift***, vol. 150 (1976), no. 3, pp. 201–220.

The place of logic in computer science education

Organizers: H. Veith, R. Zach.

▸ BYRON COOK, ALEXANDER LEITSCH, PRAKASH PANANGADEN, NICOLE SCHWEIKARDT, HELMUT VEITH, RICHARD ZACH, *The place of logic in computer science education.*

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Logic has been called the "calculus of computer science"—and yet, while any physics student is required to take several semesters of calculus, the same cannot be said about logic and students of computer science. Despite the great and burgeoning activity in logic-related topics in computer science, there has been very little interest, in North America at least, in developing a strong logic component in the undergraduate curriculum. Meanwhile, in other parts of the world, departments have set up specialized degree programs on logical methods and CS. This special session, organized under the auspices of the ASL's Committee on Logic Education, aims to explore the role of logic in the computer science curriculum. How are computer scientists trained in logic, if at all? What regional differences are there, and why? Is a greater emphasis on logic in the computer science undergraduate curriculum warranted, both from the point of view of training for research in CS and from the point of view of training for industry jobs? What should an ideal "Logic for Computer Science" course look like?

Byron Cook believes that, in the rush to create engineers and scientists, we have lost sight of the fact that an education should be broad and place emphasis on principles rather than specific skills such as Javascript. Logic is the perfect topic in this setting, as it has application in both humanities and science, and fosters a discussion about mechanics while not requiring a significant amount of technical overhead.

The Association for Computing Machinery has just chartered a new Special Interest Group on Logic and Computation (SIGLOG). Education is one of the prime concerns of this new SIG and one of the activities on the SIG's education committee will be to advocate for a greater presence of logic in the curriculum. Prakash Panangaden discusses the aims of the new SIG with particular emphasis on its educational mission.

Nicole Schweikardt will report on experiences with designing an undergraduate introductory course on logic in computer science at Goethe-University Frankfurt.

The University of Technology Vienna participates in a European Masters program in computational logic and has just started a doctoral program in Logical Methods in Computer Science. Alexander Leitsch describes these initiatives and considers lessons other departments can draw from the Vienna experience.

Presentations will be followed by a panel discussion. Materials will be available on the Committee on Logic Education website at `http://ucalgary.ca/aslcle/`.

1.4 Contributed talks

▸ ANTONIS ACHILLEOS, *Complexity bounds for Multiagent Justification Logic*.
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We investigate the complexity of systems of Multi-agent Justification Logic with interacting justifications (see [1]). The system we study has n agents, each based on some (single-agent) justification logic (we consider J, $\mathsf{J4}$, JD, $\mathsf{JD4}$, JT, LP) and a transitive, irreflexive binary relation, C. Each agent i has its own set of axioms, depending on the logic it is based on. If iCj, then we include axiom $t{:_i}\,\phi \rightarrow t{:_j}\,\phi$ (we do not include V-Verification as in [1]). Finally, it has a sufficient amount of propositional axioms and an axiomatically appropriate constant specification, which is in P. Traditionally, to establish upper complexity bounds for satisfiability for Justification Logic, we use a set of tableau rules to generate a branch and then we run the $*$-calculus on it.

A similar system for (diamond-free) modal logic was studied in [2]. We adjust appropriately the tableau for the corresponding system in [2] and the $*$-calculus can be run locally for every prefix, so we can use the same methods as in [2] to establish upper bounds. On the other hand, we can see that if we replace $\Box_i$ by $x{:_i}$ in a diamond-free modal formula (for all i), then the new formula is satisfiable iff the old one was. Thus, we can prove the same complexity bounds as in [2] - with the exception that where satisfiability for a modal logic is in NP, the corresponding justification logic has its satisfiability in Σ_2^p.

[1] Antonis Achilleos, *Complexity jumps in Multi-agent Justification Logic with interacting justifications*, Under submission

[2] ——— *Modal logics with hard diamond-free fragments*, Under submission

▸ RYOTA AKIYOSHI, *Proof-Theoretic Analysis of Brouwer's Argument of the Bar Induction*.
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In a series of papers, Brouwer had developed intuitionistic analysis, in particular the theory of choice sequences. An important theorem called the "fan theorem" plays an essential role in the development of it. The fan theorem was derived from another stronger theorem called the "bar induction", which is an induction principle on a well-founded tree. We refer to [4,5] as standard references of Brouwer's intuitionistic analysis.

Brouwer's argument in [1] contains a controversial assumption on canonical proofs of some formula. In many cases, constructive mathematicians have assumed the bar induction as axiom, hence the assumption has not been examined by them.

In this talk, we sketch an approach of Brouwer's argument via infinitary proof theory. We point out that there is a close similarity between Brouwer's argument and Buchholz' method of the Ω-rule ([2,3]). In particular, Brouwer's argument in [1] seems very close to Buchholz' embedding theorem of the (transfinite) induction axiom of ID_1 in [2], which is a theory of non-iterated inductive definition. By comparing these two arguments, we give a natural explanation of why Brouwer needed the assumption. Our conclusion is that Brouwer supposed the assumption in order to avoid the impredicativity or a vicious circle which is essentially the same as one in the Ω-rule for ID_1. In other words, the impredicativity can be explained in a very clear way from the view point of the Ω-rule. Moreover, Brouwer's argument can be formulated in a mathematically precise way by the Ω-rule. Therefore, we conclude that his introduction of the assumption is mathematically well-motivated. If time is permitting, we suggest how to carry out this idea in a mathematical way.

[1] Luitzen Egbertus Jan Brouwer, *Über Definitionsbereiche von Funktionen*, ***Mathematische Annalen***, vol. 97, 1927, pp. 60–75.

[2] Wilfried Buchholz, *The $\Omega_{\mu+1}$-Rule*, in ***Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies***, LNM 897, 1981, pp. 188–233.

[3] Wilfried Buchholz, *Explaining the Gentzen-Takeuti reduction steps*, ***Archive for Mathematical Logic***, vol. 40, pp. 255–272.

[4] Michael A. E. Dummett, ***Elements of Intuitionism***, 2nd edition, OUP, 2000.

[5] A. S. Troelstra and D. van Dalen, ***Constructivism in Mathematics***, Vol.1, North-Holland, 1988.

▸ PAVEL ALAEV, *The Δ^0_α-dimension of computable structures.*
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Let $\alpha \geqslant 1$ be a computable ordinal and $\mathfrak{A}$ be a computable structure. The Δ^0_α-dimension of $\mathfrak{A}$ is maximal $n \leqslant \omega$ such that there exist n computable presentations of $\mathfrak{A}$ without any Δ^0_α isomorphism between them. $\mathfrak{A}$ is Δ^0_α-categorical if this dimension is 1.

In [1], it was noted that if $\mathfrak{A}$ has a Σ^0_α Scott family then it is Δ^0_α-categorical. Moreover, a set of conditions $\Phi(\mathfrak{A})$ was found, under which this sufficient condition becomes necessary: if $\Phi(\mathfrak{A})$ holds then $\mathfrak{A}$ has a Σ^0_α Scott family iff it is Δ^0_α-categorical.

We prove that under a similar set of conditions $\Phi'(\mathfrak{A})$, this equivalence also holds, and, in addition, the Δ^0_α-dimension of $\mathfrak{A}$ is 1 or ω. The main part of this result is the theorem below. In addition, we fix a small error in the original formulation of $\Phi(\mathfrak{A})$.

If $\bar{a}, \bar{b}$ are tuples in $\mathfrak{A}$ of the same length, then $\bar{a} \leqslant_\alpha \bar{b}$ means that every infinite Π_α formula true on $\bar{a}$ is true on $\bar{b}$. $\mathfrak{A}$ is α-friendly if the relations $\leqslant_\beta$ are c.e. uniformly in $\beta < \alpha$. Let $\Rightarrow$ be a binary relation on finite tuples in $\mathfrak{A}$. We define a relation $\mathrm{Free}^{\Rightarrow}_\alpha(\bar{a}, \bar{c})$ on tuples in $\mathfrak{A}$ as follows:

$$\forall \beta < \alpha\ \forall \bar{a}_1\ \exists \bar{a}'\ \exists \bar{a}'_1\ \big[\, |\bar{a}| = |\bar{a}'|,\ \ \bar{c}, \bar{a}, \bar{a}_1 \leqslant_\beta \bar{c}, \bar{a}', \bar{a}'_1,\ \text{and}\ \bar{c}, \bar{a} \not\Rightarrow \bar{c}, \bar{a}' \,\big].$$

If $\Rightarrow$ is $\geqslant_\alpha$ then this definition coincides with the one in [1].

Theorem. Let $\mathfrak{A}$ be a computable α-friendly structure. Suppose that $\Rightarrow$ is a relation on finite tuples in $\mathfrak{A}$ such that
a) $\Rightarrow$ is transitive, i.e., $\bar{a} \Rightarrow \bar{b}$ and $\bar{b} \Rightarrow \bar{c}$ imply $\bar{a} \Rightarrow \bar{c}$;
b) if $g : \mathfrak{A} \to \mathfrak{A}$ is an automorphism then $\bar{a} \Rightarrow g(\bar{a})$ for every $\bar{a}$ in $\mathfrak{A}$.
If the relation $\not\Rightarrow$ is c.e. and for every $\bar{c}$ in $\mathfrak{A}$, we can effectively find $\bar{a}$ s.t. $\mathrm{Free}^{\Rightarrow}_\alpha(\bar{a}, \bar{c})$, then there exists a computable sequence $\{\mathfrak{B}_i\}_{i\in\omega}$ of computable presentations of $\mathfrak{A}$ s.t. there is no Δ^0_α isomorphism between $\mathfrak{B}_i$ and $\mathfrak{B}_j$ for $i \neq j$.

[1] C.J. Ash, *Categoricity in hyperarithmetical degrees*, ***Annals of Pure and Applied Logic***, vol. 34 (1987), no. 1, pp. 1–14.

▶ SVETLANA ALEKSANDROVA, *Uniformization in the hereditarily finite list superstructure over the real exponential field.*
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This work is concerned with the generalized computability theory, as well as properties of the real exponential field. To describe computability we use an approach via definability by $\Sigma-$formulas in hereditarily finite superstructures, which was introduced in [1].

In particular, we establish the uniformization property for $\Sigma-$predicates in the hereditarily finite list superstructure over the real exponential field. (See [2] for the structure's definition).

We shall outline the proof of the following theorem.

THEOREM 1. *For any $\Sigma-$predicate P in the hereditarily finite list superstructure over the real exponential field exists a $\Sigma-$function f with the domain $dom(f) = \{x : \exists y P(x, y)\}$ and graph $\Gamma_f \subseteq P$.*

As a corollary we obtain existence of an universal $\Sigma-$function in the same structure.

[1] ERSHOV, YU. L., ***Definability and Computability***, Consultants Bureau, New York-London-Moscow, 1996.

[2] GONCHAROV, S. S. AND SVIRIDENKO, D. I., *$\Sigma-$programming*, ***Vychislitelnye Sistemy***, 1985, no. 107, pp. 3–29.

▶ HAJNAL ANDRÉKA, ISTVÁN NÉMETI, *Weak Beth definability property for finite variable fragments.*
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Theorem Let $n > 2$. The n-variable fragment L_n of first-order logic does not have the weak Beth definability property (wBDP). Moreover, there are a theory Th and a strong implicit definition $\Sigma(D)$ for this theory such that there is no explicit definition for D even in the n-variable fragment $L^n_{\infty\omega}$ with infinite conjunctions and disjunctions, not even if we restrict the models to the finite ones. Th and Σ are in restricted L_n, i.e., do not use substitution of variables.

Discussion: Failure of wBDP for $n = 4$ was not known, failure for $n \geq 5$ was proved in [1], and for $n = 3$ in [2]. The present proof is considerably simpler than the ones in [1, 2]. Beth definability property fails for L_2, and it holds for L_n if we restrict the models to have cardinalities $< n + 2$, for all $n \geq 2$, see [3]. We conjecture that wBDP holds for L_2. If so, L_2 is a natural logic distinguishing Beth definability property from wBDP.

[1] HODKINSON, I., *Finite variable logics*, ***Bulletin of the European Association for Theoretical Computer Science***, vol. 51 (1993), pp. 111–140. With Addendum vol. 52. http://www.doc.ic.ac.uk/ imh/papers/yuri.ps.gz

[2] Simon, A., *Even the weak Beth property fails for logic with three variables*, ***Mathematical Institute of the Hungarian Academy of Sciences***, Preprint, 1993.

[3] Andréka, H., Comer, S. D., Madarász, J. X., Németi, I., Sayed-Ahmed, T., *Epimorphisms in cylindric algebras and definability in finite variable logic*, ***Algebra Universalis***, vol. 61 (2009), no. 3-4, pp. 261–282.

▸ OLGA ANTONOVA, *Aristotle's conception of demonstration and modern proof theory.*

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The history of modern mathematical proof theory begins with Beweistheorie or Hilbert's proof theory. The mathematical theories such as logicism (Frege, Russell), intuitionism (Brouwer, Heyting), set theory (Cantor, Dedekind) influenced directly the conception of proof and generally modern proof theory. The modern proof theory is based not only on mathematical theories, but also on the philosophical and logical proof theories, such as Aristotle's conception of demonstration. According to Aristotle a demonstration is a "scientific syllogism", in which the premises are true, first, immediate, more known than the conclusion, prior to the conclusion and causes of the conclusion. Aristotle's theory of demonstration impacted on the development of logic and, in particular, on the philosophical and logical conception of proof. Can we say that Aristotle's conception of demonstration is modern? Is the actual conception of proof really based on Aristotle's conception? The purpose of my talk is to analyze Aristotle's definition of demonstration and compare it with the modern approach to demonstration.

[1] Aristotle, ***Posterior analytics***, (transl. by J. Barnes), Oxford University Press, 1976.

[2] V. Hendrics et al. (eds.), ***Proof Theory: History and Philosophical Significance*** , Kluwer, 2000.

▸ MARIANNA ANTONUTTI-MARFORI, *Justifying proof-theoretic reflection principles.*

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It can be argued that by accepting the axioms of a theory as formally expressing our intuitive grasp of a mathematical structure—e.g. PA for arithmetic—we thereby implicitly commit ourselves to accepting certain other statements that are not formally provable from the axioms because of the incompleteness phenomena—such as the statement expressing the soundness of the axioms—and therefore to a fundamentally stronger theory. It follows that any formal theory that aims at capturing our pre-theoretic understanding of the natural numbers structure must admit of extensions; the question then arises as to how the axioms of arithmetic should be extended in order to construct a formal system that allows us to talk rigorously about the scope and limits of our arithmetical knowledge.

The process of recognising the soundness of the axioms is conceived of as a process of *reflection* on the given theory and the kinds of methods of proof that we recognise as correct. For this reason, the addition of proof-theoretic reflection principles as new axioms can be thought of as representing a natural way of extending PA in order to capture arithmetical knowledge.

I will distinguish two main strategies to justify the addition of reflection principles to be found in the literature (via transfinite induction, and via our truth-theoretic commitments), and I will argue that, contrary to these approaches, proof-theoretic reflection should be justified on the same fundamental grounds as our acceptance of the axioms of the initial system (see e.g. [1] and [2]). Furthermore, I will argue that on these grounds only *uniform* reflection is justified.

[1] FEFERMAN, S., *Transfinite Recursive Progressions of Axiomatic Theories*, ***The Journal of Symbolic Logic***, vol. 27 (1962), no. 4, pp. 259–316.

[2] FEFERMAN, S., *Turing in the land of O(z)*, ***The Universal Turing Machine. A Half-Century Survey*** (Rolf Herken, editor), Oxford University Press, Oxford, 1988, pp. 113–148.

[3] GÖDEL, K., *Some Basic Theorems on the Foundations of Mathematics and Their Implications*, ***Kurt Gödel. Collected Works. Volume III*** (Solomon Feferman *et alii*, editors), Oxford University Press, Oxford, 1995, pp. 304–323.

[4] MYHILL, J., *Some remarks on the notion of proof*, ***Journal of Philosophy***, vol. 57 (1960), no. 14, pp. 461–471.

▸ SERGEI ARTEMOV, TUDOR PROTOPOPESCU, *An outline of intuitionistic epistemic logic*.
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We outline an intuitionistic view of knowledge which maintains the Brouwer-Heyting-Kolmogorov semantics and is consistent with Williamson's suggestion that intuitionistic knowledge is the result of verification and that verifications do not necessarily yield strict proofs. On this view, $A \to \mathbf{K}A$ is valid and $\mathbf{K}A \to A$ is not. The former expresses the constructivity of truth, while the latter demands that verifications yield strict proofs. Unlike in the classical case where

$$\textit{Classical Knowledge} \;\Rightarrow\; \textit{Classical Truth}$$

intuitionistically

$$\textit{Intuitionistic Truth} \;\Rightarrow\; \textit{Intuitionistic Knowledge}.$$

Consequently we show that $\mathbf{K}A \to A$ is a distinctly classical principle, too strong as the intuitionistic truth condition for knowledge, "false is not known," which can be more adequately expressed by e.g., $\neg(\mathbf{K}A \wedge \neg A)$ or, equivalently, $\neg\mathbf{K}\bot$.

We construct a system of intuitionistic epistemic logic:

$$\mathsf{IEL} = \textit{intuitionistic logic } \mathsf{IPC} + \mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B) + (A \to \mathbf{K}A) + \neg\mathbf{K}\bot,$$

provide a Kripke semantics for it and prove IEL soundness, completeness and the disjunction property.

IEL can be embedded into an extension of S4, S4V, via the Gödel embedding *"box every subformula."* S4V is a bi-modal classical logic consisting of the rules and axioms of S4 for $\Box$ and D for $\mathbf{K}$, with the connecting axiom $\Box A \to \mathbf{K}A$. The soundess of the embedding is proved.

Within the framework of IEL, the knowability paradox is resolved in a constructive manner. Namely, the standard Church-Fitch proof reduces the intuitionistic knowability principle $A \to \Diamond\mathbf{K}A$ to $A \to \neg\neg\mathbf{K}A$, which is an IEL-theorem. Hence the knowability paradox in the domain of IEL disappears since neither of these principles are intuitionistically controversial. We argue that previous attempts to analyze the paradox were insufficiently intuitionistic.

▸ ARNON AVRON AND LIRON COHEN, *Predicative Mathematics via Safety Relations.*
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In [1] a new framework for predicative mathematics was developed. The main new features of this framework are that it is type-free, based on a subsystem of ZF, and the language it employs includes nothing that is not used in ordinary mathematical texts. In particular: it reflects real mathematical practice in making an extensive use of statically defined abstract set terms of the form $\{x|\varphi\}$.

In this work we show how large portions of classical analysis can be developed within that framework in a natural, predicatively acceptable way. Among other things, this includes the introduction of the natural numbers, the real line and continuous real functions, as well as formulating and proving the main classical results concerning these notions.

[1] Arnon Avron, *A New Approach to Predicative Set Theory*, ***Ways of Proof Theory*** (R. Schindler, editor), Onto Series in Mathematical Logic, onto verlag, 2010, pp. 31–63.

▸ SERIKZHAN BADAEV, SERGEY GONCHAROV, *Relativized universal numberings.*
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A numbering ν is called *universal in* a class $C(\mathfrak{F})$ of numberings of a family $\mathfrak{F}$ of sets, if $\nu \in C(\mathfrak{F})$ and every numbering of $C(\mathfrak{F})$ is reducible to ν. In the theory of numberings, a lot is known on universal numberings when $\mathfrak{F}$ is a family of sets lying in a given level of the arithmetical, or hyperarithmetical, or analytical hierarchy, or the hierarchy of Ershov, and $C(\mathfrak{F})$ is as the class of all computable numberings of $\mathfrak{F}$.

Let A be any set of natural numbers. A numbering ν of a family $\mathfrak{F}$ of A-c.e. sets is called *A-computable* if the sequence $\nu(0), \nu(1), \ldots$ is uniformly A-c.e. We will be concerned with those families $\mathfrak{F}$ of A-c.e. sets, that posses an A-computable numbering, and we will denote the class of all A-computable numberings of $\mathfrak{F}$ by $C_A(\mathfrak{F})$. W_x^A will stand for the A-c.e. set with Gödel index x.

Theorem 1. *If there exists an A-computable function g such that, for every x, $W^A_{g(x)} \in \mathfrak{F}$, and $W^A_x = W^A_{g(x)}$ if $W^A_x \in \mathfrak{F}$, then $\mathfrak{F}$ has a universal numbering in $C_A(\mathfrak{F})$.*

Theorem 2. *If $\emptyset' \leq_T A$ and $\mathfrak{F}$ has a universal numbering in $C_A(\mathfrak{F})$, then $\mathfrak{F}$ is closed under unions of increasing A-computable sequencs of sets from $\mathfrak{F}$.*

If $\mathfrak{F}$ contains the least set under inclusion then the condition $\emptyset' \leq_T A$ in Theorem 2 can be omitted.

Theorem 3. *If $\emptyset' \leq_T A$ then a finite family $\mathfrak{F}$ of A-c.e. sets has a universal numbering in $C_A(\mathfrak{F})$ if and only if $\mathfrak{F}$ contains the least set under inclusion.*

Theorem 4. *For every set A, there exists an infinite A-computable family $\mathfrak{F}$ of pairwise disjoint A-c.e. sets that has a universal numbering in $C_A(\mathfrak{F})$.*

Theorems 2 and 4 imply that the presence of the least set under inclusion in $\mathfrak{F}$ is neither necessary nor sufficient for an infinite family $\mathfrak{F}$ to have a universal numbering in $C_A(\mathfrak{F})$.

Theorem 5. *For every A, there is an infinite family $\mathfrak{F}$ with universal numbering in $C_A(\mathfrak{F})$ such that any infinite subfamily of $\mathfrak{F}$ has no Friedberg numbering.*

▸ NIKOLAY BAZHENOV, *Boolean algebras and degrees of autostability relative to strong constructivizations.*
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Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathfrak{A}$ is $\mathbf{d}$-*autostable* if, for every computable structure $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. A decidable structure $\mathfrak{A}$ is $\mathbf{d}$-*autostable relative to strong constructivizations* if every decidable copy $\mathfrak{B}$ of $\mathfrak{A}$ is $\mathbf{d}$-computably isomorphic to $\mathfrak{A}$.

Let $\mathfrak{A}$ be a computable structure. A Turing degree $\mathbf{d}$ is called the *degree of autostability* of $\mathfrak{A}$ if $\mathbf{d}$ is the least degree such that $\mathfrak{A}$ is $\mathbf{d}$-austostable. A degree $\mathbf{d}$ is the *degree of autostability relative to strong constructivizations* (*degree of SC-autostability*) of a decidable structure $\mathfrak{A}$ if $\mathbf{d}$ is the least degree such that $\mathfrak{A}$ is $\mathbf{d}$-autostable relative to strong constructivizations. Note that here we follow [1] and use the term *degree of autostability* in place of *degree of categoricity.* A great number of works (see, e.g., [2, 3, 4]) uses the term *degree of categoricity.*

Theorem 1. *Let α be a computable ordinal.*
(1) $\mathbf{0}^{(\alpha)}$ is the degree of autostability of some computable Boolean algebra;
(2) $\mathbf{0}^{(\alpha)}$ is the degree of SC-autostability of some decidable Boolean algebra.

Using the results of [3], we obtain the following corollaries.

Corollary 2. *There exists a decidable Boolean algebra without degree of SC-autostability.*

Corollary 3. *The index set of decidable Boolean algebras with degrees of SC-autostability is Π^1_1-complete.*

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[1] S. S. Goncharov, *Degrees of Austostability Relative to Strong Constructivizations*, ***Proceedings of the Steklov Institute of Mathematics***, vol. 274 (2011), no. 1, pp. 105–115.

[2] E. B. Fokina, I. Kalimullin, R. Miller, *Degrees of Categoricity of Computable Structures*, ***Archive for Mathematical Logic***, vol. 49 (2010), no. 1, pp. 51–67.

[3] B. F. Csima, J. N. Y. Franklin, R. A. Shore, *Degrees of Categoricity and the Hyperarithmetic Hierarchy*, ***Notre Dame Journal of Formal Logic***, vol. 54 (2013), no. 2, pp. 215–231.

[4] N. A. Bazhenov, *Degrees of Categoricity for Superatomic Boolean Algebras*, ***Algebra and Logic***, vol. 52 (2013), no. 3, pp. 179–187.

▸ GEGHARD BEDROSIAN AND FREDERIK HERZBERG, *Ultraproduct construction of representative utility functions with infinite-dimensional domain.*

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This paper applies model theory to macroeconomic theory. In mathematical models of macroeconomic theory, the hypothesis of a "representative agent" is ubiquitous, but the search for a rigorous justification has so far been unsuccessful and was ultimately abandoned until very recently. Herzberg (2010) constructed a representative utility function for finite-dimensional social decision problems, based on an bounded ultrapower construction over the reals, with respect to the ultrafilter induced by the underlying social choice function (via the Kirman–Sondermann (1972) correspondence). However, since the decision problems of macroeconomic theory are typically infinite-dimensional, Herzberg's original result is insufficient for many applications. We therefore generalise his result by allowing the social alternatives to belong to a general reflexive Banach space X; in addition to known results from convex analysis, our proof uses a nonstandard enlargement of the superstructure over $X \cup \mathbb{R}$, obtained by a bounded ultrapower construction with respect to the Kirman–Sondermann ultrafilter.

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▸ DAVID BELANGER, RICHARD SHORE, *A non-uniqueness theorem for jumps of principal ideals.*

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We show that for every degree $\mathbf{u}$ REA in $\mathbf{0}'$, there is a pair $\mathbf{a}_0, \mathbf{a}_1$ of distinct r.e. degrees such that $\mathbf{a}_0' = \mathbf{u} = \mathbf{a}_1'$, and such that the set $\{\mathbf{x}' : \mathbf{x} \leq \mathbf{a}_0\}$, which consists of all jumps of sets Turing-below $\mathbf{a}_0$, is equal to the corresponding set $\{\mathbf{x}' : \mathbf{x} \leq \mathbf{a}_1\}$. This defeats certain approaches to proving the rigidity of the r.e. degrees.

▶ THOMAS BENDA, *Formalizing vagueness as a doxastic, relational concept.*
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Descriptions and statements about the physical world often involve vague predicates, e.g., "x is red". It has become a common procedure to assign vague predicates intermediate truth values that are real numbers between 0 and 1. However, there is no satisfactory account what it means to be true to a given degree, which leaves doxastic degrees as the only option. Furthermore, real numbers provide an almost absurd accuracy as well as a natural metric, where in fact we want to state no more than, say, that x is rather red than not, perhaps redder or less red than some y. That suggests considering vagueness as a relational notion.

A thereby established vagueness relation is a partial order. Advantages of a relational account of vagueness are that vague predicates form a comparatively weak structure without metric and that the well-known problem of higher-order vagueness vanishes.

There is no reason not to implement doxastic degrees on the object language level. Furthermore, with the practice of evaluating vague predicates, relational vagueness should be allowed to depend on perception and epistemic as well as pragmatic context and hence be non-extensional. To set up a requisite formal language, we enclose vagueness predicates in quotation marks and perform their assessment with a background in mind which provides epistemic and pragmatic context. Thus a ternary predicate is introduced, B'Ax"Ay'b, read "I believe, with background b, Ax to at least as high a degree as Ay". Given background b, believing Ax with absolute confidence is formalized as B'Ax"0=0'b.

Such a formalization may be applied to conferring values to physical magnitudes which uses approximations and error bars. "The value of a is v" would then be vague as much as "x is red", acknowledging a fuzzy nature of experimental, particularly, macroscopic physics.

▶ ACHILLES A. BEROS, *A DNC function that computes no effectively bi-immune set.*
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In *Diagonally Non-Computable Functions and Bi-Immunity* [2], Carl Jockusch and Andrew Lewis-Pye proved that every DNC function computes a bi-immune set. They asked whether every DNC function computes an effectively bi-immune set. Several attempts have been made to solve this problem in the last few years. We construct a DNC function that computes no effectively bi-immune set, thereby answering their question in the negative. We obtain a few corollaries that illustrate how our technique can be applied more broadly.

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[2] Carl Jockusch and Andrew Lewis-Pye, *Diagonally non-computable functions and bi-immunity*, ***to appear in the Journal of Symbolic Logic***

[3] Antonín Kučera, *An alternative, priority-free, solution to Post's problem*, ***Mathematical Foundations of Computer Science***, vol. 233 (1986), pp. 493-500.

▸ KONSTANTINOS A. BEROS, *Co-analytic ideals on ω*.
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We consider a variant of the Rudin-Keisler order for ideals on ω and prove the existence of a complete co-analytic ideal with respect this order. The key tool is a parameterization of all co-analytic ideals. We obtain this parameterization via a method which yields a simple proof of Hjorth's 1996 theorem on the existence of a complete co-analytic equivalence relation. Unlike Hjorth's proof, ours does not rely on the use of the effective theory specific to Π^1_1 sets and thus generalizes under PD to other projective classes.

▸ RAVIL BIKMUKHAMETOV, *On Σ^0_2-initial segments of computable linear orders*.
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In my talk I consider the complexity of initial segments of computable linear orders. In all notations and definitions we shall adhere to [1]. A linear order $\mathcal{L} = (L, <_L)$ is *computable (X-computable)* if its domain is a computable (X-computable) set and its ordering relation is a computable (X-computable) relation. A suborder $\mathcal{I}$ of $\mathcal{L}$ is called *an initial segment* of $\mathcal{L}$ if

$$\forall x, y\ [(x <_L y\ \&\ y \in I) \Rightarrow x \in I]$$

M. Raw [2] showed that any Π^0_1-initial segment of a computable linear order has a computable presentation. On the other hand, he constructed a computable linear order with a Π^0_3-initial segment which has no computable copy. R. Coles, R. Downey and B. Khoussainov [3] showed that there is a computable linear order with a Π^0_2-initial segment which is not isomorphic to any computable linear order. Note that they obtained the previous result using an infinite injury priority method. M. V. Zubkov [4] proved the same result using only finite injury priority method. K. Ambos-Spies, S. B. S. Cooper and S. Lempp [5] showed that every Σ^0_2-initial segment of any computable linear order has a computable copy. We prove the following theorem which is a supplement to the previous result.

Theorem 1. *For any computable linear order $\mathcal{L} = (L, <_L)$ without the greatest element and for any set $M \in \Sigma^0_2$ there is a computable linear order $\tilde{\mathcal{L}} = \mathcal{A} + \eta$ such that $\mathcal{A} \cong \mathcal{L}$ and $\mathcal{A} \equiv_T M$.*

Clearly, every computable linear order with the greatest element can only be a computable (i.e., Σ_0-) initial segment. Thus, Σ^0_2-initial segments of computable linear orders contain in total all computable linear orders without the greatest elements and all Σ^0_2-degrees.

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▸ ALEXANDER C. BLOCK, *A new lower bound for the length of the hierarchy of norms.*
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A *norm* is a surjective function from the Baire space $\mathbb{R}$ onto an ordinal. Given two norms φ, ψ we write $\varphi \leq_{\mathrm{N}} \psi$ if φ continuously reduces to ψ. Then $\leq_{\mathrm{N}}$ is a preordering and so passing to the set of corresponding equivalence classes yields a partial order, the *hierarchy of norms*.

Assuming the axiom of determinacy (AD) the hierarchy of norms is a wellorder. The length Σ of the hierarchy of norms was investigated by Löwe in [1]; he determined that $\Sigma \geq \Theta^2$ (where $\Theta := \sup\{\alpha \mid \text{There is a surjection from } \mathbb{R} \text{ onto } \alpha\}$). In his talk "Multiplication in the hierarchy of norms", given at the ASL 2011 North American Meeting in Berkeley, Löwe presented a binary operation $\boxtimes$ on the hierarchy of norms such that for wellchosen norms φ, ψ the ordinal rank of $\varphi \boxtimes \psi$ in the hierarchy of norms is at least as big as the product of the ordinal ranks of φ and ψ, which implies that Σ is closed under ordinal multiplication and so $\Sigma \geq \Theta^\omega$.

In this talk I will note that in fact for wellchosen norms φ, ψ the ordinal rank of $\varphi \boxtimes \psi$ is exactly the product of the ranks of φ and ψ with an intermediate factor of ω_1. Furthermore using a stratification of the hierarchy of norms into initial segments closed under the $\boxtimes$-operation I will show that $\Sigma \geq \Theta^{(\Theta^\Theta)}$.

[1] Benedikt Löwe, *The length of the full hierarchy of norms*, ***Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino***, vol. 63 (2005), no. 2, pp. 161–168.

▸ HOVHANNES BOLIBEKYAN, *On the compactness theorem in many valued logics.*
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Nowadays many-valued logics occupy new areas of computer science. Being extensively used in various areas, theoretical investigations of different properties in such logics is a challenging area of research [1]. Firstly it worths mentioning that axiomatic systems for many valued logics are not well developed. Secondly many notions are not naturally extended in many valued logics from already existing analogues of classical or other "well-developed" non classical logics.
One of the key properties to characterize first order logic is compactness. We formulate an analogue of classical compactness theorem for arbitrary N-valued logic. To prove it overloading operators are constructed.

[1] Hahnle R. and Escalada-Imaz G., *Deduction in many-valued logics: a survey*, ***Mathware & Soft Computing IV (2), 1997, pp. 69-97.***

▸ WILL BONEY, *Computing the number of types of infinite length.*

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We show that the number of types of sequences of tuples of a fixed length can be calculated from the number of 1-types and the length of the sequences. Specifically, if $\kappa \leq \lambda$, then

$$\sup_{\|M\|=\lambda} |S^\kappa(M)| = \left(\sup_{\|M\|=\lambda} |S^1(M)| \right)^\kappa$$

We show that this holds for any abstract elementary class with λ amalgamation. No such calculation is possible for nonalgebraic types. We introduce a generalization of nonalgebraic types for which the same upper bound holds. We use this to answer a question of Shelah from [Sh:c].

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▸ BRANISLAV BORIČIĆ, MIRJANA ILIĆ, *An Intuitionistic Interpretation of Classical Implication*.

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A connection between the classical and the Heyting's logic is given by the Glivenko's Theorem: for every propositional formula A, A is classically provable iff $\neg\neg A$ is provable intuitionistically. This theorem can be understood as a possible way of intuitionistic interpretation of the classical reasoning. Embedding of the implicative fragment of classical logic into the implicative fragment of the Heyting's logic was considered by J. P. Seldin [3] and L. C. Pereira, E. H. Haeusler, V. G. Costa, W. Sanz [2]. Seldin's interpretation essentially depends on the presence of conjunction, but the second one is obtained in the pure language of implication. Here we define, in spirit of Kolmogorov's interpretation, a mapping of the pure implicational propositional language enabling to prove the corresponding result. Let $p_1, \ldots, p_n$ be a list of all propositional letters occurring in formula $A \to B$ and q any propositional letter not occurring in $A \to B$. Then the image $b(A \to B)$ of $A \to B$ is defined inductively as follows: $b(p) = (p \to q) \to q$, for each $p \in \{p_1, \ldots, p_n\}$, and $b(A \to B) = b(A) \to b(B)$. Namely, $b(A \to B)$ is obtained by replacing each occurrence of a propositional letter p in $A \to B$ by $(p \to q) \to q$, where q is a new letter.

Embedding Lemma. *For every propositional implicational formula A, A is provable in classical logic iff $b(A)$ is provable in Heyting logic.*

This is a part of our paper [1] dealing with an alternative approach to normalization of the implicative fragment of classical logic.

[1] B. Boričić, M. Ilić, *An Alternative Normalization of the Implicative Fragment of Classical Logic*, (to appear).

[2] L. C. Pereira, E. H. Haeusler, V. G. Costa, W. Sanz, *A new normalization strategy for the implicational fragment of classical propositional logic*, ***Studia Logica***, vol. 96 (2010), no. 1, pp. 95–108.

[3] J. P. Seldin, *Normalization and excluded middle I*, ***Studia Logica***, vol. 48 (1989), no. 2, pp. 193–217.

▸ MARIJA BORIČIĆ, *Models for the probabilistic sequent calculus.*

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The usual approach to treating the probability of a sentence leads to a kind of polymodal logic with iterated (or not iterated) probability operators over formulae (see [3]). On the other hand, there were some papers dealing with probabilistic form of inference rules (see [1], [2] and [4]). The sequent calculi present a particular mode of deduction relation analysis. The combination of these concepts, the sentence probability and the deduction relation formalized in a sequent calculus, makes it possible to build up sequent calculus probabilized — the system **LKprob**. Sequents in **LKprob** are of the form $\Gamma \vdash_a^b \Delta$, meaning that 'the probability of provability of $\Gamma \vdash \Delta$ is in interval $[a, b] \cap I$', where I is a finite subset of reals $[0, 1]$.

Let Seq be the set of all sequents of the form $\Gamma \vdash \Delta$. A model for **LKprob** is any mapping $p : \text{Seq} \to [0, 1]$ satisfying: *(i)* $p(A \vdash A) = 1$, for any formula A; 6 if $p(AB \vdash) = 1$, then $p(\vdash AB) = p(\vdash A) + p(\vdash B)$, for any formulas A and B; *(iii)* if sequents $\Gamma \vdash \Delta$ and $\Pi \vdash \Lambda$ are equivalent in **LK**, in sense that there are proofs for both sequents $\bigwedge \Gamma \to \bigvee \Delta \vdash \bigwedge \Pi \to \bigvee \Lambda$ and $\bigwedge \Pi \to \bigvee \Lambda \vdash \bigwedge \Gamma \to \bigvee \Delta$ in **LK**, then $p(\Gamma \vdash \Delta) = p(\Pi \vdash \Lambda)$.

We prove that our probabilistic sequent calculus **LKprob** is sound and complete with respect to the models just described.

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[2] T. Hailperin, *Probability logic*, ***Notre Dame Journal of Formal Logic***, vol. 25 (1984), pp. 198–212.

[3] Z. Ognjanović, M. Rašković, Z. Marković, *Probability logics*, ***Logic in Computer Science***, Zbornik radova 12 (20), Z. Ognjanović (ed.), Mathematical Institute SANU, Belgrade, 2009, pp. 35–111.

[4] C. G. Wagner, *Modus tollens probabilized*, ***British Journal for the Philosophy of Science***, vol. 54(4) (2004), pp. 747-753.

▸ EMANUELE BOTTAZZI, *Elementary numerosities and measures.*

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Generalizing the notion of numerosity, introduced in [1], we say that a function $\mathfrak{n}$ from the powerset of a given set Ω is an elementary numerosity if it satisfies the properties

1. the range of $\mathfrak{n}$ is the non-negative part of a non-archimedean field $\mathbb{F}$ that extends $\mathbb{R}$;
2. $\mathfrak{n}(\{x\}) = 1$ for every $x \in \Omega$;
3. $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B)$ whenever A and B are disjoint.

We have shown that every non-atomic finitely additive or sigma-additive measure is obtained from an elementary numerosity by taking its ratio to a unit. The proof of this theorem relies in showing that, given a non-atomic finitely additive or sigma-additive measure over Ω, there exists a suitable ultrafilter on Ω such that the elementary numerosity of a set can be defined as the equivalence class of a particular real Ω-sequence. A proof can also be obtained from Theorem 1 of [2], by an argument of saturation.

This result allowed to improve nonstandard models of probability, first studied in [3], that overcome some limitations of the conditional probability; further research aims towards models that avoid the Borel-Kolmogorov paradox. For this reasons, we do believe that this topic could be of particular interest not only to mathematicians but also to philosophers.

[1] V. Benci, M. Di Nasso, *Numerosities of labelled sets: a new way of counting*, ***Advances in Mathematics***, vol. 173 (2003), pp. 50–67.

[2] C.W. Henson, *On the nonstandard representation of measures*, ***Transactions of the American Mathematical Society***, vol. 172 (1972), pp. 437–446.

[3] V. Benci, L. Horsten, S. Wenmackers, *Non-Archimedean probability*, ***Milan Journal of Mathematics*** (to appear), arXiv:1106.1524.

▸ QUENTIN BROUETTE AND FRANÇOISE POINT, *Differential Galois theory in the class of formally real fields.*

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Inside the class of formally real fields, we study strongly normal extensions as defined in [1, chap. VI]. Fix L/K a strongly normal extension of formally real differential fields such that the subfield C_K of constant elements of K is real closed.

Let $\mathscr{U}$ be a saturated model of the theory of closed ordered differential fields containing L (see [3]), $\mathscr{U}$ is real closed and for $i^2 = -1$, $\mathscr{U}(i)$ is a model of DCF_0.

We denote $\mathrm{gal}(L/K)$ the group of differential K-automorphisms of L and $\mathrm{Gal}(L/K) := \mathrm{gal}(\langle L, C_{\mathscr{U}}\rangle/\langle K, C_{\mathscr{U}}\rangle)$.

Theorem 1. *The group* $\mathrm{Gal}(L/K)$*, respectively* $\mathrm{gal}(L/K)$*, is isomorphic to a definable group G in the real closed field $C_{\mathscr{U}}$, respectively C_K.*

Under the hypothesis that K is relatively algebraically closed in L, we prove that given any $u \in L \setminus K$, there exists $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(u) \neq u$.

Let $K \subseteq E \subseteq L$ be an intermediate differential field extension. As the elements of $\mathrm{Gal}(E/K)$ are not supposed to respect the order induced on $\langle E, C_{\mathscr{U}}\rangle$ by the one of $\mathscr{U}$, they do not need to have an extension in $\mathrm{Gal}(L/K)$. Therefore, we do not get a 1-1 Galois correspondence like in the classical case where C_K is algebraically closed (see [2]).

Let $\mathrm{Aut}(L/K)$ denote the subgroup of elements of $\mathrm{Gal}(L/K)$ that are increasing, let $\eta : G \to \mathrm{Gal}(L/K)$ denote a group isomorphism given by Theorem 1 and $\langle L, C_{\mathscr{U}}\rangle^r$ be the real closure of $\langle L, C_{\mathscr{U}}\rangle$ in $\mathscr{U}$.

Proposition 2. *Let G_0 be a definable subgroup of G. There is a finite tuple $\bar{d} \in \langle L, C_{\mathscr{U}}\rangle^r$ such that $\eta(G_0) \cap \mathrm{Aut}(L/K)$ is isomorphic (as a group) to* $\mathrm{Aut}(L(\bar{d})/K(\bar{d}))$.

[1] E. R. Kolchin, ***Differential algebra and algebraic groups***, Pure and Applied Mathematics, Vol. 54. Academic Press, New York-London, 1973.

[2] A. Pillay, *Differential Galois Theory I*, ***Illinois Journal of Mathematics***, vol. 42 (1998), no. 4, pp.978–699.

[3] M. Singer, *The model theory of ordered differential fields*, ***Journal of Symbolic Logic***, 43 (1978), no. 1, pp. 82–91.

▸ FRANQUI CÁRDENAS, ANDRÉS VILLAVECES, *Tameness from Woodin and unfoldable cardinals.*
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Boney in [1] showed for first time Shelah's Eventual Categoricity Conjecture from the existence of class many stronlgy compact cardinals: if K is AEC with $LS(K)$ below a strongly compact cardinals κ then K is $< \kappa$ tame. We prove similar results using Wooding cardinals and strongly unfoldable cardinals: if K is AEC with $LS(K)$ below a Woodin (strongly unfoldable) cardinal κ then K is $< \kappa$ tame.

[1] Boney Will, *Tameness from large cardinals*, ***Submitted***,

▸ ANAHIT CHUBARYAN, ARMEN MNATSAKANYAN, HAKOB NALBANDYAN, *On proof complexities of strong equal modal tautologies.*
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The research of the lengths of proofs in the systems of propositional calculus is important because of its relation to some of main problems of the common complexity theory. The investigations of proof complexity start for the systems of Classical Propositional Logic (CPL). However, natural real conclusions have constructive character and the most statements of natural and technical languages have modalities (necessary and possible). Therefore the investigation of the proofs complexities is important also for the systems of Intuitionistic Propositional Logic (IPL) and in some cases also for Modal Propositional Logic (MPL). The information about proof complexity in IPL and MPL can be important, in particular, to formalize reasoning about the way programs behave and to express dynamical properties of transitions between states.

The strong equality of tautologies in CPL and IPL, based on the notion of determinative conjunct, was introduced by first coauthor earlier (strong equality implies well-known equality but not vice versa), and the relations between the proof complexities of strong equal tautologies in different proof systems of CPL and IPL are investigated.

By analogy with the notions of determinative conjuncts in CPL, we introduce the same notion for modal tautologies. On the base of introduced modal determinative conjuncts we introduce the notion of strong equality for modal tautologies and compare different measures of proof complexity (size, steps, space and width) for them in some proof systems of MPL. We prove that 1) in some proof systems the strong equal modal tautologies have the same proof complexities and 2) there are such proof systems, in which some measures of proof complexities for strong equal modal tautologies are the same, the other measures differ from each other only by the sizes of tautologies.

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▸ WILLEM CONRADIE, ANDREW CRAIG, *Algorithmic-algebraic canonicity for mu-calculi.*
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The correspondence and completeness of logics with fixed point operators has been the subject of recent research (see [1], [2]). These works aim to develop a Sahlqvist-like theory for their respective fixed point settings. That is, they identify classes of formulas which are preserved under canonical extensions and have first-order frame correspondents.

We prove that the members of a certain class of intuitionistic mu-formulas are *canonical*, in the sense of [1]. When projected onto the classical case, our class of canonical mu-formulas subsumes the class described in [1]. Our methods use a variation of the algorithm ALBA (Ackermann Lemma Based Algorithm) developed in [3]. We show that all mu-inequalities that can be successfully processed by our algorithm, μ^*-ALBA, are canonical.

Formulas are interpreted on a bounded distributive lattice $\mathbf{A}$ with additional operations. The canonical extension of $\mathbf{A}$, denoted $\mathbf{A}^\delta$, is a complete lattice in which the completely join-irreducible elements ($J^\infty(\mathbf{A}^\delta)$) are join-dense, and the completely meet-irreducible elements ($M^\infty(\mathbf{A}^\delta)$) are meet-dense. An *admissible valuation* takes all propositional variables to elements of $\mathbf{A}A$. The algorithm aims to "purify" an inequality $\alpha \leqslant \beta$ by rewriting it as a (set of) pure (quasi-)inequalities. A pure quasi-inequality has no occurrences of propositional variables; only special variables whose interpretations range over $J^\infty(\mathbf{A}^\delta) \cup M^\infty(\mathbf{A}^\delta)$ are present. The fact that admissible and ordinary validity coincide for pure inequalities is the lynchpin for proving canonicity.

The proof of the soundness of the rules of the algorithm μ^*-ALBA rests on the order-topological properties of formulas (term functions) of the μ-calculus.

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[2] W. Conradie, Y. Fomatati, A. Palmigiano, S. Sourabh, *Algorithmic correspondence for intuitionistic modal mu-calculus*, ***Theoretical Computer Science***, to appear.

[3] W. Conradie, A. Palmigiano, *Algorithmic correspondence and canonicity for distributive modal logic*, ***Annals of Pure and Applied Logic***, vol. 163 (2012), pp. 338–376.

▸ WILLEM CONRADIE AND CLAUDETTE ROBINSON, *Hybrid extensions of S4 with the finite model property.*
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In [1] R.A. Bull characterized a class of axiomatic extensions of the modal logic S4 (the logic of the class of transitive and symmetric Kripke frames) with the finite model property. This result takes the form of a syntactic characterization of a class of formulas that may be added as axioms to S4, somewhat in the spirit of Sahlqvist's famous result in modal correspondence theory. Hybrid logics (see e.g. [2]) expand the syntax of modal logic by adding special variables, know as nominals, which are always interpreted as singletons in models and thus act as names for the states at which they hold. Additional syntactic machinery which capitalizes on the naming power of the nominals, like the satisfaction operator $@_i\phi$ or the universal modality, is often added. This makes hybrid languages significantly more expressive than their modal cousins, while retaining their good computational behaviour. In this talk we show how to extend Bull's result to three hybrid languages. The proofs we offer are algebraic and serve to illustrate the usefulness of the new algebraic semantics for hybrid logics recently introduced by the authors. Bull's proof makes essential use of the algebraic property of 'well-connectedness' which is equivalent, in the dual relational semantics, to the ability to take generated submodels. Since the truth of hybrid languages is not invariant under generated submodels, the generalization to hybrid logic is not straight-forward.

[1] R.A. Bull, *A class of extensions of the modal system S4 with the finite model property*, ***Zeitschrift für mathematische logik und grundlagen der mathematik***, vol. 11 (1965), pp. 127–132.

[2] George Gargov and Valentin Goranko, *Modal Logic with Names*, ***Journal of philosophical logic***, vol. 22 (1993), no. 6, pp. 607–636.

▸ JOHN CORCORAN AND JOSÉ MIGUEL SAGÜILLO, *Teaching independence of proposition sets.*

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In this lesson, 'independent' expresses a property of sets [of propositions] as in 'Gödel's Axiom-Set is independent'. As such, it resembles the words 'consistent', 'categorical', etc. The abstract treats only two of several senses of the adjective 'independent': *Propositionally Independent* [PropInd] and *Informationally Independent* [InfoInd]. PropInd refers to propositions per se and dates from the 1890s; InfoInd refers to information in propositions and dates from the 1990s.

This lecture builds on [1] and lectures abstracted in this Bulletin, vol. 15 (2009), pp. 244–245 and vol. 16 (2010), pp. 436–437, and p. 443. Examples employ the 1931 Gödel Axiom-Set GAX: the Zero Axiom, the Successor Axiom, and the Induction Axiom [1, pp. 13f].

A set is *propositionally independent* iff no member proposition follows from the rest. A set that is non-PropInd is redundant *itself*: it has an excess member deletable without losing information.

A set is *informationally independent* iff no information is repeated (shared between two of its members), i.e., no non-tautological consequence of one member follows from another. A set that is non-InfoInd might not be redundant *itself* but it has a *member* that is redundant: a member that has excess information.

For example, if {A, B, C} is InfoInd, then the [logically] equivalent set {(A&C),

(B&C)} is not redundant *itself*—neither member can be deleted without loss of information—but either *member* is redundant: the C can be dropped from one. {A, (B&C)} and {(A&C), B} are both equivalent to {A&C, B&C}.

InfoInd is neither necessary nor sufficient for PropInd. {0 = 0} is InfoInd but not PropInd. The Gödel Axiom-Set GAX is PropInd but not InfoInd, as shown in [1] where an InfoInd equivalent to GAX is constructed.

[1] John Corcoran, *Information recovery problems*, ***Theoria***, vol. 10 (1995), pp. 55–78.

▸ JOHN CORCORAN AND KEVIN TRACY, *Aristotle's third logic: deduction*.
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Consider the following four sentence schemata: distinct English common nouns replace placeholders.

(Every * No * Some * Not-every) S is a P.

Every instance sentence has two terms: "subject" and "predicate".

Aristotle constructed a "first logic" before constructing the familiar "syllogistic" or "second logic" [1]. Syllogistic argument constituents—premises and conclusions—are expressible using such sentences. We propose a "third logic" Aristotle could have constructed next—using three and four-noun sentences with restrictive relative clauses: 'that is an [...]'.

Every S that is an R is an M.
Every R that is an M is a P.
Every R that is an S is a P.

Instances of this "five-term" argument schema cannot be seen to be valid using the second logic. However, they can be seen to be valid using rules Aristotle could accept: expanding his rules of deduction—"conversions" and "perfect syllogisms"—could produce the following deduction schema, using notation from [2].

1.	Every S that is an R is an M.	
2.	Every R that is an M is a P.	
?	Every R that is an S is a P.	
3.	Every S that is an R is an M that is an R.	(1) Restriction Repetition
4.	Every M that is an R is a P.	(2) Subject-Restriction Conversion
5.	Every S that is an R is a P.	(3, 4) Subject-Restriction Barbara
6.	Every R that is an S is a P.	(5) Subject-Restriction Conversion

QED

This begins a series of lectures treating Aristotle's third logic.

[1] JOHN CORCORAN, *Completeness of an ancient logic*, ***Journal of Symbolic Logic***, vol. 37 (1972), pp. 692–702.

[2] ——— *Aristotle's demonstrative logic*, ***History and Philosophy of Logic***, vol. 30 (2009), pp. 1–20.

▸ JOHN CORCORAN, *Teaching course-of-values induction.*
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Let P be a property that belongs to every number whose predecessors all have it.

Clearly, P could belong to every number: if P belongs to every number, then—*a-fortiori*—P belongs to every number whose predecessors all have it.

Is the converse true? Is it the case that if P belongs to every number whose predecessors all have it, then P belongs to every number? *A-fortiori* reasoning is often non-reversible.

Does P belong to zero? It does *if* P belongs to all of zero's predecessors. No number precedes zero. *A-fortiori*, no number precedes zero but does not have P. Thus—vacuously—P belongs to all of zero's predecessors. Thus—by hypothesis—P belongs to zero.

What else can we determine about any property that—like P—belongs to every number whose predecessors all have it? Does it belong to one? Of course, since zero is the only predecessor of one. Continuing, zero and one are the only predecessors of two and they both have P. Thus two has P. By this kind of bootstrapping, we see that for any given number x, P belongs to x.

Thus, the above converse *is* true: If P is a property that belongs to every number whose predecessors all have it, then P belongs to every number. This is the *course-of-values induction principle* CVIP, also called—more revealingly—the *cumulative induction principle* CIP.

There are other ways of stating CIP or its logical equivalents.

Every property that belongs to every number whose predecessors all have it belongs to every number.

In order for a property to belong to every number, it is sufficient for it to belong to every number whose predecessors all have it.

In order for a property to belong to every number whose predecessors all have it, it is necessary for it to belong to every number.

CIP in symbols: $\forall P\,[(\forall x\,(\forall y\,(y < x \rightarrow Py) \rightarrow Px) \rightarrow \forall x\,Px]$

▸ ANDRÉS CORDÓN–FRANCO AND F. FÉLIX LARA–MARTÍN, *Π_1–induction axioms vs Π_1–induction rules: Some conservation results.*
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As proved independently by Mints, Adamowicz–Bigorajska, Kaye and Ratajczyk, if a Π_2–sentence θ is derived (over the base theory $I\Delta_0$) using m instances of parameter–free Σ_1–induction axiom scheme then θ can also be derived using at most m (nested) applications of Σ_1–induction rule. If θ is a Π_1–sentence then a similar result for Π_1–induction can be proved by exploiting the equivalence between Local Σ_2–reflection and the parameter–free Π_1–induction axiom scheme, $I\Pi_1^-$ (see [1]). However, due to the use of Local Reflection principles, the base theory used in this result must extend at least $I\Delta_0 + \exp$ and, as far as we know, no similar results for $I\Pi_1^-$ are known over plain $I\Delta_0$.

In this work we address this question. Working over $I\Delta_0$, we obtain a number of conservation results relating the number of instances of $I\Pi_1^-$ needed to derive a sentence θ, and the number and depth of *nested* applications of Π_1–induction rule needed in a derivation of θ. Several formulations of Π_1–induction rule are considered in correspondence with the quantifier complexity of the sentence θ.

Our approach is model–theoretic and uses theories of *Local Induction* as a basic tool.

[1] Beklemishev, L.D. *Reflection principles and provability algebras in formal arithmetic*. Russian Math. Surveys 60:197–268, 1999.

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The idea that the good model-theoretic and algorithmic properties of Modal Logics are due to the guarded nature of their quantification was put forward by Andreka, van Benthem and Nemeti in a series of paper in the '90 (see e.g. [1]), exploiting the satisfiability problem, the tree model property, and other similar properties of the guarded fragment of First Order Logic (GF).
Since then, further work on the guarded fragment has been done by various authors, up to the present days, in some cases enforcing this idea, in some others not. At least at a first sight, Craig interpolation is on the negative side: there are implications in GF without an interpolant in GF, while Modal Logic (and even the μ-calculus, a powerful extension of Modal Logic) enjoys a much stronger form of interpolation, the uniform one, in which the interpolant of a valid implication not only exists, but only depends on the antecedent and on the common language of antecedent and consequent. However, Hoogland and Marx ([2]) proved that Craig interpolation is restored in GF if we consider the modal character of GF with more attention, that is, if relations appearing on guards are viewed as "modalities" and the rest as "propositions", and only the latter enter in the common language. In this paper we strengthen this result by showing that GF allows a Modal Uniform Interpolation Theorem (in the sense of Hoogland and Marx).

[1] H. Andreka, I. Nemeti, J.van Benthem, *Modal Languages and Bounded Fragments of Predicate Logic*, ***Journal of Philosophical Logic***, vol. 27 (1998), no. 3, pp. 217–274.

[2] E. Hoogland, M. Marx., *Interpolation and Definability in Guarded Fragments*, ***Studia Logica***, vol. 70 (2002), no. 3, pp. 373–409.

▶ BRUNO DINIS, IMME VAN DEN BERG, *An axiomatic approach to modelling of orders of magnitude*.
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Many arguments deal informally with orders of magnitude of numbers. If one tries to maintain the intrinsic vagueness of orders of magnitude - they should be bounded, but stable under at least some additions -, they cannot be formalized with ordinary real numbers, due to the Archimedean property and Dedekind completion. Still there is the functional approach through Oh's and oh's and more generally Van der Corput's neutrices[1], both have some operational shortcomings.

Nonstandard Analysis disposes of a natural example of order of magnitude: the (external) set of infinitesimals is bounded and closed under addition[5][6]. Adopting the terminology of Van der Corput, we call a *neutrix* an additive convex subgroup of the nonstandard reals. An *external number* is the set-theoretic sum of a nonstandard real and a neutrix. The external numbers capture the imprecise boundaries of informal orders of magnitude and permit algebraic operations which go beyond the calculus of the Oh's and oh's[2]. This external calculus happens to be based more on semigroup operations than group operations, but happens to be fairly operational in concrete cases and allows for total order with a generalized form of Dedekind completion[3].

Based on joint work with Imme van den Berg, we discuss an axiomatics for the calculus of neutrices and external numbers, trying to do justice to the vagueness of orders of magnitude. In particular we consider foundational problems which appear due to the fact that some axioms are necessarily of second order, and the fact that the external calculus exceeds existing foundations for external sets[4].

[1] J.G. van der Corput, *Neutrix calculus, neutrices and distributions*, MRC Tecnical Summary Report. University of Wisconsin, 1960.

[2] B. Dinis, I. P. van den Berg, *Algebraic properties of external numbers*, J. Logic and Analysis 3:9 (2011) 1–30.

[3] B. Dinis, I. P. van den Berg, *On structures with two semigroup operations* (to appear)

[4] V. Kanovei, M. Reeken, Nonstandard Analysis, axiomatically, Springer Monographs in Mathematics (2004).

[5] F. Koudjeti, Elements of External Calculus with an aplication to Mathematical Finance, PhD thesis, Labyrinth publications, Capelle a/d IJssel, The Netherlands (1995).

[6] F. Koudjeti and I.P. van den Berg. Neutrices, external numbers and external calculus, in Nonstandard Analysis in Practice, p. 145-170. F. and M. Diener eds., Springer Universitext, 1995.

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The generic ultrafilter $\mathcal{G}_2$ forced by the partial ordering $\mathcal{P}(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin})$ is a non-p-point which is also not a Fubini product of p-points, but is a Rudin-Keisler immediate successor of its projected Ramsey ultrafilter. In [1], it was shown that $\mathcal{G}_2 \not\geq_T [\omega_1]^{<\omega}$, and hence is below the maximum Tukey type for ultrafilters, yet it is not basically generated. In [2], we show that, in fact, $\mathcal{G}_2$ is a Tukey immediate successor of its projected Ramsey ultrafilter, and moreover, the projected Ramsey ultrafilter is the only nonprincipal ultrafilter with Tukey type strictly below that of $\mathcal{G}_2$. This is done by showing that $\mathcal{P}(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin})$ contains a dense subset which in fact forms a topological Ramsey space and then proving a Ramsey-classification theorem for equivalence relations on fronts. Moreover, we generalize this to show that for all $2 \leq k < \omega$, $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ is forcing equivalent to a new topological Ramsey space $\mathcal{E}_k$ which is a generalization of the Ellentuck space. The generic ultrafilters $\mathcal{G}_k$ are non-p-points which have exactly k Tukey predecessors, as well as exactly k Rudin-Keisler predecessors.

[1] Andreas Blass, Natasha Dobrinen, Dilip Raghavan, *The next best thing to a p-point*, ***Submitted***.

[2] Natasha Dobrinen, *Generalized Ellentuck spaces and initial chains in the Tukey structure of non-p-points*, ***Preprint***.

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In [1], the following definition was introduced.

Definition 1. A Polish structure is a pair (X, G), where G is a Polish group acting faithfully on a set X so that the stabilizers of all singletons are closed subgroups of G. We say that (X, G) is small if for every $n < \omega$, there are only countably many orbits on X^n under the action of G.

Notice that, in the above definition, it is not required that X is a topological space. I will discuss some issues concerning existence of topologies on X satysfying some natural conditions. Special attention will be given to the case in which X carries a structure of a group (i. e., (X, G) is a Polish group structure).

[1] K. Krupiński, *Some model theory of Polish structures*, ***Transactions of the American Mathematical Society*** (362), 3499-3533, 2010.

▸ MARINA DORZHIEVA, *Computable numberings in Analytical Hierarchy.*
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We investigate minimal enumerations in analytical hierarchy. Enumeration $\nu \in Com^1_{n+1}$ is called minimal, if for every $\mu \in Com^1_{n+1}$ such that $\mu \leq \nu$, perfomed $\nu \equiv \mu$. One of the most important minimal numberings is Friedbergs numbering. Owings showed in [1] that there is no Π^1_1-computable Friedberg enumeration of all Π^1_1-sets using metarecursion theory. This result is obtained in classic computability theory for higher levels of analytical hierarchy:

Theorem 1. *(1) There are infinite minimal numberings of an infinite family S of Π^1_{n+1}-sets.*

(2) There is no a Π^1_{n+1}-computable Friedberg enumeration of all Π^1_{n+1}-sets.

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[1] James C. Owings, ***The meta-r.e. sets, but not the Π^1_1-sets can be enumerated without repetition***, The Journal of Symbolic Logic, Volume 35, Number 2, June 1970.

▸ MATĚJ DOSTÁL, JIŘÍ VELEBIL, *Enriched Morita equivalence of many-sorted theories.*
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When do two theories give rise to equivalent categories of models for these theories? Such theories are called Morita equivalent. This question has been studied in many contexts. In module theory this is to ask for a description of rings that have equivalent categories of modules. Throwing away the requirement for additivity, we ask when two monoids generate equivalent categories of monoid actions. In categorical algebra, two Lawvere theories are Morita equivalent if their respective categories of algebras are equivalent.

For the examples mentioned above, the characterisation of Morita equivalent theories is well-known. In [1], the authors generalise the result concerning Morita equivalent Lawvere theories to the setting of many-sorted algebras and their theories. We show that their approach can be in fact vastly generalised and present a Morita-type theorem characterising Morita equivalent theories, and make it parametric in the notion of theory. Thus we cover all the examples previously mentioned, and much more.

[1] Jiří Adámek, Manuela Sobral, and Lurdes Sousa *Morita equivalence of many-sorted algebraic theories*, ***Journal of Algebra***, vol. 297 (2006), no. 2, pp. 361–371.

▸ BENEDICT EASTAUGH, *Computational reverse mathematics and foundational analysis.*
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Reverse mathematics studies which natural subsystems of second order arithmetic are equivalent to key theorems of ordinary or non-set-theoretic mathematics. The main philosophical application of reverse mathematics proposed thus far is *foundational analysis*, which explores the limits of various weak foundations for mathematics in a formally precise manner. Richard Shore [1, 2] proposes an alternative framework in which to conduct reverse mathematics, called *computational reverse mathematics*. The formal content of his proposal amounts to restricting our attention to ω-models of RCA_0 when we prove implications and equivalences in reverse mathematics.

Despite some attractive features, computational reverse mathematics is inappropriate for foundational analysis, for two major reasons. Firstly, the computable entailment relation employed in computational reverse mathematics does not preserve justification for all of the relevant foundational theories, particularly a partial realisation of Hilbert's programme due to Simpson [3].

Secondly, computable entailment is a Π^1_1-complete relation, and hence employing it commits one to theoretical resources which outstrip those acceptable to the stronger foundational programmes such as predicativism and predicative reductionism. This argument can be formalised within second order arithmetic, making it accessible to partisans of foundational frameworks such as predicativism. In doing so we show that the existence of the set of sentences which are computably entailed is equivalent over ACA_0 to Π^1_1 comprehension.

[1] SHORE, R. A., *Reverse Mathematics: The Playground of Logic*, ***The Bulletin of Symbolic Logic***, vol. 16 (2010), no. 3, pp. 378–402.

[2] SHORE, R. A., *Reverse mathematics, countable and uncountable: a computational approach*, ***Effective Mathematics of the Uncountable***, Lecture Notes in Logic, (D. Hirschfeldt, N. Greenberg, J. D. Hamkins, and R. Miller, editors), ASL and Cambridge University Press, Cambridge, 2013, pp. 150–163.

[3] SIMPSON, S G., *Partial realizations of Hilbert's program*, ***The Journal of Symbolic Logic***, vol. 53 (1988), pp. 349–363.

▸ CHRISTIAN ESPÍNDOLA, *Semantic completeness of first order theories in constructive reverse mathematics.*

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We introduce a general notion of semantic structure for first-order theories, covering a variety of constructions such as Tarski and Kripke semantics, and prove that, over Zermelo Fraenkel set theory (ZF), the completeness of such semantics is equivalent to the Boolean Prime Ideal theorem (BPI). In particular, we deduce that the completeness of that type of semantics for non-classical theories is unprovable in intuitionistic Zermelo Fraenkel set theory IZF ([4]). Using results of Joyal ([2]) and McCarty ([3]), we conclude also that the completeness of Kripke semantics is equivalent, over IZF, to the Law of Excluded Middle plus BPI. By results in [1], none of these two principles imply each others, and so this gives the exact strength of Kripke completeness theorem in the sense of constructive reverse mathematics.

[1] BANASCHEWSKI, B. AND BHUTANI, K., *Boolean algebras in a localic topos*,***Mathematical Proceedings of the Cambridge Philosophical Society***, vol. 100 (1986), pp. 43–55.

[2] MAKKAI, M. AND REYES, G., ***First order categorical logic***, Lecture Notes in Mathematics, vol. 611, 1977.

[3] MCCARTY, D.C., *Completeness and incompleteness for intuitionistic logic*, ***Journal of Symbolic Logic***, vol. 73 (2008), no. 4, pp. 1315–1327.

[4] MYHILL, J., *Some properties of Intuitionistic Zermelo-Fraenkel set theory*, ***Proceedings of the 1971 Cambridge Summer School in Mathematical Logic***, Lecture Notes in Mathematics 337, 1973, pp. 206–231.

▸ MARAT FAIZRAHMANOV, ISKANDER KALIMULLIN, *Limitwise monotonic sets of reals.*

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We extend the limitwise monotonicity notion to the case of arbitrary computable linear ordering to get a set which is limitwise monotonic precisely in the non-computable degrees. Also we get a series of connected non-uniformity results to obtain new examples of non-uniformly equivalent families of computable sets with the same enumeration degree spectrum.

▸ TUAN-FANG FAN, CHURN-JUNG LIAU, *FUZZY BISIMULATION FOR STANDARD Gödel MODAL LOGIC.*
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The alphabet of standard Gödel modal logic $G(\Box\Diamond)$ consists of a set of propositional symbols, PV, the set of truth constants $\bar{c}$ for each rational $c \in [0,1]$, the logical connectives $\wedge$ and $\to$, and the modal operators $\Box$ and $\Diamond$. The well-formed formulas are defined by the following rules:

$$\varphi ::= p \mid \bar{c} \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in PV, c \in [0,1]$, and φ and ψ are formulas. A Kripke-Gödel model for $G(\Box\Diamond)$ is a triple $\mathfrak{M} = (W, R, V)$, where W is a set of possible worlds, $R : W \times W \to [0,1]$ is a fuzzy relation on W, and $V : W \times PV \to [0,1]$ is a truth assignment. The truth assignment can be extended to the set of all formulas in the following way:

1. $V(w, \bar{c}) = c$
2. $V(w, \varphi \wedge \psi) = \min(V(w,\varphi), V(w,\psi))$
3. $V(w, \varphi \to \psi) = V(w,\varphi) \Rightarrow V(w,\psi)$, where $\Rightarrow: [0,1]^2 \to [0,1]$ is the residuum of the min operation, i.e. $a \Rightarrow b = \max\{c \mid \min(a,c) \leq b\}$,
4. $V(w, \Box\varphi) = \inf_{u\in W}(R(w,u) \Rightarrow V(u,\varphi))$,
5. $V(w, \Diamond\varphi) = \sup_{u\in W} \min(R(w,u), V(u,\varphi))$.

A fuzzy bisimulation between two models $\mathfrak{M}_1 = (W_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, V_2)$ is any fuzzy relation $Z : W_1 \times W_2 \to [0,1]$ that satisfies (1) for any $w_1 \in W_1$ and $w_2 \in W_2$, $Z(w_1, w_2) \leq \inf_{p\in PV}(V_1(w_1,p) \Leftrightarrow V_2(w_2,p))$, where $\Leftrightarrow$ is the equivalence operation on $[0,1]$, i.e., $a \Leftrightarrow b = \min(a \Rightarrow b, b \Rightarrow a)$; and (2) $R_1 \cdot Z = Z \cdot R_2$, where $\cdot$ is the sup-min composition between fuzzy relations. A model is image-finite if for any $w \in W$, the set $\{u \mid R(w,u) > 0\}$ is finite. We prove the Hennessy-Milner style theorem for fuzzy bisimulation. That is, for any two image-finite models $\mathfrak{M}_1 = (W_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, V_2)$ and $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$Z(w_1.w_2) = \inf_{\varphi\in\mathcal{G}}(V_1(w_1,\varphi) \Leftrightarrow V_2(w_2,\varphi)),$$

where $\mathcal{G}$ denotes the set of all formulas.

▸ HADI FARAHANI, HIROAKIRA ONO, *Predicate Glivenko theorems and substructural aspects of negative translations.*
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In [1], the second author has developed a proof-theoretic approach to Glivenko theorems for substructural propositional logics. In the present talk, by using the same techniques, we will extend them for substructural predicate logics. It will be pointed out that in this extension, the following *double negation shift* scheme (DNS) plays an essential role.

$$(DNS):\ \forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$$

Among others, the following is shown, where QFLe and QFLe† are predicate extensions of FLe and FLe†, respectively (see [1]). The Glivenko theorem holds for QFLe† + (DNS) relative to classical predicate logic. Moreover, this logic is the weakest one among predicate logics over QFLe for which the Glivenko theorem holds relative to classical predicate logic. Then we will study negative translations of substructural predicate logics by using the same approach. We introduce a negative translation, called extended Kuroda translation and over QFLe it will be shown that standard negative translations like Kolmogorov translation and Gödel-Gentzen translation are equivalent to our extended Kuroda translation. Thus, we will give a clearer unified understanding of these negative translations by substructural point of view.

[1] H. Ono, *Glivenko Theorems Revisited*, ***Annals of Pure and Applied Logic***, vol. 161(2009), pp. 246–250.

[2] J. Avigad, *A variant of the double-negation translation*, ***Carnegie Mellon Technical Report CMU-PHIL***, vol. 179(2006).

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A recent approach by Beklemishev uses provability logics to represent reflection principles in formal theories and uses said principles to callibrate a theory's consistency strength [1]. There are several benefits to this approach, including semi-finitary consistency proofs and independent combinatorial statements.

A key ingredient is Japaridze's polymodal provability logic GLP_ω [4]. In order to study stronger theories one needs to go beyond GLP_ω to the logics GLP_Λ, where Λ is an arbitrary ordinal. These logics have for each ordinal $\xi < \Lambda$ a modality $\langle\xi\rangle$. Proof theoretic ordinals below Γ_0 may be represented in the closed fragment of GLP_Λ *worms* therein [2, 3]. Worms are iterated consistency statements of the form $\langle\xi_n\rangle \ldots \langle\xi_1\rangle\top$ and are well-ordered by their consistency strength.

We present a calculus for computing the order types of worms and compare the resulting ordinal representation system with standard systems based on Veblen functions. We will also discuss how larger ordinals arising from impredicative proof theory may be represented within provability logics.

[1] Beklemishev, L.D., *Provability algebras and proof-theoretic ordinals, I*, ***Annals of Pure and Applied Logic***, vol. 128 (2004), pp. 103-124.

[2] Beklemishev, L.D., *Veblen hierarchy in the context of provability algebras*, ***Logic, Methodology and Philosophy of Science, Proceedings of the Twelfth International Congress*** (P. Hájek and L. Valdés-Villanueva and D. Westerstahl, editors), King's College Publications (2005).

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▸ HARTRY FIELD, HARVEY LEDERMAN, *Prospects for a naïve theory of classes.*

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We examine the prospects for a naïve theory of classes, in which full "naïve" comprehension and an extensionality rule are maintained by weakening the background logic. Without extensionality, proving naïve comprehension consistent is formally analogous to proving naïve truth consistent, and in recent years much progress has been made on the latter question. But there is no natural analog for extensionality in the case of truth, so the question arises whether these logics for reasoning about truth can also be shown consistent with a form of extensionality. In a series of papers, and in his 2006 book ([1]), Ross Brady has presented various theories of naïve classes. We begin by providing a simpler, more accessible version of Brady's proof of the consistency of these theories. Our new presentation of Brady then makes it easy to see how Brady's result can be generalized to apply to certain logics which have a modal-like semantics given using four-valued, as opposed to three-valued worlds. (These include some logics from [2].) These "new" logics have a significant advantage over Brady's original: they validate a weakening rule (indeed, a weakening axiom) for a non-contraposable conditional. Since these laws are crucial if the conditional is to be used for restricted quantification, this is a substantial improvement.

Still, we do not think even these logics are satisfactory. The non-contraposable conditional which validates weakening in these logics is not the conditional of the extensionality rule. But there's strong intuitive motivation for the conditional in the extensionality rule to validate weakening. Otherwise, there will be "sets" which contain everything, but which are not extensionally equivalent. While Brady's logics (and the four-valued generalizations) deliver a form of extensionality, in the absence of weakening the formal rule does not capture the intuitive notion of extensionality.

[1] Ross T. Brady, ***Universal Logic***, CSLI Publications, 2006.

[2] Andrew Bacon, *A new conditional for naïve truth theory*, ***Notre Dame Journal of Formal Logic***, vol. 54 (2013), no.1, pp. 87–104.

▸ OLIVIER FINKEL, *Infinite Games Specified by 2-Tape Automata.*

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We prove that the determinacy of Gale-Stewart games whose winning sets are infinitary rational relations accepted by 2-tape Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption. Then we prove that winning strategies, when they exist, can be very complex, i.e. highly non-effective, in these games. We prove the same results for Gale-Stewart games with winning sets accepted by real-time 1-counter Büchi automata, then extending previous results obtained about these games.

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that: (a) there is a model of ZFC in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$ but σ cannot be recursive and not even in the class $(\Sigma^1_2 \cup \Pi^1_2)$; (b) there is a model of ZFC in which the game $G(L(\mathcal{A}))$ is not determined.
2. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that $L(\mathcal{A})$ is an arithmetical Δ^0_3-set and Player 2 has a winning strategy in the game $G(L(\mathcal{A}))$ but has no hyperarithmetical winning strategies in this game.
3. There exists a recursive sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $\mathcal{A}_n$, $n \geq 1$, such that all games $G(L(\mathcal{A}_n))$ are determined, but for which it is Π^1_2-complete hence highly undecidable to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$.

[1] O. Finkel. The determinacy of context-free games. *The Journal of Symbolic Logic*, 78(4):1115–1134, 2013. Preprint available from http://arxiv.org/abs/1312.3412.

[2] O. Finkel. Infinite games specified by 2-tape automata. Preprint available from http://fr.arxiv.org/abs/1312.3797.

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▸ EKATERINA FOKINA, PAVEL SEMUKHIN, AND DANIEL TURETSKY, *Degree spectra of structures under Σ_1-equivalence.*

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The properties of degree spectra of countable structures have been studied extensively in computable model theory. Recently Andrews and Miller [1] introduced and studied a notion of the degree spectra of a theory which is defined as $\mathrm{DegSp}(T) = \{\, \mathrm{Deg}(\mathcal{M}) \mid \mathcal{M} \text{ is a model of } T\}$. In particular, they constructed a theory whose spectrum is equal to a non-degenerate union of two cones, which is known to be impossible for a degree spectrum of a structure.

In our work we consider an analogous question for Σ_n-spectrum of a structure. We say two structures $\mathcal{A}$ and $\mathcal{B}$ are Σ_n-equivalent, denoted $\mathcal{A} \equiv_{\Sigma_n} \mathcal{B}$, if they satisfy the same Σ_n-sentences. Let $\mathcal{A}$ be a countable structure. The Σ_n-spectrum of $\mathcal{A}$ is defined as $\mathrm{DegSp}(\mathcal{A}, \equiv_{\Sigma_n}) = \{\, \mathrm{Deg}(\mathcal{B}) \mid \mathcal{B} \equiv_{\Sigma_n} \mathcal{A} \,\}$. The construction from [1] actually implies that there is a structure $\mathcal{A}$ such that $\mathrm{DegSp}(\mathcal{A}, \equiv_{\Sigma_2})$ is equal to a non-degenerate union of two cones. We show that this result does not hold anymore if we replace

Σ_2-equivalence by Σ_1-equivalence.

THEOREM 1. *Let $\mathcal{A}$ be a countable structure. Then* $\mathrm{DegSp}(\mathcal{A}, \equiv_{\Sigma_1})$ *cannot be equal to a non-degenerate union of two cones.*

[1] URI ANDREWS, JOSEPH S. MILLER, *Spectra of theories and structures*, ***to appear in the Proceedings of the American Mathematical Society***.

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We present a unifying, categorical approach to several constructive completeness theorems for intuitionistic (and classical) first-order theories, as well for theories in certain fragments of first-order logic, based on a theorem by A. Joyal [1, Thm 6.3.5]. We show that the notion of exploding (Tarski-) model introduced by W. Veldman [2] is adequate for certain fragments of first-order logic (as well as for classical first-order logic) and that Veldman's modified Kripke semantics arises, as a consequence, as semantics in a suitable sheaf topos. In the process we give an alternative proof of Veldman's completeness theorem, and note the equivalence of this theorem with the Fan Theorem. Finally, we show that the disjunction-free fragment is constructively complete with respect to modified Kripke semantics without appeal to the Fan Theorem, as well as without appeal to decidable axiomatizability and size restrictions on the language. This is joint work with Christian Espíndola.

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▸ SABINE FRITTELLA, *Display-type calculi for non classical logics.*
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The proposed talk gives a general outline of a line of research [1, 2, 3] aimed at providing a wide array of logics (spanning from dynamic logics to monotone modal logic through substructural logics) with display type proof systems which in particular enjoy a uniform strategy for cut elimination. We generalize display calculi in two directions: by explicitly allowing different types, and by dropping the full display property. The generalisation to a multi-type environment makes it possible to introduce specific tools enhancing expressivity, which have proved useful e.g. for a smooth proof-theoretic treatment of multi-modal and dynamic logics. The generalisation to a setting in which the full display property is not required makes it possible to account for logics, such as monotone modal logic, which admit connectives which are neither adjoints nor residuals.

Keywords: display-type calculus, multi-type calculus, proof theory, dynamic logic, monotone modal logic, substructural logic.

[1] S. FRITTELLA, G. GRECO, A. KURZ, A. PALMIGIANO AND V. SIKIMIĆ, *A proof-theoretic semantic analysis of dynamic epistemic logic*, ***Journal of Logic and Computation, Special Issue on Substructural Logic and Information Dynamics***, to

appear.

[2] ——— *Multi-type Display Calculus for Dynamic Epistemic Logic*, ***Journal of Logic and Computation, Special Issue on Substructural Logic and Information Dynamics***, to appear.

[3] S. Frittella and G. Greco, *Display-type sequent calculus for monotone modal logic*, ***AiML2014***, Submitted.

▸ ANDREY FROLOV, MAXIM ZUBKOV, *On categoricity of scattered linear orders.*
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We consider the categoricity of countable scattered linear orders. Recall that linear order is *scattered* if it has no dense suborder. A computable linear order L is *computably (Δ^0_n-, resp.) categorical* if for every computable copy L' of L there is a computable (Δ^0_n-, resp.) isomorphism between L' and L. J. Remmel [1], S. Goncharov, V. Dzgoev [2] obtained the description of computably categorical linear orders. Namely, they proved that a computable linear order is computably categorical if and only if it contains finitely many pairs of successors. Ch. McCoy [3] obtained the description of Δ^0_2-categorical computable linear order with additional conditions. We proved that if L is a computable scattered linear order such that L is a finite sum of scattered orders of rank n then L is Δ^0_{2n}-categorical. The definition of rank of scattered linear orders can be fined in [4].

[1] Remmel J. B., *Recursively categorical linear orderings*, ***Proceedings of the American Mathematical Society***, vol. 82 (1981), no. 2, pp. 387–391.

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In [2], for any $n \geq 2$, it was constructed a linear ordering L such that the spectrum $Sp(L)$ contains exactly all non-low$_n$ degrees. Recall, the *spectrum* of a linear ordering L is the class $Sp(L) = \{\deg_T(R) \mid R \cong L\}$.

R. Miller [3] constructed a linear ordering whose Δ^0_2-spectrum contains exactly all nonlow$_0$ Δ^0_2-degrees, i.e., all nonzero Δ^0_2-degrees. The Δ^0_2-*spectrum* of linear ordering L is the class $Sp(L)^{\Delta^0_2} = \{\deg_T(R) \in \Delta^0_2 \mid R \cong L\} = Sp(L) \cap \Delta^0_2$.

The author [1] constructed a linear ordering whose Δ^0_2-spectrum contains exactly all nonlow$_1$ Δ^0_2-degrees.

In [2], for any $n \geq 2$, it was constructed a linear ordering L such that $Sp(L)$ contains exactly all high$_n$ degrees. Also in [2] it was remarked that there does not exist a linear orderings L such that $Sp(L)$ is exactly all high$_n$ degrees for $n \in \{0, 1\}$.

THEOREM 1. *There exists a linear ordering L such that $Sp^{\Delta^0_2}(L) = \{\mathbf{0}'\}$. In other words, Δ^0_2-spectrum of L contains exactly all high$_0$ Δ^0_2-degrees.*

THEOREM 2. *There exists a linear ordering whose Δ^0_2-spectrum contains exactly all high$_1$ Δ^0_2-degrees.*

[1] A. FROLOV, *Δ^0_2-copies of linear orderings*, ***Algebra and Logic***, vol. 45 (2006), no. 3, pp. 201–209 (in english), pp. 69–75 (in russian).

[2] A. FROLOV, V. HARIZANOV, I. KALIMULLIN, O. KUDINOV, R. MILLER, *Spectra of high$_n$ and nonlow$_n$ degrees*, ***Journal of Logic and Computation***, vol. 22 (2012), no. 4, pp. 745–754.

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▸ HAO-CHENG FU, *A Defense of Information Economy Principle.*
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In our ordinary life it is inevitable for everyone has to adjust one's own belief state in light of new information when the new information is inconsistent with his belief state. Some philosophers such as Quine and *AGM* (Alchourrón et al.) suggested that the loss of information value should be minimized as possible whenever one confronts the inconsistency and the principle in belief revision theory is usually so-called information economy principle (*IEP* for short). Furthermore, Gärdenfors has constructed a model who recommended the idea of epistemic entrenchment to this model to explain why *IEP* works. But Rott casted some doubts on *IEP* due to the postulates of epitemic entrenchment proposed by Gärdenfors sometimes failed to realize the features of non-monotonic reasoning, i.e. it is possible that one might keep the less entrenched beliefs rather than the more ones in the process of belief change. In this paper, I want to present a game-theoretic framework to reconstruct the notion of epistemic entrenchment to avoid the challenges from Rott and prove that *IEP* is still available to be the norm to estimate the process of belief change.

Keywords: 03B42, belief change, information economy principle, epistemic entrenchment, game theory

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▸ MICHAL GARLÍK, *Model theory of bounded arithmetic and complexity theory.*
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It is well known that some problems in complexity theory can be cast as problems of constructions of expanded extensions of models of bounded arithmetic. These models are usually required to satisfy some form of bounded induction but at the same time not introduce any new lengths of strings. We shall discuss some general facts and one specific construction of this kind.

► VALENTIN GORANKO, *On the almost sure validities in the finite in some fragments of monadic second-order logic.*
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This work builds on the well-known *0-1 law* for the asymptotic probabilities of first-order definable properties of finite graphs (in general, relational structures). Fagin's proof of this result is based on a transfer between almost sure properties in finite graphs and true properties of the *countable random graph* (aka, Rado graph).

Both the transfer theorem and the 0-1 law hold in some non-trivial extensions of first-order logic (e.g., with fixed point operators) but fail in others, notably in most natural fragments of monadic second-order (MSO) and even for modal logic formulae, in terms of frame validity. The question we study here is how to characterise – axiomatically or model-theoretically – the set of *almost surely valid in the finite* formulae of MSO, i.e. those with asymptotic probability 1. This question applies likewise to every logical language where truth on finite structures is well-defined. The set of almost sure validities in the finite of a given logical language is a well-defined logical theory, containing all validities of that language and closed under all sound finitary rules of inference. Beyond that, very little is known about these theories in cases where the transfer theorem fails.

In this work we initiate a study of the theories of almost sure validity in modal logic and in the Π^1_1 and Σ^1_1 fragments of MSO on binary relational structures, aiming at obtaining explicit logical characterisations of these theories. We provide such partial characterisations in terms of characteristic formulae stating almost sure existence (for Σ^1_1) or non-existence (for Π^1_1) of bounded morphisms to special target finite graphs. Identifying explicitly the set of such finite graphs that generate almost surely valid characteristic formulae seems a quite difficult problem, to which we so far only provide some partial answers and conjectures.

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This report concerns to the problem of constructing tableau-based proof procedure for a logic of generalized truth values [1, 2].

Generalized truth values are based on the two 'sorts' of truth, ontological (we denote it as 't') and epistemic ones ('1'). They constitute a four-element lattice with natural set theoretical order and familiar binary operations: $L = (\{\varnothing, \{1\}, \{t\}, \{t, 1\}\}, \subseteq, \cap, \cup)$.

One of the most interesting feature of this structure is a definition of the unary operations. We introduce two of them: $-_t$ sends $\varnothing$ to $\{t\}$ and back, $\{1\}$ to $\{t, 1\}$ and back, while $-_1$ switches between $\varnothing$ and 1, and between $\{t\}$ and $\{t, 1\}$. This semantic structure gives rise to a propositional logic based on the language over $\{\wedge, \vee, \neg_t, \neg_1\}$ with classical binary operation and two non-classical negation-style connectives. It is worth mention that combinations $\neg_t\neg_1$ or $\neg_t\neg_1$ behave exactly like boolean negation.

For a definition of a semantic consequence relation there are several candidates, each of its own interest. We choose the simplest and most natural one: $A \vDash B$ iff the value of A is a subset os the value of B.

We propose two different tableau-style formalisation for a logic which captures a syntactical analogue of semantic logical consequence relation. One of them is more or less 'traditional' and resembles tableau systems for relevant logic FDE [3]. Another one is appropriate for designing a proof search procedure and based on well known KE formalism [3].

[1] Grigoriev O. *Bipartite truth and semi-negations.* In: ***Proceedings of 7-th International conference 'Smirnov readings in logic'***, June 22–24, Moscow, Sovremennye tetradi, 2011, pp. 54–55.

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Recent investigations in denotational semantics led Ehrhard to define a new model of λ-calculus and linear logic called *finiteness spaces* (see [1]). In finiteness spaces, types are interpreted as topological vector spaces, λ-terms and (*via* the Curry-Howard correspondence) intuitionistic proofs are interpreted as analytic functions in topological vector spaces.

However, analytic functions are infinitely differentiable, thus the question arise whether differentiation can be defined as a meaningful syntactic operation. A positive answer from a computational perspective is given by differential λ-calculus [1], an extension of λ-calculus in which the operations of differentiation and Taylor expansion of a λ-term are definable. One of the interests of differential λ-calculus lies in the analogy that can be drawn between the usual notion of differentiation (i.e. linearity in mathematical sense) and its computational meaning (i.e. linearity in computational sense).

It turns out that differentiation and the actual infinite operation involved in Taylor expansion makes sense also in a purely logical setting. The right syntax is provided by differential linear logic (**DiLL**) and analyzed in terms of differential interaction nets in [2]. **DiLL** is an extension of **LL** characterized by three new rules dealing with the ! modality: *cocontraction*, *codereliction*, and *coweakening*. The latter rules are called "co-structural" and, in a one-sided sequent calculus setting, can be considered as symmetric duals of the ?-rules. Co-structural rules give a logical status to differentiation.

In this paper, we introduce a natural deduction system for intuitionistic DiLL. We show normalization and, as corollaries, subformula, separability and introduction form properties for this system. Its relationships with the natural deduction systems for intuitionistic logic are discussed.

[1] Ehrhard, T. and L. Regnier (2003) The differential lambda-calculus. *Theoretical Computer Science*, 309(1): 1-41.

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▸ MIHA E. HABIČ, *Restricting Martin's axiom to a ccc ground model.*
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We introduce a variant of Martin's axiom, called the grounded Martin's axiom or grMA. This principle asserts that the universe is a ccc forcing extension and that MA holds for posets from the ground model. The new axiom, which emerges naturally from the analysis of the Solovay-Tennenbaum proof of the consistency of MA, is shown to have many of the desirable properties of the weaker fragments of MA. In particular, we show that grMA is consistent with a singular continuum and also that it is consistent with the left side of Cichoń's diagram collapsing to ω_1. We also show that grMA is better behaved than MA when adding generic reals. Specifically, grMA is preserved under adding a Cohen real and holds after adding a random real to a model of MA.

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In parallel with the Borel hierarchy, one can define the levels $\Sigma^0_\alpha(\Pi^1_1)$ ($\alpha < \omega_1$) of the Borel-on-coanalytic hierarchy by starting with Π^1_1 in place of the class Δ^0_1 of clopen sets. In this talk, we consider the consistency strength of determinacy for infinite perfect-information games with payoff in $\Sigma^0_\alpha(\Pi^1_1)$. This has been computed exactly for $\alpha = 0, 1$, by Martin, Harrington, and J. Simms. For $\alpha > 1$, dual results of Steel [2] and Neeman [1] have shown the strength to reside within a very narrow range in the region of a measurable cardinal κ of largest possible Mitchell order $o(\kappa)$. However, an exact equiconsistency had yet to be isolated.

We have recently completed work pinpointing the determinacy strength of levels of the Borel hierarchy of the form $\Sigma^0_{1+\alpha+3}$, showing a level-by-level correspondence between these and a family of natural Π_1 reflection principles. Combining our techniques with those of [1] and [2], we can characterize the strength of $\Sigma^0_{1+\alpha+3}(\Pi^1_1)$-DET in terms of inner models with measurable cardinals. In particular, $\Sigma^0_4(\Pi^1_1)$-DET is equivalent to the existence of a mouse satisfying $(\exists\kappa)o(\kappa) = \kappa^{++}$ plus the schema that each true Π_1 statement with parameters in $P^2(\kappa)$ reflects to an admissible set containing $P(\kappa)$.

We will also discuss progress on calculating the strength of $\Sigma^0_2(\Pi^1_1)$-DET, relating this to Mitchell's hierarchy of weak repeat point measures.

[1] Itay Neeman, *Unraveling* Π^1_1 *sets, revisited*, ***Israel Journal of Mathematics***, vol. 152 (2006), pp. 181–203.

[2] John R. Steel, *Determinacy in the Mitchell models*, ***Annals of Mathematical Logic***, vol. 22 (1982), no. 2, pp. 109–125.

▸ HARRISON-TRAINOR, MATTHEW, *Degree spectra of relations on a cone.*
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We consider structures $\mathcal{A}$ with an additional relation R. We say that two relations R and S on structures $\mathcal{A}$ and $\mathcal{B}$ respectively have the same (relativised) degree spectrum if, for sets C on a cone above $\mathbf{d}$,

$$\{R^{\tilde{\mathcal{A}}} \oplus C : \tilde{\mathcal{A}} \cong \mathcal{A} \text{ and } \tilde{\mathcal{A}} \leq_T C\} = \{S^{\tilde{\mathcal{B}}} \oplus C : \tilde{\mathcal{B}} \cong \mathcal{B} \text{ and } \tilde{\mathcal{B}} \leq_T C\}.$$

Using determinacy, these degree spectra are partially ordered. Many classes of degrees which relativise, such as the Σ^0_α degrees or α-CEA degrees, are degree spectra. This is a notion which captures solely the model-theoretic properties of the relation R. We will advocate for the naturality of this viewpoint by recasting existing results in this new language, giving new results, and putting forward new questions. Existing results of Harizanov in [3] show that there are two minimal degree spectra, the computable sets and the c.e. sets. In [1] and [2], Ash and Knight considered whether Harizanov's results could be generalised. We give a partial positive answer by showing that any degree spectrum which contains a non-Δ^0_2 degree contains all of the 2-CEA degrees. We also give an example of two incomparable degree spectra.

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▶ NADJA HEMPEL, *Around n-dependent fields.*
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The notion of n-dependent theories introduced by Shelah is a natural generalization of dependent or more frequently called NIP theories. They form a proper hierarchy of first order theories in which the case n equals to 1 coincides which NIP theories.

In my talk, I give an overview about algebraic extensions of fields defined in structures with certain properties (superstable, stable, NIP, etc.). For instance, infinite NIP fields of positive characteristic are known to be Artin-Schreier closed. I extend this result to the wider class of infinite n-dependent fields for any natural number n and present some applications to valued fields defined in this setting. Secondly, I show that non-separable closed pseudo-algebraically closed (PAC) fields have the n-independence property for all natural numbers n which is already known for the independence property (n equal to 1) due to Duret. Hence, non-separable closed PAC fields lie outside of the hierarchy of n-dependent fields.

▶ KOJIRO HIGUCHI, *The order dimensions of degree structures.*
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We investigate the *order dimensions* of several degree structures such as Turing degree structure. It may be nice if we can decompose a given degree structure into "simpler" partial orders naturally defined for the structure. Indeed, it is known that every partial order is embeddable into the product order of a family of *linear* orders. The order dimension of a given partial order is defined as the least cardinality of such a family. Thus, the order dimension of a degree structure tells us how many linear orders at least we should have so that the degree structure is embeddable into the product order of those linear orders. The concept "order dimension" was introduced by Dushnik and Miller in 1941, and it is also called Dushnik-Miller dimension. As our main results on the order dimensions of degree structures, this talk includes the following results: the order dimension of Turing degree structure is uncountable and at most the cardinality of the continuum; the order dimension of Muchnik degree structure is the cardinality of the continuum; and the order dimension of Medvedev degree structure is lying between the cardinality of the continuum and the cardinality of the power set of the continuum.

▸ ASSYLBEK ISSAKHOV, *Ideals without minimal numberings in the Rogers semilattice.*
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It is well known many infinite families of c.e. sets whose Rogers semilattice contains an ideal without minimal elements, for instance, the family of all c.e. sets, [1]. Moreover, there exists a computable family of c.e. sets whose Rogers semilattice has no minimal elements at all, [2]. In opposite to the case of the families of c.e. sets, for every computable numbering α of an infinite family $\mathfrak{F}$ of computable functions, there is a Friedberg numbering of $\mathfrak{F}$ which is reducible to α, [1]. This means that the Rogers semilattice of any computable family of total functions from level 1 of the arithmetical hierarchy contains no ideal without minimal elements.

We study computable families of total functions of any level of the Kleene-Mostowski hierarchy above level 1 and try to find elementary properties of Rogers semilattices that are different from the properties of Rogers semilattices for the families of computable functions.

THEOREM 1. *For every n, there exists a Σ^0_{n+2}-computable family of total functions whose Rogers semilattice contains an ideal without minimal elements.*

Note that every Rogers semilattice of a Σ^0_{n+2}-computable family $\mathfrak{F}$ contains the least element if $\mathfrak{F}$ is finite, [1], and infinitely many minimal elements, otherwise, [3].

Theorem 1 is based on the following criterion that extends the criterion for minimal numbering from [2].

THEOREM 2. *Let α be a numbering of an arbitrary set $\mathcal{S}$. Then there is no minimal numbering of $\mathcal{S}$ that is reducible to α if and only if, for every c.e. set W, if $\alpha(W) = S$ then there exists a c.e. set V such that $\alpha(V) = \mathcal{S}$ and, for every positive equivalence ε, either $\varepsilon \restriction W \nsubseteq \theta_\alpha$ or $W \nsubseteq [V]_\varepsilon$.*

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▸ ALEKSANDER IVANOV, *Extreme amenability of precompact expansions of countably categorical structures.*

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A group G is called **amenable** if every G-flow (i.e. a compact Hausdorff space along with a continuous G-action) supports an invariant Borel probability measure. If every G-flow has a fixed point then we say that G is **extremely amenable**. Let M be a relational countably categorical structure which is a Fraïssé limit of a Fraïssé class $\mathcal{K}$. To see whether $Aut(M)$ is amenable one usually looks for an expansion M^* of M so that M^* is a Fraïssé structure with extremely amenable $Aut(M^*)$. Moreover it is usually assumed that M^* is a **precompact** expansion of M, i.e. every member of $\mathcal{K}$ has finitely many expansions in $Age(M^*)$. Some theorems of O.Angel, A.Kechris, R.Lyone and A.Zucker describe amenability of $Aut(M)$ in this situation. It is a basic question in the subject if there is a countably categorical structure M with amenable automorphism group which does not have expansions as above.

We connect this material with the property of existence of nice enumerations, introduced by G.Ahlbrandt and M.Ziegler in 1986. We also give some interesting examples of countably categorical structures M so that $Aut(M)$ is amenable, but M does not have order expansions with extremely amenable automorphism groups.

▸ GRZEGORZ JAGIELLA, *Definable topological dynamics and real Lie groups.*
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Methods of topological dynamics have been introduced to model theory by Newelski in [3] and since then saw further development in that field by other authors. Given a model M with all types over M definable and a definable group G, we consider the category of definable flows. This category has a universal object $S_G(M)$, the space of types in G over M. It is show that the Ellis semigroup of this flow is isomorphic to $S_G(M)$ itself. It can be considered as a model-theoretic equivalent to βG, the large compactification of G.

In the talk I will describe the results from [2] that give a description of definable topological dynamics of a large class of groups interpretable in an o-minimal expansion of the field of reals along with their universal covers interpreted in a certain two-sorted structure. The results provide a wide range of counterexamples to a question by Newelski whether the Ellis group of the universal definable G-flow is isomorphic to G/G^{00} and generalize methods from [1] that provided a particular counterexample.

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▸ ANTONIS KAKAS, FRANCESCA TONI AND PAOLO MANCARELLA, *Argumentation Logic.*
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Born out of the need to formalize common sense in Artificial Intelligence (AI), Argumentation Logic (AL) brings together the syllogistic roots of logic with recent argumentation theory [1] from AI to propose a new logic based on argumentation.

Argumentation Logic is purely proof theoretic defined via a criterium of *acceptability of arguments* [2]. Arguments in AL are sets of propositional formulae with the acceptability of an argument ensuring that the argument can defend against any other argument that is inconsistent with it, under a given propositional theory. AL can be linked to Natural Deduction allowing us to reformulate Propositional Logic (PL) in terms of argumentation and to show that, under certain conditions, AL and PL are equivalent. AL separates proofs into direct and indirect ones, the latter being through the use of a restricted form of Reductio ad Absurdum (RAA) where the (direct) derivation of the inconsistency must depend on the hypothesis posed when we apply the RAA rule [3].

As such AL is able to isolate inconsistencies in the given theory and to behave agnostically to them. This gives AL as a conservative paraconsistent [4] extension of PL that does not trivialize in the presence of inconsistency. The logic then captures in a single framework defeasible reasoning and its synthesis with the strict form of reasoning in classical logic. The interpretation of implication in AL is different from that of material implication, closer to that of default rules but where proof by contradiction can be applied with them. AL has recently formed the basis to formalize psychological theories of story comprehension [5].

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▸ ISKANDER KALIMULLIN, DAMIR ZAINETDINOV, *On limitwise monotonic reducibility of Σ^0_2-sets.*
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One of the directions of research in modern computability theory focus on studying properties of limitwise monotonic functions and limitwise monotonic sets.

I. Kalimullin and V. Puzarenko [1] introduced the concept of reducibility on families of subsets of natural numbers, which is consistent with Σ-definability on admissible sets. Let $\mathcal{F}_A$ denote the families of initial segments $\{\{x \mid x < n\} \mid n \in A\}$. Accordingly to [1], we define the notion of *limitwise monotonic reducibility* of sets as a Σ-reducibility of the corresponding initial segments, namely $A \leqslant_{lm} B \Longleftrightarrow \mathcal{F}_A \leqslant_\Sigma \mathcal{F}_B$.

Let $A \equiv_{lm} B$ if $A \leqslant_{lm} B$ and $B \leqslant_{lm} A$. The *limitwise monotonic degree* (also called lm-degree) of A is $\deg(A) = \{B : B \equiv_{lm} A\}$. Let $\mathbf{S}_{lm}$ denote the class of all lm-degrees of Σ^0_2 sets. The degrees $\mathbf{S}_{lm}$ form a partially ordered set under the relation $\deg(A) \leqslant \deg(B)$ iff $A \leqslant_{lm} B$.

We prove the following theorems.

Theorem 1. *There exist infinite Σ^0_2-sets A and B such that $A \nleqslant_{lm} B$ and $B \nleqslant_{lm} A$.*

Theorem 2. *Every countable partial order can be embedded into $\mathbf{S}_{lm}$.*

Theorem 3. (jointly with M. Faizrahmanov) *There is no maximal element in $\mathbf{S}_{lm}$.*

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▸ VLADIMIR KANOVEI, *A generalization of Solovay's Σ-construction with application to intermediate models.*

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A Σ-construction of Solovay [2] is extended to the case of intermediate sets which are not necessarily subsets of the ground model, with a more transparent description of the resulting forcing notion than in the classical paper of Grigorieff [1]. As an application, we prove that, for a given name t (not necessarily a name of a subset of the ground model), the set of all sets of the form $t[G]$ (the G-interpretation of t), G being generic over the ground model, is Borel. This result was first established by Zapletal [3] by a descriptive set theoretic argument.

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▸ AHMAD KARIMI & SAEED SALEHI, *A universal diagonal schema by fixed-points and Yablo's paradox.*

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In 1906, Russell [5] showed that all the known set-theoretic paradoxes (till then) had a common form. In 1969, Lawvere [3] used the language of category theory to achieve a deeper unification, embracing not only the set-theoretic paradoxes but incompleteness phenomena as well. To be precise, Lawvere gave a common form to Cantor's theorem about power sets, Russell's paradox, Tarski's theorem on the undefinability of truth, and Gödel's first incompleteness theorem. In 2003, Yanofsky [7] extended Lawvere's ideas using straightforward set-theoretic language and proposed a universal schema for diagonalization based on Cantor's theorem. In this universal schema for diagonalization, the existence of a certain (diagonalized-out and contradictory) object implies the existence of a fixed-point for a certain function. He showed how self-referential paradoxes, incompleteness, and fixed-point theorems all emerge from the single generalized form of Cantor's theorem. Yanofsky extended Lawvere's analysis to include the Liar paradox, the paradoxes of Grelling and Richard, Turing's halting problem, an oracle version of the P=?NP problem, time travel paradoxes, Parikh sentences, Löb's Paradox and Rice's theorem. In this talk, we fit more theorems in the universal schema of diagonalization, such as Euclid's theorem on the infinitude of the primes, and new proofs of Boolos [1] for Cantor's theorem on the non-equinumerosity of a set with its powerset. We also show the existence of Ackermann-like functions (which dominate a given set of functions such as primitive recursive functions) using the schema. Furthermore, we formalize a reading of Yablo's paradox [6], the most challenging paradox in the recent years, in the framework of Linear Temporal Logic (LTL [2]) and the diagonal schema, and show how Yablo's paradox involves circularity by presenting it in the framework of LTL. All in all, we turn Yablo's paradox into a genuine mathematico logical theorem. This is the first time that Yablo's paradox becomes a (new) theorem in mathematics and logic. We also show that Priest's [4] inclosure schema can fit in our universal diagonal/fixed-point schema. The inclosure schema was used by Priest for arguing for the self-referentiality of Yablo's sequence of sentences, in which no sentence directly refers to itself but the whole sequence does so.

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▸ MARTIN KOERWIEN, *Amalgamation, characterizing cardinals and locally finite Abstract Elementary Classes.*
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We introduce the concept of a locally finite Abstract Elementary Class and develop the theory of excellence for such classes. From this we find a family of complete $L_{\omega_1,\omega}$ sentences ϕ^r such that ϕ^r is r-excellent and ϕ^r homogeneously characterizes $\aleph_r$, improving results of Hjorth [1] and Laskowski-Shelah [2] and answering a question of Souldatos. This provides the first example of an Abstract Elementary Class where the spectrum of cardinals on which amalgamation holds contains more than one interval. This work is joint with John Baldwin and Chris Laskowski.

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▸ NURLAN KOGABAEV, *The isomorphism problem for computable projective planes.*
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Estimating the complexity of the isomorphism problem for some class K of structures is one of the approaches to obtain classification theorems for computable structures in K. It is widely assumed that K has a computable classification if the isomorphism problem in K is hyperarithmetical.

For a class K of structures, closed under isomorphism, the *isomorphism problem* is the set

$$E(K) = \{\langle a, b\rangle \mid \mathcal{A}_a, \mathcal{A}_b \in K \text{ and } \mathcal{A}_a \cong \mathcal{A}_b\},$$

where $\mathcal{A}_a$ is the computable structure with computable index a.

If the set of all indices for computable members of K is hyperarithmetical, then $E(K)$ is Σ^1_1. Several classes are well-known to have maximally complicated isomorphism problems. $E(K)$ is Σ^1_1-complete under m-reducibility for each of the following classes: undirected graphs, linear orders, trees, Boolean algebras, distributive lattices, Abelian p-groups, nilpotent groups, semigroups, rings, fields, real closed fields, etc.

In the present paper we estimate the complexity of the isomorphism problem for familiar classes of projective planes and obtain the following results.

Theorem. *$E(K)$ is Σ^1_1-complete for the following classes K:*
(1) pappian projective planes;
(2) desarguesian projective planes;
(3) arbitrary projective planes.

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▸ ZOFIA KOSTRZYCKA, *On Halldén complete modal logics determined by homogeneous frames.*

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We study normal modal logics in respect of their Halldén completeness.

Definition 1. *A logic L is Halldén complete if*

$$\varphi \vee \psi \in L \ \textit{implies}\ \varphi \in L \ \textit{or}\ \psi \in L$$

for all φ and ψ containing no common variables.

Halldén complete logics are also called Halldén reasonable. The weakest normal modal logic **K** is not Halldén complete since the formula $\Box(p \wedge \sim p) \vee \Diamond(q \vee \sim q) \in$ **K** but neither disjunct is its theorem. Therefore, Halldén complete logics are either extensions of the system $\mathbf{K} \oplus \Box\bot$ or $\mathbf{D} := \mathbf{K} \oplus \Diamond\top$. The following logics are known to be Halldén complete:

- **T** and **KTB** (Kripke [5]),
- **S4** and **S5**) (McKinsey [7]),
- **S4.3** (van Benthem, Humberstone [1]).

On the other side, Halldén itself proved that logics from the interval $\mathbf{S1} - \mathbf{S3}$ are unreasonable, see [3]. Then the families of extensions of modal logics were studied in respect of Halldén completeness.

- All normal extensions of **S5** are Halldén complete (McKinsey, [7]),
- There is a continuum of Halldén complete logics in $NEXT(\mathbf{S4})$ (Chagrov, Zakharyaschev, [2]),
- There is a continuum of Halldén incomplete logics in $NEXT(\mathbf{S4})$ (Schumm, [8]),
- There is a continuum of Halldén incomplete logics in $NEXT(\mathbf{KTB}\oplus(4_2))$ (Kostrzycka, [4]).

We show how to construct Halldén complete normal extensions for some modal logics. Our approach to this problem is purely semantic. The main key-tool will be a lemma due to van Benthem and Humberstone [1].

Lemma 1. *If a modal logic L is determined by one Kripke frame, which is homogeneous, then L is Halldén complete.*

In the construction of Halldén complete logics, we are however bounded by theorem due to Lemmon [6]. We say that two logics L_1, $L_2 \in NEXT(L)$ are incomparable, if there exist two formulas φ and ψ such that $\varphi \in L_1$ but $\varphi \notin L_2$ and $\psi \in L_2$ but $\psi \notin L_1$.

Theorem 1. *Let L_1, $L_2 \in NEXT(L)$ be two incomparable logics. Then the logic $L_0 = L_1 \cap L_2$ is Halldén incomplete.*

In our talk we take advantage of the above lemma and theorem. For several normal logics we define countable many their normal extensions, which are Halldén complete, as well as uncountable many normal extensions, which are not.

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▸ BEIBUT KULPESHOV, *Some remarks on $\aleph_0$-categorical weakly circularly minimal structures.*

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The notion of *circular minimality* has been introduced and originally studied by D. Macpherson and C. Steinhorn in [1]. Here we continue studying the notion of *weak circular minimality* being its generalisation.

A *circular* order relation is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x,y,z) \rightarrow K(y,z,x))$;

(co2) $\forall x \forall y \forall z (K(x,y,z) \land K(y,x,z) \Leftrightarrow x = y \lor y = z \lor z = x)$;

(co3) $\forall x \forall y \forall z (K(x,y,z) \rightarrow \forall t[K(x,y,t) \lor K(t,y,z)])$;

(co4) $\forall x \forall y \forall z (K(x,y,z) \lor K(y,x,z))$.

A set A of a circularly ordered structure M is said to be *convex* if for any $a, b \in A$ the following holds: for any $c \in M$ with $K(a,c,b)$ we have $c \in A$ or for any $c \in M$ with $K(b,c,a)$ we have $c \in A$. A circularly ordered structure $M = \langle M, K, \dots \rangle$ is *weakly circularly minimal* if any definable (with parameters) subset of M is a finite union of convex sets [2]. Any weakly o-minimal structure is weakly circularly minimal, but the inverse is not true in general. Some of interesting examples of weakly circularly minimal structures that are not weakly o-minimal were studied in [2, 3, 4].

In [2]–[4] $\aleph_0$–categorical 1-transitive weakly circularly minimal structures have been studied, and was obtained their description up to binarity. Here we discuss some properties of $\aleph_0$–categorical weakly circularly minimal structures that are not 1-transitive. In particular, we study a behaviour of 2-formulas in such structures.

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[4] B.Sh. Kulpeshov, *Definable functions in the $\aleph_0$-categorical weakly circularly minimal structures*, **Siberian Mathematical Journal**, 50 (2009), pp. 282–301.

▸ RUTGER KUYPER, *Effective genericity and differentiable functions.*

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Recently, connections between differentiability and various notions of effective randomness have been studied. These results are typically of the form "$x \in [0,1]$ is random if and only if every function $f \in \mathcal{C}$ is differentiable at x," where $\mathcal{C}$ is some subclass of the computable functions; for example, Brattka, Miller and Nies [1] gave such characterisations for computable and Martin-Löf randomness.

In this talk we will present a complementary result for effective genericity. More precisely, our result says that $x \in [0,1]$ is 1-generic if and only if every differentiable computable function has continuous derivative at x. This result can be seen as an effectivisation of a result by Bruckner and Leonard [2].

This talk is based on joint work with Sebastiaan Terwijn [3].

[1] V. Brattka, J. S. Miller, and A. Nies, *Randomness and differentiability*, arXiv:1104.4465 [math.LO], 2011.

[2] A. M. Bruckner and J. L. Leonard, *Derivatives*, ***The American Mathematical Monthly***, vol. 73 (1966), no. 4, pp. 24–56.

[3] R. Kuyper and S. A. Terwijn, *Effective genericity and differentiability*, submitted.

► CHRIS LE SUEUR, *Determinacy of refinements to the difference hierarchy of co-analytic sets*.
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It is quite well-known result of Martin that the existence of a measurable cardinal is enough to prove the determinacy of all $\mathbf{\Pi}^1_1$ sets. The argument nicely modifies to get the determinacy of all (lightface) Π^1_1 sets from the existence of $0^\sharp$. With this argument in mind, I will discuss how the technique has been pushed since then to get more determinacy in the difference hierarchy of Π^1_1 sets, including a family of new determinacy results following from sharp-like hypotheses. To achieve this I will also demonstrate a generalised notion of computability suitable for defining the lightface Borel hierarchy in uncountable spaces.

► JUI-LIN LEE, *Explosiveness, Model Existence, and Incompatible Paraconsistencies*.
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In this talk we present that the general concept of formal inconsistencies can be well-developed for any given semantics $\models$ (no matter it is truth functional or not). Note that the concept negation is not a necessary part in our treatment. In this theory of formal inconsistencies, there are two important concepts, model existence property (i.e., w.r.t. the given inconsistency, every consistent set has a model with respect to $\models$) and explosiveness property (i.e., w.r.t. the given inconsistency, every inconsistent set is also absolutely inconsistent). Now given a semantics $\models$, it will generate a set of inconsistencies, say, $Ins_{\models} = \{I_i, \dots\}$. If a $\models$-sound proof system L has both model existence property and explosiveness for some inconsistency $I \in Ins_{\models}$, then all inconsistencies in $Ins_{\models}$ are provably equivalent in L.

Then it is natural to ask, for the classical semantics, whether there are incompactible paraconsistencies in the following sense, i.e., are there two inconsistencies I_1, I_2 (generated from classical semantics) such that there are classically sound proof systems L_1, L_2 such that in L_1 it has I_1 model existence and I_2 explosiveness but not I_1 explosiveness and not I_2 model existence. And in L_2 it has I_2 model existence and I_1 explosiveness but not I_2 explosiveness and not I_1 model existence. We will prove that the answer is positive, which shows that there are incompatible paraconsistencies.

Keywords: 03B53, model existence, explosiveness, paraconsistency

[1] Walter Carnielli, Marcelo E. Coniglio and João Marcos, "Logics of Formal Inconsistency", *Handbook of Philosophical Logic, Volume 14 (Second Edition)*, edited by D. Gabbay and F. Guenthner, pp. 15-107, Berlin: Springer, 2007.

[2] Jui-Lin Lee, "Classical model existence theorem in propositional logics", in *Perspectives on Universal Logic*, edited by Jean-Yves Béziau and Alexandre Costa-Leite, pp. 179–197, Polimetrica, Monza, Italy, 2007.

[3] Jui-Lin Lee. "Classical model existence and left resolution", Logic and Logical Philosophy Vol. 16, No. 4, 2007, pages 333–352.

▸ LAURENŢIU LEUŞTEAN, *Effective results on the asymptotic behavior of nonexpansive iterations*.

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This talk reports on an application of proof mining to the asymptotic behavior of Ishikawa iterations for nonexpansive mappings [4, 3]. *Proof mining* is a paradigm of research concerned with the extraction, using proof-theoretic methods, of finitary content from mathematical proofs. This research direction can be related to Terence Tao's proposal [6] of *hard analysis*, based on finitary arguments, instead of the infinitary ones from *soft analysis*.

We present uniform effective rates of asymptotic regularity for the Ishikawa iteration associated to nonexpansive self-mappings of convex subsets of uniformly convex Busemann geodesic space. We show that these results are obtained by a logical analysis of an asymptotic regularity proof due to Tan and Xu [5], consisting of two main steps: the first one with a classical proof, analyzed using the combination of monotone functional interpretation and negative translation, while the second one has a constructive proof, analyzed more directly using monotone modified realizability. As a consequence, our results are guaranteed by a combination of logical metatheorems for classical and semi-intuitionistic systems, proved by Gerhardy and Kohlenbach [1, 2] for different classes of spaces and adapted to uniformly convex Busemann spaces in [4].

[1] P. Gerhardy, U. Kohlenbach, *Strongly uniform bounds from semi-constructive proofs*, ***Annals of Pure and Applied Logic***, vol. 141 (2006), pp. 89–107.

[2] P. Gerhardy, U. Kohlenbach, *General logical metatheorems for functional analysis*, ***Transactions of the American Mathematical Society***, vol. 360 (2008), pp. 2615–2660.

[3] L. Leuştean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, ***Nonlinear Analysis and Optimization I: Nonlinear Analysis*** (A. Leizarowitz, B. S. Mordukhovich, I. Shafrir, A. Zaslavski, editors), Amer. Math. Soc., Providence, RI, 2010, pp. 193–209.

[4] L. Leuştean, *An application of proof mining to nonlinear iterations*,

arXiv:1203.1432v1 [math.FA], 2012; accepted for publication in Annals of Pure and Applied Logic.

[5] K.-K. Tan, H.-K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, ***J. Math. Anal. Appl.***, vol. 178 (1993), pp. 301–308.

[6] T. Tao, *Soft analysis, hard analysis, and the finite convergence principle*, 2007, available on terrytao.wordpress.com/2007/05/23/soft-//analysis-hard-analysis-and-the-finite-convergence-principle/.

▸ STEVEN LINDELL AND SCOTT WEINSTEIN, *An elementary definition for tree-width.*

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We introduce a new combinatorial parameter which naturally generalizes the notion of vertex separation number from linear layouts of graphs to layouts which are tree-like, and use this to show that the tree-width of a graph is a simple property of its normal trees – tree-like partial orders of the vertices which induce acyclic orientations of the edges. As a consequence, every graph admits a normal tree decomposition situated on its nodes which preserves its tree-width. Moreover, for graphs of fixed tree-width, this is elementary – there is a sentence of first-order logic which confirms if a given partially ordered graph determines a normal tree decomposition of width k. Our normal form is based on a generalization of normal spanning trees which are central to graph theory [1]. We say a partial order is tree-like if it has a unique minimal element, and for every element, its set of predecessors forms a chain. We refer to these chains as branches of the directed tree determined by the cover diagram. An order is *normal* for an undirected graph G if it is a tree-like partial order of the vertices in which each edge parallels a branch of the tree. Entirely analogous to the role of a linear order in situating a path-width preserving path decomposition [2], we use a normal partial order to situate a tree-width preserving tree decomposition, which we call a normal tree decomposition.

[1] Reinhard Diestel, ***Graph Theory***, 4th edition, Springer, 2010 (corrected electronic edition 2012).

[2] Nancy Kinnersley, *The vertex separation number of a graph equals its path-width*, ***Information Processing Letters***, vol. 42 (1992), no. 6, pp. 345–350.

▸ YUN LU, *Homogeneous structures and their reducts.*

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A structure is homogeneous if it is countable and every isomorphism between finite substructures extends to an automorphism. Let M be a countably infinite first order structure. A reduct is a structure N with the same domain as M, whose relations are definable without parameters in M. If a structure M is countably categorical, then its reduct is equivalent to a permutation group G of $Sym(M)$ containing $Aut(M)$ such that G is a closed subgroup of $Sym(M)$.

There is conjecture from Simon Thomas that if M is a homogeneous structure with a finite relational language, then it has finitely many reducts. In this talk, we will investigate those structures whose reducts have been classified, as well as our work on random bipartite graphs.

▶ ROBERT LUBARSKY, NORMAN PERLMUTTER, *Elementary epimorphisms between models of set theory.*
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Rothmaler [3] defined an elementary epimorphism $f : M \to N$ (between model-theoretic structures in some language) to be a homomorphism such that, for every formula ϕ in the language with parameters $n_1, \ldots, n_k$ from N true in N, there are f-preimages $m_1, \ldots, m_k$ of the n_i's such that $\phi(m_1, \ldots, m_k)$ holds in M. Here we investigate elementary epimorphisms between models of set theory, as well as the restricted notion of a Γ-elementary epimorphism, by which ϕ is restricted to a set Γ. We show that the only Π_1-elementary epimorphisms between models of ZF are isomorphisms. That result seems to be optimal, in that any of the obvious weakenings of the hypotheses allow for non-trivial such epimorphisms. For instance, there are non-trivial Σ_1-elementary epimorphisms. Also, using a result of Caicedo [1], there are non-trivial (full) elementary epimorphisms between models of ZFC$^-$, which is ZFC without Power Set. Furthermore, we study the inverse system induced by the last example, and its inverse limit. Inverse limits do not always exist, and even when they do they might not be the entire thread class [2], but in this case it is.

[1] Andrés Eduardo Caicedo, *Real-valued Measurable Cardinals and Well-orderings of the Reals*, ***Set Theory: Centre de Recerca Matemàtica Barcelona, 2003-2004*** (Joan Bagaria and Steve Todorcevic, editors), Birkhäuser, Basel, 2006, pp. 83–120.

[2] Norman Lewis Perlmutter, ***Inverse limits of models of set theory and the large cardinal hierarchy near a high-jump cardinal***, PhD dissertation, CUNY Graduate Center, Department of Mathematics, May 2013, http://boolesrings.org/perlmutter/files/2013/05/Dissertation.pdf.

[3] Philipp Rothmaler, *Elementary epimorphisms*, ***The Journal of Symbolic Logic***, vol. 70 (2005), no. 2, pp. 473–488.

▶ JUDIT X. MADARÁSZ, GERGELY SZÉKELY, *A completeness theorem for general relativity.*
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We introduce several first-order axiom systems for general relativity and show that they are complete with respect to the standard models of general relativity, i.e., to Lorentzian manifolds having the corresponding smoothness properties.

This is only a sample of our approach (see the references in [2]) to the logical analysis of special and general relativity theory in the axiomatic framework of modern mathematical logic. The aim of our research is to build a flexible hierarchy of axiom systems (instead of one axiom system only), analyzing the logical connections between the different axioms and axiomatizations. We try to formulate simple, logically transparent and intuitively convincing axioms. The questions we study include: What is believed and why? - Which axioms are responsible for certain predictions? - What happens if we discard some axioms? - Can we change the axioms, and at what price?

[1] H. Andréka, J. X. Madarász, I. Németi, and G. Székely, *An axiom system for general relativity complete with respect to Lorentzian manifolds*, arXiv:1310.1475, 2013.

[2] H. Andréka, J. X. Madarász, I. Németi, and G. Székely, *A logic road from special relativity to general relativity*, ***Synthese***, vol. 186 (2012), no. 3, pp. 633–649, arXiv:1005.0960.

▸ MACIEJ MALICKI, *Consequences of the existence of ample generics and automorphism groups of homogeneous metric structures.*
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A Polish group G has ample generics if the diagonal action of G on G^n by conjugation has a comeager orbit for every $n \in \mathbb{N}$. The existence of ample generics has very strong consequences. Every Polish group G with ample generics has the small index property (that is, every subgroup $H \leq G$ with $[G : H] < 2^\omega$ is open), the automatic continuity property (that is, every homomorphism from G into a separable group is continuous), and uncountable cofinality for non-open subgroups (that is, every countable exhaustive chain of non-open subgroups of G is finite.)

What is surprising is that all known examples of groups with ample generics are isomorphic to the automorphism group of some countable structure, and the question of whether there exists a Polish group with ample generics which is not of this form, is still open. In particular, the isometry group of the Urysohn space $\mathrm{Iso}(\mathbb{U})$, the automorphism group of the measure algebra Aut(MA), and the unitary group $U(\ell_2)$ have meager conjugacy classes. On the other hand, it is known that these groups share some of the consequence of the existence of ample generics. For example, $U(\ell_2)$ has the automatic continuity property, while Aut(MA) has the automatic continuity property, and the small index property.

Very recently, M.Sabok proposed a model theoretic approach that sheds new light on the structure of these groups, and more generally, automorphism groups of certain classes of homogeneous metric structures. In particular, he formulated a general criterion for a homogeneous metric structure X that implies that $\mathrm{Aut}(X)$ has the automatic continuity property, and he verified it for $\mathbb{U}$, MALG, and ℓ_2.

We propose a criterion that implies *all* the main consequences of the existence of ample generics: the small index property, the automatic continuity property, and uncountable cofinality for non-open subgroups, which suggests that it may be regarded as a counterpart of the notion of ample generics in the realm of homogeneous metric structures. We also verify it for $\mathbb{U}$, MALG, and ℓ_2, thus proving that the groups $\mathrm{Iso}(\mathbb{U})$, Aut(MA), $U(\ell_2)$ satisfy these properties.

▸ ALBERTO MARCONE, *Reverse mathematics of WQOs and Noetherian spaces.*
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This work in progress is joint with Emanuele Frittaion, Matthew Hendtlass, Paul Shafer, and Jeroen Van der Meeren.

If $(Q, \leq_Q)$ is a quasi-order we can equip Q with several topologies. We are interested in the Alexandroff topology $A(Q)$ (the closed sets are exactly the downward closed subsets of Q) and the upper topology $u(Q)$ (the downward closures of finite subsets of Q are a basis for the closed sets). $A(Q)$ and $u(Q)$ are (except in trivial situations) not T_1, yet they reflect several features of the quasi-order. For example, $(Q, \leq_Q)$ is a well quasi-order (WQO: well-founded and with no infinite antichains) if and only if $A(Q)$ is Noetherian (all open sets are compact or, equivalently, there is no strictly descending chain of closed sets). Moreover, if $(Q, \leq_Q)$ is WQO then $u(Q)$ is Noetherian.

Given the quasi-order $(Q, \leq_Q)$, we consider two natural quasi-orders on the powerset $\mathcal{P}(Q)$:

$$A \leq^\flat B \iff \forall a \in A\, \exists b \in B\, a \leq_Q b;$$
$$A \leq^\sharp B \iff \forall b \in B\, \exists a \in A\, a \leq_Q b.$$

We write $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ for the resulting quasi-orders, and $\mathcal{P}^\flat_f(Q)$ and $\mathcal{P}^\sharp_f(Q)$ for their restrictions to the collection of finite subsets of Q.

Goubault-Larrecq proved that if $(Q, \leq_Q)$ is WQO then $u(\mathcal{P}^\flat(Q))$ and $u(\mathcal{P}^\sharp_f(Q))$ are Noetherian, even though $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp_f(Q)$ are not always WQOs.

We study these theorems and some of their consequences from the viewpoint of reverse mathematics, proving for example:

- over RCA_0, ACA_0 is equivalent to each of "if $(Q, \leq_Q)$ is WQO then $u(\mathcal{P}^\flat(Q))$ is Noetherian", and "if $(Q, \leq_Q)$ is WQO then $A(\mathcal{P}^\flat_f(Q))$ is Noetherian";
- ACA_0 proves "if $(Q, \leq_Q)$ is WQO then $u(\mathcal{P}^\sharp_f(Q))$ is Noetherian", yet WKL_0 does not.

► JOSÉ MARTíNEZ-FERNÁNDEZ, *Non-monotonic extensions of the weak Kleene clone with constants.*
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A clone on a set A is a set of finitary functions on A that includes the projection functions and is closed for composition. It is called a clone with constants when it contains all the constant functions on A. Every truth-functional propositional language determines the clone generated by the interpretation of its operator symbols. If we consider propositional languages interpreted with a three-valued truth-functional scheme, the clones generated by the weak and strong Kleene operators are specially interesting, because Kleene logics have been applied to the study of several fields, like partial predicates, semantical paradoxes, vagueness, the semantics of programming languages, etc.

The clone with constants generated by the weak Kleene propositional operators and the constant functions will be called the weak Kleene clone and analogously for the strong Kleene clone. It is well known that the strong Kleene clone coincides with the clone of three-valued functions monotonic on the order of information (i.e., the partial order on $0, 1, 2$ determined by $2 \leq 0$, $2 \leq 1$). The aim of this paper is to determine all the clones that are extensions of the weak Kleene clone but are not included in the strong Kleene clone. Equivalently, this amounts to the characterization of all the clones that can be obtained when we add to the weak Kleene clone a set of functions that include some function non-monotonic on the order of information. Using Jablonskij's theorem that determines all three-valued maximal clones and Lau's theorem that characterizes all the three-valued submaximal clones (see [1], II5 and II14), it is easy to check that only two maximal clones (C_2 and U_2) and three submaximal clones (one of them being the strong Kleene clone) contain the weak Kleene clone. The paper will determine completely all the clones in the interval between the weak Kleene clone and U_2 and all the clones between the weak Kleene clone and C_2 that are not contained in the strong Kleene clone.

[1] Dietlinde Lau, ***Function Algebras on Finite Sets***, Springer, 2006.

▸ ALBA MASSOLO, LUIS URTUBEY, *Modelling inference in fiction.*
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As it is widely-known, fiction became a serious problem for several classical conceptions closed related to philosophy of logic (J. Woods, 2006). This was mainly due to some of the leading features of reasoning in fiction. Firstly, inference in fiction involves reasoning with incomplete information. Stories describe their characters, places, and events only in an incomplete way. Due to the fact that stories are composed by a finite set of sentences, a large amount of information about them remains unknown. Secondly, inference in fiction also involves reasoning with inconsistent information. Inconsistencies can emerge from two sources. On the one hand, information belonging to a fiction contradicts reality in many aspects. On the other hand, some stories are based on a contradiction or contain inconsistent information. This is the case of stories in which contradictions are an essential part of their plots.
In order to cope with the abovementioned features of reasoning in fiction, we propose a semantic approach of fiction based on an intuitionistic modal system. The semantic model is an adaptation of the multiple-expert semantics developed by Melvin Fitting in 1992. Firstly, we consider a propositional language to represent fictional information formally. That propositional language is interpreted in an intuitionistic modal semantics that involves two different perspectives and a partial valuation. On the one hand, these two perspectives make it possible to distinguish two sources of information involved in reasoning in fiction, i.e., fiction and reality. On the other hand, the partial valuation makes it possible to deal with incomplete information. A relation of logical consequence is defined in order to distinguish between valid and invalid inferences within the fictional context. Finally, we explore different proof-theoretical alternatives in order to characterize a deductive system for this semantic approach.

[1] John Woods, *Fictions and their Logic*, ***Philosophy of Logic*** (Dale Jacquette, editor), Elsevier, Amsterdam, 2006, pp. 1061-1126.

[2] Melvin Fitting, *Many-Valued Modal Logic II*, ***Fundamenta Informaticae***, vol. 17 (1992), no. 4, pp. 55-73.

▸ MICHAEL MCINERNEY, *Integer-valued randomness and degrees.*
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Analysing betting strategies where only integer values are allowed, perhaps for a given set F, gives an interesting variant on algorithmic randomness where category and measure intersect. We build on earlier work of Bienvenu, Stephan, and Teutsch, and study reals random in this sense, and their intricate relationship with the c.e. degrees. This is joint work with George Barmpalias and Rod Downey.

[1] Laurent Bienvenu, Frank Stephan, and Jason Teutsch, *How powerful are integer-valued martingales?*, ***Theory of Computing Systems***, vol. 51 (2010), no. 3, pp. 330–351.

▸ JEROEN VAN DER MEEREN, *The maximal order type of the trees with the gap-embeddability relation.*
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In 1985, Harvey Friedman [1] introduced a new kind of embeddability relation between finite labeled rooted trees, namely the gap-embeddability relation. Under this embeddability relation, the set of finite rooted trees with labels bounded by a fixed natural number n is a well-partial-ordering. The well-partial-orderedness of these trees (if we put a universal quantifier $\forall n$ in front) gives rise to a statement not provable in Π^1_1–CA_0.
There are still some open questions left about these famous well-partial-orderings. For example, what is the maximal order type of these sets of trees with the gap-embeddability relation? The maximal order type of a well-partial-ordering is an important characteristic of that well-partial-ordering and it captures in some sense its strength. In this talk, I will discuss some new recent developments concerning this topic.

[1] S. G. Simpson, *Nonprovability of certain combinatorial properties of finite trees*, ***Harvey Friedman's research on the foundations of mathematics***, Studies in Logic and the foundation of mathematics, (L. A. Harrington, M. D. Morley, A. Scedrov, S. G. Simpson, editors), Elsevier Science Publishers B.V., P.O. Box 1991, 1000 BZ Amsterdam, The Netherlands, 1985, pp. 87–117.

▸ NADAV MEIR, *On various strengthenings of the notion of indivisibility.*
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A structure $\mathcal{M}$ in a first order language $\mathcal{L}$ is *indivisible* if for every colouring of its universe M in two colours, there is a monochromatic substructure $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}' \cong \mathcal{M}$. Additionally, we say that $\mathcal{M}$ is *symmetrically indivisible* if $\mathcal{M}'$ can be chosen to be *symmetrically embedded* in $\mathcal{M}$ (That is, every automorphism of $\mathcal{M}'$ can be can be extended to an automorphism of $\mathcal{M}$), and that $\mathcal{M}$ is *elementarily indivisible* if $\mathcal{M}'$ can be chosen to be an elementary substructure.

The notion of indivisibility is a long-studied subject. We will present these strengthenings of the notion, examples and some basic properties. We will define a new "product" of structures which preserves these notions and use is to answer some questions presented in [1] regarding the properties and interaction between these notions.

[1] Assaf Hasson, Menachem Kojman and Alf Onshuus, *On symmetric indivisibility of countable structures*, ***Model Theoretic Methods in Finite Combinatorics*** (Martin Grohe and Johann A. Makowsky, editors), AMS, 2011, pp.417–452.

▶ JOSÉ M. MÉNDEZ, GEMMA ROBLES, FRANCISCO SALTO, *Blocking the routes to triviality with depth relevance.*
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The depth relevance condition (drc) is a strengthening of the variable-sharing property. A logic S has the drc if A and B share at least a propositional variable at the same depth in all theorems of the form $A \to B$ (cf. [1]). Logics with the drc have been used for defining non-trivial strong naïve set theories. In [3], "the class of implication formulas known to trivialize NC" is recorded. (NC abbreviates "naïve comprehension"; cf. [3], p. 435.) The aim of this paper is to show how to invalidate any member in this class by using "weak relevant model structures" (cf. [2]). Weak relevant model structures only verify logics with the drc.

[1] R. T. Brady, ***Universal Logic***, CSLI, Stanford, CA, 2006.

[2] G. Robles, J. M. Méndez, *Generalizing the depth relevance condition. Deep relevant logics not included in R-Mingle*, ***Notre Dame Journal of Formal Logic***, vol. 55 (2014), pp. 107-127.

[3] S. Rogerson, G. Restall, *Routes to triviality*, ***Journal of Philosophical Logic***, vol. 33 (2006), pp. 421-436.

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▶ OMER MERMELSTEIN, *Reducts of simple (non-collapsed) Fraïssé-Hrushovski constructions.*
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Fraïssé-Hrushovski constructions were first introduced by Hrushovski as a method for constructing strongly minimal sets that do not fit within Zilber's trichotomy conjecture. The construction can be seen as a two-step process where first a rank ω structure is constructed from a countable amalgamation class, using a variation of a Fraïssé limit construction, and then the structure is "collapsed" to a strongly minimal substructure.

In this talk we acquaint ourselves with the rank ω, non-collapsed version of the construction and its associated combinatorial geometry, and provide a general method of showing that one simple Fraïssé-Hrushovski construction is a (proper) reduct of another Fraïssé-Hrushovski construction.

► RUSSELL MILLER, JENNIFER PARK, BJORN POONEN, HANS SCHOUTENS, AND ALEXANDRA SHLAPENTOKH, *Coding graphs into fields.*
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It is well established that the class of countable symmetric irreflexive graphs is complete in computable model theory: every countable structure in a finite language can be coded into a graph in such a way that the graph has the same spectrum, the same computable dimension, and the same categoricity spectrum as the original structure, and shares most other known computable-model-theoretic properties of the original structure as well. In 2002, Hirschfeldt, Khoussainov, Shore, and Slinko collected related results and proved more, showing that many other classes of countable structures are complete in the same sense. On the other hand, classes such as linear orders, Boolean algebras, trees, and abelian groups are all known not to be complete in this way. We address the most obvious class for which this question was still open, by giving a coding of graphs into countable fields in such a way as to preserve all of these properties.

► SHEILA K. MILLER, *Budding Trees.*
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We define budding trees, show that they form a topological Ramsey space, and discuss applications. (Joint work with Natasha Dobrinen.)

► RYSZARD MIREK, *Natural Deduction in Renaissance Geometry.*
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Moritz Cantor was so impressed by the achievements of Piero della Francesca in mathematics and geometry that devoted him in his *Vorlesungen uber Geschichte der Mathematik* far more attention than to any other contemporary algebraicist. In Francesca's treatise *De prospectiva pingendi* we find the advanced geometrical exercises presented in the form of propositions. For instance, in Book 1, Proposition 8, he shows that the perspective images of orthogonals converge to a point. Proposition 12 shows how to draw in perspective a surface of undefined shape, which is located in profile as a straight line. The task is to find the image of a line perpendicular to the picture plane. But the most interesting is Proposition 13 that shows how to "degrade" a square and, more precisely the sides of the square. It is obvious that most of these propositions are used in the paintings of Francesca.

The purpose of the study is to describe these results in the form of logical system EF. Generally, the logical language is six sorted, with sorts for points, lines, circles, segments, angles, and areas. As proofs it is possible to employ the method of natural deduction. The aim is to demonstrate that such a method is the most useful for the presentation of the geometric proofs of Francesca, taking into account also the importance of *diagrams* within them.

▸ ARMEN MNATSAKANYAN, *The relation between the graphs structures and proof complexity of corresponding Tseitin graph tautologies.*
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There are many well known examples of tautologies, which require exponential proof complexities in weak systems. Some of them are graph-based formulas introduced by Tseitin in [1]. As Tseitin graph tautologies, constructed on the base of different graphs, have different proof complexities, it is interesting to investigate the relation between the structure of graphs and proof complexities of corresponding Tseitin graph tautologies. In [2] A.Urquhart constructed the sequence of graphs such that the formulas based on them are hard examples for Resolution. We describe two sufficient properties of graphs G_n on n vertices such that the formulas based on them have exponential Resolution proof steps. The network style graphs of Tseitin's formulas and graphs of Urquhart are examples of graphs with mentioned properties. If at least one of these properties is not valid for any graph, then the corresponding formula has polynomial bounded resolution refutation.

Acknowledgment. This work is supported by Grant 13-1B004 of SSC of Goverment of RA.

[1] G.S.Tseitin, *On the complexity of derivation in propositional calculus*, ***Studies in constructive mathematics and mathematical logic***, vol. 2 (1970), pp. 115–125.

[2] A.Urquhart, *Hard examples for resolution*, ***Journal of the Association for Computing Machinery***, vol. 34 (1987), pp. 209–219.

▸ ATTILA MOLNÁR, GERGELY SZÉKELY, *Modal logic of clocks: Modalizing a first-order theory of time to get a better understanding of relativity theories.*
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Goldblatt [1] proved that the modal logic **S4.2** characterizes Minkowski spacetimes; the possible worlds represent events, and the intended interpretation of the modal operator $\Diamond$ is "it is now or it will be the case in the causal future that". Unfortunately, the expressive power of this logic is very limited; the fundamental relativistic effects such as the twin paradox, time dilatation, etc. are inexpressible.

In our talk, we will modalize the first-order theory of reals to answer this challenge. The worlds, again, will represent events, while $\Diamond$ will represent "It is visible that" or "it was the case in the *lightlike separated* past that". We use only functions and relations of reals; the solely modal novelty is the presence of non-rigid designators to deal with the *clocks* of observers. This theory, beyond its expressive power could be a first step towards a connection of the axiomatic operational foundations of spacetime ([3], [4]) and the research inspired by [2] and [1] such as theories of branching spacetimes.

[1] R. Goldblatt, *Diodorean modality in Minkowski Spacetime*, ***Studia Logica***, vol. 39 (1980), issue 2-3, pp. 219–236.

[2] A. Prior, ***Past, Present, and Future***, Clarendon Press, 1967.

[3] J. Ax, *The elementary foundations of spacetime*, ***Foundations of Physics***, vol. 8 (1978), no. 7/8, pp. 507–546.

[4] H. Andréka, I. Németi, *Comparing theories: the dynamics of changing vocabulary. A case-study in relativity theory.*, ***Trends in Logic***, (to appear) (2014), pp. 29.

▸ ANDREY MOROZOV, AIZHAN SATEKBAYEVA, AND JAMALBEK TUSSUPOV,

On the existential interpretability of structures.

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We study the ∃–interpretability of constructive structures of finite predicate signatures. This definition is motivated by a kind of effective interpretability of abstract databases and leads to a good natural translation of ∃–queries.

The following definition is a restricted variant of the standard well–known definition of interpretability of structures:

Definition. *Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be two structures of finite predicate signatures and let $\langle P_1, \dots, P_k\rangle$ be the signature of $\mathfrak{A}_0$. We say that $\mathfrak{A}_0$ has a ∃–interpretation in $\mathfrak{A}_1$ if there exist*

- *$n \in \omega$ and a finite tuple of parameters $\bar{p} \in \mathfrak{A}_1$,*
- *∃–formula $U(\bar{x}, \bar{y})$, $|\bar{x}| = n$,*
- *∃–formulas $E^+(\bar{x}_0, \bar{x}_1, \bar{y})$ and $E^-(\bar{x}_0, \bar{x}_1, \bar{y})$ such that $|\bar{x}_0| = |\bar{x}_1| = n$,*
- *∃–formulas $P^+(\bar{x}_1, \dots, \bar{x}_m, \bar{y})$ and $P^-(\bar{x}_1, \dots, \bar{x}_m, \bar{y})$, for each predicate symbol P of the signature of $\mathfrak{A}_0$, where m is the arity of P with the property $|\bar{x}_1| = \cdots = |\bar{x}_m| = n$,*

such that

1. *The set $(U^{\mathfrak{A}_1}(\bar{x}))^2$ is a disjunct union of the sets $\{\langle \bar{x}_0, \bar{x}_1\rangle \mid \mathfrak{A}_1 \models E^\varepsilon(\bar{x}_0, \bar{x}_1, \bar{p})\}$, $\varepsilon \in \{+, -\}$.*
2. *For any m–ary predicate symbol P of the signature of $\mathfrak{A}_0$, the set $(U^{\mathfrak{A}_1}(\bar{x}))^m$ is a disjunct union of the sets $\{\langle \bar{x}_0, \dots, \bar{x}_m\rangle \mid \mathfrak{A}_1 \models P^\varepsilon(\bar{x}_0, \dots, \bar{x}_m, \bar{p})\}$, $\varepsilon \in \{+, -\}$.*
3. *Let $\widehat{P}_i = \{\langle \bar{x}_1, \dots, \bar{x}_m\rangle \mid \mathfrak{A}_1 \models P^+(x_1, \dots, \bar{x}_m, \bar{p})\}$, $i = 1, \dots, k$. Then the relation $E = \{\langle \bar{x}_0, \bar{x}_1\rangle \mid \mathfrak{A}_1 \models E^+(\bar{x}_0, \bar{x}_1, \bar{p})\}$ is a congruence on $\mathfrak{B} = \langle U^{\mathfrak{A}_1}(\bar{x}), \widehat{P}_1, \dots, \widehat{P}_k\rangle$ and the quotient algebra $\mathfrak{B}/E$ is isomorphic to $\mathfrak{A}_0$.*

Theorem.

1. *The ∃–interpretability generates an upper semilattice $\mathcal{L}_\exists$ in which computable structures form a principal ideal $\mathcal{L}^0_\exists$; in particular, there exists a universal computable structure, i.e., a computable structure that ∃-interprets any computable structure.*
2. *Any finite partial order is embeddable into $\mathcal{L}^0_\exists$.*

▶ WILMARI MORTON AND CLINT VAN ALTEN, *Canonical extensions and prime filter completions of poset expansions.*
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The algebraic models of substructural logics are residuated ordered algebras [2]. Embedding a residuated ordered algebra into a complete algebra of the same class has many applications in logic, e.g., the canonical extension is used to obtain relational semantics for non-classical logics [1].

The underlying sets of the algebraic structures of interest are often partially ordered. The canonical extensions of posets have been studied in [1, 2]. Upon closer inspection it can be seen that the completions in [1] and [2] are generally different. Both use a construction, first appearing in [3], based on a Galois connection between sets of filters and ideals, however, the choice of filters differs.

We investigate the construction from [3] for various choices of filters and ideals, consider the extension of operations defined on the posets and focus on some specific properties of completions obtained via this construction. Next we present a construction for completions of posets that makes use of the prime filters of the posets. We show that the completion obtained via this second construction is isomorphic to the former for a particular choice of filters.

[1] J. M. Dunn, M. Gehrke and A. Palmigiano, *Canonical extensions and relational completeness of some substructural logics*, ***The Journal of Symbolic Logic***, vol. 70 (2005), no. 3, pp. 713–740.

[2] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, ***An algebraic glimpse at substructural logics***, Studies in logic and the foundations of Mathemtaics, Volume 151 Elsevier, 2007.

[3] W. R. Tunnicliffe, *The completion of a partially ordered set with respect to a polarization*, ***Proceedings of the London Mathematical Society***, vol. 28 (1974), no. 3, pp. 13–27.

▶ MIGUEL ANGEL MOTA, *Forcing with finite conditions and preserving CH.*
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In the last years there has been a second boom of the technique of forcing with side conditions (see for instance the recent works of Asperó-Mota[1], Krueger[4] and Neeman[5] describing three different perspectives of this technique). The first boom took place in the 1980s when Todorcevic[6] discovered a method of forcing in which elementary substructures are included in the conditions of a forcing poset to ensure that the forcing poset preserves cardinals. More than twenty years later, Friedman[2] and Mitchell[3] independently took the first step in generalizing the method from adding small (of size at most the first uncountable cardinal) generic objects to adding larger objects by defining forcing posets with finite conditions for adding a club subset on the second uncountable cardinal. However, neither of these results show how to force (with side conditions together with another finite set of objects) the existence of such a large object together with the continuum being small. In this talk we will discuss new results in this area.

[1] D. Asperó and M. A. Mota, *Forcing consequences of* PFA *together with the continuum large*, Transactions of the American Mathematical Society, to appear.

[2] S. D. Friedman, *Forcing with finite conditions*, in Set Theory: Centre de Recerca Matematica, Barcelona, 2003-2004, Trends in Mathematics, pages 285-295, BirkhauserVerlag, 2006.

[3] W. Mitchel, *$I[\omega_2]$ can be the nonstationary ideal on $Cof(\omega_1)$*, Transactions of the American Mathematical Society, 361(2), pages 561-601, 2009.

[4] J. Krueger, *Coherent adequate sets and forcing square*, Fundamenta Mathematicae, to appear.

[5] I. Neeman, *Forcing with sequences of models of two types*, Notre Dame Journal of Formal Logic, to appear.

[6] S. Todorčević, *A note on the proper fircing axiom*, in Axiomatic set theory (Boulder, Colorado, 1983), volume 31 of Contemporary Mathematics, pages 209-218. American Mathematical Society, Providence, RI, 1984.

▸ JOACHIM MUELLER-THEYS, *Metalogical Extensions—Part II: First-order Consequences and Gödel.*
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The aim is conservative extension of Φ *seq* ϕ (*seq* $\in \{\models, \vdash\}$) to *metalogical consequence* Φ $seq^{\Box}\alpha$ such that, specifically: Φ $seq^{\Box}$ $\Box\phi$ iff Φ *seq* ϕ, non Φ *seq* ϕ implies Φ $seq^{\Box}\neg\Box\phi$, and Φ $seq^{\Box}\neg\Box\phi$ implies non Φ *seq* ϕ if Φ is consistent.

We will define *metalogical satisfaction* and *semantic consequence* such that $M, V \mid\models_{\Phi} \Box\alpha$ iff $\Phi \mid\models \alpha$, and we give the evident calculus *QNI*: α if α is a tautology, $\forall x\phi(x) \to \phi(t)$ if t free for x in $\phi, x \equiv x$, $\phi(x) \wedge x \equiv y \to \phi(y)$, $\Box T$; $\alpha, \alpha \to \beta/\beta, \phi \to \Psi/\phi \to \forall x\Psi$ if $x \notin$ fv $\phi, \alpha \leftrightarrow \beta/ \forall x\alpha \leftrightarrow \forall x\beta, \alpha \leftrightarrow \beta/\Box\alpha \leftrightarrow \Box\beta$, whence $\Phi \Vdash \alpha$:iff $\Phi \cup \{\neg\Box\phi : \Phi \not\models \phi\} \vdash_{\mathrm{QNI}} \alpha$.

Successive reduction $r^{\Phi}\alpha$ will be our method to proceed. Thereby we will establish that *there is only one* $seq^{\Box}$. $\Phi \mid\models \alpha$ iff $\Phi \Vdash \alpha$ will follow. Φ $seq^{\Box}\alpha$ implies Φ $seq^{\Box}\Box\alpha$, non Φ $seq^{\Box}\alpha$ implies Φ $seq^{\Box}\neg\Box\alpha$. Φ $seq^{\Box}\Box\alpha \to \alpha, \neg\Box\alpha \to \Box\neg\Box\alpha, \Box\alpha \wedge \Box(\alpha \to \beta) \to \Box\beta$. $seq^{\Box}$ does *not* produce Gödel formulae: naturally, α displays itself, and for every consistent Φ and for all α, *non* Φ $seq^{\Box}\alpha \leftrightarrow \neg\Box\alpha$. In addition, e.g., Φ $seq^{\Box}\neg\Box\bot$, and *non* Φ $seq^{\Box}\neg\Box\neg\Box\bot$ (if Φ consistent).

Immanent attempts cipher ϕ by $< \phi >$ (with respect to some Gödelisation) and try to reflect provability or truth by means of formulae $\iota = \iota(x)$. $seq^{\Box}$, uniquely achieving complete representation (transcendently, so to speak), yields the *soundness criterion*: Φ *seq* $\mathrm{I}_{\iota}(\alpha)$ must imply Φ $seq^{\Box}\alpha$, whereby the translation $\mathrm{I}_{\iota} : \mathrm{L}^{\Box} \to \mathrm{L}$ is inductively defined with $\mathrm{I}_{\iota}(\Box\alpha) := \iota(< \mathrm{I}_{\iota}(\alpha) >)$. However, *if* Φ *is sufficiently strong and consistent, then* Φ *is not soundly representable immanently.* Proof: By assumption, Φ *seq* $\sigma_0 \leftrightarrow \neg\iota(< \sigma_0 >)$ for any ι. Let $\alpha_0 := \sigma_0 \leftrightarrow \neg\Box\sigma_0$. Then Φ *seq* $\mathrm{I}_{\iota}(\alpha_0)$, but non Φ $seq^{\Box}\alpha_0$.—Sound representation of metalogic within arithmetics is impossible. Among other things, the 2^{nd} incompleteness theorem must be doubted.

Mathematization would have been unthinkable without Wilfried Buchholz.

► CARLO NICOLAI, *A note on typed truth and consistency.*
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Axiomatic theories of truth have been widely investigated in the past decades. Although they may capture quite different intuitions concerning the notion of truth, they all share a common structure: the language of a base theory—usually an arithmetical system—is expanded with resources for truth and suitable axioms governing the new vocabulary are added to it. We investigate an alternative construction: faithful to the Tarskian picture of the metatheory, we distinguish between the base theory on the one side and the theory formalizing its syntax and the truth axioms on the other. Theories constructed along these lines have already been investigated by the author in [3]. In the talk we approach the construction from a different angle: we consider the truth+syntax package as playing the role of a functor $Tr[.]$ that applies to arbitrary object theories. We will characterize this functor, answering some conjectures by Richard Heck, as a canonical, abstract consistency statement, *modulo* $I\Delta_0(\mathsf{exp})$ provable equivalence. More precisely, by resorting to well-known results of Paris, Wilkie and Visser, we show that Con_{U} (where U is an arbitrary object theory) can be seen as the unique Π^0_1-sentence σ—unique in the sense of $I\Delta_0(\mathsf{exp})$-provable equivalence—such that $Tr[\mathsf{U}]$, or variants thereof, is mutually interpretable with $\mathsf{Q}+\sigma$. By Pudlák's strengthening of Gödel's Second Incompleteness Theorem, U is not interpretable in $\mathsf{Q}+Con_{\mathsf{U}}$. Therefore any theory containing our version of a Tarski-style axiomatisation of the truth predicate will be logically stronger—at least in the sense of relative interpretability—than the theory U. In the concluding remarks, we will focus on the one hand on what happens if a similar strategy is applied to axiomatizations of the truth predicate constructed in the usual way; on the other, we consider the impact of our results on the debate around the explanatory role of the truth predicate.

[1] Petr Hájek and Pavel Pudlák, ***Metamathematics of First-Order Arithmetic***, Springer, 1998.

[2] Volker Halbach, ***Axiomatic Theories of Truth. Revised Edition***, Cambridge University Press, 2014.

[3] Carlo Nicolai and Graham E. Leigh, *Axiomatic Truth, Syntax and Metatheoretic Reasoning*, ***The Review of Symbolic Logic***, vol. 6 (2013), no. 4, pp. 613–636.

[4] Albert Visser , *Can We Make the Second Incompleteness Theorem Coordinate Free?*, ***Journal of Logic and Computation***, vol. 21 (2013), no. 4, pp. 543–560.

[5] Albert Visser , *The Formalization of Interpretability*, ***Studia Logica***, vol. 50 (1991), no. 1, pp. 81–105.

► ELENA NOGINA, *On Explicit-Implicit Reflection Principles.*
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We study reflection principles of Peano Arithmetic PA based on both proof and provability predicates (cf. [1, 2]). Let P be a propositional letter and each of $Q_1, Q_2, \ldots, Q_m$ is either '$\Box$' standing for provability in PA ([2]), or 'u:' standing for '*u is a proof of ... in* PA' ([1]), u is a fresh proof variable. Then the formula

$$Q_1Q_2\ldots Q_mP\to P$$

is called *generator*, and the set of all its arithmetical instances is the *reflection principle* corresponding to this generator. We will refer to reflection principles using their generators. It is immediate that all reflection principles without explicit proofs ($Q_i = \Box$

for all i) are equivalent to the local reflection principle $\Box P \to P$. All $\Box$-free reflection principles are provable in PA and hence equivalent to $u{:}P \to P$. Mixing explicit proofs and provability yields infinitely many new reflection principles.

Theorem 1. *Any reflection principle in* PA *is equivalent to either* $\Box P \to P$ *or* $\Box^k u{:}P \to P$ *for some* $k \geq 0$.

Theorem 2. *Reflection principles constitute a non-collapsing hierarchy with respect to their deductive strength*

$$[u{:}P \to P] \ < \ [\Box u{:}P \to P] \ < \ [\Box^2 u{:}P \to P] \ < \ \dots \ < \ [\Box P \to P].$$

The proof essentially relies on the Gödel-Löb-Artëmov logic GLA introduced in [3].

[1] S. Artemov, *Explicit provability and constructive semantics*, ***Bulletin of Symbolic Logic***, vol. 7(2001), no. 1, pp. 1–36.

[2] G. Boolos, ***The Logic of Provability***, Cambridge University Press, 1993.

[3] E. Nogina, *On logic of proofs and provability*, ***Bulletin of Symbolic Logic***, vol. 12(2006), no. 2, pp. 356.

▸ CYRUS F NOURANI, *More on Completion with Horn Filters.*
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Let $\mathrm{L}_{\mathrm{P},\omega}$ be the positive fragment obtained from the Kiesler fragment. On a subsequent paper to ASL-SLK, the author hinted that CH is not necessary to prove the proposition that every formula on the presentation P is completable with a companion closure T^*. Without CH we can prove that for Horn representations. Let us abbreviate Rasiowa-Sikorski Lemma as RSL and positive fragment consistency as PFC, respective. Now we can state the following proposition on: Define the category $\mathbf{L}_{\mathrm{P},\omega}$ to be the category with objects positive fragments and arrows the subformula preorder on formulas.

Theorem PFC+RSL implies that every positive Horn representation is completable on a Horn PFC theory.

[1] Nourani, C.F., *Positive Realizability on Horn Filters*, ***Logic Colloquium 2008***

▸ SERGI OMS, *Towards a Conditional for The Liar and the Sorites.*
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I want to present a three-valued paracomplete logic, based on the work of Hartry Field, that captures in a reasonably intuitive way how we reason under the phenomenon of vagueness in languages with a truth predicate. I claim that this is a first step towards a satisfactory logic for the Vagueness and Liar-like paradoxes where the naive theory of truth can be implemented; that is, where we can have the Intersubstitutivity Principle (IP):

> If two sentences A and B are alike except that one has a sentence C where the other has $T\ulcorner C \urcorner$, then $A \models B$ and $B \models A$.

I will use a language $\mathcal{L}$ suitable to express canonical names for its own sentences and I will extend it to a new language, $\mathcal{L}^+$, with a truth predicate, Tr. I will use models with a set W of three valued points and create a process of revision where each point is enlarged to a Kripke fixed point. The conditonals I will use will be of the following form:

$$| A \Rightarrow B |_{u,\alpha,\sigma} = \begin{cases} 1 \text{ iff } (\exists \beta < \alpha)(\forall \gamma \in [\beta, \alpha))(\forall w \in W \text{ such that } u \leq w), \\ \quad \text{if } \; | A |_{w,\gamma,\Omega} = 1 \text{ then } \; | B |_{w,\gamma,\Omega} = 1 \\ 0 \text{ iff } (\exists \beta < \alpha)(\forall \gamma \in [\beta, \alpha)), | A |_{u,\gamma,\Omega} = 1 \text{ and } \; | B |_{u,\gamma,\Omega} = 0 \\ 1/2 \text{ otherwise} \end{cases}$$

Where u is a point in the model, α is a stage on the revision process and σ is the Kripke fixed point for the truth predicate in α.

▸ SERGEY OSPICHEV, *Computable numberings in Ershov hierarchy.*
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Study the cardinality and the structure of Rogers semilattices of families of sets in different hierarchies is one of the main questions in numbering theory [1]. Here we concentrate our interest on Rogers semilattices in Ershov hierarchy [2]. The talk will cover some recent results from this field.

In work are proven

THEOREM 1. *For any nonzero ordinal notation a there is $\mathcal{S}$, infinite family of Σ_a^{-1}-sets, with only one minimal numbering.*

THEOREM 2. *For any nonzero ordinal notation a there is $\mathcal{S}$, infinite family of Σ_a^{-1}-sets, without minimal and principal numberings.*

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[1] S.S. GONCHAROV, S. BADAEV, *Theory of numberings, open problems*, ***Contemporary Mathematics*** , vol. 257, pp. 23–38.

[2] M.M. ARSLANOV, ***Ershov hierarchy***, Kazan,Kazan State University, 2007.

▸ VALERIA DE PAIVA AND LUIZ CARLOS PEREIRA, *An intuitive semantics for Full Intuitionistic Linear Logic.*
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This work describes an *intuitive* semantics in the style of Girard's well-known cigarette vending machine for Full Intuitionistic Linear Logic. Full Intuitionistic Linear Logic (FILL) was introduced by Hyland and de Paiva [1] as arising from its categorical semantics, while hinting at its independent interest as a framework for forms of parallelism in Functional Programming. The systems FILL and its intuitionistic counterpart FIL [2] show that the constructive character of logical systems is not given by syntactic size restrictions on sequent calculus, but comes about by explaining connectives in terms of intensional constructions/operations/transformations on derivations of the system. This seems to us the central message of the Brouwer-Heyting-Kolmogorov (BHK) interpretation and also of the Curry-Howard isomorphism, which we take as guiding criteria for our mathematical logic investigations. This work also aims to explain to the mythical man-on-the street what Full Intuitionistic Linear Logic is about. We were pressed on the point that, elegance of categorical constructions and esthetic criteria on proof systems notwithstanding, one should always be able to say what our logical operations mean in common words, when describing a new logical system like FILL. Initially we had no intuitive explanation for the multiplicative disjunction 'par', which now seems more understandable in terms of interactions with a 'stock-keeping' system.

[1] Hyland, Martin and de Paiva, Valeria, *Full Intuitionistic Linear Logic (extended abstract)*, ***Annals of Pure Applied Logic***, vol. 64 (1993), no. 3, pp. 273–291.

[2] Valeria de Paiva and Luiz Carlos Pereira, *A Short Note on Intuitionistic Propositional Logic with Multiple Conclusions*, ***Manuscrito - Revista Internacional Filosofia, Campinas***, vol. 28 (2005), pp. 317–329.

▶ FEDOR PAKHOMOV, *Ordinal Notations and Fundamental Sequences in Caucal Hierarchy.*
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The Caucal hierarchy of infinite graphs with colored edges is a wide class of graphs with decidable monadic theories[1]. Graphs from this hierarchy can be considered as structures with finite number of binary relations. It is known that the exact upper bound for order types of the well-orderings that lie in this class is ε_0[2]. Actually, any well-ordering from Caucal hierarchy can be used as a constructive ordinal notation system. We investigate systems of fundamental sequences for that well-orderings and the corresponding fast-growing hierarchies of computable functions.

For a well-ordering $(A, <_A)$ we can determine a system of fundamental sequences $\lambda[n]$ by a relation $Cs(x, y)$ such that

$$Cs(\alpha, \beta) \iff \alpha \text{ is a limit point of } <_A \text{ and } \beta = \alpha[n], \text{ for some } n.$$

Our principal result is that for a well-ordering with a pair of Schmidt-coherent fundamental sequences $(A, <_A, Cs_1, Cs_2)$ from Caucal hierarchy the corresponding fast-growing hierarchies $f^1_\alpha(x)$ and $f^2_\alpha(x)$ are equivalent in the following sense: for all $\alpha <_A \beta$ we have $f^1_\beta(n) > f^2_\alpha(n)$ and $f^1_\beta(n) > f^2_\alpha(n)$, for all large enough n (Schmidt-coherence is a classical condition that implies that functions from fast-growing hierarchy are strictly increasing [3]). We show that any two well-orderings with Schmidt-coherent systems of fundamental sequences from Caucal hierarchy of the same order type $< \omega^\omega$ give rise to the equivalent fast-growing hierarchies. We also prove that it is possible to extend a graph with a well-ordering from Caucal hierarchy by a Schmidt-coheren system of fundamental sequences for the well-ordering in such a way that the resulting graph will lie in Caucal hierarchy.

[1] Didier Caucal, *On Infinite Terms Having a Decidable Monadic Theory*, ***Mathematical Foundations of Computer Science 2002*** (Diks, Krzysztof and Rytter, Wojciech), vol. 2420, Springer Berlin Heidelberg, 2002, pp. 165–176.

[2] Braud Laurent, Arnaud Carayol, *Linear Orders in the Pushdown Hierarchy*, ***Lecture Notes in Computer Science*** (Samson Abramsky, et.al. eds.), vol. 6199, Springer Berlin Heidelberg, 2010, pp. 88–99.

[3] Diana Schmidt, *Built-up Systems of Fundamental Sequences and Hierarchies of Number-Theoretic Functions*, ***Archive for Mathematical Logic***, vol. 18 (1976), pp. 47–53.

▶ JÁN PICH, *Circuit lower bounds in bounded arithmetics.*
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We prove that T_{NC^1}, the true universal first-order theory in the language containing names for all uniform NC^1 algorithms, cannot prove that for sufficiently large n, SAT is not computable by circuits of size n^{2kc} where $k \geq 1, c \geq 4$ unless each function $f \in SIZE(n^k)$ can be approximated by formulas $\{F_n\}_{n=1}^{\infty}$ of subexponential size $2^{O(n^{2/c})}$ with subexponential advantage: $P_{x\in\{0,1\}^n}[F_n(x) = f(x)] \geq 1/2 + 1/2^{O(n^{2/c})}$. Unconditionally, V^0 cannot prove that for sufficiently large n, SAT does not have circuits of size $n^{\log n}$. The proof is based on an interpretation of Krajíček's proof [1] that certain NW-generators are hard for T_{PV}, the true universal theory in the language containing names for all p-time algorithms.

[1] Jan Krajíček, *On the proof complexity of the Nisan-Wigderson generator based on a hard NP∩coNP function*, ***Journal of Mathematical Logic***, vol. 11 (1), 2011, pp. 11–27.

▸ PAOLO PISTONE, *Type equations and second order logic*.
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The aim of this talk is to propose a constructive understanding of second order logic: it is argued that a better grasp of the functional content of the comprehension rule comes from the consideration of inference rules independently of logical correctness; the situation is analogous to that of computation, whose proper functional description imposes to consider non terminating (i.e. "wrong") algorithms.

The Curry-Howard correspondence allows indeed a shift from the question of provability (within a formal system) to that of typability for pure lambda terms, representing for instance recursive functions. By relying on well-known results on type inference, an equational description, independent of type systems, of the predicates required to build proofs of totality is presented: one no more focuses on what one can prove by means of a certain package of rules, but rather on what the rules needed to prove a certain formula must be like, at the level of their functional description.

This might look a bit weird at first glance: by applying this technique it is possible, in principle, to construct second order proofs of totality for all *partial* recursive functions! The assumption that every system of equations for a predicate *defines* a predicate is indeed equivalent to a naïve comprehension axiom.

The focus on typability conditions exposes a different point of view on the phenomenon of incompleteness: the lack of the relevant "diagonal" or "limit " proof is indeed explained by the lack of the relevant "diagonal" or "limit" predicates. On the other hand, on the basis of a characterization of the solvability of type equations by means of recursive techniques, it is conjectured that such a "naïve" approach to second order proofs is "complete" in the following sense: all total recursive functions are provably total in some consistent subsystem of the whole (violently inconsistent) system of equational types.

▸ DENIS PONOMARYOV, *The algorithmic complexity of decomposability in fragments of first-order logic.*
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DEFINITION 1. Let $\mathcal{T}$ be a theory and $\Delta \subseteq \mathtt{sig}(\mathcal{T})$ be a subsignature. The theory $\mathcal{T}$ is called Δ–decomposable if there exist theories $\mathcal{S}_1$ and $\mathcal{S}_2$ such that:
1) $\mathtt{sig}(\mathcal{S}_1) \cap \mathtt{sig}(\mathcal{S}_2) = \Delta$ and $\mathtt{sig}(\mathcal{S}_1) \neq \Delta \neq \mathtt{sig}(\mathcal{S}_2)$;
2) $\mathtt{sig}(\mathcal{S}_1) \cup \mathtt{sig}(\mathcal{S}_2) = \mathtt{sig}(\mathcal{T})$ and $\mathcal{T}$ is equivalent to $\mathcal{S}_1 \cup \mathcal{S}_2$.
The theories $\mathcal{S}_1$ and $\mathcal{S}_2$ are called Δ–decomposition components of $\mathcal{T}$.

We consider the algorithmic complexity of the following problems.

Let Σ and $\Delta \subseteq \Sigma$ be finite signatures. The Δ*–decomposability problem* for signature Σ is the set of indices of pairs $\langle \mathcal{T}, \Delta \rangle$, where $\mathcal{T}$ is a finite Δ–decomposable theory in signature Σ. In other words, this is the problem to decide whether a given finite set of sentences in signature Σ is Δ–decomposable. We also consider the problem of deciding whether a finite theory $\mathcal{T}$ in a finite signature Σ given by a partition $\{\sigma_1, \sigma_2, \Delta\}$ is Δ–decomposable into some components in signatures $\sigma_1 \cup \Delta$ and $\sigma_2 \cup \Delta$, respectively. We refer to this as the problem to decide whether a given theory $\mathcal{T}$ is Δ*–decomposable with a partition* $\{\sigma_1, \sigma_2\}$.

The algorithmic complexity of the Δ–decomposability problem has been studied in various calculi, ranging from expressive fragments of first-order logic [1] to classical propositional [2] and description logics [3]. The results suggested that the complexity of decomposability coincides with the complexity of entailment in the underlying logic. Although this observation was not too surprising (since, the definition of decomposability contains the logical equivalence), a general method for proving this claim was missing. We describe a method for proving that the complexity of deciding decomposability coincides with the complexity of entailment in fragments of first-order logic. We illustrate this method by showing the complexity of decomposability in signature fragments of first-order logic, i.e. those which are obtained by putting restrictions on signature.

We call a finite signature σ *complex* if it contains at least one binary predicate, or a function of arity $\geqslant 2$, or at least two unary functions.

THEOREM 2. *1) For any complex signature σ, there exists a finite extension $\Sigma \supseteq \sigma$ such that the $\varnothing$–decomposability problem for Σ is undecidable. 2) For a finite signature Σ consisting of unary predicates and constants it is coNEXPTIME-complete to decide whether a finite theory in signature Σ is Δ–decomposable with a given partition $\{\sigma_1, \sigma_2\}$.*

An extended version of the abstract containing proofs is available at:
http://persons.iis.nsk.su/en/person/ponom/papers

[1] MOROZOV A., PONOMARYOV D., *On decidability of the decomposability problem for finite theories*, ***Siberian Mathematical Journal***, vol. 51 (2010), no. 4, pp. 667–674. http://link.springer.com/article/10.1007/s11202-010-0068-6

[2] EMELYANOV P., PONOMARYOV D., *The complexity of AND-decomposition of boolean formulas*, ***Manuscript, submitted*** 2014, available at http://persons.iis.nsk.su/en/person/ponom/papers

[3] KONEV B., LUTZ C., PONOMARYOV D., WOLTER F., *Decomposing description logic ontologies*, ***Proc. twelfth international conference on the principles of knowledge representation and reasoning (KR 2010)*** (Toronto, Canada). http://www.csc.liv.ac.uk/~frank/publ/fulldecomp.pdf

▸ GEMMA ROBLES, *A Routley-Meyer semantics for Gödel 3-valued logic G3.*
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Gödel 3-valued logic G3 is the strongest of the Gödel many-valued logics introduced in [1]. Although the Routley-Meyer semantics (RM-semantics) was defined for interpreting relevant logics in the early seventies of the last century (cf. [4]), it was soon found out to be suitable for characterizing a wide family of logics regardless of their being relevant or not, due to its malleability. Still, a necessary condition for a logic S to be characterized by the RM-semantics is that Routley and Meyer's basic positive logic B_+ is included in S (cf. [4]). The aim of this paper is to provide an RM-semantics for G3 once this logic has been axiomatized as an extension of B_+ (cf. [2], [3]).

[1] K. Gödel, *Zum intuitionistischen Aussagenkalkül*, ***Anzeiger Akademie der wissenschaffen Wien, Math.-Naturwissensch, Klasse***, 69 (1933), pp. 65-69.

[2] G. Robles, *A Routely-Meyer semantics for Gödel 3-valued logic and its paraconsistent counterpart*, ***Logica Universalis***, 7 (2013), pp. 507-532.

[3] G. Robles, *A simple Henkin-style completeness proof for Gödel 3-valued logic G3*, ***Logic and Logical Philosophy***, DOI: 10.12775/LLP.2014.001, (2014).

[4] R. Routley, R. K. Meyer, V. Plumwood, R. T. Brady , ***Relevant Logics and their Rivals***, vol. 1, Atascadero, CA: Ridgeview Publishing Co., 1982.

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▸ LORENZO ROSSI, *Adding a conditional to Kripke's theory of truth.*
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Take Peano arithmetic as base theory, let $\mathcal{L}$ be its language and let $\mathcal{L}_T$ be $\mathcal{L}$ plus a fresh predicate T for "... is true". In Kripke's models ([2]), for φ a sentence of $\mathcal{L}_T$ and $\ulcorner\varphi\urcorner$ its code, φ and $T\ulcorner\varphi\urcorner$ have the same truth-value. The logic of Kripke's construction is weak. Field [1] adds a primitive conditional $\rightarrow$ (different from the material one), providing a semantics that preserves Kripke's theory and validates desirable laws. However, Field's construction has a very high computational complexity.

Can we equip Kripke's theory with a non-trivial conditional given by a simple model? Let $\mathcal{L}_T^{\rightarrow} := \mathcal{L}_T \cup \{\rightarrow\}$ (for a new connective $\rightarrow$). Via an inductive construction, I define a monotone operator Υ that incorporates the Kripkean jump and acts on triples of sets of sentences of $\mathcal{L}_T^{\rightarrow}$ $\langle A, B, C\rangle$. A (B, C) represents the sentences we suppose to have value 1 (0, $\frac{1}{2}$). Given $\langle A, B, C\rangle$ as input, Υ yields a new triple $\langle A', B', C'\rangle$, and in each of A', B', and C' also sentences of the form $\varphi \rightarrow \psi$ are introduced, while Kripke's evaluation is preserved. The process grows monotonically up to a fixed point $\langle A_\infty, B_\infty, C_\infty\rangle$, that interprets the sentences of $\mathcal{L}_T^{\rightarrow}$. This semantics is partial: some sentences have no value. Unlike in Kripke's models, if φ is valued $\frac{1}{2}$, this is a *positive* semantic information: no sentence is valued $\frac{1}{2}$ *simply because* it has not value 1 nor 0. Consistent fixed points validate interesting principles, such as $\varphi \leftrightarrow T\ulcorner\varphi\urcorner$, for large classes of $\mathcal{L}_T^{\rightarrow}$-sentences. This construction is general and can model distinct intuitions about well-behaved conditionals: as an example, I apply it to the sentences that are grounded in $\mathcal{L}_T$.

[1] Field, Hartry (2008), *Saving truth from paradox*, Oxford University Press, Oxford.

[2] Kripke, Saul (1975), "Outline of a theory of truth", *Journal of Philosophy*, 72, 19, 690-750.

▸ SAEED SALEHI, *A characterization for diagonalized–out objects*.
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Cantor's Diagonal Argument came out of his third proof for the uncountability of the set of real numbers (see e.g. [2]). Unlike the first and second proofs, the diagonal argument can also show the non-equinumerosity of a set with its powerset. In modern terms the proof is as follows: for a function $F: A \to \mathscr{P}(A)$, where $\mathscr{P}(A) = \{B \mid B \subseteq A\}$ is the powerset of A, the anti-diagonal set $D_F = \{a \in A \mid a \notin F(a)\}$ is not in the range of F because if it were, say $D_F = F(\alpha)$, then $\alpha \in D_F \leftrightarrow \alpha \notin F(\alpha) \leftrightarrow \alpha \notin D_F$ contradiction. This argument shows up also in Russell's Paradox, the set of sets which do not contain themselves, $R = \{x \mid x \notin x\}$, and in Turing's non-recursively-enumerable set $\overline{K} = \{n \in \mathbb{N} \mid n \notin W_n\}$ where W_n is the domain of the n^{th} recursive function φ_n (i.e., $W_n = \{x \in \mathbb{N} \mid \exists y : \varphi_n(x) = y\}$) by which one can show the algorithmic unsolvability of the halting problem (of a given algorithm on a given input). There are, in fact, many other instances of the diagonal arguments in wide areas of mathematics from logic and set theory to computability theory and theory of computational complexity.

In this talk, we examine this argument in more detail and discuss some other proofs (e.g. [4, 5]) of Cantor's theorem (on the non-equinumerosity of a set with its powerset). By introducing a generalized diagonal argument, we show that all other proofs should fit in this generalized form, which is roughly as follows: for a function $g : A \to A$ the generalized anti-diagonal set $D_F^g = \{g(a) \mid g(a) \notin F(a)\}$ is not in the range of F because if it were, say $D_F^g = F(\alpha)$, then $g(\alpha) \in D_F^g \leftrightarrow g(\alpha) \notin F(\alpha) \leftrightarrow g(\alpha) \notin D_F^g$ contradiction. For the argument to go through, the function g should satisfy some conditions; and we will prove that every subset of A (say $B \subseteq A$) that is not in the range of F (for all $a \in A$, $B \neq F(a)$ holds) should somehow be in this generalized anti-diagonal form ($B \cap g[A] = D_F^g$) for some suitable function g which satisfies those conditions; cf. [1, 3]. We will argue that this provides a characterization for diagonal proofs and indeed characterizes the objects whose existence are proved by a kind of diagonal(izing out) argument.

[1] Jacob C. E. Dekker, *Productive Sets*, ***Transactions of the American Mathematical Society***, vol. 78 (1955), no. 1, pp. 129–149.

[2] John Franks, *Cantor's Other Proofs that $\mathbb{R}$ is Uncountable*, ***Mathematics Magazine***, vol. 83 (2010), no. 4, pp. 283–289.

[3] Bruce M. Horowitz, *Sets Completely Creative via Recursive Permutations*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 24 (1978), no. 25–30, pp. 445–452.

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► SAM SANDERS, *Reverse Mathematics, more explicitly.*
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The program *Reverse Mathematics* ([4]) can be viewed as a classification of theorems of ordinary, i.e. non-set theoretical, mathematics from the point of view of *computability*. Working in Kohlenbach's *higher-order Reverse Mathematics* ([1]), we study an alternative classification of theorems of ordinary mathematics, namely based on the central tenet of Feferman's *Explicit Mathematics* ([2, 3]) that *a proof of existence of an object is converted into a procedure to compute said object.* Nonstandard Analysis is used in an essential way.

Our preliminary classification gives rise to the *Explicit Mathematics theme* (EMT). Intuitively speaking, the EMT states a standard object with certain properties can be computed by a functional if and only if this object *merely exists classically* with the same *nonstandard* properties. Besides theorems of classical mathematics, we also consider intuitionistic objects, like the fan functional ([1, p. 293]).

Acknowledgement. This research is generously sponsored by the John Templeton Foundation and the Alexander Von Humboldt Foundation.

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[4] S. Simpson, *Subsystems of Second-order Arithmetic*, ***2nd ed., Perspectives in Logic***, Cambridge University Press, Cambridge, 2009

► LUCA SAN MAURO, *Towards a theory of computably enumerable graphs.*
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In recent literature, the theory of computably enumerable equivalence relations (ceers) has been widely investigated (see, for instance, [1], [2]). One of the most fruitful approaches is to study them considering the degree structure generated by the following reducibility: Given two ceers R and S, we say that R is reducible to S ($R < S$) if there is a computable function f s.t., for every x, y, $x\,R\,y \Leftrightarrow f(x)\,S\,f(y)$.

In this talk, we propose to make use of this reducibility within a more general context than that of ceers, namely in the study of (simply undirected) c.e. graphs. Our presentation is divided in two parts.

Firstly, we focus on computable graphs. While the theory of computable equivalence relations is quite trivial ([1]), in this context the situation is more intricate. We provide a partial characterization for the computable case.

Secondly, we move to universal graphs. Let U be defined as follows: $e\,U\,i \Leftrightarrow e \in W_i \vee i \in W_e$. We prove that, for any c.e. graph G, $G < U$.

More generally, recall that there is a unique random graph RG s.t. every countable graph G can be embedded as an induced subgraph of RG ([3]). This fact depends on a specific property $(*)$ of RG (see ([3]) for the definition of $(*)$). Hence, it is natural to ask for some analogue of $(*)$ in our context – specially after noticing that $(*)$ fails for U. We discuss several candidates for this role.

[1] U. Andrews, S. Lempp, J. S. Miller, K. M. Ng, L. San Mauro, A. Sorbi, *Universal computably enumerable equivalence relations*, ***Journal of Symbolic Logic***, to appear

[2] S. Gao, P. Gerdes, *Computably Enumerable Equivalence Relations*, ***Studia Logica***, February 2001, Volume 67, Issue 1, pp 27-59 vol. 67 (2001), no. 1, pp. 27–59.

[3] P. Cameron, *The random graph*, ***The Mathematics of Paul Erdős, II (2nd ed.)*** (R. L. Graham, J. Nešetřil and S. Butler, ed.), Springer, Publisher's address, 2013, pp. 353–378.

▸ ANDREY SARIEV, *Definability of* $\mathbf{0}'$ *in the structure of the* ω*-enumeration degrees.*
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In this paper we find a first order formula which defines the first jump of the least element in the structure of ω-enumeration degrees.

[1] I.N. Soskov, *The ω-enumeration degrees*, ***Journal of Logic and Computation***, to appear.

[2] I.N. Soskov, H. Ganchev, *The jump operator on the ω-enumeration degrees*, ***Annals of Pure and Applied Logic***, to appear.

▸ NOAH SCHWEBER, *Computability in generic extensions.*
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In this talk we will explore connections between computable structure theory and generic extensions of the set-theoretic universe, V. Recall the definition of *Muchnik reducibility* for countable structures: $\mathcal{A} \leq_w \mathcal{B}$ if every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$. We will begin by introducing the notion of *generic Muchnik reducibility*, $\leq_w^*$: we say $\mathcal{A} \leq_w^* \mathcal{B}$ for uncountable structures $\mathcal{A}, \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in some (=every) generic extension $V[G]$ in which $\mathcal{A}$ and $\mathcal{B}$ are both countable. We will discuss the basic properties and give some examples of generic Muchnik (non-)reducibilities among natural uncountable structures.

We will then turn our attention to *generic presentability*. Roughly speaking, an object $\mathcal{X}$ is generically presentable if a "copy" of $\mathcal{X}$, up to the appropriate equivalence relation, exists in every generic extension of the universe by some fixed forcing notion. Solovay [Sol70] showed that all generically presentable *sets* (up to equality) already exist in the ground model; we will investigate the situation for *countable structures* (up to isomorphism) and *infinitary formulas* (up to semantic equivalence). We will present two Solovay-type results (and some consequences): (1) any structure generically presentable by a forcing not making ω_2 countable has a copy in V, and (2) (under CH) any structure generically presentable by a forcing not collapsing ω_1 has a *countable* copy in V. Time permitting, we will discuss a contrasting result coming from work by

Laskowski and Shelah [LS93].

This is joint work with Julia Knight and Antonio Montalban [KMS].

[KMS]Julia Knight, Antonio Montalbán, and Noah Schweber, ***Computable structures in generic extensions***, In preparation.

[LS93]M. C. Laskowski and S. Shelah, *On the existence of atomic models*, ***J. Symbolic Logic***, 58(4):1189-1194, 1993.

[Sol70]Robert M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, ***Ann. of Math (2)***, 92:1-56, 1970.

► PAUL SHAFER, *Every non-zero honest elementary degree has the cupping property.*
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If $a < b$ are elements of a lattice, then we say that *a cups to b* if there is a $c < b$ such that $a \cup c = b$. In [1], Kristiansen proves that if $\mathbf{a} <_{\mathrm{E}} \mathbf{b}$ in the lattice of honest elementary degrees and $\mathbf{a}$ is significantly above $\mathbf{0}$ (that is, there is a function elementary in $\mathbf{a}$ that majorizes every elementary function), then $\mathbf{a}$ cups to $\mathbf{b}$. We improve this result by relaxing the restriction that $\mathbf{a}$ is significantly above $\mathbf{0}$ to simply that $\mathbf{a}$ is non-zero: if $\mathbf{a}$ and $\mathbf{b}$ are honest elementary degrees with $\mathbf{0} <_{\mathrm{E}} \mathbf{a} <_{\mathrm{E}} E\mathbf{b}$, then $\mathbf{a}$ cups to $\mathbf{b}$. This answers a question in [2].

[1] Lars Kristiansen, *Subrecursive degrees and fragments of Peano arithmetic*, ***Archive for Mathematical Logic***, vol. 40 (2001), no. 5, pp. 365–397.

[2] Lars Kristiansen, Jan-Christoph Schlage-Puchta, and Andreas Weiermann, *Streamlined subrecursive degree theory*, ***Annals of Pure and Applied Logic***, vol. 163 (2012), no. 6, pp. 698–716.

► ALEX SIMPSON, *A sheaf model of randomness.*
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In this talk I will present some properties of the universe of sets from the perspective of a particular sheaf topos, which I call the *random topos*. This is a boolean topos, hence a model of classical set theory, whose properties make it a natural home for developing a version of probability theory based on random elements.

An important feature of the topos is a fundamental notion of *independence*. This gives rise to a canonical definition of random element: an element (e.g., from the interval $[0, 1]$) is defined to be *random* if it is contained in all measure 1 subsets that are indpendent of it. This definition can be used to support the development of theories of probability and measure, in which all sets are measurable (though not necessarily Lebesgue measurable), and measures are κ-additive for any aleph, κ. (Of course the Axiom of Choice fails, though Dependent Choice holds.)

The above results closely mirror work of van Lambalgen from 1992 [1]. However, our approach differs from his in two main respects. The first is that our model is a sheaf topos built over a site of probability spaces. Because of this, statements about randomness get translated, by Kripke-Joyal semantics in the topos, into statements in standard (Kolmogorov-style) probability theory. Second, the notion of independence that we use can be understood prior to and separately from the definition of randomness. Independence in our sense corresponds roughly to “no information in common”.

In contrast, van Lambalgen's notion of independence has a definition of randomness built into it.

[1] MICHIEL VAN LAMBALGEN, *Independence, Randomness and the Axiom of Choice*, ***Journal of Symbolic Logic***, vol. 57 (1992), no. 4, pp. 1274–1304.

▸ ALEXANDRA SOSKOVA, *Degree spectra of sequences of structures.*
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There is a close parallel between classical computability and the effective definability on abstract structures. For example, the Σ^0_{n+1} sets correspond to the sets definable by means of computable infinitary Σ_{n+1} formulae on a structure $\mathfrak{A}$. In his last paper, Soskov gives an analogue for abstract structures of Ash's reducibilities between sets of natural numbers and sequences of sets of natural numbers. He shows that for every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure $\mathfrak{M}$ such that the sequences that are ω-enumeration reducible to $\vec{\mathfrak{A}}$ coincide with the c.e. in $\mathfrak{M}$ sequences. He generalizes the method of Marker's extensions for a sequence of structures. Soskov demonstrates that for any sequence of structures its Marker's extension codes the elements of the sequence so that the n-th structure of the sequence appears positively at the n-th level of the definability hierarchy. The results provide a general method given a sequence of structures to construct a structure with n-th jump spectrum contained in the spectrum of the n-th member of the sequence. As an application a structure with spectrum consisting of the Turing degrees which are non-low$_n$ for all $n < \omega$ is obtained. Soskov shows also an embedding of the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures.

We apply these results and generalize the notion of degree spectrum with respect to an infinite sequence of structures $\vec{\mathfrak{A}}$ in two ways as Joint spectra of $\vec{\mathfrak{A}}$ and Relative spectra of $\vec{\mathfrak{A}}$. We study the set of all lower bounds of the generalized notions in terms of enumeration and ω-enumeration reducibility.

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▸ DANIEL T. SOUKUP, *Davies-trees in infinite combinatorics.*
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Davies-trees are special sequences of countable elementary submodels which played important roles in generalizing arguments using the Continuum Hypothesis to pure ZFC proofs. The most notable application of this technique is probably Jackson and Mauldin's solution to the Steinhaus tiling problem [3].

The aim of this talk is to introduce Davies-trees and to point out several new applications in infinite combinatorics. Such include simple proofs to the following results: the plane is the union of $n + 2$ "clouds" provided that the continuum is at most $\aleph_n$ [1]; every uncountably chromatic graph contains k-connected uncountably chromatic subgraphs for each finite k [2].

Our belief is that Davies-trees did not get their well deserved attention despite the fact that they provide an easily applicable tool for logicians and set theorists.

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▸ WOJCIECH STADNICKI, *A descriptive set theoretical axiomatization of the Mathias model.*
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We investigate a series of axioms, which capture the combinatorial core of the Mathias model. These axioms are formulated in terms of games with Borel sets and functions, without explicitly refering to forcing. In this way we derive a descriptive set theoretical axiomatization of the Mathias model. We consider some properties of this model, in particular values of cardinal coefficients. We derive them directly from our axioms.

One of those axioms implies that $\mathfrak{h}((\mathcal{P}(\omega)/\mathrm{fin})^2) = \omega_1$ (see [1]), where $\mathfrak{h}(\mathbb{P})$ is the distributivity of $\mathbb{P}$. Moreover, it gives $\mathfrak{h}((\omega)^\omega, \leq^*) = \omega_1$ (see [2]), where $(\omega)^\omega$ is the set of infinite partitions of ω. For $X, Y \in (\omega)^\omega$ we say $X \leq^* Y$ iff almost every piece of X is a union of pieces of Y.

Although we concentrate on the Mathias model, our methods are more general. One can produce an analogous axiomatization of other models obtained by the iteration of suitably definable proper forcing.

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▸ VLADIMIR STEPANOV, *Truth theory for logic of self-reference statements as a quaternion structure.*
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Let $\boldsymbol{P(x)}$ be a predicate formula of a fragment of the type-free second-oder language without ∀- and ∃-quantors, in which predicates can take other predicate as arguments. Let $\boldsymbol{P(x)}$ be constructed by $\leftrightarrow$ $\neg$ from atomic predicate $\boldsymbol{Tr(x)}$, which satisfies Tarsky axiom: $\boldsymbol{Tr(x) \leftrightarrow x}$.

The self-reference might be expressed with the help of the fixed-point axiom. As for us, for the same aim we would use the quantor of self-reference $\mathsf{S}\boldsymbol{x}$ combined with the axiom of self-reference [1]: $\mathsf{S}\boldsymbol{xP(x)} \leftrightarrow \boldsymbol{P(}\mathsf{S}\boldsymbol{xP(x))}$.

The logic which there are only those formulas which contain biconditional $(\leftrightarrow)$ and negation $(\sim)$ is the three Cartesian direct power of classical propositional logic $\boldsymbol{C^2}$. The characteristic matrix of that logic is

$$\underline{\mathbf{M}}^{\mathbf{c}}_{\mathbf{8}} = (\mathbf{M}^{\mathbf{c}}_{\mathbf{2}})^{\mathbf{3}} = < \{\mathbf{T}, \mathbf{V}, \mathbf{A}, \mathbf{K}, \sim\mathbf{K}, \sim\mathbf{A}, \sim\mathbf{V}, \sim\mathbf{T}\ \}, \sim,\ \underline{\leftrightarrow},\ \{\mathbf{T}\} >.$$

Here $\mathbf{T}$=true, $\mathbf{V}$=truthteller, $\mathbf{A}$=liar, $\mathbf{K} = (\mathbf{V}\underline{\leftrightarrow}\mathbf{A})$. In thus certain multiple-valued logic $\underline{\mathbf{M}}^{\mathbf{c}}_{\mathbf{8}}$ the truth table for connection of biconditional $(\underline{\leftrightarrow})$ represents the Cayley table for the Klein four group (see below).

$\leftrightarrow$	**T**	**V**	**A**	**K**
T	**T**	**V**	**A**	**K**
V	**V**	**T**	**K**	**A**
A	**A**	**K**	**T**	**V**
K	**K**	**A**	**V**	**T**

The Klein four group

$\mathbf{V^2} = \mathbf{A^2} = \mathbf{K^2} = \\ = \mathbf{VAK} = \mathbf{T}$

are replaced with

$\mathbf{V^2} = \mathbf{A^2} = \mathbf{K^2} = \\ = \mathbf{VAK} = \sim\mathbf{T}$

$\leftrightarrow_Q$	**T**	**V**	**A**	**K**
T	**T**	**V**	**A**	**K**
V	**V**	$\sim$**T**	**K**	$\sim$**A**
A	**A**	$\sim$**K**	$\sim$**T**	**V**
K	**K**	**A**	$\sim$**V**	$\sim$**T**

The quaternion group

Thus received the quaternion group allows us to make the following hypothesis:
The Quaternion Hypothesis: We postulate that truth space of self-reference statements is a quaternion structure, so that the units { **V**, **A**, **K** } represent dimensions of truth space of properly self-reference statements , while the scalar **T** represents a classical statements, and the space units obey the product rules given by W. R.Hamilton in 1843. This property we try to use for recording estimates of logical formulas in the form of a quaternion: $\mathbf{Q} = a_0\mathbf{T} + a_1\mathbf{V} + a_2\mathbf{A} + a_3\mathbf{K}$. Here $a_0 \div a_3$ take the values 1, $\sim$, 0, which means that the component may be positive or negative occurrence, or may not have it all.

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Artemov [1, 2] established an arithmetical interpretation for the Logics of Proofs $\mathsf{LP}_{\mathsf{CS}}$, which yields a classical provability semantics for the modal logic $\mathsf{S4}$. These Logics of Proofs are parameterized by so-called constant specifications CS that state which axioms can be used in the reasoning process, and the arithmetical interpretation relies on the constant specifications being finite. In our paper [3], we remove this restriction by introducing weak arithmetical interpretations that are sound and complete for a wide class of constant specifications, including infinite ones. In particular, they interpret the full Logic of Proofs LP.

This is joint work with Roman Kuznets.

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[3] Roman Kuznets and Thomas Studer, *Weak arithmetical interpretations for the logics of proofs*, submitted.

▸ ALEXEY STUKACHEV, *Dynamic logic on approximation spaces.*
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We present recent results on a version of dynamic logic [2, 4, 5] suitable to describe properties of approximation spaces [1, 3, 6], with the set of finite (compact) elements considered as a structure (typical example is the set of rational numbers within the set of real numbers). We consider the case when this structure generates on a whole approximation space an induced structure in a way definable in dynamic logic. One of the natural questions is to describe properties (model-theoretic, effective, etc.) of structures induced this way.

We apply this general technique to the topics studied in [7, 8].

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[4] Yu.L. Ershov, ***Definability and Computability***, Plenum, New York, 1996.

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[7] A.I. Stukachev, Effective Model Theory: an Approach via Σ-Definability, ***Lecture Notes in Logic***, v. 41 (2013), 164-197.

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► NOBU-YUKI SUZUKI, *Some properties related to the existence property in intermediate predicate logics.*

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We discuss relationships between the *existence property* (EP) and its weak variants in intermediate predicate logics.

In [2], we provided a negative answer to Ono's problem **P52** in intermediate predicate logics (*Does EP imply the disjunction property?* [1]), and presented some related results.

To solve the problem, we considered two variants of EP in super-intuitionistic predicate logics, and used them to construct counterexamples in intermediate logics. One variant is an extreme EP; namely, for every $\exists x A(x)$, $\mathbf{L} \vdash \exists x A(x)$ implies that there exists a *fresh* individual variable v such that $\mathbf{L} \vdash A(v)$. This property is so extreme that none of intermediate predicate logics has it. However, if we restrict $\exists x A(x)$ to a sentence, EP implies this property, which we call the *sentential existence property* (sEP). Another one is a weak variant of EP; an intermediate predicate logic $\mathbf{L}$ is said to have the *weak existence property* (wEP), if for every $\exists x A(x)$ that contains no free variables other than $v_1, v_2, \ldots, v_n$, $\mathbf{L} \vdash \exists x A(x)$ implies $\mathbf{L} \vdash \bigvee_{i=1}^{n} A(v_1)$. Then, it is easy to see that EP implies wEP, and wEP implies sEP.

In the present talk, we show that the converses of these implications do not hold in intermediate predicate logics.

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[2] N.-Y. Suzuki, *A negative solution to Ono's problem P52: Existence and disjunction properties in intermediate predicate logics*, submitted.

▸ MÁTÉ SZABÓ, *C. I. Lewis' Influence on the Early Work of Emil Post.*
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Post's paper [2] from 1921 contains the first published proof of the completeness of the propositional subsystem of *Principia Mathematica* and a decision procedure for it. His unsuccessful attempts in the following years to extend his results to the whole of *Principia Mathematica* lead him to anticipate the Incompleteness and Undecidability results of Gödel and Turing [3]. Being deeply influenced by Lewis' 'Heterodox view' [1], Post considered logical systems as "purely formal developments" to "reach the highest generality possible." This "preoccupation with the outward forms of symbolic expressions" allowed, according to Post, for "greater freedom of method and technique." It made his developments recognizably different from the others, but it was in part "perhaps responsible for the fragmentary nature of [his] development." Moreover, Post views the logical process as "*Essentially Creative*"; that makes "the mathematician much more than a kind of clever being who can do quickly what a *machine* could." Post interprets this conclusion as being contrary to Lewis' view. In my talk I will summarize Lewis' 'Heterodox view' and make transparent his influence on Post's early work. At the end I will show that Post's interpretation of his conclusion is not in conflict with Lewis' views as expressed in [1].

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[2] E. Post, *Introduction to a General Theory of Elementary Propositions*, ***American Journal of Mathematics***, vol. 43 (1921), no. 3, pp. 163–185.
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▸ MAKOTO TATSUTA, AND WEI-NGAN CHIN, *Completeness of second-order separation logic for program verification.*
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This paper extends the separation logic given in [2] to second-order logic and investigates the system. Assertions are extended by $X(e, \ldots, e)$ with a second-order variable X and second-order universal quantification $\forall X A$. Since higher-order separation logic has been actively studied, for example, in [1], this system is interesting.

Since the system has the inference rule

$$\frac{\{A_1\}P\{B_1\}}{\{A\}P\{B\}}\ (conseq) \qquad (A \to A_1 \text{ true},\ B_1 \to B \text{ true})$$

the completeness is relative completeness with respect to true assertions in the standard model.

The expressiveness theorem is proved by extending [3] to second-order logic. In particular, the heapcode translation is extended as follows:

$$\mathrm{HEval}_{X(\vec{t})}(m) = X(\vec{t}, m),$$
$$\mathrm{HEval}_{\forall X A}(m) = \forall X \mathrm{HEval}_A(m).$$

Expressiveness Theorem. For every program P and assertion A, there is a formula W such that for any store s, any heap h, and any second-order assignment σ, W is true at (s, h) with σ if and only if (s, h, σ) is in the weakest precondition for P and A.

Completeness Theorem. If $\{A\}P\{B\}$ is true in the standard model, then $\{A\}P\{B\}$ is provable in the system.

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[2] J.C. Reynolds, Separation Logic: A Logic for Shared Mutable Data Structures, In: *Proceedings of LICS2002* (2002) 55–74.

[3] M. Tatsuta, W.N. Chin, and M.F. Al Ameen, Completeness of Pointer Program Verification by Separation Logic, In: *Proceeding of SEFM 2009* (2009) 179–188.

▸ HSING-CHIEN TSAI, *Finite Inseparability of Elementary Theories Based on Connection.*
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Consider a first-order language L. For any L-formula α, let $\#\alpha$ stand for the Gödel number of α. An L-theory T is finitely inseparable if and only if there is a recursive function f such that for any two disjoint recursively enumerable sets A and B such that $\{\#\alpha : \alpha \text{ is a valid sentence in } L\} \subseteq A$ and $\{\#\alpha : \alpha \text{ is an } L\text{-sentence refuted by some finite model of } T\} \subseteq B$, $f(a, b) \notin A \cup B$, where a and b are indices of A and B respectively. It is easy to see that finite inseparability implies undecidability and the former is strictly stronger than the latter. Let C be a binary predicate and I will show the finite inseparability of the theory axiomatized by the following three axioms: (1) $\forall x Cxx$; (2) $\forall x \forall y (Cxy \to Cyx)$; (3) $\forall x \forall y ((x \neq y \wedge Cxy) \to \exists z (Cxz \wedge \neg Cyz))$. Making use of the said result, I will also show the finite inseparability of the theory axiomatized by (1), (2), (4) $\forall x \forall y (\forall z (Cxz \leftrightarrow Cyz) \to x = y)$ and (5) for any formula $\alpha, \exists x \alpha \to \exists y \forall z (Cyz \leftrightarrow \exists u (\alpha \wedge Cuz))$. The foregoing theory contains exactly the mereological part and the quasi-Boolean part of Clarke's system. There is still one more part of Clarke's system, that is, the quasi-topological part. It is still unknown whether the full Clarke's system is finitely inseparable or not. However, such a system does have finite models and some of them are of a peculiar kind. Based on this observation, I conjecture that the full Clarke's system is also finitely inseparable.

Keywords: AMS classification 03B25, decidability, undecidability, finite inseparability, mereology, mereotopology

▸ TOSHIMICHI USUBA, *Reflection principle of list-chromatic number of graphs.*
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Let $G = \langle V, \mathcal{E} \rangle$ be a simple graph, that is, V is a non-empty set of vertexes and

$\mathcal{E} \subseteq [V]^2$ is a set of edges. The *list chromatic number of* G, $\mathrm{List}(G)$, is the minimal (finite or infinite) cardinal κ such that for every function F on V with $|F(x)| = \kappa$ for $x \in V$, there is a function f on V satisfying that $f(x) \in F(x)$ and if $x\mathcal{E}y$ then $f(x) \neq f(y)$. The *coloring number of* G, $\mathrm{Col}(G)$, is the minimal (finite or infinite) cardinal κ such that there is a well-ordering $\triangleleft$ on V such that $|\{y \in V : y \triangleleft x, y\mathcal{E}x\}| < \kappa$ for every $x \in V$. It is known that $\mathrm{List}(G) \leq \mathrm{Col}(G) \leq |V|$.

The *reflection principle of coloring number of graphs*, RP(Col), is the assertion that every graph with uncountable coloring number has a subgraph of size $\aleph_1$ with uncountable coloring number. This principle wad studied in [1] and [2], and it was appeared that this principle is a very strong large cardinal property. On the other hand, Komjáth [4] showed the consistency of the statement that $\mathrm{Col}(G) = \mathrm{List}(G)$ for every graph G with infinite coloring number. Using his result, Fuchino and Sakai [3] proved that the standard model with RP(Col) also satisfies the *reflection principle of list-chromatic number of graphs*, RP(List), which assets that every graph with uncountable list-chromatic number has a subgraph of size $\aleph_1$ with uncountable list-chromatic number. They also constructed a model in which RP(Col) holds but RP(List) fails. These results suggest the natural question: Does RP(List) imply RP(Col)?

In this talk, we prove the following consistency results, which show that RP(List) does not imply RP(Col), and the bounded version of RP(List) is not a large cardinal property:

1. Suppose GCH. Let λ be a cardinal $> \omega_1$. Then there is a poset which preserves all cardinals, and forces that "RP(List) restricted to graphs of size $\leq \lambda$ holds".
2. Relative to a certain large cardinal assumption, it is consistent that RP(List) holds but RP(Col) fails.

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[2] S. Fuchino, H. Sakai, L. Soukop, T. Usuba, *More about the Fodor-type reflection principle*, ***preprint***,

[3] S. Fuchino, H. Sakai, *On reflection and non-reflection of countable list-chromatic number of graphs*, ***RIMS Kôkyûroku***, vol.1790 (2012), pp. 31–44.

[4] P. Komjáth, *The list-chromatic number of infinite graphs*, ***Israel Journal of Mathemathics***, vol. 196 (2013), no. 1, pp. 67–94.

▸ SEBASTIEN VASEY, *Indiscernible extraction and Morley sequences.*
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We present a new proof of the existence of Morley sequences in simple theories. We avoid using the Erdős-Rado theorem and instead use Ramsey's theorem. The proof shows that the basic theory of forking in simple theories can be developed inside $\left\langle H((2^{2^{|T|}})^+), \in \right\rangle$ without using the axiom of replacement, answering a question of Grossberg, Iovino and Lessmann, as well as a question of Baldwin.

▸ STEFAN V. VATEV, *Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures.*
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For an infinite sequence of sets $\mathscr{R} = \{R_n\}_{n\in\omega}$ and a set X, we write $\mathscr{R} \leq_{c.e.} X$ if for every n, R_n is computably enumerable in $X^{(n)}$, uniformly in n. Soskov [4] considered the following redicibility between sequences of sets

$$\mathscr{R} \leq_\omega \mathscr{P} \text{ iff } (\forall X \subseteq \mathbb{N})[\mathscr{P} \leq_{c.e.} X \;\Rightarrow\; \mathscr{R} \leq_{c.e.} X].$$

This reducibility naturally induces an equivalence relation, whose equivalence classes are called ω-enumeration degrees. They form an upper semi-lattice, which have been extensively studied by a number of researchers at Sofia University over the past decade.

In this talk we discuss how to encode an infinite sequence of sets $\mathscr{R}$ into a single countable structure $\mathcal{N}_\mathscr{R}$, preferably in a finite language, such that the Turing degree spectrum of $\mathcal{N}_\mathscr{R}$ is the set

$$Sp(\mathcal{N}_\mathscr{R}) = \{d_T(X) \mid \mathscr{R} \text{ is c.e. in } X\}.$$

We present two such methods. The first one was studied by Soskov [3] and is based on the so-called Marker's extensions [2]. The other approach is based on the idea of coding each set R_n by a sequence of pairs of computable structures [1]. We conclude that for any two infinite sequences of sets $\mathscr{R}$ and $\mathscr{P}$ we can build countable structures $\mathcal{N}_\mathscr{R}$ and $\mathcal{N}_\mathscr{P}$ such that

$$\mathscr{R} \leq_\omega \mathscr{P} \iff Sp(\mathcal{N}_\mathscr{P}) \subseteq Sp(\mathcal{N}_\mathscr{R}).$$

In other words, the ω-enumeration degrees are embeddable into the Muchnik degrees generated by spectra of structures.

[1] C. J. Ash, J. F. Knight, *Pairs of Recursive Structures*, ***Annals of Pure and Applied Logic***, vol. 46, (1990), pp. 211–234.

[2] David Marker, *Non Σ_n Axiomatizable Almost Strongly Minimal Theories*, ***Journal of Symbolic Logic***, vol. 54, (1989), no. 3, pp. 921–927.

[3] Ivan N. Soskov, *Effective Properties of Marker's Extensions*, ***Journal of Logic and Computation***, vol. 23 (2013), no. 6, pp. 1335–1367.

[4] Ivan N. Soskov, *The ω-enumeration degrees*, ***Journal of Logic and Computation***, vol. 17 (2007), pp. 1193–1217.

▸ PAULO VELOSO, SHEILA VELOSO, AND MARIO BENEVIDES, *On graph calculus approach to modalities.*

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We introduce a graphical approach to modalities. We employ formal systems where graphs are expressions that can be manipulated so as to mirror reasoning at the semantical level. This visual approach is flexible and modular providing decision procedures for several normal logics. Promising cases are the application of this approach to PDL for structured data [1] and to memory logics [2].

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[2] Areces, C., Figueira, S. Mera, S. : Completeness Results for Memory Logics. LNCS 5407, pp. 16–30, Springer (2009)

▸ ROGER VILLEMAIRE, *An ordinal rank characterising when Forth suffices.*
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In the original proof that countable dense linear orders are isomorphic, Cantor maps elements in a single direction, contrary to the now common back-and-forth method. He then relies on specific properties of dense linear orders to show that his mapping is indeed onto and hence an isomorphism. This map construction method have been named *Forth* by P. J. Cameron, who, settling a question of A. Mathias, constructed an $\aleph_0$-categorical structure for which Forth fails to yield an onto mapping. In [1] Cameron considered homogeneous structures, for which the Forth construction always build an onto mapping (*Forth suffice* in his terminology). In particular he gave a necessary condition for Forth to suffice. McLeish [2] introduced another necessary condition, more general that Cameron's, but still not sufficient.

This talk will present a necessary and sufficient condition for Forth to suffice in terms of a new ordinal rank. We will emphasise that the rank is derived from a combination of a smallest and a greatest fixpoint (of monotone operators), while McLeish implicitly used a single fixpoint. We will also highlight the existence of homogeneous structures for all possible countable ordinal ranks, with a construction using unions of wreath powers.

[1] P. J. Cameron, ***Oligomorphic Permutation Groups***, London Mathematical Society Lecture Note Series, Cambridge University Press, 1990.

[2] S. J. McLeish, *The forth part of the back and forth map in countable homogeneous structures*, ***Journal of Symbolic Logic***, vol. 62 (1997), no. 3, pp. 873–890.

▸ ANTONIO VINCENZI, *On the logical use of implicit contradictions.*

The basic idea is that (assuming that the logic languages are not rigid) the counterexamples of the *Robinson* property can be considered as an implicit generalization of the usual antinomian contradictions. Since the Robinson property is very rare, these contradictions are not pathological. On the other hand, they can be used in some generalizations of the *'by absurdum'* strategy that concern properties more_subtile than the truth of a statement.

Mathematically, the use of implicit contradictions has a positive impact on *Abstract Model Theory.* For this consider pairs ($\mathscr{L}$,ST)'s in which $\mathscr{L}$ is a model–theoretic logic and ST is its underlying set–theory (see [BF] and [B], respectively) and work in a context where these contradictions can be solved by the relative form ROB(($\mathscr{L}$,ST),($\mathscr{L}^+$,ST$^+$)) of the Robinson property and where *Robinson = Interpolation + Compactness.*

Then, assuming that a logic operation is *formally pure* if it cannot self–referentially negate itself, the counterexamples of *Interpolation* can be characterized by the following Purity Theorem. ($\mathscr{L}$,ST) *has Interpolation iff the* ($\mathscr{L}$,ST)*–proofs are formally pure.*

Instead, the counterexamples of the *Compactness* can be characterized by the results

related to the following
Compactification Conjecture. *If* $[\lambda,\lambda]$–COMP($\mathscr{K}$,ST) *fails then there is a set–theory* ST^{+}=ST+*strong axiom(s) in which 'cofinality λ' is absolute and* $[\lambda,\lambda]$–COMP(($\mathscr{K}$,ST), ($\mathscr{K}$,ST^{+})) *holds.*
Metamathematically, since the pure proofs can be formalized by *Gentzen–style proof–systems that do not introduce new symbols*, the first result is a technical specification of the purity aim of Proof Theory related to the complexity of proof systems. The second kind of results is a technical instrument for studying the *interaction between logics and set–theoretic universes.*
Philosophically, implicit contradictions, being non–pathological, solvable and incompatible with pure formalization are good ingredients for a mathematical description of the *Hegel's Dialectic Logic.*

[BF]Barwise J., Feferman S., *Model–Theoretic Logics*, ***Springer 1985.***
[B]Barwise J., *Admissible Sets and Structure*, ***Springer 1975***

▸ ALEXEY VLADIMIROV, *Some partial conservativity properties for Intuitionistic Set Theory with the principle UP.*
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Let $\mathbb{ZFI}2C$ be usual intuitionistic Zermelo-Fraenkel set theory in two-sorted language (where sort 0 is for natural numbers, and sort 1 is for sets).

Axioms and rules of the system are: all usual axioms and rules of intuitionistic predicate logic, intuitionistic arithmetic, and all usual proper axioms and schemes of Zermelo-Fraenkel set theory for variables of sort 1, namely, axioms of Extensionality, Infinity, Pair, Union, Power set, Infinity; schemes Separation, Transfinite Induction as Regularity, and Collection as Substitution.

It is well-known that both $\mathbb{ZFI}2C$ and $\mathbb{ZFI}2C + DCS$ (where DCS is a well-known principle Double Complement of Sets) have some important properties of effectivity: disjunction property DP, numerical existence property (but not full existence property!) and also that the Markov Rule, the Church Rule, and the Uniformization Rule are admissible in it. Such collection of existence properties shows that these theories are sufficiently constructive theories.

On the other hand, $\mathbb{ZFI}2C + DCS$ contains the classical theory $\mathbb{ZF}2$ (i.e. $\mathbb{ZFI}2C + LEM$) in the sense of Gödel's negative translation. Moreover, a lot of important mathematical reasons may be formalized in $\mathbb{ZFI}2C + DCS$, so, we can formalize and decide in it a lot of informal problems about transformation of a classical reason into intuitionistical proof and extraction of a description of a mathematical object from some proof of it's existence.

So, $\mathbb{ZFI}2C + DCS$ can be considered as a basic system of Explicit Set Theory. We can extend it by a well-known intuitionistic principles, such that Markov Principle M, Extended Church Principle ECT, and the Principle UP.

It is well-known that both $\mathbb{ZFI}2C + DCS + M + ECT$, and $\mathbb{ZFI}2C + DCS + M$ has the same effectivity properties as $\mathbb{ZFI}2C$ and $\mathbb{ZFI}2C + DCS$.

It is known also that $\mathbb{ZFI}2C + DCS + M + ECT$ is conservative over the theory $\mathbb{ZFI}2C + DCS + M$ w.r.t. class of all formulae of kind

w.r.t. class of all formulae of kind $\forall a \exists b \vartheta(a;b)$, where $\vartheta(a;b)$ is a arithmetical negative (in the usual sense) formula. We also have that $\mathbb{ZFI}2C + M + ECT$ is conservative over the theory $\mathbb{ZFI}2C + M$ w.r.t. class of all formulae of kind $\forall a \exists \vartheta(a;b)$, where ECT is the usual schema of the Extended Church Thesis.

The Principle $UP : \forall x \exists a \psi(x;a) \to \exists a \forall x \psi(x;a)$ is a well-known specifical intuitionistic principle.It claims that we can't define effectivly non-trivial function from sets to natural numbers. It havs been studed in intuitionistic type theory.

In the article we prove that $\mathbb{ZFI}2C + DCS + M + CT + UP$ is conservative over the theory $\mathbb{ZFI}2C + DCS + M$ w.r.t. class of all formulae of kind $\forall a \exists b \vartheta(a;b)$, where $\vartheta(a;b)$ is a arithmetical negative (in the usual sense) formula. Sure, we also prove that $\mathbb{ZFI}2C + M + ECT$ is conservative over the theory $\mathbb{ZFI}2C + M$ w.r.t. class of all formulae of kind $\forall a \exists \vartheta(a;b)$.

We also prove that the theories $\mathbb{ZFI}2C + DCS + M + CT + UP$, $\mathbb{ZFI}2C + DCS + M + UP$,$\mathbb{ZFI}2C + DCS + UP$, and $\mathbb{ZFI}2C + UP$ have the same effectivity properties as $\mathbb{ZFI}2C$ and $\mathbb{ZFI}2C + DCS$.

▶ LINDA BROWN WESTRICK, *A computability approach to three hierarchies.*
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We analyze the computable part of three hierarchies from analysis and set theory. The hierarchies are those induced by the Cantor-Bendixson rank, the differentiability rank of Kechris and Woodin, and the Denjoy rank. Our goal is to identify the descriptive complexity of the initial segments of these hierarchies. For example, we show that for each recursive ordinal $\alpha > 0$, the set of Turing indices of computable $C[0,1]$ functions that are differentiable with rank at most α is $\Pi_{2\alpha+1}$-complete. Similar results hold for the other hierarchies. Underlying of all the results is a combinatorial theorem about trees. We will present the theorem and its connection to the results.

▶ PIOTR WOJTYLAK, WOJCIECH DZIK, *Almost structurally complete consequence operations extending* **S4.3**.
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Generalizing well-known results by R.Bull and K.Fine we proved in [2]

THEOREM 1. *(i) Each finitary consequence operation Cn extending* **S4.3** *has a finite basis (over some $L \in \mathrm{NExt}(\mathbf{S4.3})$) consisting of finitary passive rules;*
(ii) Each finitary consequence operation Cn extending **S4.3** *coincide on finite sets with a finitely approximable modal consequence operation.*

Let us recall that a consequence operation Cn is finitely approximable if $Cn = \overrightarrow{\mathbb{K}}$ for some class $\mathbb{K}$ of finite matrices. In the proof of the above Theorem 1 we used our earlier result, from [1],

THEOREM 2. *Each modal formula unifibale in* **S4** *has a projective unifier there. Consequently, each modal consequence operation extending* **S4.3** *is complete with respect to admissible finitary impassive rules (= is almost structurally complete in the finitary case of inferential rules).*

In case of infinitary rules, we have no projective unification, nor (any variant of) structural completeness, for **S4.3**. We prove in [3]

THEOREM 3. *A modal consequence operation Cn extending* **S4.3** *is almost structurally complete (with respect to infinitary rules) iff Cn is finitely approximable.*

We also provide an uniform basis, consisting of infinitary rules, for all admissible rules of any $L \in \mathrm{NExt}(\mathbf{S4.3})$. This rule basis is uncountable and contains, as a sample,

$$\frac{\{\Box(\alpha_i \leftrightarrow \alpha_j) \to \alpha_0 : 0 < i < j\}}{\alpha_0}$$

It also follows that the lattice of all almost structurally complete extensions of **S4.3** is a (complete) sublattice of the lattice of all consequence operations over **S4.3**, isomorphic to the lattice of all finitary extensions of **S4.3**.

[1] Dzik, W., Wojtylak, P., *Projective Unification in Modal Logic*, ***Logic Journal of the IGPL*** 20(2012) No.1, 121–153.

[2] Dzik, W., Wojtylak, P., *Modal consequence relations extending* **S4.3**. *An application of projective unification.*, ***Notre Dame Journal of Formal Logic*** (to appear).

[3] Dzik, W., Wojtylak, P., *Almost structurally complete consequence operations extending* **S4.3**., (in preparation).

▸ TIN LOK WONG, *Some applications of the Arithmetized Completeness Theorem to second-order arithmetic.*

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Gödel's Completeness Theorem is one of the most fundamental results in mathematical logic. When formalized in arithmetic, it is often referred to as the Arithmetized Completeness Theorem (ACT). The ACT is a surprisingly powerful machinery for constructing nonstandard models of arithmetic. For instance, it has been known [1, 2] that 'all possible kinds' of extensions of a model of Peano arithmetic can, in a sense, be realized using the ACT. We find new applications of the ACT in the context of second-order arithmetic. These include an alternative proof of Harrington's theorem [3] that WKL_0 is Π^1_1-conservative over RCA_0.

[1] Kenneth Mc Aloon, *Completeness theorems, incompleteness theorems and models of arithmetic*, ***Transactions of the American Mathematical Society***, vol. 239 (1978), pp. 253–277.

[2] James H. Schmerl, *End extensions of models of arithmetic*, ***Notre Dame Journal of Formal Logic***, vol. 33 (1992), no. 2, pp. 216–219.

[3] Stephen G. Simpson, ***Subsystems of Second Order Arithmetic***, Perspectives in Logic, Cambridge University Press, 2009.

▸ MITKO YANCHEV, *Complexity of generalized grading with inverse relations and intersection of relations.*

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The language of Graded Modal Logic (GML, Kit Fine, 1972) is an extension of the classical propositional modal language with counting (or *grading*) modal operators $\Diamond_n$, for $n \geq 0$, which have purely quantitative meaning. S. Tobies proves (Tobies, 2000) that the satisfiability problem for the graded modal language is PSPACE-complete.

The language of Majority Logic (MJL, Pacuit and Salame, 2004) augments the graded modal language with some qualitative capabilities. Two extra unary modal operators, M and W, are added. In Kripke models $M\varphi$ says that more than half of all accessible worlds satisfy φ, what represents the simplest case of *rational grading*.

The language of Presburger Modal Logic (PML, Demri and Lugiez, 2006) is a many-relational modal language with independent relations, having the so-called *presburger constraints*, which can express both integer and rational grading. Demri and Lugiez show that the satisfiability for the PML language is PSPACE-complete, what strengthens the main result of Tobies, and answers the open question about MJL.

At that time a generalization of modal operators for rational grading in the spirit of the majority operators is given (Tinchev and Y., 2006), and it is used in the language of Generalized Graded Modal Logic (GGML, Tinchev and Y., 2010). New unary grading operators are considered, M^r and W^r, where r is a rational number in (0,1). These operators distinguish the part of accessible worlds having some property.

The generalized rational grading operators are expressible by presburger constraints, so the PSPACE completeness of the satisfiability for the generalized graded modal language is a consequence of that for PML. On the other hand an independent proof using a specific technique for exploring the complexity of rational grading is given (Y., 2011). The presence of separate integer and rational grading operators, and the use of the technique developed for the latter allow following a common way for obtaining complexity results as in less, so in more expressive languages with rational grading. In particular, complexity results—from polynomial to PSPACE—for a range of description logics, syntactic analogs of fragments of GGML, are obtained (Y., 2012, 2013).

In this talk we consider many-relational generalized graded modal language adopting *inverse relations* and *intersection of relations*. Rational grading operators are $\lceil\sigma\rceil^r$ and $\lfloor\sigma\rfloor^r$, where σ is an intersection of (possibly inverse) relations. We show that the satisfiability problem for this expressive modal language with generalized grading keeps the PSPACE complexity.

▸ AIBAT YESHKEYEV, *On Jonsson sets and some their properties.*
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Let L is a countable language of first order. Let T - Jonsson perfect theory complete for existential sentences in the language L and its semantic model is a C.

We say that a set X - Σ-definable if it is definable by some existential formula.

a) The set X is called Jonsson in theory T, if it satisfies the following properties:

1. X is Σ-definable subset of C;

2. $Dcl(X)$ is the support of some existentially closed submodel of C.

b) The set X is called algebraically Jonsson in theory T, if it satisfies the following properties:

1. X is Σ-definable subset of C; 2. $Acl(X)$ is the support of some existentially closed submodel of C.

Using these definitions of the Jonsson sets we can get relatively invariant properties of the similarity of the Jonsson theories on arbitrary subsets of the semantic model.

We say that two sets are Jonsson (equivalent, categorical, syntactically similar, semantically similar) to each other, respectively, if will be (Jonsson equivalent, categorical, stable, similar syntactically, semantically similar) their corresponding theories of the models, which are obtained by the corresponding closures of these sets.

For example: two Jonsson sets syntactically similar to each other, if syntactically similar the theories obtained as their respective closures. In the case when obtained theories will be not Jonsson theories, we will consider correspondingly syntactically similarity [1] of the the elementary theories of existentially closed models which are closures of these sets.

If $\forall\exists$-consequences of arbitrary theories are Jonsson theories, in this case we can consider the Jonsson fragment of such theories and we will try to build results for them in the Jonsson's technic manner. As part of these newly introduced definitions, consider and try to describe the Jonsson strongly minimal set. This in turn will lead to a series of new formulations of the problem, such as a refinement regarding both kinds (countable, uncoutable) of the categoricity under this newly introduced subjects.

All undefined concepts about Jonsson theories in this thesis can be found in [2].

[1] Mustafin T. G., *On similarities of complete theories*, ***Logic Colloquium '90: proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic*** (Finland, July 15-22), 1990, pp. 259–265.

[2] Yeshkeyev A.R., ***Jonsson Theories***, Publisher of the Karaganda state university, 2009.

▸ PEDRO ZAMBRANO, ANDRÉS VILLAVECES, *Uniqueness of limit models in metric abstract elementary classes under categoricity and some consequences in domination and orthogonality of Galois types.*
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Abstract Elementary Classes (AECs) corresponds to an abstract framework for studying non first order axiomatizable classes of structures. In [2], Grossberg, VanDieren and Villaveces studied uniqueness of limit models as a weak notion of superstability in AECs.

In [3], Hirvonen and Hyttinen gave an abstract setting similar to AECs to study classes of metric structures which are not axiomatizable in continuous logic [1], called Metric Abstract Elementary Classes (MAECs).

In this work, we will talk about a study of a metric version of limit models as a weak version of superstability in categorical MAECs [5], and some consequences of uniqueness of limit models in domination, orthogonality and parallelism of Galois types ([4]).

[1] I. Ben-Yaacov, A. Berenstein, C. W. Henson and A. Usvyatsov, *Model theory for metric structures*, ***Model theory with Applications to Algebra and Analysis*** (Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie, editors), vol. 2, Cambridge Univ Press, 2008, pp. 315–427.

[2] R. Grossberg, M. VanDieren and A. Villaveces, *Uniqueness of Limit Models in Classes with Amalgamation*, ***Submitted***.

[3] Å. Hirvonen and T. Hyttinen, *Categoricity in homogeneous complete metric spaces*, ***Arch. Math. Logic***, vol. 48 (2009), pp. 269–322.

[4] A. Villaveces and P. Zambrano, *Around Independence and Domination in Metric Abstract Elementary Classes, under Uniqueness of Limit Models*, ***Accepted at Math. Logic Quarterly***, (2014).

[5] A. Villaveces and P. Zambrano, *Limit Models in Metric Abstract Elementary Classes: The categorical case*, ***Submitted at Math. Logic Quarterly***, (2013)

2 Logic, Algebra and Truth Degrees (LATD) 2014

2.1 Tutorials

▸ FRANZ BAADER, *Fuzzy Description Logics.*
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Description logics (DLs) are a well-investigated family of logic-based knowledge representation formalisms, which can be used to formalize the important notions of a given application domain using terminological axioms. Fuzzy variants of DLs were introduced in order to deal with applications where precise definitions of the relevant notions are not possible. Though fuzzy DLs have been investigated for more than 20 years, it became clear only recently that certain frequently used terminological axioms (called general concept inclusions, GCIs) may cause undecidability in fuzzy DLs.

The tutorial will provide a brief introduction into Description Logics and the use of tableau-based algorithms to decide important inference problems such as satisfiability and subsumption in DLs. In particular, it will consider how such algorithms can deal with GCIs. Subsequently, the tutorial will introduce fuzzy DLs and show how tableau-based algorithms can be extended to decide inference problems for these logics, but also point out why GCIs cannot be handled in the same way as for crisp DLs. Finally, it will demonstrate that the presence of GCIs actually leads to undecidability for many fuzzy DLs.

▸ VINCENZO MARRA, *The more, the less, and the much more: An introduction to Łukasiewicz logic as a logic of vague propositions, and to its applications.*
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In the first talk of this tutorial I offer an introduction to Łukasiewicz propositional logic that differs from the standard ones in that it does not start from real-valued valuations as a basis for the semantical definition of the system. Rather, I show how a necessarily informal but rigorous analysis of the semantics of certain vague predicates naturally leads to axiomatisations of Łukasiewicz logic. It is then the deductive system itself, now motivated by the intended semantics in terms of vagueness, that inescapably leads to magnitudes — the real numbers or their non-Archimedean generalisations. In the second talk I show through examples how the availability of such an intended semantics, far from being an ornamental addition to the literature, is in fact a *sine qua non* to deploy Łukasiewicz logic in applications of genuine importance. In closing, if time allows, I revisit Hájek's Programme in many-valued logic in light of our discussion of Łukasiewicz logic.

2.2 Invited talks

▸ SILVIO GHILARDI, *Step frame analysis in single- and multi-conclusion calculi.*

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(This contribution is joint work with Nick Bezhanishvili). We introduce semantic and algorithmic methods for establishing a variant of the analytic subformula property (called the bounded proof property, bpp) [4, 3] for modal propositional logics. Our methodology originated from tools and techniques developed on one side within the algebraic/coalgebraic literature dealing with free algebra constructions [1, 8, 7, 6] and on the other side from classical correspondence theory in modal logic. The main result states that the bpp and fmp (the finite model property) can be characterized as dual embeddability properties of finite two-sorted frames (called 'step frames') into standard Kripke frames.

The methodology has been recently extended to multi-conclusion rules [5] in order to cope with some canonical axiomatizations of universal classes. This extension allowed to establish both the bpp and fmp for the class of stable modal logics [2], i.e., for those logics whose corresponding frames are closed under homomorphic images.

[1] S. Abramsky. A Cook's tour of the finitary non-well-founded sets. In *Essays in honour of Dov Gabbay*, pages 1-18. College Publications, 2005.

[2] G. Bezhanishvili, N. Bezhanishvili, and R. Iemhoff. Stable canonical rules. 2014. ILLC Prepublication Series Report PP-2014-08.

[3] N. Bezhanishvili and S. Ghilardi. The Bounded proof property via step algebras and step frames. Technical Report 306, Department of Philosophy, Utrecht University, 2013. To appear in Annals of Pure and Applied Logic.

[4] N. Bezhanishvili and S. Ghilardi. Bounded Proofs and Step Frames. In *Proc. Tableaux 2013*, number 8123 in Lecture Notes in Artificial Intelligence, pages 44-58, 2013.

[5] N. Bezhanishvili and S. Ghilardi. Multiple-conclusion rules, hypersequents syntax and step frames. In *Proc. of AimL 2014*, 2014. Available also from author's webpage aat ILLC Prepublication Series Report PP-2014-05.

[6] N. Bezhanishvili, S. Ghilardi, and M. Jibladze. Free modal algebras revisited: the step-by-step method. In *Leo Esakia on Duality in Modal and Intuitionistic Logics*, Trens in Logic. Springer, 2014.

[7] D. Coumans and S. van Gool. On generalizing free algebras for a functor. *Journal of Logic and Computation*, 23(3):645-672, 2013.

[8] S. Ghilardi. An algebraic theory of normal forms. *Annals of Pure and Applied Logic*, 71:189-245, 1995.

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▸ MELVIN FITTING, *The Range of Realization*.
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Justification logics are explicit versions of modal logics. Modal operators are replaced with justification terms, representing specific steps in a formal proof. Justification logics are connected with modal logics via Realization Theorems, which say that necessity operators in a modal theorem can be replaced with justification terms to produce an explicit version of the theorem, a version that is provable in the corresponding justification logic. Negative boxes become variables, positive boxes become terms built up from variables, thus revealing a hidden input-output structure to modal theorems. The first Realization Theorem connected modal S4 with justification LP (logic of proofs). For a long time, the only examples of justification logics were for modal logics closely related to S4. But now it is becoming clear that the phenomenon is a much more general one than had been supposed.

I will discuss the historical origin of Justification Logics, and their corresponding Realization Theorems. Then I will bring things up to date. The range of modal logics which have a justification counterpart, and a connection via a Realization Theorem, is much larger than had been anticipated. At the moment, it is known that all modal logics axiomatized by Geach formulas fall into this category. This class is infinite, and includes many standard modal logics. It seems likely that this extends to Sahlquist formulas, but there is no proof at the moment. This is very much a work in progress.

▶ GEORGE METCALFE, *First-order logics and truth degrees.*
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Classical first-order logic enjoys a number of key properties – notably prenex forms, a deduction theorem, Skolemization, and Herbrand theorems – that render it a particularly suitable formalism for knowledge representation and (automated) reasoning. With the introduction of further truth degrees, however, such properties may be lost or require new formulations. The aim of this talk is to explore the relationships between these key properties in the context of first-order semilinear logics, focussing in particular on first-order Gödel logic and Lukasiewicz logic, paradigmatic logics, respectively, of order and continuity.

▶ DALE MILLER, *Combining Intuitionistic and Classical Logic: a proof system and semantics.*
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While Gentzen's sequent calculus provides a framework for developing the proof theory of both classical and intuitionistic logic, it did not provide us with one logic that combines them. There are, of course, a number of ways to relate classical and intuitionistic logic: for example, intuitionistic logic can be translated into classical logic with the addition of a modality and classical logic can be embedded into intuitionistic logic using negative translations. Here we consider the problem of building proof systems and semantics for a logic in which classical and intuitionistic connectives mix freely. Our solution, the logic of Polarized Intuitionistic Logic, employs a polarization (red/green) of formulas and two entailment judgments. We give a Kripke semantics and a sequent calculus for this logic for which soundness and completeness holds. The sequent calculus proof system mixes elements of Gentzen's LJ proof system and Girard's LC proof system.

This talk is based on joint work with Chuck Liang and the paper "Kripke Semantics and Proof Systems for Combining Intuitionistic Logic and Classical Logic" in the *Annals of Pure and Applied Logic* 164(2), pp. 86-111 (2013).

▶ DANA S. SCOTT, *Geometry without points.*
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Ever since the compilers of Euclid's Elements gave the "definitions" that "a point is that which has no part" and "a line is breadthless length", philosophers and mathematicians have worried that the the basic concepts of geometry are too abstract and too idealized. In the 20th century writers such as Husserl, Lesniewski, Whitehead, Tarski, Blumenthal, and von Neumann have proposed "pointless" approaches. A problem more recent authors have emphasized it that there are difficulties in having a rich theory of a part-whole relationship without atoms and providing both size and geometric dimension as part of the theory. A solution will be proposed using the Boolean algebra of measurable sets modulo null sets along with relations derived from the group of rigid motions in Euclidean n-space. (This is a preliminary report on on-going joint work with Tamar Lando, Columbia University.)

▸ ALASDAIR URQUHART, *Relevance logic: problems open and closed.*
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I discuss a collection of problems in relevance logic. The main problems discussed are: the decidability of the positive semilattice system, decidability of the fragments of **R** in a restricted number of variables, and the complexity of the decision problem for the implicational fragment of **R**. Some related problems are discussed along the way.

2.3 Contributed talks

▸ STEFANO AGUZZOLI, DENISA DIACONESCU, TOMMASO FLAMINIO, *A method for generalizing finite automata arising from Stone-like dualities.*
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We start our investigation by first providing a dictionary for translating deterministic finite automata [7] (DFA henceforth) in the language of classical propositional logic. The main idea underlying our investigation is to regard each DFA as a finite set-theoretical object, applying the finite slice of Stone duality [8] in order to move from DFA to algebra, and finally using the algebraizability of classical logic to introduce the formal objects which arise by this "translation" and that we call *classical fortresses* (for FORmula, TheoRy, SubstitutionS). We show that classical fortresses accept exactly the same languages as finite automata, that is, regular languages [9].

It is known that if one tries to describe the behavior of DFA using a logical language, by Büchi-Elgot-Trakhtenbrot Theorem [2, 5, 11], one comes up with a formalization in the monadic fragment of classical second-order logic. Hence, it is worth to point out that we address a different problem: we do not aim at *describing* DFA using logic, but at introducing logico-mathematical objects – classical fortresses – capable to *mimic* them through the mirror of the Stone duality.

Definition 1. Let Σ be a finite alphabet and let $V = \{v_1, \ldots, v_n\}$ be a finite set of propositional variables[1]. A *classical fortress in n variables* over Σ is a triple

$$\mathcal{F} = \langle \varphi, \{\sigma_a\}_{a\in\Sigma}, \Theta\rangle,$$

where

1. φ is a formula in $Form(V)$,
2. for each $a \in \Sigma$ the map $\sigma_a : V \to Form(V)$ is a substitution,
3. Θ is a prime theory in the variables V.

A classical fortress $\mathcal{F} = \langle \varphi, \{\sigma_a\}_{a\in\Sigma}, \Theta\rangle$ *accepts* a word $w = a_1 \ldots a_k \in \Sigma^*$, denoted by $\mathcal{F} \Vdash w$, if $\Theta \models \varphi[\sigma_{a_1} \circ \ldots \circ \sigma_{a_k}]$. The *language* of a classical fortress $\mathcal{F}$ is hence the set of all words accepted by $\mathcal{F}$, that is, $\mathcal{L}(\mathcal{F}) = \{w \in \Sigma^* \mid \mathcal{F} \Vdash w\}$.

The following theorems establish the relation between DFA and classical fortresses:

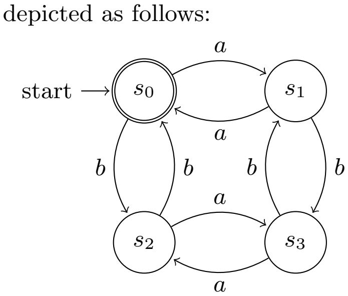
Theorem 2. *For every complete DFA A with 2^n states, there exists a classical fortress in n variables $\mathcal{F}_A$ such that $\mathcal{L}(A) = \mathcal{L}(\mathcal{F}_A)$.*

Theorem 3. *For every classical fortress in n variables $\mathcal{F}$, there exists a complete DFA $A_{\mathcal{F}}$ with 2^n states such that $\mathcal{L}(\mathcal{F}) = \mathcal{L}(A_{\mathcal{F}})$.*

Summing up, classical fortresses constitute another descriptor for regular languages:

Theorem 4. *A language $\mathcal{L}$ is a regular if and only if there is a classical fortress $\mathcal{F}$ such that $\mathcal{L}(\mathcal{F}) = \mathcal{L}$.*

We also provide an algorithm showing how to move from automata to classical fortress and *viceversa*. As an example, let us consider the complete deterministic automaton A with 2^2 states depicted as follows:



The language accepted by A is

$$\mathcal{L}(A) = \{w \mid w \text{ has both an even number of } a\text{'s and an even number of } b\text{'s}\}$$

The fortress $\mathcal{F}_A = \langle \varphi, \{\sigma_a\}_{a\in\Sigma}, \Theta\rangle$ in the variables $\{v_1, v_2\}$, can hence be defined starting from A in the following way:

$\varphi = \neg v_1 \wedge \neg v_2,$

	$\sigma_{\mathbf{a}}$	$\sigma_{\mathbf{b}}$
v_1	v_1	$\neg v_1$
v_2	$\neg v_2$	v_2

,

$\Theta = \langle \neg v_1 \wedge \neg v_2 \rangle$[2].

Besides being propositional logical descriptors for regular languages, classical fortresses are also an efficient and robust formalism for providing alternative and intuitive proofs for the closure properties of regular languages. In this setting, we provide easy proofs for the classical results stating that the class of regular languages is closed under the usual set theoretical operations of union, intersection and complementation.

Classical fortresses, as objects specified in classical propositional logic on a finite

[1] We denote by $Form(V)$ the set of formulas built only from variables in V in classical propositional logic.
[2] We denote by $\langle\Gamma\rangle$ the deductive closure of a set of formulas Γ.

number of variables, allow an easy generalization to any non-classical logical setting. In theoretical terms, in fact, given a propositional logical calculus L, one can easily adapt the definition of classical fortress to the frame of L, introducing in this way a notion of *L-fortress* and hence studying the language accepted by such an object. Therefore, one can raise the following natural question:

What is the reflection of L-fortresses in the theory of automata?

This task, which is not always viable, allows to introduce a notion of L-automata as the corresponding computational counterpart of L-fortresses, and to characterize the class of languages accepted by L-automata. A logic L allows such a turn-about, only if L enjoys the following, informally stated, properties:

1. L is algebraizable in the sense of [1], its algebraic semantics being denoted by $\mathbb{L}$.
2. $\mathbb{L}$ is locally finite and, in particular, for every n, the n-freely generated $\mathbb{L}$-algebra is finite.
3. There is a Stone-like duality between the finite slice of $\mathbb{L}$ and a target category $\mathbb{C}$ which plays the same rôle as $\mathbb{S}et_{fin}$ does in Stone duality.

Gödel logic [6, 3] is an algebraizable many-valued logic whose equivalent algebraic semantics – the variety $\mathbb{G}$ of Gödel algebras – is the subvariety of Heyting algebras defined by the so called *prelinearity equation*: $(x \to y) \vee (y \to x) = 1$. Moreover, $\mathbb{G}$ is locally finite, and for each n, the free n-generated Gödel algebra $F_n(\mathcal{G})$ is finite. The finite slice $\mathbb{G}_{fin}$ of $\mathbb{G}$ has been shown to be dually equivalent to the category $\mathcal{F}_{fin}$ of *finite forests* and order preserving open maps [4]. The latter dual categorical equivalence is in fact a Stone-like theorem for finite Gödel algebras, and this makes Gödel logic suitable to attempt a generalization of classical fortresses and DFA.

Formulas of Gödel logic are defined as usual in the signature $\{\wedge, \vee, \to, \perp, \top\}$ and they will be denoted by lower case greek letters $\varphi, \psi, \ldots$. Greek capital letters $\Theta, \Gamma \ldots$ denotes theories and, if Θ is a theory and φ is a formula, $\Theta \models_G \varphi$ means that φ is a consequence of Θ in Gödel logic.

A *Gödel fortress* on n variable can be defined naturally: it is a triple

$$\mathcal{F}_G = \langle \varphi, \{\sigma_a\}_{a \in \Sigma}, \Theta \rangle,$$

where φ is a formula, for all $a \in \Sigma$, σ_a is a substitution, and Θ is a prime theory. Obviously, φ and Θ are formalized in Gödel language with n variables.

Unlike the case of classical logic, in Gödel logic, prime theories are not maximal. To each prime Gödel theory Θ, we can indeed associate a maximal chain (ordered by inclusion) of prime theories $\Theta_1 \supset \Theta_2 \supset \cdots \supset \Theta_h = \Theta$ and hence define a notion of *graded acceptance.*

Definition 5. Let $\langle \varphi, \{\sigma_a\}_{a \in \Sigma}, \Theta \rangle$ be a Gödel-fortress and let $\Theta_1 \supset \Theta_2 \supset \cdots \supset \Theta_h = \Theta$ be the maximal chain of inclusion of prime theories included into Θ. A word $w = a_1 \ldots a_k \in \Sigma^*$ is *j-accepted* if $\Theta_j \models_G \varphi[\sigma_{a_1} \circ \ldots \circ \sigma_{a_k}]$.

Driven by the notion of *graded* acceptance, we can hence define a generalization of regular languages.

Definition 6. Let Σ be an alphabet. An *onion* of languages over Σ is a sequence $\mathcal{O} = L_1 \supseteq L_2 \supseteq \ldots \supseteq L_h$, where $L_j \subseteq \Sigma^*$ for all $j = 1, \ldots, h$. Further, a onion $\mathcal{O}$ is *regular*, if each L_j is a regular language.

Now, let $\mathcal{F}_G = \langle \varphi, \{\sigma_a\}_{a \in \Sigma}, \Theta \rangle$ be a Gödel fortress. Then $\mathcal{F}_G$ accepts an onion $\mathcal{O} = L_1 \supseteq L_2 \supseteq \ldots \supseteq L_h$ if for all $w \in \Sigma^*$,

$$w \in L_j \text{ iff } \Theta_j \models_G \varphi[\sigma_{a_1} \circ \ldots \circ \sigma_{a_k}].$$

THEOREM 7. *The class of onions recognized by Gödel fortresses is exactly the class of regular onions.*

Gödel automata can hence be introduced exploring the Stone-like duality between the finite slice $\mathbb{G}_{fin}$ of $\mathbb{G}$ and the finite slice $\mathcal{F}_{fin}$ of the category of forests and order-preserving open maps between them.

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▸ HUGO ALBUQUERQUE, JOSEP MARIA FONT, AND RAMON JANSANA, *Generalizing the Leibniz and Suszko operators.*

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§1. Introduction. Let $\mathcal{S}$ be a (sentential) logic. For every algebra $\boldsymbol{A}$ (of the similarity type of the logic), the set of all the $\mathcal{S}$-filters of $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The lattice of congruences of the algebra $\boldsymbol{A}$ is denoted by $\mathrm{Con}\boldsymbol{A}$. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is *compatible with* $F \subseteq A$ when $\langle a, b\rangle \in \theta$ and $a \in F$ imply $b \in F$, for all $a, b \in A$.

The *Leibniz operator of* $\boldsymbol{A}$ is the map $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ defined as:

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) := \max\{\theta \in \mathrm{Con}\boldsymbol{A} : \theta \text{ is compatible with } F\} \qquad \text{for every } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}.$$

The *Suszko operator of* $\boldsymbol{A}$ is the map $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ defined as:

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) := \bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}, F \subseteq G\} \qquad \text{for every } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}.$$

These two operators, particularly the Leibniz operator, have been fundamental tools in the task of building the recent apparatus of abstract algebraic logic and one of its main classifications of logics, the Leibniz hierarchy; see [4, 8, 9].

The two operators show some parallelisms and some differences. For instance, they both satisfy Correspondence Theorems for suitable kinds of matrix homomorphisms, under suitable assumptions (the Leibniz operator is known to work particularly well for protoalgebraic logics); the Suszko operator is always order preserving, while the Leibniz operator is order preserving if and only if the logic is protoalgebraic; the Leibniz operator always commutes with inverse images of surjective homomorphisms, while the Suszko operator does not in general.

Moreover, many of the classes of the Leibniz hierarchy have characterizations in terms of properties of the Leibniz operator, while only the class of truth-equational logics has so far been characterized in terms of properties of the Suszko operator.

This contribution is based on [1]. In this paper we introduce a common generalization of the two operators; we study its general properties; and we apply them to the particular cases of both operators; in particular we find new characterizations of several classes of the Leibniz hierarchy (see Figure 1) in terms of properties of the Leibniz and the Suszko operators.

§2. Compatibility operators in general.

Definition 1. An $\mathcal{S}$-*compatibility operator on* an algebra $\boldsymbol{A}$ is a map $\nabla^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ such that $\nabla^{\boldsymbol{A}}(F)$ is compatible with F, that is, such that $\nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Such an operator is *order preserving* when $F \subseteq G$ implies $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$, for all $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

This notion was first considered (without a specific name) by Czelakowski in [5]. The Leibniz operator is the largest $\mathcal{S}$-compatibility operator, while the Suszko operator is the largest order preserving $\mathcal{S}$-compatibility operator (as said before, the Leibniz operator is not in general order preserving).

A preliminary study is made of general properties of the compatibility operators and of the order preserving ones. In the course of this study the following notions are introduced:

Definition 2. Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The $\nabla^{\boldsymbol{A}}$-*class of* F is $[\![F]\!]^{\nabla^{\boldsymbol{A}}} := \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}$. The set F is a $\nabla^{\boldsymbol{A}}$-*filter* when $F = \min[\![F]\!]^{\nabla^{\boldsymbol{A}}}$. The set of all $\nabla^{\boldsymbol{A}}$-filters of $\boldsymbol{A}$ is denoted by $\mathcal{F}i^{\nabla^{\boldsymbol{A}}}_{\mathcal{S}}\boldsymbol{A}$.

A homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is $\nabla^{\boldsymbol{A}}$-*compatible with* F when $\ker(h) \subseteq \nabla^{\boldsymbol{A}}(F)$.

The notion of $\nabla^{\boldsymbol{A}}$-filter is inspired in (but does not coincide with) that of "Leibniz filter", introduced and studied for protoalgebraic logics in [6], and extended to arbitrary logics in [7]. In the paper the general properties of the $\nabla^{\boldsymbol{A}}$-filters are studied.

Note that an homomorphism h is $\boldsymbol{\Omega}^{\boldsymbol{A}}$-compatible with F if and only if the congruence $\ker(h)$ is compatible with F in the ordinary sense; thus, this is also a generalization of one of the older notions.

A *family of* $\mathcal{S}$-*compatibility operators* is a class $\nabla = \{\nabla^{\boldsymbol{A}} : \boldsymbol{A} \text{ an algebra}\}$ such that for every $\boldsymbol{A}$, $\nabla^{\boldsymbol{A}}$ is an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. Particular cases are the families $\boldsymbol{\Omega} := \{\boldsymbol{\Omega}^{\boldsymbol{A}} : \boldsymbol{A} \text{ an algebra}\}$, called simply *the Leibniz operator*, and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}} := \{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \boldsymbol{A} \text{ an algebra}\}$, called simply *the Suszko operator*. For a property that concerns a single algebra, we say that a family ∇ *globally* has it when for every algebra $\boldsymbol{A}$, the operator $\nabla^{\boldsymbol{A}}$ has that property.

The following is a weakening of the property that an operator family commutes with inverse images of surjective homomorphisms (a property that the Leibniz operator has but the Suszko operator has not; see below).

Definition 3. A family ∇ of $\mathcal{S}$-compatibility operators is *coherent* when for every surjective homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$, any of the following conditions, which are equivalent, is satisfied:

- If h is $\nabla^{\boldsymbol{A}}$-compatible with $h^{-1}(G)$, then $\nabla^{\boldsymbol{A}}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{\boldsymbol{B}}(G)\big)$, for all $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.
- If h is $\nabla^{\boldsymbol{A}}$-compatible with F, then $h\big(\nabla^{\boldsymbol{A}}(F)\big) = \nabla^{\boldsymbol{B}}\big(h(F)\big)$, for all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

The Leibniz operator and the Suszko operator are coherent families of $\mathcal{S}$-compatibility operators. Now we have all the elements needed to state the first main result of the paper:

Theorem 4 (General Correspondence Theorem). *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators. For every surjective homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is $\nabla^{\boldsymbol{A}}$-compatible with F, then h induces an order-isomorphism between $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ and $[\![h(F)]\!]^{\nabla^{\boldsymbol{B}}}$, whose inverse is given by h^{-1}.*

This generalizes and extends several results obtained in [2, 5, 6, 7] either for the Leibniz operator and protoalgebraic logics, or for the Suszko operator and arbitrary logics.

§3. Applications to the Leibniz hierarchy. After the general study, the paper specializes many points for the Leibniz operator and the Suszko operator. As a final output, some characterizations of several classes in the Leibniz hierarchy in terms of properties of the two operators have been obtained.

The first group of characterizations uses the following two properties:

Definition 5. A family ∇ of $\mathcal{S}$-compatibility operators *commutes with inverse images of (surjective) homomorphisms* when for every (surjective) homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, $\nabla^{\boldsymbol{A}}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{\boldsymbol{B}}(G)\big)$.

These are actually strengthenings of the property of coherence: every family with either property is coherent. Recall that the Leibniz operator always commutes with inverse images of surjective homomorphisms.

Theorem 6. *Let $\mathcal{S}$ be a logic.*
1. *$\mathcal{S}$ is protoalgebraic if and only if the Suszko operator commutes with inverse images of surjective homomorphisms;*
2. *$\mathcal{S}$ is equivalential if and only if the Suszko operator commutes with inverse images of homomorphisms;*
3. *$\mathcal{S}$ is truth-equational if and only if the Suszko operator is globally injective;*
4. *$\mathcal{S}$ is weakly algebraizable if and only if the Suszko operator is globally injective and commutes with inverse images of surjective homomorphisms;*
5. *$\mathcal{S}$ is algebraizable if and only if the Suszko operator is globally injective and commutes with inverse images of homomorphisms.*

The second group requires more definitions. The specialization of the notions of Definition 2 to the Suszko operator produces the notion of a *Suszko filter*; for every algebra $\boldsymbol{A}$, the set of all the Suszko filters over $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}^{\mathrm{Su}}\boldsymbol{A}$.

Both the Leibniz operator and the Suszko operator have been used in the literature to define two classes of algebras that play the rôle of *the algebraic counterpart of a logic* $\mathcal{S}$ in different situations:

$$\mathsf{Alg}^{*}\mathcal{S} := \{\boldsymbol{A} : \text{there is some } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \mathrm{Id}_{\boldsymbol{A}}\}.$$

$$\mathsf{Alg}\mathcal{S} := \{\boldsymbol{A} : \text{there is some } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Id}_{\boldsymbol{A}}\}.$$

Finally, given an arbitrary algebra $\boldsymbol{A}$ and an arbitrary class K of algebras, the following set of so-called *relative congruences* is considered:

$$\mathrm{Con}_{\mathsf{K}}\boldsymbol{A} := \{\theta \in \mathrm{Con}\boldsymbol{A} : \boldsymbol{A}/\theta \in \mathsf{K}\}$$

This set was first introduced in abstract algebraic logic by Blok and Pigozzi [3], in order to prove their characterization of (finitary and finitely) algebraizable logics as those for which, in any algebra $\boldsymbol{A}$, the Leibniz operator is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, where K is a quasivariety, which then happens to be the equivalent algebraic semantics of $\mathcal{S}$ and to coincide with $\mathsf{Alg}\mathcal{S}$; the order in the two sets is set inclusion.

The following results should be regarded in this spirit, i.e., they characterize further classes in the Leibniz hierarchy by properties that include that some operator is (or restricts to) an order isomorphism.

The first one concerns the Leibniz operator and, hence, the congruences relative to the class $\mathsf{Alg}^{*}\mathcal{S}$:

THEOREM 7. *Let $\mathcal{S}$ be a logic.*
1. *$\mathcal{S}$ is protoalgebraic if and only if for every $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ restricts to an order isomorphism between $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^{*}\mathcal{S}}\boldsymbol{A}$;*
2. *$\mathcal{S}$ is equivalential if and only if the Leibniz operator commutes with inverse images of homomorphisms and for every $\boldsymbol{A}$, the operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ restricts to an order isomorphism between $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^{*}\mathcal{S}}\boldsymbol{A}$;*
3. *$\mathcal{S}$ is weakly algebraizable if and only if for every $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^{*}\mathcal{S}}\boldsymbol{A}$;*
4. *$\mathcal{S}$ is algebraizable if and only if the Leibniz operator commutes with inverse images of homomorphisms and for every $\boldsymbol{A}$, the operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^{*}\mathcal{S}}\boldsymbol{A}$.*

The second one concerns the Suszko operator and, hence, the congruences relative to the class $\mathsf{Alg}\mathcal{S}$:

THEOREM 8. *Let $\mathcal{S}$ be a logic.*
1. *$\mathcal{S}$ is protoalgebraic if and only if for every $\boldsymbol{A}$, the Suszko operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ restricts to an order isomorphism between $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$;*
3. *$\mathcal{S}$ is truth-equational if and only if for every $\boldsymbol{A}$, the operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order embedding of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ into $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$;*
4. *$\mathcal{S}$ is weakly algebraizable if and only if for every $\boldsymbol{A}$, the operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$;*
5. *$\mathcal{S}$ is algebraizable if and only if the Suszko operator commutes with inverse images of homomorphisms and for every $\boldsymbol{A}$, the operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.*

A few of the points in Theorems 6, 7 and 8 are essentially known, and are reproduced here in order to highlight how the new ones match the existing framework.

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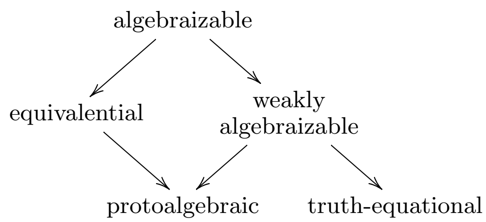

FIGURE 1. The portion of the Leibniz hierarchy referred to in the text. "→" means "included in" or "implies".

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▸ CLINT VAN ALTEN, *Embedding partially ordered sets into distributive lattices*.
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For many logical systems there exists an associated class of ordered algebraic models in which many of the properties of implication, conjunction and disjunction of the logic are captured by the order relation and its corresponding meets and joins. The study and classification of such order structures can offer insights into the corresponding logics, and standard order-theoretic techniques may be applied to various problems. In particular, the technique of 'completing' a partially ordered set, that is, embedding a partially ordered set into a (complete) lattice, has many applications in logic. For example, for creating complete lattice models, for finite model property constructions and for axiomatizing subreduct classes, to name a few.

Within the class of logics, a dichotomy exists between those that are 'distributive', i.e., satisfy the distributive laws, and those that are not. Logics that are distributive include: classical, intuitionistic, relevant and modal logics, as well as those fuzzy logics such as Monoidal t-norm Logic, Basic Logic and Many-valued Logic, whose ordered-algebraic model classes are generated by chains. Logics from the non-distributive class include a large class of substructural logics such as Full Lambek Calculus, Linear Logic and, generally, logics whose algebraic models are classes of (non-distributive) residuated lattices.

In this talk we discuss embeddings, or completions, of partially ordered sets into *distributive* lattices. To be precise what we mean by an embedding in our context we give the following definition.

A partially ordered set $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ can be *embedded* into a lattice $\mathbf{L} = \langle L, \leq^{\mathbf{L}} \rangle$ if there exists an injective map $\mu : P \to L$ (called an *embedding*) such that for any $a, b \in P$,

$$\text{if } a \leq^{\mathbf{P}} b \text{ then } \mu(a) \leq^{\mathbf{L}} \mu(b),$$
$$\text{if } a \not\leq^{\mathbf{P}} b \text{ then } \mu(a) \not\leq^{\mathbf{L}} \mu(b),$$

and finite existing meets and joins are *preserved* by μ, i.e., for all finite $A \subseteq P$,

$$\text{if } \bigwedge^{\mathbf{P}} A \text{ exists then } \mu(\bigwedge^{\mathbf{P}} A) = \bigwedge^{\mathbf{L}} \mu[A],$$
$$\text{if } \bigvee^{\mathbf{P}} A \text{ exists then } \mu(\bigvee^{\mathbf{P}} A) = \bigvee^{\mathbf{L}} \mu[A].$$

Every partially ordered set can be embedded into a (complete) lattice, and there exist a number of general methods for doing so, such as: ideal completion, filter completion, MacNeille completion and canonical extension.

Not every partially ordered set can be embedded into a *distributive* lattice, however. For example, if the partially ordered set contains within it a copy of either of the basic non-distributive lattices N_5 or M_3, then this copy must be preserved in every lattice in which the partially ordered set is embedded, and hence no such lattice can be distributive.

In this talk we address the following problem: Characterize the class of all partially ordered sets that can be embedded into a *distributive lattice*.

A partially ordered set $\mathbf{P}$ can be embedded into a lattice $\mathbf{L}$ if, and only if, $\mathbf{P}$ is order-isomorphic to a substructure of $\mathbf{L}$, i.e., there exists a subset $X \subseteq L$ such that $\mathbf{P}$ is isomorphic to the partially ordered set $\langle X, X^2 \cap \leq^{\mathbf{L}} \rangle$. Another way of viewing the above problem, therefore, is to characterize the class of all partially ordered sets that are substructures of distributive lattices.

Furthermore, since the distributive lattices are precisely those lattices embeddable into products of chains, we may view the above problem as characterizing the class of partially ordered sets embeddable into chain-products, in such a way that existing finite meets and joins are preserved. (If we omit the requirement of preserving meets and joins, then the problem is known as the 'encoding problem' for partially ordered sets.)

We show, by example, that the absense of a copy of N_5 or M_3 in a partially ordered set is not sufficient for embedding the partially ordered set into a distributive lattice. Neither is it sufficient that the distributive laws hold in all cases in which the relevant meets and joins exist.

The following characterization holds: a partially ordered set is embeddable into a distributive lattice if, and only if, for every pair of elements a, b such that $a \not\leq b$ in the partially ordered set, there exists a 'prime filter' of the partially ordered set that contains a and not b. By a *prime filter* we mean an upward closed subset that is closed under existing finite meets and whose complement is closed under existing finite joins. This condition is a second-order condition in that it quantifies over subsets of the partially ordered set. A natural question to ask is if there exists an equivalent first-order condition.

To answer the above question we consider the following decision problem:

DistPoset: Given a finite partially ordered set, determine if it can be embedded into a distributive lattice.

We prove that this decision problem is NP-hard by showing that there exists a polynomial reduction of the classical NP-complete problem 3SAT into **DistPoset**. In addition, we show that **DistPoset** is in NP by presenting an NP-algorithm based on the above-mentioned second-order characterization. Thus, **DistPoset** is an NP-complete problem. It follows that no first-order axiomatization exists for the class of partial ordered sets embeddable into a distributive lattice, except in the case that P = NP.

▸ ROB ARTHAN, PAULO OLIVA, *On Pocrims and Hoops*.
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§1. Introduction. Recall, e.g. from [3], that a *pocrim* is a structure for the signature $(0, +, \multimap)$ such that if we define $x \geq y$ by $x \multimap y = 0$, then *(i)* the $(0, +)$-reduct is a partially ordered commutative monoid with 0 as least element[3] and *(ii)* $x + y \geq z$ iff $x \geq y \multimap z$. A *hoop* is then defined as a pocrim satisfying the following identity, which we call *commutativity of weak conjunction*:

$$[\mathsf{cwc}] \qquad x + (x \multimap y) = y + (y \multimap x).$$

Hoops have been quite widely studied and have many good properties. However, many of these properties depend on identities for which elementary proofs are very hard to find. Bosbach's original work on the subject in the 1960s [4] is a *tour de force* of equational reasoning involving many ingenious instantiations of the hoop axioms. The present authors have had some success using the Prover9 automated theorem prover to find elementary proofs of identities over hoops. Analysis of the machine-generated proofs has given us some insight into methods for elementary reasoning in hoops [1]. Nonetheless, finding elementary proofs in hoops remains very far from easy. In the work reported here we give an indirect model-theoretic method for proving identities over hoops that succeeds in many useful cases.

We then undertake an algebraic analysis of the double negation translations of Kolmogorov, Gentzen and Glivenko that attempt to represent classical logic in intuitionistic logic. The relevant algebraic structures are bounded pocrims. We give a semantic formulation relative to a class of bounded pocrims of Troelstra's requirements for a correct double negation translation. We find that the Kolmogorov translation is correct for bounded pocrims, while there are classes of bounded pocrims that make the Gentzen translation correct and the Glivenko translation incorrect and *vice versa*. When we restrict attention to bounded hoops, we can use our method for proving identities to show that the double negation mapping is a hoop endomorphism, from which it follows that all three translations agree and hence are correct for bounded hoops.

§2. Identities in Hoops. A *bounded pocrim* is a structure for the signature $(0, 1, +, \multimap)$ whose $(0, +, \multimap)$-reduct is a pocrim satisfying $1 + x = 1$. In a bounded pocrim, we define $\neg x = x \multimap 1$ and $\delta(x) = \neg\neg x$. An *involutive pocrim* is a bounded pocrim satisfying $\delta(x) = x$. We write $\mathbb{B}$ for the (unique) pocrim on the two-element universe $\{0, 1\}$. Clearly $\mathbb{B}$ is involutive. If $\mathbf{C}$ and $\mathbf{D}$ are pocrims, their *ordinal sum*,

[3]Strictly speaking, we are working with *dual* pocrims and hoops, since we write the monoid operation additively and arrange the ordering so that 0 is the least element. Our work was, in part, motivated by an interest in continuous logic, where the additive notation is standard. Our $x \multimap y$ is often written as $y \dot{-} x$ or $y \ominus x$ in the literature. We prefer $x \multimap y$ because of the analogy with logical implication.

$\mathbf{C} \frown \mathbf{D}$, is the pocrim $(C \sqcup (D \setminus \{0\}), 0, +, \multimap)$ where $+$ and $\multimap$ extend the given operations on C and D to the disjoint union $C \sqcup (D \setminus \{0\})$ in such a way that whenever $c \in C$ and $0 \neq d \in D$, $c + d = d$. $\mathbb{B} \frown \mathbb{B}$ is the simplest example of a non-involutive pocrim. A *Wajsberg* pocrim is one satisfying $(x \multimap y) \multimap y = (y \multimap x) \multimap x$.

The following theorem is proved by a straightforward application of Blok and Ferreirim's characterization of subdirectly irreducible hoops [3]. It provides an invaluable tool for verifying identities in hoops. It is really two theorems, one for all hoops and one for bounded hoops.

THEOREM 1 ([2]). *Let ϕ be an identity in the language of a (bounded) hoop in the variables $x_1, \ldots, x_n$. Then ϕ is valid in the class of all (bounded) hoops iff $\phi(x_1, \ldots, x_n)$ holds under any interpretation of $x_1, \ldots, x_n$ in a (bounded) hoop $\mathbf{H}$ that can be expressed as an ordinal sum $\mathbf{S} \frown \mathbf{F}$ where $\mathbf{S}$ is subdirectly irreducible and Wajsberg, where $\mathbf{H}$ is generated by $x_1, \ldots, x_n$ and where $S = \{0\}$ iff $H = \{0\}$.*

The equational theory of subdirectly irreducible Wajsberg hoops is equivalent to that of the hoop $(\mathbb{R}^{\geq 0}, 0, +, \multimap)$, comprising the additive monoid of non-negative real numbers with $x \multimap y = \mathsf{max}(y - x, 0)$. The equational theory of subdirectly irreducible Wajsberg and bounded hoops is equivalent to that of the hoop $([0, 1], 0, 1, \dot{+}, \multimap)$ with the same definition of $\multimap$ and with $x \dot{+} y = \mathsf{min}(x + y, 1)$ (and so, in particular, such a hoop is involutive). Thus Theorem 1 reduces the decision problem for identities in hoops to special cases that often turn out to be particularly simple.

For example consider the identity $\neg kx \multimap \delta(x) \multimap x = 0$ for $0 < k \in \mathbb{N}$. Clearly this holds in any involutive hoop. It also holds in any hoop of the form $\mathbf{S} \frown \mathbb{B}$ (since in such a hoop, either $x = 1$ or $\neg kx = 1$). This covers all cases of a bounded hoop $\mathbf{H} \cong \mathbf{S} \frown \mathbf{F}$ with $\mathbf{H}$ generated by a single element of S and with $\mathbf{S}$ Wajsberg and subdirectly irreducible. (Note that if x generates $\mathbf{S}$ *qua* hoop, it generates $\mathbf{S} \frown \mathbb{B}$ *qua* bounded hoop.) Hence, by Theorem 1, $\neg kx \multimap \delta(x) \multimap x = 0$ holds in any bounded hoop. In [1], we give an explicit inductive construction of a proof of this identity in a logical system $\mathbf{ŁL_i}$ that is sound and complete for the theory of bounded hoops. Inspection of the lemmas that make up inductive step reveals 19 intricate applications of the axiom of $\mathbf{ŁL_i}$ that corresponds to the hoop identity [cwc].

THEOREM 2. *Let $\mathbf{H}$ be a hoop. The set of idempotent elements of $\mathbf{H}$ forms a subhoop.*

The only difficulty in the proof is showing that $x \multimap y$ is idempotent if x and y are idempotent. This is done using Theorem 1 to prove $i(x) \multimap i(y) \multimap i(x \multimap y) = 0$, where we write $i(x)$ for $x \multimap x + x$ (so that x is idempotent iff $i(x) = 0$).

Jipsen and Montagna [6] show that the idempotent elements in a GBL-algebra form a subalgebra. However, GBL-algebras are lattice-ordered, whereas hoops are only semilattices, in general. So Jipsen and Montagna's proof does not carry over to hoops. On the other hand, our proof breaks down in the non-commutative case.

§3. Double Negation Translations. If $\mathbf{P}$ is a pocrim, let $N = \mathsf{im}(\neg) = \{\neg x \mid x \in P\}$. Since $\delta(\neg x) = \neg x$, $\mathsf{im}(\delta) = N$. Clearly $\{0, 1\} \subseteq N$ and N is closed under $\multimap$, since $\neg x \multimap \neg y = \neg(\neg x + y)$. In general, N is not closed under addition and hence is not a subpocrim and δ does not respect either $+$ or $\multimap$.

The situation in a hoop is much more satisfactory. If $\mathbf{H}$ is a bounded hoop, the *involutive replica*, $\mathsf{IR}(\mathbf{H})$, of $\mathbf{H}$ is defined to be $\mathbf{H}/\theta$, where θ is the smallest congruence[4] such that $x\ \theta\ \delta(x)$ for all $x \in H$. $\mathbf{H} \mapsto \mathsf{IR}(\mathbf{H})$ is the objects part of a functor from the category of bounded hoops to the category of involutive hoops and every homomorphism from $\mathbf{H}$ to an involutive hoop factors uniquely through $\mathsf{IR}(\mathbf{H})$.

[4]Hoops have an equational axiomatization due to Bosbach [4] and hence the quotient of a hoop by a congruence is again a hoop.

THEOREM 3. *If* $\mathbf{H}$ *is a bounded hoop, then the double negation mapping,* δ*, is a hoop endomorphism. Moreover, if* $\pi : \mathbf{H} \to \mathsf{IR}(\mathbf{H})$ *is the natural projection, then* π *factors as* $\pi = \iota \circ \delta$ *where* $\iota : \mathsf{im}(\delta) \to \mathsf{IR}(\mathbf{H})$ *is an isomorphism.*

To prove this, we apply Theorem 1 to show that δ is a homomorphism. We then show that the kernel congruence of this homomorphism is the same as the congruence θ in the definition of $\mathsf{IR}(\mathbf{H})$.

Beginning with Kolmogorov, logicians have studied *double negation translations* that represent classical logic in intuitionistic logic. Kolmogorov's translation inductively replaces every subformula of a formula by its double negation. Subsequent authors have devised more economical translations: Gentzen applies double negation to atomic formulas only, while Glivenko just applies double negation at the outermost level of a formula without changing its internal structure.

We undertake an algebraic analysis of the various double negation translations. We work in a language, $\mathcal{L}$, built from a countable set of variables $\mathsf{Var} = \{V_1, V_2, \dots\}$, the constant 1 (falsehood) and the binary connectives $\multimap$ (implication) and $\otimes$ (conjunction). We write $A^\perp$ for $A \multimap 1$ and 0 for $1 \multimap 1$. The *standard interpretation* of $\mathcal{L}$ interprets $\multimap$, $\otimes$ and 1 as $\rightarrow\!\!\bullet$, $+$ and 1 respectively. We define intuitionistic affine logic, $\mathbf{AL_i}$, and what we call intuitionistic Łukasiewicz logic, $\mathbf{ŁL_i}$, which are sound and complete for bounded pocrims and bounded hoops respectively, under the standard interpretation.

We will view a double negation translations as a non-standard semantics and so we need a framework to compare semantics. Let Poc_1 be the category of bounded pocrims and homomorphisms and let Set be the category of sets. Given any set X, let $H_X : \mathsf{Poc}_1 \to \mathsf{Set}$ be the functor that maps a pocrim $\mathbf{P}$ to $\mathsf{Hom}_{\mathsf{Set}}(X, P)$, i.e., the set of all functions from X to P, and maps a homomorphism $h : \mathbf{P} \to \mathbf{Q}$ to $f \mapsto h \circ f : \mathsf{Hom}_{\mathsf{Set}}(X, P) \to \mathsf{Hom}_{\mathsf{Set}}(X, Q)$. Now let $\mathsf{Ass} = H_{\mathsf{Var}}$ and $\mathsf{Sem} = H_{\mathcal{L}}$. We define a *semantics* to be a natural transformation $\mu : \mathsf{Ass} \to \mathsf{Sem}$.

So given a bounded pocrim $\mathbf{P}$, $\mathsf{Ass}(\mathbf{P})$ denotes the set of assignments $\alpha : \mathsf{Var} \to P$, while $\mathsf{Sem}(\mathbf{P})$ denotes the set of all possible functions $s : \mathcal{L} \to P$. A semantics μ is a family of functions $\mu_{\mathbf{P}}$ indexed by bounded pocrims $\mathbf{P}$ such that $\mu_{\mathbf{P}} : \mathsf{Ass}(\mathbf{P}) \to \mathsf{Sem}(\mathbf{P})$ and such that for any homomorphism $f : \mathbf{P} \to \mathbf{Q}$ the following diagram commutes.

$$
\begin{array}{ccc}
\mathsf{Ass}(\mathbf{P}) & \xrightarrow{\mathsf{Ass}(f)} & \mathsf{Ass}(\mathbf{Q}) \\
\Big\downarrow{\scriptstyle \mu_{\mathbf{P}}} & & \Big\downarrow{\scriptstyle \mu_{\mathbf{Q}}} \\
\mathsf{Sem}(\mathbf{P}) & \xrightarrow{\mathsf{Sem}(f)} & \mathsf{Sem}(\mathbf{Q})
\end{array}
$$

The standard interpretation of $\mathcal{L}$ then corresponds to the *standard semantics* μ^{S} defined as follows:

$$
\begin{aligned}
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(V_i) &= \alpha(V_i) \\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(1) &= 1 \\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(A \otimes B) &= \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(A) + \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(B) \\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(A \multimap B) &= \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(A) \rightarrow\!\!\bullet\, \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(B)
\end{aligned}
$$

The Kolmogorov translation corresponds to a semantics μ^{K} defined like μ^{S}, but applying double negation to everything in sight:

$$\mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(V_i) = \delta(\alpha(V_i))$$
$$\mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(1) = 1$$
$$\mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(A \otimes B)) = \delta(\mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(A) + \mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(B))$$
$$\mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(A \multimap B)) = \delta(\mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(A) \rightarrow\!\!\bullet\, \mu^{\mathsf{K}}_{\mathbf{P}}(\alpha)(B))$$

The Gentzen and Glivenko translations correspond to semantics obtained by composing the standard semantics with double negation:

$$\mu^{\mathsf{Gen}} = \mu^{\mathsf{S}} \circ \delta^{\mathsf{Var}}$$
$$\mu^{\mathsf{Gli}} = \delta^{\mathcal{L}} \circ \mu^{\mathsf{S}}$$

where δ^X denotes the natural transformation from $H_X = \mathsf{Hom}_{\mathsf{Set}}(X, \cdot)$ to itself with $\delta^X_{\mathbf{P}} = f \mapsto \delta \circ f$.

Let $\mathcal{C}$ be a class of bounded pocrims, we say that a semantics μ is a *double negation semantics* for $\mathcal{C}$ if the following conditions hold:

(DNS1): If $\mathbf{P} \in \mathcal{C}$ is involutive, then $\mu_{\mathbf{P}} = \mu^{\mathsf{S}}_{\mathbf{P}}$.

(DNS2): Given a formula A, if, for every involutive $\mathbf{P} \in \mathcal{C}$ and every $\alpha : \mathsf{Var} \to P$, we have $\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(A) = 0$, then, for every $\mathbf{P} \in \mathcal{C}$ and every $\alpha : \mathsf{Var} \to P$, we have $\mu_{\mathbf{P}}(\alpha)(A) = 0$.

(DNS3): $\delta^{\mathcal{L}} \circ \mu = \mu$.

The above definition can be shown to agree with the usual syntactic definition of a double negation translation due to Troelstra under certain conditions on the class $\mathcal{C}$.

Theorem 4. *The Kolmogorov semantics, μ^{K}, is a double negation semantics for the class of all pocrims.*

The proof is by induction over proof trees in $\mathbf{AL_i}$.

Theorem 5. *The Kolmogorov semantics, μ^{K}, the Gentzen semantics, μ^{Gen}, and the Glivenko semantics, μ^{Gli}, are double negation semantics for any class of hoops.*

The proof uses Theorem 3 to show that the three semantics agree in hoops.

Theorem 6. *There are finite pocrims $\mathbf{L}_3$, $\mathbf{P}_4$, $\mathbf{Q}_4$ and $\mathbf{Q}_6$ such that:*
(i) *The Gentzen semantics, μ^{Gen}, is a double negation semantics for the class of pocrims $\{\mathbf{L}_3, \mathbf{P}_4\}$, but the Glivenko semantics, μ^{Gli}, is not.*
(ii) *The Glivenko semantics, μ^{Gli}, is a double negation semantics for the class of pocrims $\{\mathbf{Q}_4, \mathbf{Q}_6\}$, but the Gentzen semantics, μ^{Gen}, is not.*

$\mathbf{P}_4$ and $\mathbf{Q}_6$ were found using the Mace4 tool to find small examples that refute certain identities (the subscripts in the names give the order of the pocrims). See [2] for the operation tables of these pocrims. For *(i)*, $\mathbf{P}_4$ refutes $\delta(\delta(x) \rightarrow\!\!\bullet\, x) = 0$ and this can be shown to invalidate μ^{Gli}. $\mathbf{L}_3$ is $\mathsf{im}(\delta : \mathbf{P}_4 \to \mathbf{P}_4)$ and can be seen by inspection of the operation tables to be an involutive subpocrim. This can be shown to imply that μ^{Gen} is a double negation semantics for $\{\mathbf{P}_4, \mathbf{L}_3\}$. For *(ii)*, $\mathbf{Q}_6$ invalidates μ^{Gen} because it refutes $\delta(\delta(x+y) \rightarrow\!\!\bullet\, x+y) = 0$. $\mathbf{Q}_4$ is the involutive replica $\mathsf{IR}(\mathbf{Q}_6)$ and this can be shown to imply that μ^{Gli} is a double negation semantics for $\{\mathbf{Q}_4, \mathbf{Q}_6\}$.

With a little more work, Theorem 6 can be used to show the existence of logics extending $\mathbf{AL_i}$ in which the syntactic Gentzen translation meets Troelstra's requirements on a double negation translation but the syntactic Glivenko translation does not and *vice versa.*

Galatos and Ono [5] have studied the Glivenko translation from a different perspective in the case of residuated lattices. It would be interesting to attempt to combine their approach with ours.

We would like to thank George Metcalfe for comments and pointers to the literature.

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▸ ARNON AVRON, *A semi-relevant, paraconsistent dual of Łukasiewicz Logic.*
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Paraconsistent logics are logics which allow non-trivial inconsistent theories. In other words: unlike in classical logic, in paraconsistent logics a single contradiction does not necessarily implies everything. Fuzzy logics, on the other hand, are logics which are based on the idea of *degrees of truth*, according to which the truth-value assigned to a proposition that involves imprecise concepts (like "tall" or "old") might not be one of the two classical values 0 and 1, but any real number between them.

Now none of the standard fuzzy logics as they are presented in the literature (see [2] for extensive surveys) is paraconsistent.[5] The reason is that their consequence relation is based on preserving absolute truth (i.e. $T \vdash \varphi$ iff every legal valuation that assigns 1 to all elements of T assigns 1 to φ as well). In order to develop useful paraconsistent fuzzy logics it is necessary to replace this consequence relation of the standard fuzzy logics by a less strict one, and the obvious way to do so is to use as the set of designated values a set which is more comprehensive than just $\{1\}$.

The main goal of this paper is to present a paraconsistent dual (called **FT**) of Łukasiewicz Logic $Ł_\infty$ which reflects the above idea. Like $Ł_\infty$, the semantics of **FT** is based on taking the real unit interval [0,1] as the set of truth-values, and it interprets there $\wedge$, $\vee$, and $\neg$ exactly like $Ł_\infty$ does. Moreover, like in $Ł_\infty$ (and other standard fuzzy logics), the interpretation of the connective $\to$ (denoted here again by $\to$) satisfies the basic condition that $a \to b$ is designated iff $a \leq b$. However, while $Ł_\infty$ has a single designated value, **FT** has a single *non-designated* value. Another way in which **FT** is a dual to $Ł_\infty$ is in the way it relates to the basic structural rules of Gentzen: while $Ł_\infty$ accepts the implicational axioms which correspond to the weakening rule and the permutation (or exchange) rule, but reject the one that corresponds to contraction, **FT** accepts the latter axiom but rejects the former two.

Another important feature of **FT** is that it belongs to Anderson and Belnap's family

[5]Recently fuzzy logics which are degree-preserving (rather than truth-preserving as the standard fuzzy logics) and *are* paraconsistent were investigated in [3]. Unlike most paraconsistent logics (including **FT**, the one investigated here), those logics are not only paraconsistent, but also *paracomplete*. In contrast to **FT** those logics are partially $\neg$-explosive with respect to $\neg q \vee q$, and their official implication does not respect MP.

of relevant and semi-relevant logics, since the Hilbert-type system which is proved in this paper to be strongly sound and complete for it is obtained by extending Anderson and Belnap's favorite system **E** (or just the weaker system **T**) with the following three axioms schemas (the first two of which are valid in all fuzzy logics ever studied, while the last reflects our very liberal choice of the set of designated values):

[Mi] $\varphi \to (\varphi \to \varphi)$ (Mingle)

[Li] $(\varphi \to \psi) \vee (\psi \to \varphi)$ (Linearity)

[DP] $\varphi \vee (\varphi \to \psi)$ (Disjunctive Peirce)

Now **FT** itself cannot be taken as a relevant logic, since it does not have the variable-sharing property. However, it can be viewed as a semi-relevant system, since it satisfies the same criterion of semi-relevance as the well-known semi-relevant system **RM**: if $\vdash_{\mathbf{FT}} \varphi \to \psi$ then either φ and ψ share an atomic variable, or both $\neg\varphi$ and ψ are classical tautologies. At this point it is worth noting that like **FT**, **RM** too can be viewed as a paraconsistent fuzzy logic, since it is paraconsistent, as well as strongly sound and complete for a semantics which is again based on taking the real unit interval [0,1] as the set of truth-values, and interpreting there $\wedge$ and $\vee$ (but not $\neg$ and $\to$) exactly like Ł_∞. However, in this semantics for **RM** the set of designated values is neither $\{1\}$ (like in Ł_∞) or (0,1] (like in **FT**), but [1/2,1]. Another important property that **FT** shares with **RM** (while most strictly relevant systems lack it) is its being *decidable*.

On our way to introduce and investigate **FT** we introduce and investigate a weaker (but still decidable) system, **TMP**, which has an interest of its own. Proof-theoretically, **TMP** is obtained from **FT** by deleting the linearity axiom [Li]. This Hilbert-type system is strongly sound and complete for the following class of structures:

Definition 1. A **TMP**-*structure is a tuple* $\mathcal{M} = \langle A, \leq, 0, 1, \neg, \to\rangle$ *such that:* $\langle A, \leq, 0, 1, \neg\rangle$ *is a bounded De-Morgan lattice, i.e.:* $\langle A, \leq, 0, 1\rangle$ *is a bounded distributive lattice where 0 is the minimal element and 1 is the maximal one, and* $\neg$ *is a De-Morgan involution on it (i.e. it satisfies* $\neg\neg a = a$ *and* $a \leq b \Rightarrow \neg b \leq \neg a$ *for every* $a, b \in A$*);*
0 is meet-irreducible in $\mathcal{M}$*, i.e.* $a \wedge b = 0$ *iff either* $a = 0$ *or* $b = 0$ *(where, as usual,* $\wedge$ *denotes the meet operation on the lattice* $\langle A, \leq\rangle$*, and* $\vee$ *– the join operation);*
The binary operation $\to$ *on* A *is defined as follows:*

$$a \to b = \begin{cases} \neg a \vee b & a \leq b \\ 0 & a \not\leq b \end{cases}$$

The set $\mathcal{D}$ *of designated values in* v *is taken to be* $\{a \in A \mid a \neq 0\}$.

A general semantics for **FT** (relative to which it is again strongly sound and complete) is given by the **TMP**-structures in which the relation $<$ is linear.

Obviously, the semantics of **TMP** described above is again based on the idea of ordered truth-degrees. The main difference between it and the standard general semantics of fuzzy logics (including **FT**) is that in the semantics of **TMP** the order relation of the truth-degrees is not demanded to be linear. As a result, **TMP** does have the variable-sharing property, and so unlike **FT** it belongs to the family of relevant logics, not just the semi-relevant ones. However, in contrast to the most known relevant logics (like **E**, **R**, and **T**), **TMP** has the advantages of being strongly decidable and having the finite model property.

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► ARNOLD BECKMANN AND NORBERT PREINING, *Separating intermediate predicate logics of well-founded and dually well-founded structures by monadic sentences.*
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There are at least three good reasons for studying predicate logics defined by linear Kripke frames with constant domains: These logics are typical examples of intermediate predicate logics, that is logics that lie between classical and intuitionistic logic [8, 12], and bare relation to linear-time temporal logic [14, 11, 15]. Furthermore, they are linked to one of the three main t-norm based logics called Gödel logics [7]: The logics defined by *countable* linear Kripke frames with constant domains coincide with the set of all Gödel logics [4]. Finally, they have interesting connections to the theory of linear orders. For example, studying countable closed linear orderings with respect to continuous monotone embeddability has lead to the surprising (because it is in contrast to other cases like Intuitionistic or Modal logic) result that there are only countably many Gödel logics [3].

The original motivation for this work was to understand how much we can express in the world of linear Kripke frames with constant domain, if the language is restricted to one of the *simplest* reasonable first-order fragments which extends propositional logic, namely the fragment of first-order formulas based on exactly one monadic predicate symbol. Very early guesses, that there are only four such logics ("What can we express more than infima and suprema and their order?"), were soon overthrown. In fact, our results in this work show that there are countably infinite many such logics.

More specifically, we will show the following theorem:

Theorem 1 ([5]). *For any ordinals $0 < \alpha < \beta < \omega^\omega$, the logics defined by α and β as well-founded linear Kripke frames with constant domains can be separated by a first-order sentence which uses only one monadic predicate symbol. The same holds if we take ordinals as* dually *well-founded Kripke frames.*

Related work The study of the relation between Kripke frames and ordinals carries a long tradition, and results related to ours have been obtained in [10], which in turn is related to [9]. Similar result have been obtained in [13]. Minari [9] showed that any ordinal ξ less then ω^ω is Kripke-definable, which in his interpretation means that there is a formula separating the logics of the Kripke frames based on ξ and $\xi + 1$.

We improve this result by firstly providing a formula in the monadic fragment with only one predicate symbol, and secondly, by separating any two logics of Kripke frames based on two different ordinals less than ω^ω. This also explains why the formulas we are providing are dependent on both ordinals.

Minari [9] also discusses the definability of ordinals larger then ω^ω: He shows that no ordinal bigger then ω_1 is Kripke-definable (based on Löwenheim-Skolem), and conjectures that no ordinal between ω^ω and ω_1 is definable.

Relation to quantified propositional logic The monadic fragment under discussion can be seen as the linear fragment of Gabbay's 2h logic [6], the second order propositional logic, which could also be called intuitionistic quantified propositional logic. The semantics of this logic is based on Kripke frames with the addition that the set of possible interpretations for atomic propositions is not necessarily the full set, but any arbitrary subset of the sets of all upsets of the Kripke frame (set D in [6]). Note that the restriction to evaluate atomic propositions into a restricted set does not apply to the extension to compound formulas. Each first order quantification, as its variable is only occurring within one monadic predicate symbol, can be replaced by a corresponding propositional quantification. In this way, each particular model of the second order propositional logic $\mathsf{2h}$ can be simulated by one particular model and evaluation of monadic first order linear Kripke logic. Thus, counter models can be translated from second order propositional logic $\mathsf{2h}$ to monadic first order logics of linear Kripke frames with constant domains, and vice versa.

A less direct relation exists to *quantified propositional Gödel logics* [2, 1], where the full set of truth values can act as possible interpretation for atomic propositions. In this case, counter models can be translated from quantified propositional Gödel logic to monadic first order Gödel logic, but not vice versa.

The present work also exploits and continues the connection between logics of linear Kripke frames and Gödel logics, obtained in [4]. Due to the fact that evaluations in Kripke frames are governed by special rules with respect to the order — in other words, evaluations in Kripke frames are based on upsets — evaluations in Gödel logics have a much simpler structure. Furthermore we view our results as part of a wider research programme which connects the theory of linear orders to investigations of logics. In particular, we are interested in the question which order theoretic notion resembles the structure of logics best.

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▸ LIBOR BĚHOUNEK, *Church-style type theories over finitary weakly implicative logics*.
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In [2, 3], a minimalistic simple type theory in the vein of Church [4] and Henkin [6] has been introduced, which constitutes the type-theoretic core of Novák's fuzzy type theory [7] and is adaptable to a broad class of underlying non-classical logics. In this contribution we construct Church–Henkin simple type theories for finitary weakly implicative logics [5]. The resulting type theory $\mathrm{TT}(L)$ over a given finitary weakly implicative logic L is the minimal (extensional, substitution-invariant) type theory which is closed under the rules of λ-conversion (e.g., [1]) and the intersubstitutivity of equals, whose propositional fragment coincides with L, and whose sound and complete Henkin semantics consists of Henkin-style general models over all L-algebras (or any class of L-algebras for which L is complete). The soundness and completeness theorem for $\mathrm{TT}(L)$ is obtained by a schematic adaptation of the proof for the ground theory TT_0 of [2, 3], for each L from the class of logics. The type theory $\mathrm{TT}(L)$ thus can be viewed as the Church–Henkin higher-order logic L.

Due to space restrictions, the construction of $\mathrm{TT}(L)$ can only be sketched here; the details and proofs will be given in a full paper on the topic (under preparation).

Recall from [5] that a *weakly implicative logic* is a substitution-invariant Tarski consequence relation over a propositional language containing a connective $\to$, which is closed under the rules of modus ponens, the transitivity and reflexivity of $\to$, and the congruence of all propositional connectives with respect to simultaneous bidirectional implication. A *finitary* weakly implicative logic is a weakly implicative logic that has an axiomatic system consisting only of finitary derivation rules (incl. axioms, or nullary rules). This fairly general class of logics contains most well-known propositional and modal logics presentable by Hilbert-style calculi. Further on, let L be a weakly implicative logic in a propositional language $\mathcal{L}$ with a finitary axiomatic system A_L, and let K be a class of L-algebras for which L is complete (e.g., the class of all L-algebras).

The syntax of the higher-order logic $\mathrm{TT}(L)$ follows the syntax of Church's classical type theory of [4]: it uses the usual hierarchy of complex types over the primitive types o (for truth values) and ε (for elements). The type of functions from the domain of type β to the domain of type α is denoted by $(\alpha\beta)$. The types are usually marked as subscripts of expressions, with omitted parentheses grouping to the left. The logical vocabulary of $\mathrm{TT}(L)$ contains the constants $=_{o\alpha\alpha}$ for each type α and $\bar{c}_{o\overset{n}{\dots}o}$ for each n-ary propositional connective $c \in \mathcal{L}$ (infix notation will be used for $=_{o\alpha\alpha}$ and $\bar{\to}_{ooo}$). *Formulae* (or 'λ-terms') are formed from constants and variables of all types by the usual constructions of function application, written $(A_{\alpha\beta}B_\beta)_\alpha$, and λ-abstraction, written $(\lambda x_\beta A_\alpha)_{\alpha\beta}$. Formulae of type o are called *propositions; theories* are sets of propositions. The notions of subformula, free and bound (by λ) variable, and substitutability are defined as usual.

The axiomatic system of $\mathrm{TT}(L)$ consists of the following schemata of derivation rules:

$$
\begin{aligned}
A_o &\ /\ A_o[B_\alpha/x_\alpha]\\
A_o, B_\alpha =_{o\alpha\alpha} B'_\alpha &\ /\ A_o[B'_\alpha /\!\!/ B_\alpha]\\
&\ /\ (\lambda x_\beta A_\alpha)B_\beta =_{o\alpha\alpha} A_\alpha[B_\beta/x_\beta]\\
F_{\alpha\beta}x_\beta =_{o\alpha\alpha} G_{\alpha\beta}x_\beta &\ /\ F_{\alpha\beta} =_{o(\alpha\beta)(\alpha\beta)} G_{\alpha\beta}\\
A_o \Rightarrow_{ooo} B_o, B_o \Rightarrow_{ooo} A_o &\ /\ A_o =_{ooo} B_o\\
D_o^{\varphi_1}, \ldots, D_o^{\varphi_n} &\ /\ D_o^{\psi} \quad \text{for each rule } \varphi_1, \ldots, \varphi_n \,/\, \psi \text{ of } A_L,\ n \geq 0,
\end{aligned}
$$

where $A_o[B_\alpha/x_\alpha]$ denotes the formula arising from the formula A_o by replacing *all* free occurrences of a variable x_α in A_o with a formula B_α substitutable for x_α in A_o; $A_o[B'_\alpha /\!\!/ B_\alpha]$ denotes the formula arising from the formula A_o by replacing a *single* occurrence of a subformula B_α in A_o with a formula B'_α substitutable for this occurrence of B_α in A_o; the formulae $F_{\alpha\beta}$ and $G_{\alpha\beta}$ do not contain free x_β; and the translation D_o^φ of an $\mathcal{L}$-formula φ is defined recursively as follows:

- D_o^p is a variable x_o^p for every propositional variable p of L
- $D_o^{c(\varphi_1,\ldots,\varphi_n)}$ is the formula $\bar{c}_{o\ldots o} D_o^{\varphi_1} \ldots D_o^{\varphi_n}$ for each n-ary connective $c \in \mathcal{L}$.

The notions of proof and provability are defined as usual; a theory is *consistent* if it does not prove all propositions.

The Henkin-style semantics of $\mathrm{TT}(L)$ is defined in the following manner. A *basic frame* over non-empty sets M_o and M_ε is a system $M = \{M_\alpha\}_{\alpha \in \mathit{Types}}$ such that $\emptyset \neq M_{\alpha\beta} \subseteq M_\alpha^{M_\beta}$ for all types α, β. A *frame* $\mathbf{M} = \langle M, D, E, I\rangle$ is a basic frame M equipped with: (i) a set $D \subsetneq M_o$ of *designated* truth values; (ii) a system $E = \{E_\alpha\}_{\alpha \in \mathit{Types}}$ of functions $E_\alpha\colon M_\alpha^2 \to M_o$ interpreting $=_{o\alpha\alpha}$, such that $E_\alpha(m, m') \in D$ iff $m = m'$ for all $m, m' \in M_\alpha$; and (iii) an interpretation I of the logical constants $\bar{c}_{o\ldots o}$ for each propositional connective $c \in \mathcal{L}$, such that $\mathbf{A} = \langle M_o, \{I(\bar{c}_{o\ldots o})\}_{c \in \mathcal{L}}\rangle$ is an L-algebra from the class K (we say that $\mathbf{M}$ is a frame *over* the algebra $\mathbf{A}$). As usual, *valuations* and *interpretations* in a frame $\mathbf{M}$ are assignments of elements of M_α to variables and extralogical constants of each type α. A semantic value $\mathbf{M}_v^J(A_\alpha)$ of a formula A_α in a frame $\mathbf{M}$ under an interpretation J and a valuation v is defined by the following Tarski conditions:

- $\mathbf{M}_v^J(x_\alpha) = v(x_\alpha)$ for each variable x_α
- $\mathbf{M}_v^J(d_\alpha) = J(d_\alpha)$ for each extralogical constant d_α
- $\mathbf{M}_v^J(A_{\alpha\beta}B_\beta) = \mathbf{M}_v^J(A_{\alpha\beta})\big(\mathbf{M}_v^J(B_\beta)\big)$
- $\mathbf{M}_v^J(\lambda x_\beta A_\alpha) = F\colon M_\beta \to M_\alpha$ such that $F(m) = \mathbf{M}_{v[x_\beta\colon m]}^J(A_\alpha)$, where $v[x_\beta\colon m]$ is the valuation such that $v[x_\beta\colon m](y_\beta) = m$ if y_β is x_β, and $v(y_\beta)$ otherwise.

A (Henkin) *model* $\mathbf{M}^J$ is an interpretation J in a frame $\mathbf{M}$ such that the semantic values of all formulae are defined under all valuations. A proposition A_o is *valid* in a model $\mathbf{M}^J$ if $\mathbf{M}_v^J(A_o) \in D$ under all valuations v in $\mathbf{M}$. A model $\mathbf{M}^J$ is a *model of* a theory T if all $A_o \in T$ are valid in $\mathbf{M}^J$. A theory T *entails* A_o, written $T \models A_o$, if all models of T are also models of A_o.

The following Strong Soundness and Completeness Theorem for $\mathrm{TT}(L)$ can be proved by a schematic adaptation of the proof of the strong soundness and completeness theorem for the ground type theory TT_0 (axiomatized by the first four rule schemata of $\mathrm{TT}(L)$ above) of [2] and employing the strong soundness and completeness of L with respect to K:

1. T is consistent iff T has a Henkin model over an L-algebra $\mathbf{A} \in K$. Consequently, $T \models A_o$ iff $T \vdash A_o$, and $T \models A_o$ iff $T' \models A_o$ for a finite $T' \subseteq T$ (compactness).
2. The propositional fragment of $\mathrm{TT}(L)$ coincides with L.

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▸ NICK BEZHANISHVILI, NICK GALATOS, LUCA SPADA, *Canonical formulas for k-potent residuated lattices.*
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The problem of axiomatising intermediate logics (i.e. axiomatic extensions of Intuitionistic logic) has been extensively studied. One of the first general methods was developed by Jankov [8]. For each finite subdirectly irreducible Heyting algebra A, Jankov designed a formula that encodes the structure of A. The main property of the Jankov formula $\chi(A)$ is that a Heyting algebra B refutes $\chi(A)$ iff A is isomorphic to a subalgebra of a homomorphic image of B. There is a continuum of intermediate logics axiomatised by Jankov formulas, however, not every intermediate logic is axiomatisable by these formulas. In fact, Jankov formulas axiomatise exactly the splitting logics and their joins in the lattice of intermediate logics. In particular, if an intermediate logic L corresponds to a locally finite variety of Heyting algebras, then it behaves well with regard to Jankov formulas: L is axiomatised over Intuitionistic logic by Jankov formulas and all extensions of L are also axiomatised over L by Jankov formulas.

Model-theoretic analogues of Jankov formulas were later developed by de Jongh [5] for intermediate logics and by Fine [6] for modal logics. Zakharyaschev generalised Fine's approach, developed the model-theoretic theory of canonical formulas [9] and showed that each intermediate logic is axiomatisable by canonical formulas.

Recently [1, 2, 3] developed a generalisation of Jankov formulas, which provides a purely algebraic account of Zakharyaschev's canonical formulas. Although the variety of Heyting algebras is not locally finite, it has two well-behaved locally finite reducts: distributive lattices and implicative semilattices. The key idea is, given a Heyting algebra A, to encode in a Jankov-like formula the full structure of the e.g., $\vee$-free reduct of A, and only partially the behaviour of $\vee$. This can be achieved by postulating that $\vee$ is preserved for only those pairs of elements of A that belong to some designated subset D of A^2. The key result of [1] is that their canonical formulas also axiomatise all intermediate logics.

In this work we generalise the above result to $\mathbf{FL}_{ew}^k$, logics whose equivalent algebraic semantics is given by the class of k-potent, commutative, integral, residuated lattices (hereafter k-RL, for short). These structures are simply commutative residuated lattices in which the top element is also the neutral element of the multiplication and such that $x^{k+1} = x^k$ holds [7]. Notice that Heyting algebras are exactly the 2-potent commutative, integral, residuated lattices. The main result of this communication is that every variety of k-RL can be axiomatised by canonical formulas. More specifically we prove that if B is any subdirectly irreducible k-RL, then, for every formula φ such that $B \not\models \varphi$ there exist canonical formulas γ_i, $i \leq m$, such that $B \not\models \bigwedge_{i \leq m} \gamma_i$.

The strategy of the proof is quite similar to [3] and proceeds in two steps.

Step 1. Given a formula φ axiomatising a proper extension of $\mathbf{FL}_{ew}^k$, we associate to it a finite system of finite algebras $A_1, .., A_m$ and sets $D_i^{\wedge}, D_i^{\rightarrow} \subseteq A^2$ for $i \leq m$, such that for every subdirectly irreducible k-RL B:

$$(\star_1) \qquad B \not\models \varphi \quad \Longleftrightarrow \quad \exists i \leq m \, \exists C \quad A_i \rightarrowtail_D C \twoheadleftarrow B$$

where $\rightarrowtail_D$ indicates a homomorphisms of the $\vee$-free reducts of C and A_i that preserves designated meets in $D^{\wedge}$ and designated arrows in $D^{\rightarrow}$.

To build the system $A_1, ..., A_m$ associated to φ we use the Finite Embeddability Property of k-RL [4]. Given any formula φ that is not a consequence of $\mathbf{FL}_{ew}^k$ we proceed as follows. If $\mathbf{FL}_{ew}^k \not\vdash \varphi(X_1, \ldots, X_n)$, then φ fails on the free generators $X_1, \ldots, X_n$ of the free n-generated k-RL $\mathcal{F}(n)$. Therefore, $\varphi(X_1, \ldots, X_n) \neq 1$ in $\mathcal{F}(n)$. Let S be the $(\cdot, \vee)$-subalgebra of $\mathcal{F}(n)$ generated by the partial subalgebra $\mathsf{Sub}_{\varphi(n)}(\varphi)$ of all polynomials of $\varphi(X_1, ..., X_n)$. In other words:

$$S := \left\{ \left(c_{11}^{p_{11}} \cdot ... \cdot c_{1l}^{p_{1l}} \right) \vee ... \vee \left(c_{m1}^{p_{m1}} \cdot ... \cdot c_{ml}^{p_{ml}} \right)) \mid 0 \leq p_{ij} \leq k \text{ and } c_{ij} \in \mathsf{Sub}_{\varphi(n)}(\varphi) \right\} \ .$$

The $(\cdot, \vee)$-algebra S can be endowed with the structure of a finite k-RL and $\varphi(X_1, \ldots, X_n) \neq 1$ in S (for details on this construction see [4]).

Let $A_1, \ldots, A_m$ and $h_i \colon S \twoheadrightarrow A_i$ be the list of subdirectly irreducible k-RLs that are homomorphic images of S and such that $A_i \models \varphi(h_i(X_1), ..., h_i(X_n)) \neq 1$, together with their canonical quotient maps. We set

$$D_i^{\wedge} := \{(a, b) \in \left(\mathsf{Sub}_{A_i}(\varphi(h_i(X_1), ..., h_i(X_n))) \right)^2 \mid a \wedge b \in \mathsf{Sub}_{A_i}(\varphi)\},$$
$$D_i^{\rightarrow} := \{(a, b) \in \left(\mathsf{Sub}_{A_i}(\varphi(h_i(X_1), ..., h_i(X_n))) \right)^2 \mid a \rightarrow b \in \mathsf{Sub}_{A_i}(\varphi)\} \ .$$

We call $\{(A_i, D_i^{\wedge}, D_i^{\rightarrow}) \mid 1 \leq i \leq m\}$ the *system associated with* φ.

Step 2. We now associate to each finite, subdirectly irreducible k-RL A and two sets $D_i^{\wedge}, D_i^{\rightarrow} \subseteq A^2$ a canonical formula $\gamma(A, D_i^{\wedge}, D_i^{\rightarrow})$ such that the following holds.

$$(\star_2) \qquad \exists i \leq m \, \exists C \quad A_i \rightarrowtail_D C \twoheadleftarrow B \quad \Longleftrightarrow \quad \exists i \leq m \quad B \not\models \gamma(A_i, D_i^{\wedge}, D_i^{\rightarrow}) \ .$$

For each $a \in A$, we introduce a new variable X_a, and set

$$\begin{aligned}\Gamma := \quad & (X_0 \leftrightarrow \bot) \wedge (X_1 \leftrightarrow \top) \wedge \\ & \bigwedge \{X_{a \cdot b} \leftrightarrow X_a \cdot X_b \mid a, b \in A\} \ \wedge \\ & \bigwedge \{X_{a \vee b} \leftrightarrow X_a \vee X_b \mid a, b \in A\} \ \wedge \\ & \bigwedge \{X_{a \to b} \leftrightarrow X_a \to X_b \mid (a, b) \in D^{\to}\} \\ & \bigwedge \{X_{a \wedge b} \leftrightarrow X_a \wedge X_b \mid (a, b) \in D^{\wedge}\} \\ & \text{and} \\ \Delta := \quad & \bigvee \{X_a \to X_b \mid a, b \in A \text{ with } a \not\leq b\} \ .\end{aligned}$$

Finally, we define the *canonical formula* $\gamma(A, D^{\wedge}, D_{\to})$ associated with A, $D^{\wedge}$, and $D^{\to}$ as

$$\gamma(A, D^{\wedge}, D_{\to}) = \Gamma^k \to \Delta \ .$$

THEOREM 1. *If* $\mathbf{FL}^k_{ew} \not\vdash \varphi(X_1, \ldots, X_n)$*, then there exist*

$$(A_1, D_1^{\wedge}, D_1^{\to}), \ldots, (A_m, D_m^{\wedge}, D_m^{\to})$$

such that each A_i is a finite subdirectly irreducible k-RL, $D_i^{\wedge}, D_i^{\to} \subseteq A_i^2$, and for each subdirectly irreducible k-RL B, we have:

$$B \models \varphi(X_1, \ldots, X_n) \textit{ if, and only if, } B \models \bigwedge_{i=1}^{m} \gamma(A_i, D_i^{\wedge}, D_i^{\to}).$$

PROOF. It is enough to combine $(\star_1)$ and $(\star_2)$ above. ⊣

COROLLARY 2. *Each extension of FL^n_{ew} L is axiomatisable by canonical formulas. Furthermore, if L is finitely axiomatisable, then L is axiomatisable by finitely many canonical formulas.*

PROOF. Let L be a extension of FL^n_{ew}. Then L is obtained by adding $\{\varphi_j \mid j \in J\}$ to $\mathbf{FL}^k_{ew}$ as new axioms. We can safely assume to be in the non-trivial case for which $\mathbf{FL}^k_{ew} \not\vdash \varphi_j$ for each $j \in J$. We claim that the extension L is axiomatised by the canonical formulas of the systems associated with the φ_j's. Indeed, by Birkhoff's subdirect decompositions theorem, it is enough to check that for each subdirectly irreducible algebra B and for each $j \in J$, there exist $(A_{j1}, D_{j1}^{\wedge}, D_{j1}^{\to}), \ldots, (A_{jm}, D_{jm}^{\wedge}, D_{jm}^{\to})$ such that $B \models \varphi_j$ iff $B \models \bigwedge_{i=1}^{m_j} \gamma(A_{ji}, D_{ji}^{\wedge}, D_{ji}^{\to})$. But this is entailed at once by Corollary 1. In particular, if L is finitely axiomatisable, then L is axiomatisable by finitely many canonical formulas. ⊣

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▶ MATTEO BIANCHI, *Trakhtenbrot theorem and first-order axiomatic extensions of MTL.*

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In [Tra50], B.A. Trakhtenbrot showed that in classical logic the set of first-order tautologies associated to finite models is not recursively enumerable: moreover, it is known that such set is Π_1-complete (in [Vau60, BGG01] it is shown that the theorem works also with languages containing at least a binary predicate, and without equality). This result implies the fact that the completeness w.r.t. finite models does not hold, in first-order logic (indeed, the set of theorems of classical predicate logic is Σ_1-complete). One can ask if a similar result holds also in non-classical logics, for example many-valued logics. A first answer was given in [Háj99] by P. Hájek, that generalized Trakhtenbrot theorem to the first-order versions of Łukasiewicz, Gödel and Product logics (with respect to their standard algebras): that paper was published in 1999, and from then a much larger family of many-valued logics has been introduced, namely the monoidal t-norm based logic MTL and its axiomatic extensions ([EG01, CHN11]).

Differently to what happens in classical logic, in these many-valued logics we do not have necessarily a single totally ordered algebraic structure in which we can evaluate the truth-values of a formula: in particular,

Definition 1. Let L be an axiomatic extension of MTL. If there exists an L-chain $\mathcal{A}$ such that L is complete w.r.t $\mathcal{A}$, then L enjoys the single chain completeness (SCC).

Not all the axiomatic extensions of MTL enjoy this property: in [Mon11] an extensive study has been done, about the SCC.

In the first-order versions of the axiomatic extensions of MTL, we need to restrict to totally ordered algebras: indeed, if not, the soundness does not necessarily holds, see [EGHM03, Example 5.4] for a counterexample over Gödel logic. This is not by chance, but it is a consequence of the fact that such logics are axiomatized in the way to have the completeness w.r.t. the class of all chains (such development of first-order logics has many connections with the works of Mostowski and Rasiowa, as explained in [Háj06]). So, here the analysis of single chain completeness becomes even more justified, than in the propositional case. However, such a study is also (much) harder than in the propositional case, as pointed out in [Mon11].

In this talk we show a generalized version of Trakhtenbrot theorem for the first-order axiomatic extensions of MTL. We work on a countable language containing only predicates, with at least a binary one: let us call $\mathbf{P}$ the set of all predicates. In first-order axiomatic extension of MTL we restrict to chains, and the notion of model is defined as follows: given an MTL-chain $\mathcal{A}$, an $\mathcal{A}$-model is a structure $\mathbf{M} = \langle M, \{r_P\}_{P\in\mathbf{P}}\rangle$, where M is a non-empty set, and for every n-ary P, $r_P : M^n \to A$ is a fuzzy relation.

Variables are interpreted as elements of M, and formulas with an inductive Tarskian like definition (details will be given during the talk). A model $\mathbf{M}$ is finite whenever M is a finite set.

Our main result is the following.

THEOREM 2. *Let L be an axiomatic extension L of MTL whose corresponding variety is generated by a chain: for every generic L-chain $\mathcal{A}$ the set $fTAUT_{\forall}^{\mathcal{A}}$ (the set of first-order tautologies associated to the finite $\mathcal{A}$-models) is Π_1. Moreover, if in addition L is an axiomatic extension of BL or an axiomatic extension of SMTL or an axiomatic extension of WNM, then for every generic L-chain $\mathcal{A}$ the set $fTAUT_{\forall}^{\mathcal{A}}$ is Π_1-complete.*

As a corollary, we have that if L is one of BL, BL_n, Ł, Ł_n, G, G_n, Π, SMTL, SBL, SBL^n, SBL_n,WNM, NM, NMG, RDP, DP, and $\mathcal{A}$ is a generic L-chain, then $\mathrm{fTAUT}_{\forall}^{\mathcal{A}}$ is Π_1-complete.

Let L be an axiomatic extension of MTL. Another interesting property is the completeness of L$\forall$ (the first-order version of L) with respect to the finite $\mathcal{A}$-models of an L-chain $\mathcal{A}$: if such a chain exists, we say that L$\forall$ enjoys the fSCC (single chain completeness w.r.t. finite models of a chain). In the classical case the only chain is the two element boolean algebra $\mathbf{2}$, and since $\mathrm{fTAUT}_{\forall}^{\mathbf{2}}$ is Π_1-complete (by Trakhtenbrot theorem), then the fSCC fails to hold (because the set of first-order theorems is Σ_1-complete).

Interestingly, we have a negative result also for the first order version of all the axiomatic extensions of MTL.

THEOREM 3. *For every axiomatic extension L of MTL, the fSCC fails to hold, for L$\forall$.*

We conclude by discussing the expansions of MTL with the Δ operator. The Δ operator is an additional connective, firstly introduced in [Baa96], whose algebraic semantics is the following, for every MTL-chain $\mathcal{A}$, and $x \in \mathcal{A}$: $\Delta(x) = 1$ if $x = 1$, and $\Delta(x) = 0$ otherwise.

We have two main negative results.

THEOREM 4. *Let L be an axiomatic extension of MTL_{Δ} whose corresponding variety is generated by an L-chain. If $TAUT_L$ (the set of theorems of L) is decidable, then for every generic L-chain $\mathcal{A}$ it holds that $fTAUT_{\forall}^{\mathcal{A}}$ is Π_1-complete.*

We have a negative result also concerning the fSCC, analogously to Theorem 3.

THEOREM 5. *Let L be an axiomatic extension of MTL_{Δ} such that $TAUT_L$ is decidable. Then the fSCC fails to holds, for L$\forall$.*

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▸ FRODE BJØRDAL, *Truth degrees in the interval [-1,1] for the librationist system £*.
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The librationist foundational system, now named £, is published most completely in [1], and we send the reader there for a rather precise account of the system and its semantics. £ gives a comprehensive and fully type free account on how to deal with the paradoxes and at the same time gives a foundation for mathematics and semantics without compromising any classical logical theorems. [1] established that £ is stronger than the Big Five of the Reverse Mathematics Program. More recent work, presented in [2] and as a lecture at the adjoint conference *Universal Logic 2013*, has shown that if $ZF\Omega^-$ $=$*ZF minus extensionality plus 'there are omega inaccessible cardinals'* is consistent then £ has an interpretation of $ZF\Omega^-$ which £ believes is a standard (i.e. well founded) model of $ZF\Omega^-$. Moreover, £ then has an interpretation of ZF with extensionality given theorem 1 of [4] which has it that a system S slightly weaker than ZF minus extensionality - with collection in lieu of replacement - has an interpretation of ZF with extensionality.

We here concentrate upon how £ deals with the paradoxes in a novel manner, and we present an external way of thinking about the situation which cannot be matched by the theses of £. This external viewpoint involves the definition of a series of novel concepts, and an upshot will be that formulas of £ can be assigned truth degrees represented by an infinite discrete subset of the rational interval [-1,1].

In its most pure form £ is a set theoretic system with the set theoretic language minus the identity sign plus set brackets for a set forming variable-cum-formula operating operator; identity is defined à la Leibniz-Russell, and £ is highly non-extensional. For various purposes we in some situations add *sort* constants such as a truth predicate, but that need not concern us here. In unpublished superseded accounts £ was understood as a non-adjunctive paraconsistent system. But it turns out that we in our reasoning from the outside about £ best think of connectives as acting upon *valencies*.

The valency of a sentence is the set of ordinals where it holds in the Herzbergerian style semi inductive semantics with the *librationist twist* that formulas unbounded under the closure ordinal are the ones taken as designated and true, and not only those formulas stably in as from some ordinal below the closure ordinal. The *valor* of a sentence is the least upper bound of its valency. The *contravalence* of a sentence is the closure ordinal Ϙ (archaic Greek Koppa) minus the valency of that sentence, and the *ambovalence* of two sentences is the intersection of their valencies. Induced set theoretic definitions introduce the concepts of *velvalence*, *subvalence of ... under ,,,* and *homovalence* for veljunction, subjunction (material conditional) and equijunction (material biconditional), respectively. A sentence is true iff its valor is the closure ordinal Ϙ, and a sentence is false iff its negjunction (negation) is true. Connectives of £ are *valency functional*: Let $v(A)$ and $v(B)$ be the valencies of A and B; then the valency of $\neg A$ is the contravalency of A, the valency of the conjunction $A \wedge B$ is the ambovalenc of A and B, the valency of the veljunction $A \vee B$ is the velvalence of A and B, the valency of the subjunction $A \rightarrow B$ is the subvalence of A under B and the valency of the equijunction $A \leftrightarrow B$ is the homovalency of A and B. In the special and preferable case of non-paradoxical sentences valency functionality of connectives induce their truth functionality.

A sentence dictates its valor, and its valency is the way the valor is dictated. We take two sentences to contradict each other iff they are contravalent and dictate differently. Two sentences are complementary iff they are contravalent and dictate the same, i.e. thence the closure ordinal. For an example, let r be Russell's set $\{x : x \notin x\}$ and *Russel's sentence* be $r \in r$. Russell's sentence and its negjunction $r \notin r$ dictate the same in complementary ways.

Let us agree that a theory is *contrasistent* iff it has a thesis A as well as its negjunction $\neg A$ as a thesis. We take a theory to be inconsistent iff it has a thesis of the form $A \wedge \neg A$. A theory is trivial iff all formulas are theorems. Trivial systems and inconsistent theories with simplification (conjunction elimination) are contrasistent. £ is contrasistent, but neither trivial nor inconsistent. Moreover, unlike in paraconsistent approaches, all theses of classical logic remain theses of £, and £ has no thesis which contradicts classical logic. £, which is a super (semi) formal system is not recursively axiomatizable, but a lot of informative prescription schemas ("axiom schemas") and regulations ("inference rules") are isolated; importantly, modus ponens is not an unexceptional regulation, and it is in the novelty of regulations that £ most deviates from and, as we think, supersedes classical approaches.

Let there be a discrete infinite subset S of the rational interval [0,1] so that both $0 \in S$ and $1 \in S$, and so that there is an order preserving bijection h from Ϙ+1 to S; this can happen if there are appropriate layerings of Cauchy sequences of rational numbers in S to mimic the order type of Ϙ+1 when the members of S are taken as ordered by their size as rational numbers. We assume S to contain all rational numbers in the interval [0,1] that have less than ten decimals, so that for practical discernments S contains all interesting numbers in the interval; if one should want to strengthen or relax the preciseness required of practical discernments, one adjusts the decimal proviso accordingly. We use h as a function from the valor of a sentence to $S \subset$ [0,1]. For any sentence A, let $\overline{v}(A)$ be the valor of A. We take the *truth-value* of a sentence A to be $h(\overline{v}(A))$. We take the *truth-degree* $d(A)$ of a sentence A to be $h(\overline{v}(A)) - h(\overline{v}(\neg A))$. We have three cases, viz. (1) positive truth degrees, (2) negative truth degrees and (3) truth degree zero. (1) A sentence has positive truth degree just if its truth value is 1 and the truth value of its negjunction is different from 1. Sentences which are classical logical theorems have truth degree 1, but already some sentences of arithmetic have a truth degree a tiny bit less than 1, as it turns out. There will be sentences with a positive truth degree close to zero; the situation remains to be catalogized. (2) These cases are symmetrical with the previous ones. (3) Paradoxical sentences as Russell's sentence introduced above or, by way of example on account of a phenomenon related to *McGee's paradox* adapted from [5], some instances of the schema $\forall x(a \in \{y : \alpha(x, y)\}) \to a \in \{y : \forall x\alpha(x, y)\}$ have truth value 1 while also the negjunction of the sentence has truth value 1. These sentences thus have truth degree 0. In £, nonnegative truth degrees are the designated truth degrees. One should note well that connectives are not truth degree functional. For example, $d(r \in r) = d(r \notin r) = 0$ whereas $d(r \in r \vee r \notin r) = 1$.

There may be a variety of options with respect to how the introduced notions should be interpreted. One implausible suggestion would be to take the truth degree as a measure of the informativeness of a sentence; with such a way of thinking sentences with truth degree zero are taken to carry zero information whereas tautologies are taken to be fully informative, and sentences with intermediate truth degrees are taken to have some intermediate informativeness. Perhaps there are no reasonable interpretations of truth degrees different from -1, 0 and 1; in that case we should only concern ourselves with whether a sentence has negative, zero or positive truth degree. Regardless of interpretative choice, £ is above all concerned with the sentences with a positive truth degree, i.e. true non-paradoxical sentences. But all sentences with a non-negative truth degree are instrumental in isolating the sentences with positive truth degree, and it is not yet fully understood how instrumental paradoxical sentences may be in isolating sentences with positive truth degrees. If there are any interesting relationships with truth degrees in fuzzy set theory or related theories they remain occult to this author.

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▸ MATTIA BONGINI, AGATA CIABATTONI, FRANCO MONTAGNA, *Proof Search and Co-NP Completeness for Many-Valued Logics.*
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The invertibility of rules (the premises are derivable whenever their conclusions are) in a proof system is an important feature for guiding proof search and turns out to be very useful to settle the computational complexity of the formalized logic.

For many-valued logics, calculi with invertible rules have been provided for all *finite-valued logics*. This does not apply to infinite-valued logics where, excepting Gödel logic [4, 2], proof search oriented calculi, when available, require some ingenuity; this is for instance the case of the calculi for Łukasiewicz and Product logic, see e.g. [12].

An important step towards the automated construction of proof search was done in [4], with the introduction of *sequents of relations* (disjunctions of semantic predicates over formulas) and of *projective* logics. Intuitively, a logic is projective if for each connective $\Box$, the value of $\Box(x_1, \ldots, x_n)$ is equal to a constant or to one of the $x_1, \ldots, x_n$, depending on some relations $R_1, \ldots, R_k$ between $x_1, \ldots, x_n$. Such relations consitute a *partition of the unit*, in the sense that, for all $x_1, \ldots, x_n$, exactly one of $R_i(x_1, \ldots, x_n)$ holds.

For instance, Gödel logic G is projective. Indeed, using $x \leq y$, $y < x$ as partition of the unit, connectives are defined by cases as

$$x\&y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x, \end{cases} \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x, \end{cases}$$

Given a projective logic, starting from a formula expressing that Φ is a theorem, usually, $1 \leq \Phi$, we may reduce our formulas producing a tree, whose leaves are disjunctions of relations over atomic formulas. If the set of valid formulas of this form is in P, the logic turns out to be in Co-NP.

The methodology was extended in [9] to *semi-projective logics* in which the value of each $\Box(x_1, \ldots, x_n)$ can also be a term of the form $p(x_i)$ with p unary function symbol. Semi-projective logics capture, for instance, Nilpotent and Weak Nilpotent Minimum logic, the relevant logic *RM* and n-contractive BL-logics (i.e. Hajek's Basic Fuzzy Logic BL extended with n-contraction).

Based on the results in [6], this talk describes a methodology to introduce relational hypersequent calculi for a wider class of logics (hyperprojective logics) which include Łukasiewicz logic, product logic and Hájek's logic *BL*.

Hyperprojective logics are similar to projective logics, but the relations used in the reductions involve *multisets* of formulas and not just on formulas. Hence, conditions in the definition of projective logics are modified accordingly.

For every connective $\Box$ and for every relation $R(\mu, \Phi, \mu_1, \ldots, \mu_k)$, where $\Phi = \Box(\phi_1, \ldots, \phi_n)$ and $\mu, \mu_1, \ldots, \mu_k$ are multisets of formulas of L, there is a partition of the unit $C_i(\nu_1, \ldots, \nu_h), i = 1, \ldots, m$ such that the reductions look like:
If $C_i(\nu_1, \ldots, \nu_h)$ holds, then $R(\mu, \Phi, \mu_1, \ldots, \mu_n)$ reduces to $R(\mu, \gamma, \mu_1, \gamma_1, \ldots, \mu_n, \gamma_n)$, $i = 1, \ldots, k$, where $\nu_1, \ldots, \nu_h, \gamma, \gamma_1, \ldots, \gamma_n$ are multisets whose elements are (constants or) among $\phi_1, \ldots, \phi_n$.

The original definition of hyperprojective logic [6] is slightly more general and captures for instance semiprojective logics.

Starting from hypersequents of the form $1 \leq \Phi$ and applying the above reductions, we obtain a tree, called the *reduction tree* of Φ. Reductions can also be read backwards, that is, if a hypersequent H reduces to $H_1, \ldots, H_n$, then we can introduce the rule $\frac{H_1 \ldots H_n}{H}$. In this way, we obtain a proof system which will be denoted by $\mathbb{H}L$.

For example, in product logic where the comma represents product, formulas of the form $\phi\&\psi$ reduce to ϕ, ψ, a multiset of formulas.

Reductions for $\to$ are slightly more complex. For instance, $\phi \to \psi \leq \gamma$ is equivalent to $\psi \leq \phi, \gamma$. Thus after this reduction the components of $\phi \to \psi$ occur in different places in the sequent of relations.

Using the reductions repeatedly, we arrive at leaves the form $\mu \leq \nu$ or $\mu < \nu$, where ν, μ are multisets of atomic formulas of product logic. Since comma is interpreted as product, there is a P-time algorithm for checking whether a disjunction of such relations is valid (in the reals) or not.

An easy but interesting result on hyperprojective logics is the following:

THEOREM 1. *Any hyperprojective logic L is decidable.*

We now identify sufficient conditions on the r-hypersequent calculus $\mathbb{H}L$ in order that a logic L is Co-NP complete. First of all, any substructural logic is Co-NP hard, [11], and hence, we only have to worry about Co-NP containment.

A first condition is uniformity of contexts. We say that a proof system for a hyperprojective logic has a *uniform set of rules* if the contexts (i.e., the cases in the reduction by cases) of a reduction of a compound formula $\Phi = \Box(\phi_1, \ldots, \phi_n)$ only depend on $\Box, \phi_1, \ldots, \phi_n$ and do not depend on the relation where Φ appears or on the position of Φ inside the relation.

Uniformity allow us to reduce *all* instances of a formula together. Hence, once a formula of maximal complexity is reduced it will no longer appear in the reduction tree. It follows that the number of nodes in the reduction tree starting from $1 \leq \Phi$ does not exceed the number of subformulas of Φ.

Since we may replace two distinct partitions of the unit by a common refinement, for *any* hyperprojective logic we can always get uniformity for free.

The next ingredient of Co-NP completeness is resource-boundedness. A rule is said to be *resource-bounded* if the total number of occurrences of any subformula ϕ_i in the reduction of a formula $\Phi = \Box(\phi_1, \ldots, \phi_n)$ does not exceed the number of occurrences of such formula in Φ.

Unlike uniformity, resource boundedness is not for free, and when it is not satisfied, the size of nodes in a branch of the reduction tree starting from a formula can be exponential in the size of the formula.

A final condition for the Co-NP containment of a hyperprojective logic is that the set of valid disjunctions of relations between multisets with atomic formulas only is in P (such disjunctions will be called *simple* formulas).

THEOREM 2. *Let L be a hyperprojective logic whose r-hypersequent calculus $\mathbb{H}L$ has uniform and resource bounded rules. Suppose further that the set of valid simple formulas is in Co-NP. Then the set of theorems of L is in Co-NP.*

Examples. The following logics fall into the scope of Theorem 2: (1) All projective logics, provided their axioms are in P.

(2) Product Logic, Łukasiewicz Logic and *BL* are examples of hyperprojective logics having uniform rules. For *BL* we use sequents of the form $x \prec y$ and $x \ll y$ as in [7]. Unlike other proof systems, our system for *BL* is resource bounded and uniform.

(3) The previous theorem on Co-NP completeness also includes semiprojective logics like Gödel logic with an involutive negation, Nilpotent Minimum NM, Weak Nilpotent Minimum WNM, and R-Mingle RM.

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▸ FÉLIX BOU, *Introducing an exotic MTL-chain*.
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In this short explanation the algebraic signature $\langle \cdot, \to, \wedge, \vee, 0, 1 \rangle$ is used for $\mathrm{FL_{ew}}$-algebras, where $\cdot$ stands for the fusion (sometimes also called the multiplicative conjunction or intensional conjunction), $\to$ for the residuum (also called implication), $\wedge$ for the meet, $\vee$ for the join, 0 for the minimum and 1 for the maximum. The order associated to the lattice operations is denoted by $\leq$. We recall that two famous subvarieties of $\mathbb{FL}_{\mathrm{ew}}$ are the variety $\mathbb{MTL}$ of MTL-algebras [4] and the variety $\mathbb{BL}$ of BL-algebras [5]. The variety $\mathbb{MTL}$ is the subvariety of $\mathbb{FL}_{\mathrm{ew}}$ generated by its chains, and so lately its elements have also been called semilinear $\mathrm{FL_{ew}}$-algebras (e.g., [6]). On the other hand, $\mathbb{BL}$ is the subvariety of $\mathbb{MTL}$ characterized by the following divisibility equation (or identity)

(divisibility) $$x \wedge y = x \cdot (x \to y) \ ,$$

and it is well known to be the subvariety of $\mathbb{MTL}$ generated by continuous t-norms [3]. It is worth saying that while BL-algebras are at present very well understood (see [1] and the recent survey [2]), this is not at all the case neither with MTL-algebras nor with MTL-chains (see [6]).

It is trivial that there are equations (e.g., the very divisibility one) which distinguish

$\mathbb{MTL}$ from $\mathbb{BL}$, i.e., equations which hold in all BL-algebras but fail in some MTL-algebra. In this contribution we want to address this question under the restriction of only allowing equations in the positive fragment. The *positive fragment* is the one given by just considering the operations $\cdot$, $\wedge$, $\vee$, 0 and 1. Thus, the positive fragment does not allow the use of $\rightarrow$ (and neither the usual negation $\neg$ nor addition $+$). The terms in the positive fragment will be called *positive terms*; and analogously, *positive equations* refer to equations in the positive fragment. The main problem we are interested is the following:

Problem. Are $\mathbb{MTL}$ and $\mathbb{BL}$ equationally distinguishable in the positive fragment? That is, is there some positive equation which holds in $\mathbb{BL}$ but not in $\mathbb{MTL}$?

The answer to this question is affirmative. Indeed, the following result holds.

Main Theorem. The equation

$$(x_1 \cdot x_4 \cdot x_7) \wedge (x_2 \cdot x_5 \cdot x_8) \wedge (x_3 \cdot x_6 \cdot x_9) \ \leq \ (x_1 \cdot x_2 \cdot x_3) \vee (x_4 \cdot x_5 \cdot x_6) \vee (x_7 \cdot x_8 \cdot x_9)$$

is valid in $\mathbb{BL}$, but fails in $\mathbb{MTL}$.

The failure of this equation in $\mathbb{MTL}$ has been proved by the author exhibiting a concrete counterexample: the 36-element involutive IMTL-chain whose fusion table is shown later in this abstract. It is worth noticing that the size of this chain is too big to be found using a brute-force attack; and indeed, the more interesting part of this research is the explanation of the methodology employed to find this *exotic* MTL-chain.

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Table 1: The Caley table of $\cdot$ in the exotic algebra $\boldsymbol{E}$

```
#FUSION
fus_table = [
  # 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35     |
  #----------------------------------------------------------------------------------------------------------------|---
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], # |  0
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1], # |  1
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2], # |  2
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3], # |  3
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 3, 4], # |  4
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5, 5, 5, 5], # |  5
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 5, 5, 5, 5, 6], # |  6
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 5, 5, 7, 7, 7], # |  7
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 3, 5, 5, 7, 7, 8], # |  8
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 3, 4, 5, 6, 7, 7, 9], # |  9
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,10,10,10,10,10,10,10,10,10,10], # | 10
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,10,10,10,10,10,10,10,10,10,11], # | 11
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3,10,10,10,10,10,10,10,12,12,12], # | 12
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5, 5,10,10,10,10,10,13,13,13,13,13], # | 13
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 5, 5, 5,10,10,10,10,10,13,13,13,13,14], # | 14
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 5, 5, 5,10,10,10,10,10,13,13,13,14,15], # | 15
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3, 5, 7, 7,10,10,10,12,12,13,13,16,16,16], # | 16
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 3, 3, 5, 7, 8,10,10,11,12,12,13,13,16,16,17], # | 17
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18], # | 18
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,19], # | 19
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,20,20,20], # | 20
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 3,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,20,20,21], # | 21
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 3,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,20,21,22], # | 22
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5, 5, 5, 5,18,18,18,18,18,18,18,18,18,18,18,18,18,23,23,23,23,23], # | 23
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 5, 5, 5, 7, 7,18,18,18,18,18,18,18,18,18,18,18,20,20,23,23,24,24,24], # | 24
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 5, 5, 5, 7, 8,18,18,18,18,18,18,18,18,18,18,19,20,20,23,23,24,24,25], # | 25
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,10,10,10,10,10,10,10,10,18,18,18,18,18,18,18,18,26,26,26,26,26,26,26,26,26,26], # | 26
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,10,10,10,10,10,10,10,10,18,18,18,18,18,18,18,18,26,26,26,26,26,26,26,26,26,27], # | 27
  [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 1,10,10,10,10,10,10,10,11,18,18,18,18,18,18,18,19,26,26,26,26,26,26,26,26,26,28], # | 28
  [ 0, 0, 0, 0, 0, 0, 0, 3, 3, 3,10,10,10,10,10,10,12,12,18,18,18,18,18,18,20,20,26,26,26,26,26,26,26,29,29,29], # | 29
  [ 0, 0, 0, 0, 0, 0, 1, 3, 3, 4,10,10,10,10,10,10,12,12,18,18,18,18,18,18,20,20,26,26,26,26,26,26,27,29,29,30], # | 30
  [ 0, 0, 0, 0, 0, 5, 5, 5, 5, 5,10,10,10,13,13,13,13,13,18,18,18,18,18,23,23,23,26,26,26,26,26,31,31,31,31,31], # | 31
  [ 0, 0, 0, 0, 1, 5, 5, 5, 5, 6,10,10,10,13,13,13,13,13,18,18,18,18,18,23,23,23,26,26,26,26,27,31,31,31,31,32], # | 32
  [ 0, 0, 0, 3, 3, 5, 5, 7, 7, 7,10,10,12,13,13,13,16,16,18,18,20,20,20,23,24,24,26,26,26,29,29,31,31,33,33,33], # | 33
  [ 0, 0, 1, 3, 3, 5, 5, 7, 7, 7,10,10,12,13,13,14,16,16,18,18,20,20,21,23,24,24,26,26,26,29,29,31,31,33,33,34], # | 34
  [ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35], # | 35
]
```

► LEONARDO CABRER, *Classification of germinal MV-algebras.*
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When studying local properties of maps it is natural to consider two maps equivalent if they coincide on a neighbourhood of a point. More precisely, given a topological space X and a point $x \in X$, two functions $f, g\colon X \to Y$ are *locally equivalent at* x if there is a neighbourhood U of x such that $f{\restriction}U = g{\restriction}U$. The class of maps locally equivalent to f at x is called the *germ* of f at x. Applying the concept of germ equivalence to MV-algebras of McNaughton maps, in [7, §4.3] Mundici defines the notion of *germinal* ideal (see also [6]). We devote this paper to the study of germinals ideal of MV-algebras.

For every $n = 1, 2, \dots$, let $\mathcal{M}([0,1]^n)$ denote the MV-algebra of piecewise affine linear continuous functions $f\colon [0,1]^n \to [0,1]$, such that each piece of f has integer coefficients. $\mathcal{M}([0,1]^n)$ is the n-generator free MV-algebra, and the free generators are the coordinate map $\pi_i\colon [0,1]^n \to [0,1]$ with $i = 1, \dots, n$ (see [3, §3] and the references therein). For every $x \in [0,1]^n$, let the MV-ideals $\mathfrak{h}_x$ and $\mathfrak{o}_x$ of $\mathcal{M}([0,1]^n)$ be defined by:

$$\mathfrak{h}_x = \{f \in \mathcal{M}([0,1]^n) \mid f(x) = 0\}, \text{ and}$$

$$\mathfrak{o}_x = \{f \in \mathcal{M}([0,1]^n) \mid f(U \cap [0,1]^n) = \{0\} \text{ for some open } U \subseteq [0,1]^n \text{ containing } x\}.$$

The ideals $\mathfrak{h}_x$ and $\mathfrak{o}_x$ are called the *maximal ideal* (or sometimes, the *hull*) and *germinal ideal* of $\mathcal{M}([0,1]^n)$ at x, respectively.

The assignment $x \mapsto \mathfrak{h}_x$ determines a bijection between points of $[0,1]^n$ and maximal ideals of $\mathcal{M}([0,1]^n)$ (see [3, Proposition 3.4.7]). Moreover, $\mathcal{M}([0,1]^n)/\mathfrak{h}_x$ is isomorphic to the subalgebra S of $[0,1]$ generated by the coordinates of x. More precisely, if $x = (x_1, \dots, x_n)$ the map $[f]_{\mathfrak{h}_x} \mapsto f(x)$ determines an isomorphism between $\mathcal{M}([0,1]^n)/\mathfrak{h}_x$ and the subalgebra of $[0,1]$ whose universe is $S = (x_1\mathbb{Z} + \cdots + x_n\mathbb{Z} + \mathbb{Z}) \cap [0,1]$. Every finitely generated simple MV-algebra is isomorphic to $\mathcal{M}([0,1]^n)/\mathfrak{h}_x$ for some $n = 1, 2 \dots$ and $x \in [0,1]^n$.

From these observations it follows that for any $x = (x_1, \dots, n) \in [0,1]^n$ and $y = (y_1, \dots, y_m) \in [0,1]^m$, the MV-algebras $\mathcal{M}([0,1]^n)/\mathfrak{h}_x$ and $\mathcal{M}([0,1]^m)/\mathfrak{h}_y$ are isomorphic if and only if the groups

$$G_x = x_1\mathbb{Z} + \cdots + x_n\mathbb{Z} + \mathbb{Z} \quad \text{and} \quad G_y = y_1\mathbb{Z} + \cdots + y_m\mathbb{Z} + \mathbb{Z}$$

coincide. Moreover, denoting by Γ the categorical equivalence [5] between unital lattice-ordered (abelian) groups and MV-algebras, for each $x \in [0,1]^n$ we can write

$$\mathcal{M}([0,1]^n)/\mathfrak{h}_x \cong \Gamma(G_x, 1). \tag{1}$$

Germinal ideals have a more complicated description. For instance, if $x = \frac{1}{5}$ and $y = \frac{2}{5}$, then $G_x = \frac{1}{5}\mathbb{Z} = G_y$, but $\mathcal{M}([0,1])/\mathfrak{o}_x \not\cong \mathcal{M}([0,1])/\mathfrak{o}_y$. This is observed in [7, Example 5.5].

An MV-algebra A is said to be *germinal* if there exist $n = 1, 2, \dots$ and $x \in [0,1]^n$ such that $A \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_x$. Our final result provides a necessary and sufficient condition for $x, y \in [0,1]^n$ to satisfy $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$.

The aim of this paper is to give a complete classification of germinal MV-algebras. As an application, we will settle the fifth one of the eleven problems in [7].

§1. $\mathbb{Z}$-maps and integer affine transformations. Given sets $X, Y \subseteq [0,1]^n$ we say that $\eta\colon X \to Y$ is a $\mathbb{Z}$-*map* if η is continuous and piecewise (affine) linear, with finitely many linear pieces, each piece having integer coefficients. $\mathbb{Z}$-maps appear naturally as duals of homomorphisms of finitely presented MV-algebras (see [4]). A

$\mathbb{Z}$-map η is called a *$\mathbb{Z}$-homeomorphism* if it is a homeomorphism and its inverse η^{-1} is also a $\mathbb{Z}$-map.

From [1, Theorem 3.1] (see also [7, Theorem 8.7]) we have:

THEOREM 1. *For any $n = 1, 2, \ldots$ and $x, y \in [0,1]^n$ the following conditions are equivalent:*

(i) $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$.

(ii) For some open sets $U, V \subseteq \mathbb{R}^n$ with $x \in U$ and $y \in V$, there is a $\mathbb{Z}$-homeomorphism $\eta\colon U \cap [0,1]^n \to V \cap [0,1]^n$ such that $\eta(x) = y$.

If $x, y \in [0,1]^n$ are such that $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_x$, it is easy to see that x is in the interior of $[0,1]^n$ if and only if y is. Since each proper face of $[0,1]^n$ is $\mathbb{Z}$-homeomorphic to $[0,1]^k$ for some $k \leq n$, it is enough to classify germinal ideals of points in the interior of $[0,1]^n$. The following immediate consequence of Theorem 1 is a key tool in the proof of our main result:

COROLLARY 2. *For any $n = 1, 2, \ldots$ and points x, y in the interior of $[0,1]^n$ the following conditions are equivalent:*

(i) $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$*;*

(ii) *there exist an $n \times n$-matrix $\mathcal{A}$ and a $b \in \mathbb{Z}^n$ such that*

 (a) $\mathcal{A}x + b = y$*;*

 (b) *$\mathcal{A}$ has integer coefficients; and*

 (c) *the determinant of $\mathcal{A}$ is 1 or -1.*

§2. Classification of germinal MV-algebras. To present our result we first need to recall some definitions (see for example [3] or [7]).

The *denominator* $\operatorname{den}(x)$ of a rational point $x \in \mathbb{Q}^n$ is the least common denominator of its coordinates. The *homogeneous correspondent* of a rational point $y \in \mathbb{Q}^n$ is the integer vector $\widetilde{y} = \operatorname{den}(y)(y, 1) \in \mathbb{Z}^{n+1}$. Given rational points $v_0, \ldots, v_k \in \mathbb{Q}^n$, their convex hull $\operatorname{conv}(v_0, \ldots, v_t) \subseteq \mathbb{R}^n$ is said to be *regular* (or *unimodular*) if the set $\{\tilde{v}_0, \ldots, \tilde{v}_t\}$ of homogeneous correspondents of the vertices of T can be extended to a base of the free abelian group $\mathbb{Z}^{n+1}$.

A subset F of $\mathbb{R}^n$ is a *rational affine subspace* of $\mathbb{R}^n$ if E is the affine hull of some rational points in $\mathbb{R}^n$, i.e., there exist $v_0, \ldots, v_e \in \mathbb{Q}^n$ such that $E = \operatorname{aff}(v_0, \ldots, v_e)$.

Let $F \subseteq \mathbb{R}^n$ be an e-dimensional rational affine space, $e = 0, \ldots, n$. If $0 \leq e < n$ we define

$$c_F = \min\{\operatorname{den}(v) \mid v \in \mathbb{Q}^n \setminus F \text{ and } \exists v_0, \ldots, v_e \in F \text{ with } \operatorname{conv}(v, v_0, \ldots, v_e) \text{ regular}\}.$$

If $e = n$ we fix $c_F = 1$.

For each $x \in \mathbb{R}^n$ let

$$F_x = \bigcap \{F \subseteq \mathbb{R}^n \mid x \in F \text{ and } F \text{ is a rational affine space}\}.$$

The following is the restriction to $[0,1]^n$ of the main result in [2] where a complete classification of the orbits of the n-dimensional affine group over the integers acting on $\mathbb{R}^n$ is provided.

THEOREM 3. *Fix $n = 1, 2, \ldots$. For all $x, y \in [0,1]^n$ the following conditions are equivalent:*

(i) *There exist an $n \times n$-matrix $\mathcal{A}$ with integer coefficients having determinant 1 or -1, and a vector $b \in \mathbb{Z}^n$ such that $\mathcal{A}x + b = y$;*

(ii) $(G_x, c_{F_x}) = (G_y, c_{F_y})$.

Combining Corollary 2 and Theorem 3 we obtain:

Corollary 4 (Classification of germinal MV-algebras). *For any* $n = 1, 2, \dots,$ *and* x *and* y *lying in the interior of* $[0,1]^n$ *the following conditions are equivalent:*

(i) $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$;
(ii) $(G_x, c_{F_x}) = (G_y, c_{F_y})$.

§3. Solution of Mundici's fifth problem. In [7, §20.3], Mundici presented a list of eleven problems about MV-algebras. The fifth problem reads as follows:

> *Conjecture:* For any $n = 2, 3, \dots$ and rational points x, y in the interior of $[0,1]^n$, if $\mathrm{den}(x) = \mathrm{den}(y)$, then $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$.

For any rational point $x \in [0,1]^n$,

$$\mathrm{den}(x) = m \text{ if and only if } \mathcal{M}([0,1]^n)/\mathfrak{h}_x \cong \text{Ł}_m,$$

where Ł_m denotes the $(m+1)$-element MV-chain. Therefore, Mundici's conjecture can be rewritten as follows:

> For any $n = 2, 3, \dots$ and rational points x, y in the interior of $[0,1]^n$, if $\mathcal{M}([0,1]^n)/\mathfrak{h}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{h}_y$, then $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$.

We prove a stronger version of this conjecture. The proof relies on the following result (see [2, Lemmas 8 and 14]).

Lemma 5. *For each* $n = 1, 2, \dots$ *and each* $x \in [0,1]^n$*, if* $\mathrm{rank}(G_x)$ *the rank of the group* G_x *is not* n*, then* $c_{F_x} = 1$.

Finally, combining (1), Lemma 5 and Corollary 4 we obtain the following result:

Theorem 6. *Fix* $n = 1, 2, \dots$ *and (not necessarily rational) points* x, y *lying in the interior of* $[0,1]^n$*. If* $\mathrm{rank}(G_x) \neq n$*, the following conditions are equivalent:*

(i) $\mathcal{M}([0,1]^n)/\mathfrak{h}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{h}_y$;
(ii) $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$.

In particular, when $n \geq 2$ *and* x, y *are rational points, we have* $G_x = \frac{1}{\mathrm{den}(x)}\mathbb{Z}$ *and* $G_y = \frac{1}{\mathrm{den}(y)}\mathbb{Z}$*. Thus* $\mathcal{M}([0,1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0,1]^n)/\mathfrak{o}_y$ *if and only if* $\mathrm{den}(x) = \mathrm{den}(y)$.

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▸ MING CHAI, *Monitoring distributed systems with multi-valued LTL*.
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For a safety critical system, one of the utmost cares is taken for providing correctness guarantee for the system. For this purpose, various methods are used at different stages of systems' life cycles, including specification, design, programming, implementation and maintenance. Unfortunately, full information of a system is not always available when attempting these methods (i.e., the underlying system may contain uncertainties). Kleene's three-valued logic and its extensions have been introduced for specifying uncertainties contained in systems.

Model based methods (e.g., model checking and model based testing) are introduced into some safety critical domains for system quality control. These techniques are typically achieved with the aid of models of the underlying systems. Although model based techniques continuously grow more powerful, their practical applications are heavily hindered by the following reasons.

- When checking a high complexity system, model based techniques could suffer from the so-called state explosion problems;
- when checking a black-box system, a model of the system may not be available;
- the checking object of model based techniques is a model of the underlying system, not the system itself.

A recent trend has started to explore a lightweight formal verification solution named *runtime verification* [1], where a system's behaviour is checked against correctness properties. Runtime verification is performed by using a *monitor*. This is a device or a piece of software that reads the behaviour of a system and gives a satisfaction verdict as the result. Unlike standard model based techniques, runtime verification only checks the current execution of the underlying system. Therefore, it does not suffer from the state explosion problem when dealing with a large system. Furthermore, runtime verification does not need a model of the system. It is well suited to check black-box systems. Finally, the checking object of runtime verification is the system itself. Thus, the possibility of introducing additional errors in the modelling is excluded.

The problem of monitoring a system is essentially to find the answer for a query: "does the system hold the correctness property?". The answer is achieved according to some knowledge of the system, i.e., an observed system's execution. When monitoring a distributed system, the following two dimensions of uncertainty arise.

On one hand, unknown future of an observed execution can cause uncertainty in the monitoring result. An execution can be observed by a monitor only up to a certain moment. It is extended while the system is running. For instance, let $\tau = (open, read, write, write)$ be an observed trace and "a file will be eventually closed after it is open" a monitoring property. Since we do not know whether the event "close" will be actually executed in the future, the monitoring result is not adequately expressed by a boolean value (true or false) at this point in time.

On the other hand, uncertain timing of events can cause uncertainty in the monitoring result. An execution is built according to time stamps of events. For a distributed system, the order of causally unrelated executions of different components is not always determined when the system does not have a global clock. This can cause race conditions, which are difficult to catch and eliminate by testing or model checking. A race condition impacts monitoring results as well. It can result in inconsistency between an observed trace and the actual behaviour of a distributed system. Therefore, the monitoring results are not always certain. For instance, given a distributed system consisting of two components, each of which concurrently writes a data to a file. A monitor reads, e.g., a trace (write1, write2). However, due to asynchronicity, the actual behaviour of this system could be (write2, write1). Since we don't know which component writes the file first, the monitoring result for the property "write1 is executed before write2" is uncertain.

One solution for the first uncertainty could be to restrict the evaluation to completed traces, e.g. post-mortem dumps. However, in many contexts intermediate results are desirable. For the second uncertainty, one could restrict properties for race condition, e.g., accept only the correct event order for critical races. Unfortunately, such restrictions will make it difficult to build a monitor. Thus, we use the five-valued logic to faithfully present satisfaction relations between executions and properties in monitoring results.

Linear temporal logic (LTL) is a well accepted and expressive formal language used for specifying correctness properties for runtime verification.

Let AP be a finite set of atomic propositions. Then $\Sigma = 2^{AP}$ is a (finite) alphabet of *events*. A trace τ over Σ is an element of Σ^*. The concatenation of traces τ and τ' is denoted by $\tau \circ \tau'$. The length of τ is $|\tau|$, and ε is the empty trace of length 0.

We assume reader familiar with LTL [2]. We here only present our notations. The LTL language consists of propositions (p_1, p_2), boolean operators ($\neg$ and $\vee$) and temporal operators $\mathbf{U}$ ("until") and $\mathbf{X}$ ("next"). Given an LTL formula φ, we use the following shorthands: $(\varphi_1 \wedge \varphi_2)$ standards for $\neg\,(\neg\varphi_1 \vee \neg\,\varphi_2)$, $(\varphi_1 \rightarrow \varphi_2)$ stands for $\neg\,(\varphi_1 \vee \varphi_2)$, $\mathbf{F}\,\varphi$ stands for $(\top\ \mathbf{U}\ \varphi)$ and $\mathbf{G}\,\varphi$ stands for $(\neg\mathbf{F}\,\neg\,\varphi)$. For a trace τ and an LTL formula φ, the satisfaction relation $\tau \models \varphi$ is defined as in [2].

For any given τ and φ, there are exactly two possible answers to the question "does τ satisfy φ?". We say that $\mathsf{eval}(\tau \models \varphi) \in \{\mathit{true}, \mathit{false}\}$, where $\mathsf{eval}(\tau \models \varphi) \triangleq \mathit{true}$ if $\tau \models \varphi$, and $\mathsf{eval}(\tau \models \varphi) \triangleq \mathit{false}$ if $\tau \not\models \varphi$.

When monitoring a distributed system, an observed event may actually be executed at a different global time (according to the observer) than a local time recorded in the time stamp of the event. A set of traces can be implied according to an observed trace, and one in the set is the actual execution of the system.

Given a non-empty set of traces $\mathcal{T}$, we define $\mathsf{eval}(\mathcal{T} \models \varphi) \triangleq \bigcup_{\tau \in \mathcal{T}} \mathsf{eval}(\tau \models \varphi)$. The truth value of $\mathsf{eval}(\mathcal{T} \models \varphi)$ is in $E_3 \triangleq \{T, F, ?\}$, where $T \triangleq \{true\}$, $F \triangleq \{false\}$ and $? \triangleq \{true, false\}$.

Let $\mathcal{T} \circ \tau' \triangleq \{\tau \circ \tau' \mid \tau \in \mathcal{T}\}$ be the concatenation of $\mathcal{T}$ and τ'. We define an open semantics for LTL on finite trace sets as follows.

Definition 1. The *open semantics* for an LTL formula φ on a set of traces $\mathcal{T}$ is defined as $[\![\mathcal{T} \models \varphi]\!] \triangleq (\mathsf{c} \rightsquigarrow \mathsf{c}')$, where
$\mathsf{c} \triangleq \mathsf{eval}(\mathcal{T} \models \varphi)$, and $\mathsf{c}' \triangleq (\bigcup_{\tau' \in \Sigma^*} \mathsf{eval}((\mathcal{T} \circ \tau') \models \varphi))$.

Since $\mathcal{T} = \mathcal{T} \circ \varepsilon$, we have $\mathsf{eval}(\mathcal{T} \models \varphi) = \mathsf{eval}(\mathcal{T} \circ \varepsilon \models \varphi)$. Since $\varepsilon \in \Sigma^*$, it holds that $\mathsf{eval}(\mathcal{T} \circ \varepsilon \models \varphi) \subseteq (\bigcup_{\tau' \in \Sigma^*} \mathsf{eval}((\mathcal{T} \circ \tau') \models \varphi))$. Therefore, if $[\![\mathcal{T} \models \varphi]\!] = (\mathsf{c} \rightsquigarrow \mathsf{c}')$, then $\mathsf{c} \subseteq \mathsf{c}'$ must hold.

Thus, there are only five possible truth values in the range of $[\![\mathcal{T} \models \varphi]\!]$. We denote these values by $E_5 \triangleq$ {true (*tt*), false (*ff*), possible true (*pt*), possible false (*pf*), unknown (*uk*)}, with the definition $tt \triangleq (T \rightsquigarrow F)$; $ff \triangleq (F \rightsquigarrow F)$; $uk \triangleq (? \rightsquigarrow ?)$; $pt \triangleq (T \rightsquigarrow ?)$ and $pf \triangleq (F \rightsquigarrow ?)$. For example, the truth value $(? \rightsquigarrow F)$ is not a truth value of $[\![\mathcal{T} \models \varphi]\!]$

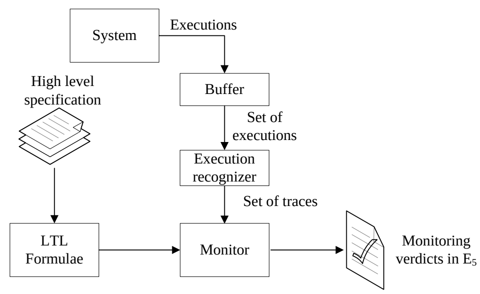

FIGURE 1. A Framework for Monitoring Systems

because $? \not\subseteq F$. Given $\mathsf{e}_1, \mathsf{e}_2 \in E_5$ with $\mathsf{e}_1 = \mathsf{c}_1 \rightsquigarrow \mathsf{c}'_1$ and $\mathsf{e}_2 = \mathsf{c}_2 \rightsquigarrow \mathsf{c}'_2$, we define $\mathsf{e}_1 \vee \mathsf{e}_2 \triangleq ((\mathsf{c}_1 \vee \mathsf{c}_2) \rightsquigarrow (\mathsf{c}'_1 \vee \mathsf{c}'{}_2))$, and $\neg\, \mathsf{e}_1 \triangleq ((\neg \mathsf{c}_1) \rightsquigarrow (\neg \mathsf{c}'_1))$. The five-valued truth tables can then be calculated in a straightforward manner by adopting Kleene's three value truth tables.

The framework of our runtime verification approach for distributed systems is shown in Fig. 1. In this framework, correctness properties are from the high level specification, and are expressed with LTL formulae. The buffer collects low level executions from the system. It has a timer, which resets to 0 when a new execution is observed. If the timer equals to the maximal time delay, then the buffer sends the set of collected executions to the execution recognizer. The execution recognizer converts received executions into a set of high-level event traces, which can be recognized by the monitor. The monitor consists of LTL formulae and five-valued LTL checking algorithms. It presents satisfaction verdicts as monitoring results.

We use the five-valued LTL to monitor a concrete example from the European Train Control System (ETCS). In ETCS level 2, the Raido Block Center (RBC) is responsible for providing movement authorities to allow the safe movement of trains. If a train requests to enter a new RBC area, the RBC of the leaving area (i.e., the handing over RBC, denoted with RBC_{HOV}) sends a request message (denoted with *Req*) to the RBC of the entering area (i.e., the accepting RBC, denoted with RBC_{ACC}). If the entering area is not occupied by another train (the route state is "clear"), then RBC_{ACC} permits the request by sending a route related information (denoted with *RRI*), and set the route state to "occupied". After the train has been running a safety distance, the accepting RBC set the route state to "clear" again.

We consider the case that two trains from different routes try to enter the same RBC area (Fig. 2). If the two trains request to enter the accepting RBC area at almost the same time, a race condition arises.

A timed event is abstracted with a pair (e, t), where $e \in \Sigma$ and $t \in \mathbb{R}$ being the time of the event emitted by a system. We assume that trains do not have a global clock, and the maximal time delay of an event is $\Delta t = 5$. We denote the route state "clear" with an event C, and the route state "occupied" with $\neg C$.

We build a behaviour according to the example given in the specification SUBSET-039 (FIS for the RBC/RBC handover). It is divided into the following three sets of events with monitoring.

$\mathfrak{b}_1 = \{(\{Req(1), C\}, 0), (\{Req(2), C\}, 1), (\{Req(1), C\}, 2)\ (\{Req(2), C\}, 3)\}$;

$\mathfrak{b}_2 = \{(RRI(2), 10), (RRI(2), 12), (RRI(2), 14), (Req(1), 16), (Req(1), 17)\}$;

$\mathfrak{b}_3 = \{(C, 40)\}$.

The behaviour $(\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3)$ can be convert to a sequence of trace sets $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2\mathcal{T}_3$.

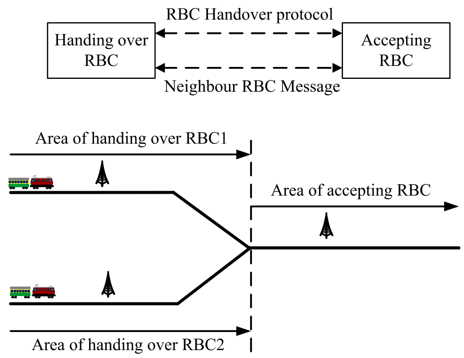

FIGURE 2. A case study: two trains from different routes try to enter a same RBC area

We consider the following properties.

- Property 1: an RBC$_{HOV}$ sends a request to the RBC$_{ACC}$, and if the route is clear, then the RBC$_{ACC}$ sends RRI to the RBC$_{HOV}$, and sets the route occupied, i.e., $\varphi_1 = (\mathit{Req}(i) \wedge C) \wedge \mathbf{F}\ (\mathit{RRI}(i) \wedge \neg C)$; and
- property 2: if RBC$_{ACC}$ sends an RRI to a RBC$_{HOV}$, it can not send it to another RBC$_{HOV}$ until the route is clear, i.e., $\varphi_2 = \mathbf{G}\ (\mathit{RRI}(i) \rightarrow (\neg \mathit{RRI}(i')\ \mathbf{U}\ C))$, with $i \neq i'$.

We also inject some errors into the executions, and get behaviours as follows.
$\mathfrak{b}'_1 = \{(\{\mathit{Req}(1), C\}, 0, 5), (\{\mathit{Req}(2), C\}, 1, 5)\}$;
$\mathfrak{b}'_2 = \{(\mathit{RRI}(1), 8, 5), (\mathit{Req}(2), 10, 5)\}$;
$\mathfrak{b}'_3 = \{(\mathit{RRI}(2), 17, 5)\}$.
The trace set sequence for this run is denoted with $\mathcal{T}' = \mathcal{T}'_1\mathcal{T}'_2\mathcal{T}'_3$. The online monitoring results of $\mathcal{T}$ and $\mathcal{T}'$ with respect to φ_1 and φ_2 are as follows.

	$\mathcal{T}$			$\mathcal{T}'$		
	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}'_1$	$\mathcal{T}'_2$	$\mathcal{T}'_3$
φ_1	*pf*	*uk*	*uk*	*pf*	*uk*	*tt*
φ_2	*pt*	*pf*	*pt*	*pt*	*pf*	*ff*

With the aid of work [4], a rewriting algorithm is developed for five-valued LTL based monitoring, and is implemented in Maude. This is a high performance system providing a rewriting environment, and is able to execute 2 million rewrites per second. We created a long trace set sequence by repeating 100 times $\mathcal{T}$, and checking it against φ_2. The monitoring process can be done by using only 0.28 million rewrites.

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▸ KAREL CHVALOVSKÝ, *Undecidability of consequence relation in Full Non-associative Lambek Calculus.*
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Substructural logics can be seen as logics lacking some structural rules when presented in the form of sequent calculi. Lambek introduced in [6] a calculus, which is now called after him, where all the normal structural rules, i.e. contraction, exchange, and weakening, are missing. Hence a sequent is a pair of a structure and a formula, where the structure is a sequence of formulae. Lambek also introduced in [7] a non-associative variant of this calculus, we call it **NL**, where the structure is a binary tree with leaves labelled by formulae. Not surprisingly, these systems have various motivations, e.g. Lambek's original motivations come from linguistics.

The language of **NL** contains product and two implications as the only connectives. However, it is common in substructural logics to consider also additive join, which will be important for us, and meet. In this way we obtain the Full Non-associative Lambek Calculus (**FNL**), see e.g. [3, 5].

A natural question to ask is whether provability in these systems is algorithmically tractable. It is known, see [3], that provability in **FNL** is decidable in polynomial space, but we prove that the (finitary) consequence relation in **FNL** is undecidable. This is somewhat surprising since the consequence relation is known to be decidable in **NL**, see [2], and the distributive **FNL**, see [3]. In the former case it is decidable even in polynomial time.

In fact, we show that the problem is undecidable even for the fragment with sequents containing only product and join by encoding the halting problem for 2-tag systems in this particular language. Moreover, using some transformations we get that even more restricted sequents are sufficient, whence the problem can be expressed using sequents containing only an implication and join. Furthermore, the construction still works if the structural rules of exchange and contraction are added. Note that if the rule of weakening is added then this problem is decidable, see [1].

As **FNL** is complete with respect to lattice-ordered residuated groupoids, see e.g. [5], it follows that the word problem for them is undecidable. In fact, this also holds for join-semilattices expanded by a groupoid operation (product) where this operation distributes over join, because our construction requires only product and join. Moreover, the idempotency and commutativity of join is not needed in full generality. Similarly, we can formulate our results in terms of term rewriting systems.

We recall that 2-tag systems, which were proposed by Post in [9], are very simple abstract machines that operate on finite words (sequences of letters). A 2-tag system is given by a finite alphabet of letters $\mathcal{A}$ and a production function π from $\mathcal{A}$ to finite words over $\mathcal{A}$. Fix $\mathcal{A}$ and π, the computation of this given 2-tag system on a word w is defined as follows. If $|w| < 2$ we terminate. Otherwise, we examine the first letter in

w, which is some letter a. Then we delete the first two letters in w and append $\pi(a)$ to the rest of the word after the last letter and obtain a new word. We repeat this process until it is possible, i.e. we can run forever or at some point we terminate—we obtain a word with less than two letters. In the later case we say that the 2-tag system terminates on w. It is well-known, see [4], that the halting problem for a 2-tag system, i.e. whether it terminates on a given w, is generally undecidable. For a comprehensive treatment one may refer to the book [8] by Minsky.

The encoding of 2-tag systems in **FNL** works as follows. We represent words by formulae containing only product. Although product is non-associtive, it is easy to mark letters in such a formula as deleted, by changing them to some other symbols, and append letters to its end. The real problem is to perform these steps correctly—the right steps are performed in the right order. We get around this problem by using join, because product distributes over join and therefore they play nicely together. We should emphasize that the construction is a bit technical, but the proofs are not particularly complicated.

We conclude by noting that join is not needed to prove that the consequence relation is undecidable in the associative case. There is a long tradition of similar results in this much more important and natural case since the word problem for semigroups was proved to be undecidable, independently, by Markov and Post.

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▸ AGATA CIABATTONI, NIKOLAOS GALATOS, AND REVANTHA RAMANAYAKE, *Embeddings into BiFL-algebras and conservativity.*
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A *BiFL$_e$-algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \to, +, 0, -)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ and $(A, +, 0)$ are commutative monoids and the following residuation and dual residuation conditions hold:

$$x \cdot y \leq z \Leftrightarrow y \leq x \to z \quad \text{and} \quad z \leq x + y \Leftrightarrow z - y \leq x.$$

A *FL$_e^+$-algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \to, +, 0)$ where the second (dual) residuation condition above is replaced by $x + (y \wedge z) = (x + y) \wedge (x + z)$.

We show that every FL$_e^+$-algebra can be embedded into a BiFL$_e$-algebra. Also, we will provide some equational properties that are preserved by this embeding, thus yielding the conservativity of the corresponding logics. We will use the method of residuated frames, introduced in [2]. The construction of the BiFL$_e$-algebra is partly inspired by proof-theoretic considerations in [1].

We note that BIFL$_e$ is a natural symmetrized version of the well known logic FL$_e$ (which is coincided with inuitionistic linear logic without exponentials), that avoids the stipulation of involutiveness (hence avoiding going all the way to linear logic without exponentials). As we consider extensions with structural rules, our results cover that of the conservativity of bi-intuitionistic logic over intuitionistic and one of our motivations was to extend this result to the substructural setting. A second motivation was to investigate the interaction between a residuated and a dually residuated pair of connectives; as expected the situation is much more involved than cases where two residuated pairs are considered together, such as in the case of distrubutive FL. Finally, this work sets the basis for a deeper understanding of connections between display logic, nested sequent calculi and residuated frames, which are explained in another forthcoming paper.

§1. The residuated frame. Given an FL$_e^+$-algebra $\mathbf{A}$ we will define a structure $\mathbf{W_A} = (W, W', N, \circ, \varepsilon, \gg, \oplus, \epsilon, \ll)$.

We define the set W by the following grammar:

$$W := W, A \mid W < A \mid \varepsilon$$

We assume that comma is a commutative monoid operation with ε as its neutral element. Elements of W of the form $w < a$ and ε are called *proper*. For convenience we extend the multiplication of A to $A \cup \{\varepsilon\}$ by $a \cdot \varepsilon = \varepsilon \cdot a = a$, for $a \in A$; we also define $\varepsilon \to a = a$. Then every element of W is of the form p, a, where p is proper and $a \in A \cup \{\varepsilon\}$; recall that p, ε equals p.

We define the set W' to be given by the grammar:

$$W' := P > A \mid \epsilon,$$

where P is the set of proper elements of W. We write a for $\varepsilon > a$.

We will now define (multisorted) hyperoperations $\circ$, $\oplus$, $\gg$ and $\ll$ on W and W'. These should not be confused with the formal constructs of comma, $>$ and $<$ used to define the elements of W and W'.

Recall that a *hyperoperation* $\circ$ on W is a function from $W \times W$ to $\mathcal{P}(W)$. If $x \circ y$ is a singleton $\{z\}$, we will simply write $x \circ y = z$, adopting the standard operational notation. For subsets X, Y of W we define $X \circ Y = \bigcup\{x \circ y : x \in X, y \in Y\}$. We say that a structure $(W, \circ, \varepsilon)$ is *commutative hypermonoid*, if $\circ$ is associative, commtative and has ε as a unit element: $x \circ \varepsilon = x = \varepsilon \circ x$, $x \circ y = y \circ x$ and $x \circ (y \circ z) = (x \circ y) \circ z$.

We define the (hyper)operation $\circ$ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we extend it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$; here we write X, a for the set $\{x, a : x \in X\}$.

We define $\oplus$ on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and $\emptyset$ otherwise. Also, $w' \oplus \epsilon = \epsilon \oplus w' = w'$.

For $(p, a) \in W$ and $p' > a' \in W'$, we define $(p, a) \gg (p' > a') = (p \circ p') > (a \to a')$. Also, $(p, a) \ll (p' > a') = (p, a) < a'$, if $p' = \varepsilon$; and $\emptyset$ otherwise; also $(p, a) \ll \epsilon = (p, a)$.

For $x \in W$ and $a \in A$ we define $x^+[a]$ as follows by induction on the structure of x.
$(\varepsilon)^+[a] := a$,
$(x, b)^+[a] := x^+[b \to a]$,
$(x < b)^+[a] := x^+[b + a]$.

Finally, we define the relation $N \subseteq W \times W'$ by:

$$(p, a)\ N\ (p' > a) \quad \Leftrightarrow \quad 1 \leq (p \circ p')^+[a \to a'].$$

Here we write $1 \leq X$ for $(\forall x \in X)(1 \leq x)$. Also, we write $X\ N\ Y$ for: $x\ N\ y$ for all $x \in X$ and $y \in Y$.

Lemma 1. *The structure $(W, W', N, \circ, \varepsilon, \gg, \oplus, \epsilon, \ll)$ is a commutative bi-residuated frame, namely $(W, \circ, \varepsilon)$ and $(W', \oplus, \epsilon)$ are commutative hypermonoids and the following conditions hold*

$$x \circ y\ N\ z \Leftrightarrow y\ N\ x \gg z \quad \textit{and} \quad z\ N\ x \oplus y \Leftrightarrow z \ll y\ N\ x.$$

§2. The dual algebra. Here we describe a general construction on an arbitrary commutative bi-residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \gg, \oplus, \epsilon, \ll)$ that yields a BiFL$_e$ algebra $\mathbf{W}^+$ as follows.

For subsets X, Y of W and Z of W' we define
$X^{\rhd} = \{z \in W : X \Rightarrow z\}$, $Z^{\lhd} = \{x \in W : x\ N\ Z\}$, $\gamma(X) = X^{\rhd\lhd}$,
$X \cup_\gamma Y = \gamma(X \cup Y)$, $X \circ_\gamma Y = \gamma(X \circ Y)$, $X \to Y = \{w \in W : X \circ \{w\} \subseteq Y\}$, $X + Y := (X^{\rhd} \circ Y^{\rhd})^{\lhd}$, $X - Y = X \ll Y^{\rhd}$.

Then we define $\mathbf{W}^+ = (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \circ_\gamma, \gamma(\varepsilon), \to, +, \epsilon^{\lhd}, -)$.

Lemma 2 (cf. [2]). *For every commutative biresiduated frame $\mathbf{W}$, the structure $\mathbf{W}^+$ is a BiFL$_e$-algebra.*

Since multiplication distributes over join, every equation in the language of $(\vee, \cdot, 1)$ can be written as a conjunction of equations of the form $t_0 \leq t_1 \vee \ldots \vee t_n$, where each t_i is a product of variables. Without loss of generality we can assume that t_0 is linear (each variable occurs at most once).

(To see this replace each variable x in t_0 appearing m times by $x_1 \vee \ldots \vee x_m$, where the x_i's are fresh variables. From the resulting inequality we can keep from the left-hand side only a term where all the x_i's occur and obtain an equivalent inequality; for the equivalence set $x_1 = \ldots = x_m = x$, for one direction, and use properties of $\vee$ on the left-hand side for the other direction.)

The above equation is equivalent to the propery: for all z

$$\frac{t_1 \le z \quad \dots \quad t_n \le z}{t_0 \le z}$$

If we replace $\cdot$ by $\circ$ and 1 by ε in the t_i's we obtain terms in W, which we denote by t_i°. We call a condition of the form

$$\frac{t_1^\circ \; N \; z \quad \dots \quad t_n^\circ \; N \; z}{t_0^\circ \; N \; z}$$

a left rule. This should be read as an implication from the conjunction of the assumptions in the numerator to the conclusion in the denominator. In a similar way we define right rules from $(\wedge, +, 0)$-equations.

THEOREM 3 (cf. [2]). *If the above left, or right, rule holds in a frame* $\mathbf{W}$*, then the above equation holds in* $\mathbf{W}^+$.

§3. The Gentzen frame. Returining now to our concrete commutative bi-residuated frame $\mathbf{W_A}$, we note that for all $a, b \in A$, $x, x' \in W$ and $z, z' \in W'$ following conditions are satisfied. We refer to the pair $(\mathbf{W_A}, \mathbf{A})$ as a Gentzen frame.

$$\frac{x \; N \; a \quad a \; N \; z}{x \; N \; z} \; (cut) \qquad \frac{}{a \; N \; a} \; (id)$$

$$\frac{a \circ b \; N \; z}{a \cdot b \; N \; z} \; (L\cdot) \qquad \frac{x \; N \; a \quad x' \; N \; b}{x \circ x' \; N \; a \cdot b} \; (R\cdot)$$

$$\frac{a \; N \; z \quad b \; N \; z'}{a + b \; N \; z \oplus z'} \; (L+) \qquad \frac{x \; N \; a \oplus b}{x \; N \; a + b} \; (R+)$$

$$\frac{x \; N \; a \quad b \; N \; z}{a \to b \; N \; x \gg z} \; (L\to) \qquad \frac{x \; N \; a \gg b}{x \; N \; a \to b} \; (R\to)$$

$$\frac{a \; N \; z}{a \wedge b \; N \; z} \; (L\wedge) \qquad \frac{b \; N \; z}{a \wedge b \; N \; z} \; (L\wedge) \qquad \frac{x \; N \; a \quad x \; N \; b}{x \; N \; a \wedge b} \; (R\wedge)$$

$$\frac{a \; N \; z \quad b \; N \; z}{a \vee b \; N \; z} \; (L\vee) \qquad \frac{x \; N \; a}{x \; N \; a \vee b} \; (R\vee) \qquad \frac{x \; N \; b}{x \; N \; a \vee b} \; (R\vee)$$

THEOREM 4 (cf. [2]). *The map* $a \mapsto \{a\}^{\lhd}$ *is an embedding from* $\mathbf{A}$ *to* $\mathbf{W}^+_{\mathbf{A}}$.

The logics BiFL_e and FL_e^+ are defined based on the consequence relations associated with BiFL_e-algebras and FL_e^+-algebras. So for example, a formula/term ϕ is a theorem of BiFL_e iff the equation $1 \le \phi$ is valid in all BiFL_e-algebras.

It follows from the theorem above that BiFL_e is conservative over FL_e^+.

THEOREM 5. *Every* FL_e^+*-algebra can be embedded into a* $BiFL_e$*-algebra.*

§4. Preservation of equations A left structural rule is called *non-ameliorating* if none of its non-trivial 1-metavariable instances on the left have a single variable on the left of the conclusion. Namely, all of its non-trivial 1-metavariable instrances have at least one occurence of $\circ$ in the conclusion. The definition for right rules is similar.

For example, the rules

$$\frac{x \; N \; z}{x \circ x \; N \; z} \qquad \frac{x \; N \; z \quad y \; N \; z}{x \circ y \; N \; z}$$

$$\frac{x \circ x \circ x \; N \; z}{x \circ x \; N \; z} \qquad \frac{x \circ x \circ x \; N \; z \quad x \circ x \circ y \; N \; z \quad x \circ y \circ y \quad N \; z \quad y \circ y \circ y \; N \; z}{x \circ y \; N \; z}$$

are non-ameliorating. The first two correspond to the equivalent equations $x^2 \leq x$ and $xy \leq x \vee y$. The last two correspond to the equivalent equations $x^2 \leq x^3$ and $xy \leq x^3 \vee x^2 y \vee xy^2 \vee y^3$. Note that the 1-metavariable instances on the left (obtained by either $x = \varepsilon$ or $y = \varepsilon$ are trivial.

However, the following rules are ameliorating.

$$\frac{x \circ x \; N \; z}{x \; N \; z} \qquad \frac{x \circ x \; N \; z \quad \varepsilon \; N \; z}{x \; N \; z} \qquad \frac{x \circ y \circ x \; N \; z}{x \circ y \; N \; z}$$

The first two correspond to the equations $x \leq x^2$ and $x \leq x^2 \vee 1$. The terminology is justified by the fact that the non-trivial 1-metavariabe instances of the non-ameliorating rules do not lead from premises with $\circ$ to a simpler conclusion without $\circ$. On the contrary, rules like contraction ameliorate (some of) the premises by transforming them into a conclusion without $\circ$.

THEOREM 6. *If* $\mathbf{A}$ *satisfies a non-ameliorating left or right rule, then so does* $\mathbf{W_A}$*, hence also* $\mathbf{W_A^+}$*. The corresponding extensions of the logic are conservative.*

COROLLARY 7. *If* $\mathbf{A}$ *satisfies left mingle* ($x \leq x \cdot x$) *or right mingle* ($x + x \leq x$)*, then so does* $\mathbf{W_A}$.

§5. Extensions containing Grishin (b) rule. Assuming that $\mathbf{A}$ satisfies the Grishin (b) equation $x(y+z) \leq xy + z$, we modify the definition of the operations on the frame.

We define $\circ$ recursively on the structure of the elements by the conditions $p \circ \varepsilon = \varepsilon \circ p = p$,

$$(w < a) \circ (w' < a') = \{((w < a) \circ w') < a', ((w' < a') \circ w) < a\},$$

and $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define $\oplus$ on W' by $(p > a) \oplus (p' > a') = (p \circ p') > (a + a') \in W'$. Also, $w' \oplus \epsilon = \epsilon \oplus w' = w'$.

For $(p, a) \in W$ and $p' > a'$, we define $(p, a) \gg (p' > a') = (p \circ p') > (a \to a')$ and $(p, a) \ll (p' > a') = (p \circ p') < (a \to a') \in P \subseteq W$.

Given a left rule, the corresponding right rule is obtained by replacing $t_i^{\circ} \; N \; z$ by $x \; N \; t_i$, where x is a variable for elements of W and t_i^+ is obtained by replacing $\circ$ by $+$ and ε by ϵ.

THEOREM 8. *If* $\mathbf{A}$ *satisfies Grishin (b), as well as the left and right versions of a rule, then so does* $\mathbf{W_A}$*, hence also* $\mathbf{W_A^+}$*. The corresponding extensions of the logic are conservative.*

COROLLARY 9. *If* $\mathbf{A}$ *satisfies Grishin (b) and contraction* ($x \cdot x \leq x$ *and* $x \leq x + x$)*, then so does* $\mathbf{W_A}$.

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▸ ALEX CITKIN, *A meta-logic of multiple-conclusion rules.*
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Our goal is to introduce a framework for working with generalized multiple-conclusion rules (in propositional logics) containing asserted and rejected propositions. The idea to consider rules that have more than one conclusion goes back to Carnap's notion of junctives (cf. [1]). In this same book Carnap also considers syntax means for rejection of junctives. Later, in the 1970-th T. Smiley and D. Shoesmith and also D. Scott introduced and studied the multiple-conclusion consequence relations.

The presence of multiple conclusions and rejected propositions makes it difficult - if not impossible - to use the regular syntactic and semantic means. The presence of many conclusions also requires to clarify the notion of admissible rule (cf. [5, 3] and the Section 2.3 below). The presence of rejected proposition in the rules makes it unclear how to use regular algebraic means and, therefore, makes it hard to use the framework introduced in [5]. In order to overcome the difficulties with algebraic semantics, we are using syntactic means: introducing a meta-logic that allows us to work with such generalized multiple-conclusion rules.

One of the advantages of using refutation system is that recursive axiomatizability leads to decidability (cf. [7]). The syntactical approach to decidability is especially important in cases when a logic does not have a good algebraic semantics, which often happens in fuzzy logics. In Section 2.3, we discuss this in more details and we will illustrate it by proving the decidability of some logics.

In Section 2.3, we discuss different definitions of admissibility in the case of multiple-conclusion rules (and in the Section 2.3 we will extend these definitions to generalized rules). Then, in the Section 2.3, we introduce a framework for working with generalized multiple-conclusion rules.

§1. Refutation and decidability. If a logic $\mathcal{L}$ is defined by a *regular deductive system*, that is, by a pair $\langle Ax; \mathcal{R}\rangle$, where Ax is a set of axioms and $\mathcal{R}$ is a set of rules, we can use derivation in order to obtain formulas from $\mathcal{L}$. We can add a set of anti-axioms and some rules for refutation (like modus tollens, for instance), and obtain a deductive system that will allow us to use derivation and also obtain the formulas not valid in $\mathcal{L}$. The idea of such deductive system belongs to Łukasiewicz and was developed further by R. Suszko and his collaborators, T. Skura, V. Goranko.

A logic $\mathcal{L}$ is *recursively axiomatizable* (or *recursively L-axiomatizable*) if $\mathcal{L}$ can be defined by a deductive system containing recursive sets of axioms and rules (respectively, axioms, anti-axioms and rules). From this point forward we will assume that the set of rules of a deductive system contains the rule of substitution and the rest of the rules are structural (in the sense of [6]).

THEOREM 1. *If a logic $\mathcal{L}$ is recursively axiomatizable and the lattice of all extensions of $\mathcal{L}$ (relative to set-theoretic meets and closed joins) is atomic with finite numbers of atoms, then $\mathcal{L}$ is recursively L-axiomatizable and, hence, decidable.*

Often, we link the decidability of a propositional logic with the finite model property (fmp). It is well known that every finitely axiomatizable logic with the fmp is decidable. But if $\mathcal{L}$ does not have the fmp while all proper extensions of $\mathcal{L}$ enjoy the fmp, that is, $\mathcal{L}$ is maximal among logics without the fmp, then the lattice of all extensions of $\mathcal{L}$ is atomic. Hence, the following generalization holds.

Corollary 1. *If a logic $\mathcal{L}$ is recursively axiomatizable and is maximal among logics without the fmp (for instance Kuznetsov-Gerčiu logic), then $\mathcal{L}$ is finitely Ł-axiomatizeble and, therefore, is decidable.*

The same is also true for the logics maximal among not-locally-tabular logics.

§2. Admissibility. In this section we consider regular (deductive) systems (see e.g. [5]). If S is a system, then $\mathsf{L}(S)$ denotes a set of all theorems of S. If S is a system and r is a rule, by $S + \mathsf{r}$ we denote a system S extended by the rule r.

There are different ways of how to define admissibility of a rule r in a given logic L ((A) and (C) are due to [5]), namely:

(A) A rule r is admissible in L if $\mathsf{L}(S+\mathsf{r}) = \mathsf{L}(S)$ for some system S such that $\mathsf{L}(S) = \mathsf{L}$
(B) A rule r is admissible in L if $\mathsf{L}(S+\mathsf{r}) = \mathsf{L}(S)$ for any system S such that $\mathsf{L}(S) = \mathsf{L}$
(C) A rule is admissible in L if for each substitution whenever all premisses are in L one of its conclusions is in L

The rules admissible in the sense of (A) will be called *conservative* (fully admissible; [3]), the rules satisfying (B) will be called *strongly conservative*, while the rules admissible in the sense of (C) will be called *admissible* (strictly admissible; [3]). In the case of single-conclusion rules the sets of conservative, strongly conservative and admissible rules coincide. With multiple-conclusion rules, it is not any more the case (even in logics as simple as classical).

Recall that a non-trivial algebraizable logic is *tabular* if the corresponding variety is generated by a finite algebra.

Theorem 2. *Suppose L is a non-trivial tabular logic. Then there is an infinite set of non-equivalent conservative over L rules that are not admissible in $\mathcal{L}$. In particular, there is an infinite set of non-equivalent rules conservative but not admissible in classical propositional logic (CPL).*

Let us observe that admissible rules enjoy the following property: if r_1 and r_2 are admissible in a logic L, then the consequence relation defined by these two rules still has L as its set of the theorems. For conservative rules, it is not the case: for instance, if L is a logic of the 10-element single-generated Heyting algebra the rules $\mathsf{r}_1 := P \vee Q/P, Q$ and $\mathsf{r}_2 := (\neg\neg P \to P) \to (P \vee \neg P)/\neg\neg P \vee \neg P$ are conservative over L (and the latter even is admissible in L), while rules $\mathsf{r}_1, \mathsf{r}_2$ allow to derive over L the Scott's formula that is not valid in L. Hence, in the multiple-conclusion environment the admissible rules are conservative, but not necessarily strongly conservative. On the other hand, the rule $P \vee Q/P, Q$ is strongly conservative over CPL, but it obviously is not admissible.

Theorem 3. *The following holds:*

(a) *There are strongly conservative rules that are not admissible;*
(b) *There are admissible rules that are not conservative.*

§3. Meta-Logics for generalized rules. We consider a propositional language with finite set $\mathcal{C}$ of finitely-ary connectives (not containing the following signs that we preserve for use in a meta-language $\oplus, \ominus, \odot, \dot\wedge, \dot\vee, \dot\to, \dot-, \bot, \top, \vdash$) and we use Frm to denote the set of (propositional) formulas built in a usual way using connectives from $\mathcal{C}$ and the (propositional) variables from the infinitely countable set Var. A mapping $\sigma : \mathsf{Var} \to \mathsf{Frm}$ is called a *substitution* and in a natural way σ can be extended to a mapping $\mathsf{Frm} \to \mathsf{Frm}$. By Σ we denote the set of all substitutions, and ε denotes the trivial substitution which maps every formula to itself. If S is a set of formulas we say that S is *closed under substitutions* if $A \in S$ entails $\sigma(A) \in S$ for any $\sigma \in \Sigma$. And S is *closed under reverse substitution* if $\sigma(A) \in S$ entails $A \in S$ for any $\sigma \in \Sigma$.

3.1. Logics. In order to include the rejected propositions, we generalize the notion of logic in the following way.

Definition 4. We will call a *logic* an ordered pair $\mathcal{L} = \langle \mathsf{L}^+; \mathsf{L}^- \rangle$, where $\mathsf{L}^+, \mathsf{L}^- \subseteq \mathsf{Frm}$, L^+ is closed under substitution and L^- is closed under reverse substitution. Formulas from L^+ are *asserted propositions* of $\mathcal{L}$ and formulas from L^- are *rejected propositions* of $\mathcal{L}$. If $\mathsf{L}^+ \cap \mathsf{L}^- = \emptyset$ the logic is called *consistent*. If $\mathsf{L}^+ \cup \mathsf{L}^- = \mathsf{Frm}$ the logic is called *full*. A full and consistent logic is called *standard*.

Accordingly, a matrix semantics can be constructed by defining *r-matrix* as an ordered triple $\mathcal{M} := \langle \mathscr{A}; \mathsf{A}^+, \mathsf{A}^- \rangle$, where $\mathscr{A}$ is an algebra (with operations from $\mathcal{C}$) and A^+ and A^- are the sets of *distinguished* and *rejected elements*. Every r-matrix in a natural way defines a logic $\mathcal{L}(\mathcal{M}) := \langle \mathsf{L}(\mathcal{M}^+), \mathsf{L}(\mathcal{M}^-) \rangle$, where $\mathsf{L}(\mathcal{M})^+ := \{A \mid \nu(A) \in \mathsf{A}^+$ for every valuation $\nu\}$ and $\mathsf{L}(\mathcal{M})^- := \{A \mid \nu(A) \in \mathsf{A}^-$ for some valuation $\nu\}$.

Remark 3.1. The introduced above r-matrices look similar to Mlinowski's q-matrices (see e.g. [4]), but we define the matrix consequence relation differently.

3.2. Meta-Language. The expressions of form $\oplus A$ and $\ominus A$, where $A \in \mathsf{Frm}$, are respectively called *positive atomic statements* and *negative atomic statements* (that sometimes are called signed formulas). The *statements* are defined by induction: the atomic statements are statements, if α, β are statements, then $\alpha \dot{\wedge} \beta, \alpha \dot{\vee} \beta, \alpha \dot{\to} \beta, \dot{-}\, \alpha, \top, \bot$ are statements. We denote by St the set of all statements. And we denote by St^+ a set of all positive statements, that is, the statements containing only the positive atomic statements and $\top$. Accordingly, by St^- we denote the set of all negative statements.

For a given logic $\mathcal{L}$ one can treat any substitution σ as a valuation $\sigma_{\mathcal{L}} : \mathsf{St} \to \mathbf{2}$ in the 2-element Boolean algebra $\mathbf{2} := \langle \{\bot, \top\}; \dot{\wedge}, \dot{\vee}, \dot{\to}, \dot{-} \rangle$ by letting

$$\sigma_{\mathcal{L}}(\oplus A) = \top \text{ if and only if } A \in \mathsf{L}^+ \text{ and } \sigma_{\mathcal{L}}(\ominus A) = \top \text{ if and only if } A \in \mathsf{L}^-.$$

A statement α is said to be *valid in a logic* $\mathcal{L}$ (in written $\mathcal{L} \vDash \alpha$) if the $\varepsilon_{\mathcal{L}}(\alpha) = \top$.

3.3. Meta-Logic. On the set of statements we define in a regular way a notion of derivation: the axiom schemata we obtain from the the schemata of (CPL) by replacing variables with variables for statements. For instance, from the schema $p \to (q \to p)$ we obtain $\alpha \dot{\to} (\beta \dot{\to} \alpha)$ where α and β range over St. The inference meta-rules are: for every $\alpha, \beta \in \mathsf{St}$ and every $A \in \mathsf{Frm}$ and every $\sigma \in \Sigma$

$$\alpha, \alpha \dot{\to} \beta \vdash \beta \quad (MP) \qquad \oplus A \vdash \oplus\sigma(A) \quad (SB) \qquad \ominus\sigma(A) \vdash \ominus A \quad (RS)$$

Definition 5. A set $\mathscr{T} \subseteq \mathsf{St}$ closed with respect to meta-rules (MP),(SB),(RS) is called *a theory*.

Proposition 1. *Given a theory $\mathscr{T}$, the pair*

$$\mathcal{L}(\mathscr{T}) := \langle \{A \in \mathsf{Frm} \mid \oplus A \in \mathscr{T}\}; \{A \in \mathsf{Frm} \mid \ominus A \in \mathscr{T}\} \rangle$$

forms a logic.

The logic $\mathcal{L}(\mathscr{T})$ is called a *logic of theory* $\mathscr{T}$ and we also will say that a theory $\mathscr{T}$ *defines a logic* $\mathcal{L}(\mathscr{T})$ or that $\mathscr{T}$ is a *theory of* L. Thus, every theory uniquely defines a logic. On the other hand, the following holds.

Proposition 2. *For every logic $\mathcal{L}$ there is at least one theory defining $\mathcal{L}$.*

3.4. Rules. We can clarify the different notions of admissibility in the following way.

Definition 6. The statements of form $\alpha_1 \dot{\wedge} \ldots \dot{\wedge} \alpha_n \dot{\to} \beta_1 \dot{\vee} \ldots \dot{\vee} \beta_m$ are called *rules.*

Proposition 3. *Given a theory $\mathscr{T}$, every statement is interderivable with a conjunction of rules.*

Definition 7. A rule is *admissible* in a given logic $\mathcal{L}$ if it is valid in $\mathcal{L}$. A rule α is c*conservative* over $\mathcal{L}$ if there is a theory of $\mathcal{L}$ containing α. A rule α is *strongly conservative* over $\mathcal{L}$ if every theory of $\mathcal{L}$ can be extended to a theory of $\mathcal{L}$ containing α. A rule α is *derivable* in $\mathcal{L}$ if α belongs to every theory of $\mathcal{L}$.

For a logic $\mathcal{L}$ by $\mathcal{A}(\mathcal{L}), \mathcal{C}(\mathcal{L}), \mathcal{S}(\mathcal{L}), \mathcal{D}(\mathcal{L})$, we denote respectively the sets of all admissible, all conservative, all strongly conservative, and all derivable over $\mathcal{L}$ rules. Then

$$\mathcal{D}(\mathcal{L}) \subseteq \mathcal{A}(\mathcal{L}) \subseteq \mathcal{C}(\mathcal{L}) \text{ and } \mathcal{D}(\mathcal{L}) \subseteq \mathcal{S}(\mathcal{L}) \subseteq \mathcal{C}(\mathcal{L})$$

3.5. Standard Logics. Let us consider standard logics.

Proposition 4. *A logic $\mathcal{L}$ is standard if and only if in $\mathcal{L}$ are admissible rules:*

$$\dot{-} \oplus P \dot{\to} \ominus P \quad (C_1) \qquad \oplus P \dot{\to} \dot{-} \ominus P \quad (C_2)$$

It is easy to see that (by contraposition) the rules (C_1) and (C_2) are equivalent (in the meta-logic) with the following rules

$$\dot{-} \ominus P \dot{\to} \oplus P \quad (C_3) \qquad \ominus P \dot{\to} \dot{-} \oplus P \quad (C_4)$$

Proposition 5. *If $\mathcal{L}$ is a standard logic and $\mathscr{T}$ is its theory containing (C_1) and (C_2), then every conservative over $\mathcal{L}$ rule from $\mathscr{T}$ is admissible.*

The proof of the above statement is based on the observation that using (C_1) - (C_4) one can show that every rule is inter-derivable with single-conclusion rule. And conservative single-conclusion rules are admissible.

For the logics with implication $\to$ the following rule deserves special attention, because it allows to use modus ponens for $\to$:

$$\oplus(A \to B) \vdash \oplus A \dot{\to} \oplus B. \quad \text{(IMP)}$$

As an example, let us consider standard intermediate logics.

Theorem 8. *Every standard intermediate logic can be L-axiomatized by Zakharyaschev's canonical formulas (for the definition of the canonical formulas see* [2]*).*

Remark 3.2. We cannot use the axiomatization suggested by T. Smiley [8], for the rule $\mathsf{r} := \ominus A \dot{\wedge} \ominus B \dot{\to} \ominus (A \vee B)$ together with rule (RS) gives inconsistent logic. Indeed, if there is a formula A such that $\ominus A$ is valid, by (RS), the statements $\ominus p$, where p is any variable, are valid. Likewise, due to there is a formula $\neg A$ such that $\ominus \neg A$ is valid, a statement $\ominus \neg p$ is valid. And, by r, we have $\ominus(p \vee \neg p)$, i.e. the logic is inconsistent.

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▸ PILAR DELLUNDE, ÀNGEL GARCíA-CERDAÑA, AND CARLES NOGUERA, *Advances on elementary equivalence in model theory of fuzzy logics.*

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Elementary equivalence is a central notion in classical model theory that allows to classify first-order structures. It was defined by Tarski in [10] who, together with Vaught, also proved fundamental results on elementary extensions and elementary chains in [11]. Later it has received several useful characterizations, among others, in terms of systems of back-and-forth, and has yielded many important results like the general forms of Löwenheim-Skolem theorems. (For general surveys on the subject and historical overviews we refer the reader to [1, 8].)

In the context of fuzzy predicate logics, the notion of elementarily equivalent structures was defined in [7]. There the authors presented a characterization of conservative extension theories using the elementary equivalence relation (see Theorems 6 and 11 of [7]). A related approach is the one presented in [9] where models can be elementary equivalent *in a degree d*. Following the definitions of [7], a few recent papers have contributed to the development of model theory of predicate fuzzy logics (see e.g. [4, 3]). However, the understanding of the central notion of elementary equivalence is still far from its counterpart in classical model theory. The present contribution intends to provide some advances towards this goal. After some preliminaries on first-order fuzzy logics in the first section, we list some of our new results in Section 2.

§1. The framework. In the following let L be a fixed core semilinear logic in a propositional language $\mathcal{L}$ (i.e. an expansion of the logic SL of [2], possibly with additional connectives with a congruence property, that is complete with respect to a semantics of linearly ordered algebras). The language of a first-order extension of L is defined in the same way as in classical first-order logic. A *predicate language* $\mathcal{P}$ is a triple $\langle Pred_{\mathcal{P}}, Func_{\mathcal{P}}, Ar_{\mathcal{P}}\rangle$, where $Pred_{\mathcal{P}}$ is a non-empty set of *predicate symbols,* $Func_{\mathcal{P}}$ is a set (disjoint with $Pred_{\mathcal{P}}$) of *function symbols,* and $Ar_{\mathcal{P}}$ is the *arity function,* assigning to each predicate or function symbol a natural number called the *arity* of the symbol. The function symbols f with $Ar_{\mathcal{P}}(f) = 0$ are called *object* or *individual constants.* The predicates symbols P for which $Ar_{\mathcal{P}}(P) = 0$ are called *truth constants.*

$\mathcal{P}$-terms and (atomic) $\mathcal{P}$-formulae of a given predicate language are defined as in classical logic (note that the notion of formula also depends on propositional connectives in $\mathcal{L}$). A $\mathcal{P}$*-theory* is a set of $\mathcal{P}$-formulae. The notions of free occurrence of a variable, substitutability, open formula, and closed formula (or, synonymously, *sentence*) are defined in the same way as in classical logic. Unlike in classical logic, in fuzzy logics without involutive negation the quantifiers $\forall$ and $\exists$ are not mutually definable and have to be both primitive symbols.

There are several variants of the first-order extension of a propositional fuzzy logic L that can be defined. Following Hájek's approach in [5, 6] and the general presentation of [2], we restrict to logics of models over linearly ordered algebras and introduce the first-order logics L$\forall$ and L$\forall^{\mathrm{w}}$ (respectively, complete w.r.t. all models or w.r.t. witnessed models). The logic L$\forall$ in language $\mathcal{P}$ has the following axioms:

(P) The axioms of L

($\forall$1) $(\forall x)\phi(x) \to \phi(t)$, where the $\mathcal{P}$-term t is substitutable for x in ϕ

($\exists$1) $\phi(t) \to (\exists x)\phi(x)$, where the $\mathcal{P}$-term t is substitutable for x in ϕ

($\forall$2) $(\forall x)(\chi \to \phi) \to (\chi \to (\forall x)\phi)$, where x is not free in χ

($\exists$2) $(\forall x)(\phi \to \chi) \to ((\exists x)\phi \to \chi)$, where x is not free in χ

($\forall$3) $(\forall x)(\chi \vee \phi) \to \chi \vee (\forall x)\phi$, where x is not free in χ.

The deduction rules of L$\forall$ are those of L plus the rule of *generalization:*

(Gen) $\langle \phi, (\forall x)\phi \rangle$.

The logic L$\forall^{\mathrm{w}}$ is the extension of L$\forall$ by the axioms:

(C$\forall$) $(\exists x)(\phi(x) \to (\forall y)\phi(y))$

(C$\exists$) $(\exists x)((\exists y)\phi(y) \to \phi(x))$.

A $\mathcal{P}$-*structure* is $\langle A, \mathbf{M}\rangle$ where A is an L-algebra and $\mathbf{M} = \langle S, \langle P_{\mathbf{M}}\rangle_{P\in\mathbf{P}}, \langle f_{\mathbf{M}}\rangle_{f\in\mathbf{F}}\rangle$, where M is a non-empty domain; $P_{\mathbf{M}}$ is an n-ary fuzzy relation, i.e. a function $S^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$ with $n \geq 1$ and an element of A if P is a truth constant; $f_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary $f \in \mathbf{F}$ with $n \geq 1$ and an element of M if f is an object constant.

Let $\langle A, \mathbf{M}\rangle$ be a $\mathcal{P}$-structure. An $\mathbf{M}$-*evaluation* of the object variables is a mapping v which assigns to each variable an element from S. Let v be an $\mathbf{M}$-evaluation, x a variable, and $a \in M$. Then $\mathrm{v}[x{\to}a]$ is an $\mathbf{M}$-evaluation such that $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each $y \neq x$.

Let $\langle A, \mathbf{M}\rangle$ be a $\mathcal{P}$-structure and v an $\mathbf{M}$-evaluation. We define *values* of $\mathcal{P}$-terms and *truth values* of $\mathcal{P}$-formulae in $\mathbf{M}$ for an evaluation v as:

$$\begin{array}{rcll}
||x||^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
||f(t_1,\ldots,t_n)||^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(||t_1||^{\mathbf{S}}_{\mathrm{v}},\ \ldots,\ ||t_n||^{\mathbf{M}}_{\mathrm{v}}), & \text{for } f \in \mathbf{F} \\
||P(t_1,\ldots,t_n)||^{\mathbf{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(||t_1||^{\mathbf{M}}_{\mathrm{v}},\ \ldots,\ ||t_n||^{\mathbf{M}}_{\mathrm{v}}), & \text{for } P \in \mathbf{P} \\
||c(\phi_1,\ldots,\phi_n)||^{\mathbf{M}}_{\mathrm{v}} & = & c^A(||\phi_1||^{\mathbf{M}}_{\mathrm{v}},\ \ldots,\ ||\phi_n||^{\mathbf{M}}_{\mathrm{v}}), & \text{for } c \in \mathcal{L} \\
||(\forall x)\phi||^{\mathbf{M}}_{\mathrm{v}} & = & \inf_{\leq^A}\{||\phi||^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}, & \\
||(\exists x)\phi||^{\mathbf{M}}_{\mathrm{v}} & = & \sup_{\leq^A}\{||\phi||^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}. &
\end{array}$$

If the infimum or supremum does not exist, we take its value as undefined. We say that $\langle A, \mathbf{M}\rangle$ is *safe* iff $||\phi||^{\mathbf{M}}_{\mathrm{v}}$ is defined for each $\mathcal{P}$-formula ϕ and each $\mathbf{M}$-evaluation v. $\langle A, \mathbf{M}\rangle$ is a *model* of a set of formulae Γ if it is safe and for every $\phi \in \Gamma$, $||\phi||^{\mathbf{M}}_{\mathrm{v}} \in F^A$ (where F^A is the filter of designated elements of the algebra A). If $\phi(x_1, \ldots, x_n)$ has $x_1, \ldots, x_n$ as free variables and $d_1, \ldots, d_n \in M$, by $||\phi(d_1, \ldots, d_n)||^{A}_{\mathbf{M}}$ we denote the truth value for any evaluation v such that $v(x_i) = d_i$ for each i. Finally, we call $\langle A, \mathbf{M}\rangle$ a *witnessed model* if all interpretations of quantifiers are actually maxima or minima reached by elements of the domain.

The semantical notion of consequence is defined in the usual way (every model of the premises is also a model of the conclusion) and corresponding completeness theorems are proved (see [7]).

§1. Results on elementary equivalence and elementary substructures. In this section we give a compressed sample list of the kind of results we can achieve regarding elementary substructures and elementary equivalence for predicate fuzzy logics. See [3, 4, 7] for any unexplained notion.

Definition 1 ([4]). Let $\langle A, \mathbf{N}\rangle$ be a $\mathcal{P}$-structure, $K \subseteq N$, $e_1, \dots, e_n \in K$, and $\phi(x, y_1, \dots, y_n)$ a $\mathcal{P}$-formula. We denote by $X^{\langle A,\mathbf{N}\rangle}_{\phi,e_1,\dots,e_n,K}$ the following subset of A: $\{||\phi(d, e_1, \dots, e_n)||^A_{\mathbf{N}} \mid d \in K\}$. It is said that a subset Y of A is *definable with parameters in* $\langle A, \mathbf{N}\rangle$ if there are $K \subseteq N$, $e_1, \dots, e_n \in K$, and a $\mathcal{P}$-formula $\phi(x, y_1, \dots, y_n)$ such that $Y = X^{\langle A,\mathbf{N}\rangle}_{\phi,e_1,\dots,e_n,K}$.

Definition 2. The *cardinality* of $\langle B, \mathbf{M}\rangle$ is the cardinality of the domain M, denoted by $|M|$.

Definition 3. We denote by $p(B)$ the minimum cardinal γ satisfying that, for every $X \subseteq B$ definable with parameters in $\langle B, \mathbf{M}\rangle$ such that its infimum and supremum exist, there is a $Y \subseteq X$ of cardinality $\leq \gamma$, which also has infimum and supremum and such that $\inf X = \inf Y$ and $\sup X = \sup Y$.

Theorem 4 (Downward Löwenheim-Skolem Theorem). *Let $\langle B, \mathbf{M}\rangle$ be an infinite $\mathcal{P}$-structure. Assume that every subset of B definable with parameters in $\langle B, \mathbf{M}\rangle$ has infimum and supremum. Then, for every cardinal κ with $\max\{p(B), |\mathcal{P}|, \omega\} \leq \kappa \leq |M|$ and every $Z \subseteq M$ with $|Z| \leq \kappa$, there is $\langle B, \mathbf{O}\rangle$ which is an elementary substructure of $\langle B, \mathbf{M}\rangle$ of cardinality $\leq \kappa$ and $Z \subseteq O$.*

Theorem 5 (Upward Löwenheim-Skolem Theorem). *For every $\langle B, \mathbf{M}\rangle$ and every $\kappa \geq \max\{|M|, |\mathcal{P}|\}$, there is a structure $\langle B, \mathbf{O}\rangle$ of cardinality κ such that $\langle B, \mathbf{M}\rangle$ is elementary mapped in $\langle B, \mathbf{O}\rangle$.*

Theorem 6 (Upward Löwenheim-Skolem Theorem for relational languages). *Assume that $\mathcal{P}$ is a purely relational predicate language. For every $\mathcal{P}$-structure and every $\kappa \geq \max\{|M|, |\mathcal{P}|\}$, there is $\langle B, \mathbf{O}\rangle$ of cardinality κ such that $\langle B, \mathbf{M}\rangle$ is an elementary substructure of $\langle B, \mathbf{O}\rangle$.*

Definition 7. We say that two $\mathcal{P}$-structures $\langle B_1, \mathbf{M}_1\rangle$ and $\langle B_2, \mathbf{M}_2\rangle$ are *elementary equivalent* (in symbols: $\langle B_1, \mathbf{M}_1\rangle \equiv \langle B_2, \mathbf{M}_2\rangle$) if for every $\mathcal{P}$-sentence σ, $||\sigma||^{B_1}_{\mathbf{M_1}} \in F^{B_1}$ iff $||\sigma||^{B_2}_{\mathbf{M_2}} \in F^{B_2}$.

Definition 8. Two $\mathcal{P}$-structures over the same chain $\langle B, \mathbf{M}_1\rangle$ and $\langle B, \mathbf{M}_2\rangle$ are *filter-strongly elementary equivalent* (in symbols: $\langle B, \mathbf{M}_1\rangle \equiv^{fs} \langle B, \mathbf{M}_2\rangle$) if for each $\mathcal{P}$-sentence σ, $||\sigma||^{B}_{\mathbf{M_1}} \in F^{B}$ iff $||\sigma||^{B}_{\mathbf{M_2}} \in F^{B}$ and, in this case, $||\sigma||^{B}_{\mathbf{M_1}} = ||\sigma||^{B}_{\mathbf{M_2}}$.

Definition 9. We say that two $\mathcal{P}$-structures over the same chain $\langle B, \mathbf{M}_1\rangle$ and $\langle B, \mathbf{M}_2\rangle$ are *strongly elementary equivalent* (in symbols: $\langle B, \mathbf{M}_1\rangle \equiv^{s} \langle B, \mathbf{M}_2\rangle$) if for every $\mathcal{P}$-sentence σ, $||\sigma||^{B}_{\mathbf{M_1}} = ||\sigma||^{B}_{\mathbf{M_2}}$.

Example 10. The notions of elementary equivalent and strongly elementary equivalent structures are different. Consider Gödel–Dummett logic G, a predicate language with only one monadic predicate P and take two structures over the standard Gödel chain, $\langle [0,1]_{\mathrm{G}}, \mathbf{M}_1\rangle$ and $\langle [0,1]_{\mathrm{G}}, \mathbf{M}_2\rangle$. The domain in both cases is the set of all natural numbers N and the interpretation of the predicate is:

$$P_{\mathbf{M_1}}(n) = \begin{cases} \frac{3}{4} - \frac{1}{n} & \text{if } n \geq 2 \\ 0 & 0 \leq n \leq 1 \end{cases} \qquad \text{and} \qquad P_{\mathbf{M_2}}(n) = \begin{cases} \frac{1}{2} - \frac{1}{n} & \text{if } n \geq 2 \\ 0 & 0 \leq n \leq 1. \end{cases}$$

On the one hand, $||(\exists x)P(x)||_{\mathbf{M_1}} = \frac{3}{4}$ but $||(\exists x)P(x)||_{\mathbf{M_2}} = \frac{1}{2}$, so the structures are not strongly elementary equivalent. On the other hand, elementary equivalence still holds. Take g as any non-decreasing bijection from $[0,1]$ to $[0,1]$ such that $g(\frac{3}{4}) = \frac{1}{2}$, $g(1) = 1$, $g(0) = 0$, and for every $n \in \mathsf{N}$ $g(\frac{3}{4} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{n}$. g is a G-homomorphism preserving suprema and infima. Then we can consider the σ-mapping $\langle g, Id\rangle$ and apply [3, Proposition 8] to obtain that $\langle [0,1]_{\mathrm{G}}, \mathbf{M}_1\rangle \equiv \langle [0,1]_{\mathrm{G}}, \mathbf{M}_2\rangle$.

Definition 11. Let $S(t)$ be the set of subterms of t that are not variables. We define by induction the *nested rank* of φ, denoted by $NR(\varphi)$, as follows.

- For every n-ary predicate R of $\mathcal{P}$, $NR(R(t_1, \dots, t_n)) = |\bigcup_{1\leq i\leq n} S(t_i)|$.
- For every $n \geq 1$, every $\mathcal{P}$-formulae $\phi_1, \dots, \phi_n$ and every n-ary connective $\lambda \in \mathcal{L}$,

$$NR(\lambda(\phi_1, \dots, \phi_n) = \max\{NR(\phi_1), \dots, NR(\phi_n)\} + 1.$$

- For any 0-ary connective $\lambda \in \mathcal{L}$, $NR(\lambda) = 0$.
- For every $\mathcal{P}$-formula φ, $NR((\forall x)\varphi) = NR((\exists x)\varphi) = NR(\varphi) + 1$.

Definition 12. Given $\mathcal{P}$-structures $\langle B_1, \mathbf{M_1}\rangle$ and $\langle B_2, \mathbf{M_2}\rangle$, we write $\langle B_1, \mathbf{M_1}\rangle \equiv_n \langle B_2, \mathbf{M_2}\rangle$ whenever for every $\mathcal{P}$-sentence σ with $NR(\sigma) \leq n$, $||\sigma||^{B_1}_{\mathbf{M_1}} \in F^{B_1}$ iff $||\sigma||^{B_2}_{\mathbf{M_2}} \in F^{B_2}$.

Definition 13. A pair $\langle T, R\rangle$ is a *partial relative relation* between $\langle B_1, \mathbf{M}_1\rangle$, $\langle B_2, \mathbf{M}_2\rangle$ if

1. $T \subseteq B_1 \times B_2$ such that $dom(T) = B_1$ and $rg(T) = B_2$. For each n-ary λ, if $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in T$, then $\langle \lambda^{B_1}(a_1, \dots, a_n), \lambda^{B_2}(b_1, \dots, b_n)\rangle \in T$. For every $a \in B_1$ and $b \in B_2$, such that $\langle a, b\rangle \in T$, $a \in F^{B_1}$ iff $b \in F^{B_2}$.
2. $R \subseteq M_1 \times M_2$ and if $\langle d_1, e_1\rangle, \dots, \langle d_n, e_n\rangle \in R$, then for each n-ary P, $\langle ||P(d_1, \dots, d_n)||^{B_1}_{\mathbf{M_1}}, ||P(e_1, \dots, e_n)||^{B_2}_{\mathbf{M_2}}\rangle \in T$.

Definition 14. We say that two structures $\langle B_1, \mathbf{M}_1\rangle$ and $\langle B_2, \mathbf{M}_2\rangle$ are *n-finitely relatives via* $\langle I_m \mid m \leq n\rangle$ (we write $\langle B_1, \mathbf{M}_1\rangle \sim_n \langle B_2, \mathbf{M}_2\rangle$) if

1. Every I_m is a non-empty set of partial relative relations,
2. *(Forth condition)* For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$ and any $d \in M_1$, there is a relation $\langle T, R'\rangle \in I_m$, such that $R \subseteq R'$ and $d \in dom(R')$.
3. *(Back condition)* For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$ and any $e \in M_2$, there is a relation $\langle T, R'\rangle \in I_m$, such that $R \subseteq R'$ and $e \in rg(R')$.
4. For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$, and any constant c of $\mathcal{P}$,$\langle T, R \cup \{\langle c_{\mathbf{M_1}}, c_{\mathbf{M_2}}\rangle\}\rangle \in I_m$.
5. For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$, any n-ary function symbol f of $\mathcal{P}$, and any $\langle d_1, e_1\rangle, \dots, \langle d_n, e_n\rangle \in R$, $\langle T, R \cup \{\langle f_{\mathbf{M_1}}(d_1, \dots, d_n), f_{\mathbf{M_2}}(e_1, \dots, e_n)\rangle\}\rangle \in I_m$.

Theorem 15 (Back and forth). *If $\mathcal{P}$ is finite and $\langle B_1, \mathbf{M}_1\rangle$, $\langle B_2, \mathbf{M}_2\rangle$ are witnessed, then for each $n \in \omega$, $\langle B_1, \mathbf{M}_1\rangle \equiv_n \langle B_2, \mathbf{M}_2\rangle$ iff $\langle B_1, \mathbf{M}_1\rangle \sim_n \langle B_2, \mathbf{M}_2\rangle$.*

We will also discuss characterizations in terms of back and forth of the other notions of elementary equivalence we have introduced.

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▸ WOJCIECH DZIK AND PIOTR WOJTYLAK, *Admissible rules and almost structural completeness in some first-order modal logics.*

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Almost Structural Completeness is proved and the form of admissible rules is found for some first-order modal logics extending S4.3. Bases for admissible rules are also investigated.

A logic is structurally complete if all (structural) rules which are admissible are also derivable in it. Many logics are not structurally complete because the only rules that are admissible but not derivable are passive. A rule $r: \quad \varphi_1, \ldots, \varphi_k/\psi$ is *passive* in a logic L if $\sigma(\varphi_1, \ldots, \varphi_k) \nsubseteq L$, for every substitution σ, that is r can not be applied to theorems of L. For example the following rule $P_2: \quad \Diamond p \wedge \Diamond\neg p/\bot$ is passive in modal logics extending S4. A logic is *almost structurally complete* if every (structural) rule which is admissible but not passive, is also derivable in it.

W.A.Pogorzelski and T.Prucnal [8] introduced substitutions for atomic formulas in first-order logic (which are homomorphisms of the language algebra modulo bounded variables). They showed that classical first-order logic (in the standard formalization: with Modus Ponens and Generalization rules) is not stucturally complete, but the system extended with a (non-structural) rule of substitution for atomic formulas is stucturally complete. It was shown in [2] that classical first-order logic in the standard formalization is almost structurally complete.

Let L be a first-order language (for simplicity: without identity, functions and constant symbols) containing infinitely many predicate symbols P_j, for each arity $n \geq 0$, with a special 0-ary predicate symbol $\bot$ that denotes syntactic falsehood . Formulas are built up using countably many variables v_i, propositional connective of implication $\to$, the quantifier $\forall$ and the modal connective of necessity $\Box$; the rest of connectives $\wedge, \vee, \neg, \leftrightarrow, \Diamond$ and the quantifier $\exists$ are defined in the usual way (for the basic formalization we follow S.Ghilardi Part II of [1] and also [7]). The symbol $\underline{x}$ denotes the string of variables $x_1, \ldots, x_n$, and $\varphi(\underline{x})$ denotes a formula that contains free variables only among $x_1, \ldots, x_n$. Symbols $P_j(x_1, \ldots, x_k)$ or $P_j(\underline{x})$, where P_j is a $k-$ary predicate symbol, will be called *atomic formulas* and the set of all atomic formulas will be denoted by At. The set of all atomic formulas occuring in a formula φ will be denoted by $At(\varphi)$.

We will use substitutions for atomic formulas in 1st order logic defined in [8]. According to the definition a substitution for atomic formulas $\varepsilon : At \to L$ can be extended to an endomorphism (modulo renaming of bounded variables) of the language algebra $\mathcal{L} = (L, \to, \perp, \Box, \forall_{x_i}, i \in N)$. Given a propositional normal modal logic $\mathbf{L}$ the (smallest) quantified modal logic $\mathbf{QL}$ corresponding to $\mathbf{L}$ is given by the axiom schemes from $\mathbf{L}$ plus the axioms:

$$\forall_{x_i}\varphi \to \varphi(y/x_i) \text{ where } y \text{ is free for } x \text{ in } \varphi$$

and the rule

$$\frac{\psi \to \varphi(x_i)}{\psi \to \forall_{x_i}\varphi(x_i)}, \text{ where } x_i \text{ is not a free variable in } \psi,$$

equivalently, the additional axiom $\forall_{x_i}(\psi \to \varphi(x_i)) \to (\psi \to \forall_{x_i}\varphi(x_i))$, where x_i is not a free in ψ, and the Rule of Generalization can be used, RG :

$$\frac{\varphi}{\forall_{x_i}\varphi}$$

moreover the logic is closed on Modus Ponens and on the Necessitation Rule RN:

$$\frac{\varphi}{\Box\varphi}$$

We assume that $\vdash$ denotes a consequence relation with Modus Ponens, Generalisation, and Necessitation as the only rules, in particular $\Gamma \vdash_{\mathbf{L}} \varphi$ means that a formula φ can be derived (proved) from the set of formulas Γ by means of axioms of the logic $\mathbf{L}$ using Modus Ponens, Generalization, and Necessitation rules.

By the definition, for every substitution $\varepsilon : At \to L$, $\Gamma \vdash_{\mathbf{L}} \varphi \Rightarrow \varepsilon[\Gamma] \vdash_{\mathbf{L}} \varepsilon\varphi$. We will make use of so called *Barcan Formula* BF: $\forall_x\Box\varphi \to \Box\forall_x\varphi$ and the extensions of $\mathbf{QL}$ with BF is denoted by $\mathbf{QL}$+BF. It is known that $\mathbf{QS5}$+BF =$\mathbf{QS5}$.

§1. Projective unifiers. Almost structural completeness of QS5. We extend the ideas of unifiers and projective unifiers of Ghilardi [6] from modal logics to first order modal logics. A *unifier* for a formula φ in a first order logic $\mathbf{L}$ is a substitution $\varepsilon\colon At \to L$ such that $\vdash_{\mathbf{L}} \varepsilon(\varphi)$. In this case a formula φ is called *unifiable.*

A substitution $\varepsilon\colon At \to L$ is a *projective unifier* for a formula φ in a 1-st order logic $\mathbf{L}$ if it is a unifier for φ and for every $P_i(\underline{x}) \in At(\varphi)$, $\underline{x} = x_1, \ldots, x_n$,

$$\varphi \vdash_{\mathbf{L}} \varepsilon(P_i(x_1, \ldots, x_n)) \leftrightarrow P_i(x_1, \ldots, x_n)$$

We say that a logic $\mathbf{L}$ enjoys projective unification if for every unifiable formula in $\mathbf{L}$ a projective unifier exists. Observe that if a logic enjoys projective unification then it is almost structurally complete.

Let $\forall\varphi$ denotes the universal quantification $\forall_{\underline{x}}\varphi(\underline{x})$ over all free variables $\underline{x}$ of φ.

THEOREM 1. *Every axiomatic extension of* $\mathbf{QS5}$ *enjoys projective unification and, hence, it is almost structurally complete.*

Let $\alpha_0 := \Box\neg\varphi_1 \vee \Box\varphi_1$,
$\alpha_1 := \Box\neg\varphi_1 \vee \Box(\varphi_1 \vee \neg\varphi_2) \vee \Box(\varphi_1 \vee \varphi_2)$,
$\alpha_2 := \Box\neg\varphi_1 \vee \Box(\varphi_1 \vee \neg\varphi_2) \vee \Box(\varphi_1 \vee \varphi_2 \vee \neg\varphi_3) \vee \Box(\varphi_1 \vee \varphi_2 \vee \varphi_3)$, etc.
and let $\mathbf{S5_n} = \mathbf{S5} + \alpha_n$, $n \geq 0$.

COROLLARY 2. *The logics* **QS5** *and* $\mathbf{QS5_n}$, *$n \geq 0$, are almost structurally complete.*

However the scope of projective unification in logics extending **QS4**+BF is limited.

THEOREM 3. *If a logic* **QL** *is an axiomatic extension of* $\mathbf{QS4} + \mathrm{BF}$ *and* **QL** *enjoys projective unification, then* $\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi) \in \mathbf{QL}$, *that is,* **QL** *is an (axiomatic) extension of* $\mathbf{QS4.3} + \mathrm{BF}$.

§2. Admissible Rules. Passive Rules. We use the symbol $\forall_{\underline{x}}\varphi(\underline{x})$ to denote $\forall_{x_1} \ldots \forall_{x_n}\varphi(x_1, \ldots, x_n)$. One can estimate whether a formula φ is (non-)unifiable by the following:

THEOREM 4. *Let* **L** *be an axiomatic extension of* **QS4**, *and for a formula* φ, $At(\varphi) = \{P_1(\underline{x_{k_1}}), \ldots, P_n(\underline{x_{k_n}})\}$. *Then the following conditions are equivalent:*

- φ *is not unifiable in* **L**,
- $\varphi \vdash_{\mathbf{L}} (\Diamond\exists_{\underline{x_{k_1}}} P_1(\underline{x_{k_1}}) \wedge \Diamond\exists_{\underline{x_{k_1}}}\neg P_1(\underline{x_{k_1}})) \vee \cdots \vee (\Diamond\exists_{\underline{x_{k_n}}} P_n(\underline{x_{k_n}}) \wedge \Diamond\exists_{\underline{x_{k_n}}}\neg P_n(\underline{x_{k_n}}))$

The expression on the right of $\vdash_{\mathbf{L}}$ will be denoted by $UB(P_1, \ldots, P_n)$ (*Upper Bound*).

LEMMA 5. *Let* **L** *be an axiomatic extension of* $\mathbf{QS4.3} + \mathrm{BF}$ *(in particular of* **QS5**). *If φ is not unifiable in* **L**, *then there is a formula ψ such that*

$$\varphi \vdash_{\mathbf{L}} \Diamond\exists_{\underline{x1}}\psi \wedge \Diamond\exists_{\underline{x}}\neg\psi$$

COROLLARY 6. *The following rule*

$$\frac{\Diamond\exists_{\underline{x}}\psi \wedge \Diamond\exists_{\underline{x}}\neg\psi}{\bot}$$

forms a basis for all passive rules in every axiomatic extension of $\mathbf{QS4.3} + \mathrm{BF}$ *and a basis for all admissible rules in* **QS5**.

Note that the rule: $\Diamond p \wedge \Diamond\neg p/\bot$ was shown by Rybakov [9] to be a basis for admissible rules in each propositional modal logic containing **S4.3** .

COROLLARY 7. *A modal consequence relation* $\vdash_{\mathbf{QS5}}$ *extended by the rule*

$$\frac{\Diamond\exists_{\underline{x}}\psi \wedge \Diamond\exists_{\underline{x}}\neg\psi}{\bot}$$

is structurally complete.

Note: there are other extensions of **QS4.3**+BF for which the analogous results hold.

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Recently, the literature on social choice theory has seen important generalisations of the classical Arrovian problem of preference aggregation culminating in the new field of judgment aggregation (for a survey see List and Puppe 2009). An essential feature of these generalisations is the extension of the problem of aggregation from the aggregation of preferences to the aggregation of arbitrary information represented by judgments on a set of propositions, the "agenda", on the truth value of which a collectivity (like a panel of experts) has to make a decision. It thus seemed natural to exploit the potential of model theory which, broadly speaking, studies the relation between abstract structures and statements about them (for an introduction to model theory see Bell and Slomson 1969) and to analyse the problem of judgment aggregation as the problem of aggregating the models that satisfy these judgments (see Herzberg and Eckert 2012, following Lauwers and Van Liedekerke 1995). In a model theoretic perspective, the aggretation problem as it underlies Arrovian impossibility results can be related to the well known fact (see Bell and Machover 1977, p. 174) that a product of individual models (e.g. a profile of individual preference relations) may not share the first order properties of its factor models (e.g. transitivity). For this reason the direct product construction is often modified by using another boolean algebra than $\mathbf{2} = \{0, 1\}$ and in particular the powerset algebra over the index set as an algebra of truth values (see e.g. Bell 2005). This approach was first applied to social welfare functions in Skala 1978 as one of the many attempts to overcome Arrow's dictatorship result and is here extended to the problem of judgment aggregation. In this note we extend it to the problem of judgement aggregation. While the major body of the literature on judgment aggregation studies the (in)consistency between properties of the aggregation rule and properties of the agenda, the significance of our simple possibility result consists in stressing the importance of the set of truth values and its algebraic structure.[6] This significance is closely related to a property of order preservation of mappings between the powerset algebra over the set of individuals and the algebra of truth values.

Fix an arbitrary set A, and let $\mathcal{L}$ be a language consisting of constant symbols for all elements a of A as well as (at most countably many) predicate symbols P_n, $n \in \mathbf{N}$. We shall denote the arity of P_n by $\delta(n)$ (for all $n \in \mathbf{N}$).

Let $\mathcal{S}$ be the set of atomic formulae in $\mathcal{L}$, and let $\mathcal{T}$ be the *boolean closure* of $\mathcal{S}$, i.e. the closure of $\mathcal{S}$ under the logical connectives $\neg$, $\wedge$, $\vee$.

The relational structure $\mathfrak{A} = \langle A, \langle R_n \;:\; n \in \mathbf{N}\rangle\rangle$ is called a *realisation of* $\mathcal{L}$ *with domain* A or an $\mathcal{L}$*-structure with domain* A if and only if the arities of the relations R_n correspond to the arities of the predicate symbols P_n, that is, if $R_n \subseteq A^{\delta(n)}$ for each n.

An $\mathcal{L}$-structure $\mathfrak{A}$ is a *model* of the theory T if $\mathfrak{A} \models \varphi$ for all $\varphi \in T$, i.e. if all sentences of the theory hold true in $\mathfrak{A}$ (with the usual Tarski definition of truth).

A *boolean-valued model* for $\mathcal{L}$ is a mapping which assigns to each $\mathcal{L}$-formula λ a truth value $\|\lambda\|$ in some arbitrary complete boolean algebra $\mathbf{B} = \langle B, \sqcup, \sqcap, ^*, 0_B, 1_B\rangle$ in such a way that boolean connectives and logical connectives commute:
$\|\neg\lambda\| = \|\lambda\|^*$; $\|\phi \vee \varphi\| = \|\phi\| \sqcup \|\varphi\|$; $\|\phi \wedge \varphi\| = \|\phi\| \sqcap \|\varphi\|$ (see Jech 1989).

boolean-valued models stand in a natural relation to products of models, like they play a role in aggregation theory.

Let Ω be the collection of models of T with domain A.

Let I be a (finite or infinite) set. Elements of I will be called *individuals*, elements of Ω^I will be called *profiles* and will be denoted by $\underline{\mathfrak{A}} := \langle \mathfrak{A}_i \rangle_{i \in I}$.

Remark 1. *Observe that any such profile* $\underline{\mathfrak{A}} \in \Omega^I$ *as a mapping* $I \to \Omega$ *induces a map from the set of* $\mathcal{L}$*-formulae to the powerset algebra* $P(I) = \left\langle 2^I, \cup, \cap, \complement, \varnothing, I \right\rangle$[7]*, which maps any* $\mathcal{L}$*-formula* λ *to the coalition of all individuals whose models satisfy* λ*, i.e.* $\{i \in I : \mathfrak{A}_i \models \lambda\}$.

We now call a boolean-valued map f which assigns to each profile $\underline{\mathfrak{A}} \in \Omega^I$ and each formula λ a truth value $\|\lambda\|_f^{\underline{\mathfrak{A}}}$ in some arbitrary complete boolean algebra $\mathbf{B} = \langle B, \sqcup, \sqcap, ^*, 0_B, 1_B\rangle$ a **boolean-valued aggregation rule** (BVAR) if and only if it is a boolean-valued model as a function of the formula argument, that is, if one has for all $\mathcal{L}$-formulæ λ, ϕ, ψ and all profiles $\underline{\mathfrak{A}} \in \Omega^I$,
$\|\neg\lambda\|_f^{\underline{\mathfrak{A}}} = \left(\|\lambda\|_f^{\underline{\mathfrak{A}}}\right)^*$; $\|\phi \vee \varphi\|_f^{\underline{\mathfrak{A}}} = \|\phi\|_f^{\underline{\mathfrak{A}}} \sqcup \|\varphi\|_f^{\underline{\mathfrak{A}}}$; $\|\phi \wedge \varphi\|_f^{\underline{\mathfrak{A}}} = \|\phi\|_f^{\underline{\mathfrak{A}}} \sqcap \|\varphi\|_f^{\underline{\mathfrak{A}}}$.

The following properties are reformulations of standard conditions for judgment aggregation rules in the framework of BVARs.

In particular, the non-dictatorship condition can be expressed in the following way:

Definition 1. A BVAR f is **non-dictatorial** if there exists no individual $i \in I$ such that for any $\mathcal{L}$-formula λ and any profile $\underline{\mathfrak{A}} \in \Omega^I$
$\mathfrak{A}_i \models \lambda \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} = 1_B$.

Obviously, non-dictatorship is only relevant if the set I consists of at least two individuals, which will be assumed throughout.

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[6] Among the relatively few many-valued extensions of judgment aggregation Pauly and Hees 2006, Dokow and Holzman 2010, and Herzberg 2014 deserve to be noted. Closest in spirit to our possibility result is, however, Duddy and Piggins 2013 which establishes a characterization of the possibility/impossibility boundary in the framework of t-norms.

[7] Wherein $\complement D = I \setminus D$ for all $D \subseteq I$.

Intuitively, non-dictatorship in the framework of BVARs guarantees that there exists no individual who can ensure for her judgments the highest truth degree. On the other hand, the intuitively appealing Pareto principle requires that unanimous agreement be respected by a judgment aggregation rule:

Definition 2. A BVAR f is **paretian** if for any $\mathcal{L}$-formula λ and any profile $\underline{\mathfrak{A}} \in \Omega^I$
$\{i \in I : \mathfrak{A}_i \models \lambda\} = I \Rightarrow \|\lambda\|_f^{\mathfrak{A}} = 1_B$.

Central to aggregation problems are independence conditions of various strength:

Definition 3. A BVAR f is **independent** if for any $\mathcal{L}$-formula λ and any pair of profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$
$\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}'_i \models \lambda\} \Rightarrow \|\lambda\|_f^{\mathfrak{A}} = \|\lambda\|_f^{\mathfrak{A}'}$.

Definition 4. A BVAR f is **neutral** if for any $\mathcal{L}$-formulas λ, λ' and any profile $\underline{\mathfrak{A}} \in \Omega^I$
$\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}_i \models \lambda'\} \Rightarrow \|\lambda\|_f^{\mathfrak{A}} = \|\lambda'\|_f^{\mathfrak{A}}$.

Definition 5. A BVAR f is **systematic** if it is independent and neutral, i.e. if for any $\mathcal{L}$-formulas $\lambda, \lambda'$and any profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$
$\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}'_i \models \lambda'\} \Rightarrow \|\lambda\|_f^{\mathfrak{A}} = \|\lambda'\|_f^{\mathfrak{A}'}$.

The property of systematicity might appear strong at first look but it is well-known in the literature on judgment aggregation that it is implied by the independence property and a condition of logical richness known as total blockedness, i.e. if any formula is related to any other one by a sequence of conditional entailments.

The framework of BVARs allows to use the partial order structure $\langle P(I), \subseteq\rangle$ of the powerset algebra $P(I)$ over the set of individuals (the "coalition algebra") resp. of the algebra of truth values $\langle \mathbf{B}, \leqslant\rangle$ for the formulation of conditions on aggregation rules. In particular, the monotonicity property can be formulated in a natural way as such an order preservation property:

Definition 6. A BVAR f is **monotonic** if for any $\mathcal{L}$-formula λ and any pair of profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$
$\{i \in I : \mathfrak{A}_i \models \lambda\} \subsetneq \{i \in I : \mathfrak{A}'_i \models \lambda\} \Rightarrow \|\lambda\|_f^{\mathfrak{A}} \leqslant \|\lambda\|_f^{\mathfrak{A}'}$.

Monotonicity is known to be an important property of aggregation rules because it guarantees non-manipulability, i.e. the impossibility for any individual to increase the collectively assigned truth value of a formula by signalling its negation.

The conjunction of monotonicity and independence (known in the judgment aggregation literature as monotone independence, see Nehring and Puppe 2010) can now be formulated as an order preservation property of the aggregation rule with respect to the partial orders of the coalition algebra and the algebra of truth values.

Proposition 1. *A BVAR f satisfies* ***monotone independence*** *(i.e. is monotonic and independent) if and only if for all profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$ and any formula $\lambda \in \mathcal{T}$*

$$\{i \in I : \mathfrak{A}_i \models \lambda\} \subseteq \{i \in I : \mathfrak{A}'_i \models \lambda\} \Rightarrow \|\lambda\|_f^{\mathfrak{A}} \leqslant \|\lambda\|_f^{\mathfrak{A}'} . \tag{1}$$

A natural BVAR F can now be defined by assigning to any $\mathcal{L}$-formula λ and any profile $\underline{\mathfrak{A}} \in \Omega^I$ precisely the subset of individuals in whose models it holds true, i.e. $\|\lambda\|_F^{\mathfrak{A}} = \{i \in I : \mathfrak{A}_i \models \lambda\}$. Thus, the algebra of truth-values is simply identified with the coalition algebra.

This construction immediately leads to the following possibility result:

Theorem 7. *The BVAR F is a neutral, paretian and non-dictatorial judgment aggregation rule which satisfies monotone independence.*

The main interest of this simple boolean-valued construction consists in highlighting the implications for the aggregation problem of the structure of the set of truth values and the significance of the condition of order preservation with respect to the powerset algebra over the set of individuals and the algebra of truth values (for a deeper exploration of the relation between judgment aggregation rules and boolean algebra homomorphisms see Herzberg 2010).

This significance is closely related to a property of homomorphisms of boolean algebras. Note that systematicity (i.e. the conjunction of independence and neutrality) permits a decomposition of any BVAR as $h \circ F$. One can show that this h is a homomorphism and thus order-preserving, whence neutrality and independence already entail monotonicity.

THEOREM 8. *A neutral BVAR which satisfies (monotone) independence induces a homomorphism h_f of the coalition algebra $P(I) = \langle 2^I, \cup, \cap, \complement, \varnothing, I\rangle$ to its co-domain, the boolean algebra of truth values $\mathbf{B} = \langle B, \sqcup, \sqcap, ^*, 0_B, 1_B\rangle$*

Now there is a connection between the homomorphy among boolean algebras and the source of dictatorship, viz. the existence of an ultrafilter on the set of individuals: For, any ultrafilter on a finite set is the collection of all supersets of a singleton – the dictator –, and 2-valued homomorphisms have an ultrafilter as its shell (see e.g. Bell and Slomson 1969):

LEMMA 9. *Let $g : A \to B$ be a homomorphism between boolean algebras. Then the shell of g, i.e. the set $\{x \in A : g(x) = 1_B\}$ is a filter. If B is the two-valued algebra $\mathbf{2} = \{0, 1\}$ of truth values, then the shell $g^{-1}\{1_B\}$ of g is an ultrafilter.*

With the help of such a purely algebraic result, we obtain in the BVAR framework a typical Arrow-style dictatorship result, as a simple corollary of the previous theorem:

COROLLARY 10. *Let f be a neutral BVAR which satisfies (monotone) independence and has co-domain $\mathbf{2} = \{0, 1\}$. If the set I of individuals is finite, then f is a dictatorship.*

We have thus described a framework for boolean-valued judgment aggregation. While the major body of the literature on judgment aggregation draws attention to inconsistencies between properties of the agenda and properties of the aggregation rule, the simple (im)possibility results in this paper highlight the role of the set of truth values and its algebraic structure. In particular, it is shown that central properties of aggregation rules can be formulated as homomorphy or order-preservation conditions on the mapping between the powerset algebra over the set of individuals and the algebra of truth values. This is further evidence that the problems in aggregation theory are driven by information loss, which in our framework is given by a coarsening of the algebra of truth values.

The mathematical description of this information loss in Theorem 8 can also be formulated as an algebraic factorization result. Let $\vdash$ be the provability relation of classical first-order logic, let $T \subseteq \mathcal{L}$ be consistent (possibly empty), and let $\equiv$ denote provable equivalence given T (i.e., $\phi \equiv \psi$ if and only if both $T \cup \{\phi\} \vdash \psi$ and $T \cup \{\psi\} \vdash \phi$). As is well-known, this is an equivalence relation, and the set of equivalence classes $\mathcal{L}/\equiv$ forms a boolean algebra (with representative-wise negation, conjunction and disjunction as complement, meet and join, respectively), the Lindenbaum algebra of T. It is obvious that for any BVAR f, the map

$$H_f : \mathcal{L}/\equiv \times \Omega^I \to B, \quad \langle [\lambda]_\equiv, \underline{\mathfrak{A}}\rangle \mapsto f(\lambda, \underline{\mathfrak{A}})$$

is well-defined. It is also clear that for any $\underline{\mathfrak{A}} \in \Omega^I$, $H_f(\cdot, \underline{\mathfrak{A}})$ is a homomorphism, due to the definition of the boolean operations on the Lindenbaum algebra and the definition of a boolean-valued map. Given any profile $\underline{\mathfrak{A}} \in \Omega^I$, we have the following commutative diagram of boolean homomorphisms:

$$\begin{tikzcd} \mathcal{L}/\equiv \arrow[r, "H_{F,\underline{\mathfrak{A}}}"] \arrow[d, "H_{f,\underline{\mathfrak{A}}}"'] & P(I) \arrow[dl, "h"] \\ B & \end{tikzcd}$$

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► JOSEP MARIA FONT AND TOMMASO MORASCHINI, *On logics of varieties and logics of semilattices.*
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§1. Introduction. A basic problem in algebraic logic is *how to associate a class of algebras with a given logic in a meaningful way.* Abstract algebraic logic provides three such general procedures:

- The class of the ***Leibniz-reduced algebras*** of $\mathcal{L}$, denoted by $\mathsf{Alg}^*\mathcal{L}$, which is the class of algebraic reducts of the reduced matrix models of $\mathcal{L}$.
- The ***algebraic counterpart*** of $\mathcal{L}$, denoted by $\mathsf{Alg}\mathcal{L}$, which is the class of its Tarski-reduced algebras; i.e., the algebraic reducts of the reduced generalized matrix models of $\mathcal{L}$. That this class really deserves this name is argued in [1, 2].
- The ***intrinsic variety*** of $\mathcal{L}$, denoted by $\mathbb{V}\mathcal{L}$, which is the variety generated by the Lindenbaum-Tarski algebra of $\mathcal{L}$; it corresponds to the equations $\alpha \approx \beta$ such that for all formulas $\varphi(x, \vec{z})$, $\varphi(\alpha, \vec{z}) \dashv\vdash_{\mathcal{L}} \varphi(\beta, \vec{z})$.

In general the classes $\mathsf{Alg}^*\mathcal{L}$ and $\mathsf{Alg}\mathcal{L}$ need not form a variety. There are logics for which the three classes are different, although for a large class of logics (for instance, for all the protoalgebraic ones, but also for others) it is true that $\mathsf{Alg}^*\mathcal{L} = \mathsf{Alg}\mathcal{L}$, and for others (for instance, for all finitary selfextensional logics with either conjunction or a uniterm deduction-detachment theorem) $\mathsf{Alg}\mathcal{L} = \mathbb{V}\mathcal{L}$.

Having these general procedures at hand, the dual problem of *how to associate a logic with a given class of algebras* can be precisely formulated. In particular, given a variety V (of an arbitrary language), we are lead to the following three questions:

Question 1: Is there a logic $\mathcal{L}$ such that $\mathsf{Alg}^*\mathcal{L} = \mathsf{V}$?

Question 2: Is there a logic $\mathcal{L}$ such that $\mathsf{Alg}\mathcal{L} = \mathsf{V}$?

Question 3: Is there a logic $\mathcal{L}$ such that $\mathbb{V}\mathcal{L} = \mathsf{V}$?

In this contribution we address them in general, in the framework of abstract algebraic logic, and for the particular case of the variety of semilattices. Some of the results reported on here will appear in [3, 4].

§2. The questions in general. It is well-known that $\mathsf{Alg}^*\mathcal{L} \subseteq \mathsf{Alg}\mathcal{L} \subseteq \mathbb{V}\mathcal{L}$ and that $\mathbb{V}(\mathsf{Alg}^*\mathcal{L}) = \mathbb{V}(\mathsf{Alg}\mathcal{L}) = \mathbb{V}\mathcal{L}$, where for any class K of algebras, $\mathbb{V}(\mathsf{K})$ is the variety generated by K. Therefore, any positive solution to Question 1 solves also Questions 2 and 3, and any positive solution to Question 2 solves Question 3. The variety of commutative semigroups shows that Question 1 cannot be answered in general in the positive. We show that Question 2 (and hence Question 3) can be so, by introducing an (up to now) unusual logic.

Definition 1. Let V be a variety. $\mathcal{L}_{\mathsf{V}}$ is the logic defined by the class of matrices $\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{V}, F \subseteq A\big\}$.

Note that if the variety V is trivial, then the logic $\mathcal{L}_{\mathsf{V}}$ is the almost inconsistent logic.

The main general results on the logic $\mathcal{L}_{\mathsf{V}}$ are gathered in the following statement.

Theorem 2. *Let V be a non-trivial variety.*

1. *The logic $\mathcal{L}_{\mathsf{V}}$ is fully selfextensional, filter-distributive, unitary and has no theorems.*
2. *The logic $\mathcal{L}_{\mathsf{V}}$ is neither conjunctive nor disjunctive.*
3. *If $\boldsymbol{A} \in \mathsf{V}$ is subdirectly irreducible, then $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{L}_{\mathsf{V}}$.*
4. $\mathbb{V}\mathcal{L}_{\mathsf{V}} = \mathsf{Alg}\mathcal{L}_{\mathsf{V}} = \mathsf{V}$. *That is, $\mathcal{L}_{\mathsf{V}}$ is a solution to Questions 2 and 3.*
5. *If Question 1 has a solution, then $\mathcal{L}_{\mathsf{V}}$ is one, and the weakest.*
6. *The logic $\mathcal{L}_{\mathsf{V}}$ is neither protoalgebraic nor truth-equational.*
7. *If $\mathbb{V}\mathcal{L} = \mathsf{V}$, then $\mathcal{L}_{\mathsf{V}} \leqslant \mathcal{L}$. That is, $\mathcal{L}_{\mathsf{V}}$ is the weakest logic solving Question 3.*

In particular, this provides a handful of (relatively) natural examples of logics that lie outside the Leibniz hierarchy. The converse of point 7 of Theorem 2 does not hold in general (the variety of Heyting algebras provides an easy counterexample); however, in the case of the variety of semilattices it does hold (Theorem 6).

In point 4 of Theorem 2 we see that Questions 2 and 3 always have a positive answer. Together with the fact that in general it is $\mathsf{Alg}\mathcal{L}$ that deserves the title of "algebraic counterpart" of the logic $\mathcal{L}$, this suggests the following definition: Given a variety V, a logic $\mathcal{L}$ is ***a logic of*** V when $\mathsf{Alg}\mathcal{L} = \mathsf{V}$. The logics of a variety V form a poset when ordered with respect to deductive strength, which we denote as follows.

$$\boldsymbol{\mathcal{L}og}(\mathsf{V}) := \big\langle \{\mathcal{L} : \mathcal{L} \text{ is a logic of } \mathsf{V}\}, \leqslant \big\rangle.$$

This poset is closed under meets of arbitrary non-empty families, and its minimum is $\mathcal{L}_{\mathsf{V}}$. In case V is the variety $\mathbb{V}\big(\{\boldsymbol{A}\}\big)$ generated by a single algebra $\boldsymbol{A}$, we will write $\boldsymbol{\mathcal{L}og}(\boldsymbol{A})$ instead of $\boldsymbol{\mathcal{L}og}\big(\mathbb{V}\big(\{\boldsymbol{A}\}\big)\big)$.

One of the few known general properties of the poset $\boldsymbol{\mathcal{L}og}(\mathsf{V})$ for an arbitrary variety V, is that if an algebraizable logic (or, more generally, a logic whose truth predicate is universally definable with parameters in the class of its reduced models [5]) belongs to it, then it is maximal in it. However, something more can be said when stronger conditions on V are assumed; for instance, for varieties generated by a *primal algebra*.

At first sight, the most natural and general way of constructing examples of logics of $\mathbb{V}(\boldsymbol{A})$ seems to be to consider logics determined by a *generalized matrix* $\langle \boldsymbol{A}, \mathcal{C} \rangle$, whose algebraic reduct is $\boldsymbol{A}$ itself. Let $\boldsymbol{\mathcal{C}}(A)$ be the poset of all non-trivial (i.e., different from $\{A\}$ and $\{\emptyset, A\}$) closure systems over A, ordered under set-theoretic inclusion. We show that each of these closure systems defines a logic of $\mathbb{V}(\boldsymbol{A})$, and that different closure systems define different logics.

Theorem 3. *Let $\boldsymbol{A}$ be a non-trivial primal algebra. The map $\mathcal{C} \longmapsto \vdash_{\langle \boldsymbol{A}, \mathcal{C} \rangle}$ that associates with each closure system $\mathcal{C} \in \boldsymbol{\mathcal{C}}(A)$, the logic determined by the generalized matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$, is a well-defined order reversing embedding of $\boldsymbol{\mathcal{C}}(A)$ into $\boldsymbol{\mathcal{L}og}(\boldsymbol{A})$. Moreover, the logic $\vdash_{\langle \boldsymbol{A}, \mathcal{C} \rangle}$ is finitely equivalential if and only if $\emptyset \notin \mathcal{C}$.*

From this it follows that, given a non-trival primal algebra $\boldsymbol{A}$, there are at least as many logics of $\mathbb{V}(\boldsymbol{A})$ as non-trivial closure systems over A.

The logics defined in this way by ordinary matrices behave even better.

Theorem 4. *Let $\boldsymbol{A}$ be a non-trivial primal algebra and $\mathcal{L}$ a logic. The following conditions are equivalent.*

(i) *$\mathcal{L} \in \boldsymbol{\mathcal{L}og}(\boldsymbol{A})$ and is maximal in this poset.*
(ii) *$\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$.*
(iii) *$\mathcal{L}$ is the logic determined by $\langle \boldsymbol{A}, F \rangle$, for some $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.*

If a logic $\mathcal{L}$ is algebraizable, then $\mathsf{Alg}^{*}\mathcal{L} = \mathsf{Alg}\mathcal{L}$ and this class coincides with the equivalent algebraic semantics for $\mathcal{L}$. Therefore, we conclude that in the case of varieties generated by a primal algebra, Question 1 can be answered in the positive too, and in a non-unique way; actually, we can determine the number of algebraizable logics having the given variety as equivalent algebraic semantics.

Corollary 5. *Let $\boldsymbol{A}$ be a non-trivial primal algebra. There are exactly $|\mathcal{P}(A)| - 2$ algebraizable logics whose equivalent algebraic semantics is $\mathbb{V}(\boldsymbol{A})$.*

§3. The logics of semilattices. Another way of gaining more information is to restrict the analysis to the logics associated with a concrete variety. As an example of this approach, we focus on logics associated with ***the variety of semilattices***, denoted as SL; this variety is of particular interest, as it constitutes the ordered skeleton of a large number of algebras coming from the field of non-classical logic. Thus, we work in the simple language of semilattices $\langle\cdot\rangle$, of type $\langle 2\rangle$. For some constructions it is useful to regard semilattices as meet-semilattices, in the sense that, given $\boldsymbol{A} \in \mathsf{SL}$ and $a, b \in A$, we consider the order relation given by $a \leqslant b$ if and only if $a = a \cdot b$.

The main general results obtained can be summarized as follows.

THEOREM 6. *Let $\mathcal{L}$ be non-trivial. The following conditions are equivalent.*

(i) *$\mathcal{L}$ is a logic of semilattices; i.e.,* $\mathsf{Alg}\mathcal{L} = \mathsf{SL}$.

(ii) $\mathbb{V}\mathcal{L} = \mathsf{SL}$.

(iii) $\mathcal{L}_{\mathsf{SL}} \leqslant \mathcal{L}$.

Moreover, if $\mathcal{L}$ is a logic of semilattices, then $\mathcal{L}$ has no theorems, it is neither protoalgebraic nor truth-equational, and it is selfextensional.

From Theorem 2 it follows that the unanswered Question 1, in the case of semilattices, can be equivalently formulated as follows.

Question 4: Is it true that $\mathsf{SL} \subseteq \mathsf{Alg}^{*}\mathcal{L}_{\mathsf{SL}}$? That is, is it true that for every semilattice $\boldsymbol{A}$ there is some $F \subseteq A$ such that $\Omega^{\boldsymbol{A}}F = \mathrm{Id}_A$?

Even if this problem remains OPEN in general, we have been able to identify a large class of semilattices belonging to $\mathsf{Alg}^{*}\mathcal{L}_{\mathsf{SL}}$, namely the so-called *semilattices with sectionally finite height* (semilattices all whose elements have finite height); in particular this includes all finite semilattices and all semilattices with finite height. Moreover, in [3] we have shown that in each such semilattice the Leibniz operator establishes a bijection between a certain family of subsets (called *clouds*) and the set of all congruences of the semilattice (weakening what happens for algebraizable logics, where the Leibniz operator is not just a bijection, but an order isomorphism).

Finally we focus on the poset of logics of semilattices $\boldsymbol{\mathcal{L}og}(\mathsf{SL})$. Two logics in this set deserve special attention: the logics $\mathcal{CPC}_\wedge$ and $\mathcal{CPC}_\vee$, which are respectively the $\{\wedge\}$-fragment and the $\{\vee\}$-fragment of classical propositional logic. These two logics have a particular location in the poset $\boldsymbol{\mathcal{L}og}(\mathsf{SL})$.

THEOREM 7.

1. *$\mathcal{CPC}_\wedge$ and $\mathcal{CPC}_\vee$ belong to $\boldsymbol{\mathcal{L}og}(\mathsf{SL})$, and are its only maximal elements.*
2. *If $\mathcal{L} \in \boldsymbol{\mathcal{L}og}(\mathsf{SL})$ and $\mathcal{L} \nleq \mathcal{CPC}_\wedge$, then $\mathcal{L} = \mathcal{CPC}_\vee$.*
3. *If $\mathcal{L} \in \boldsymbol{\mathcal{L}og}(\mathsf{SL})$ and $\mathcal{CPC}_\wedge \cap \mathcal{CPC}_\vee < \mathcal{L}$, then either $\mathcal{L} = \mathcal{CPC}_\wedge$ or $\mathcal{L} = \mathcal{CPC}_\vee$.*

Moreover, the poset $\boldsymbol{\mathcal{L}og}(\mathsf{SL})$ is atomless. In order to see this we consider other logics in the family; for this, we consider an infinite sequence of different variables $\langle x_n : n \in \mathbb{N}\rangle$, and for each $n \in \mathbb{N}$ we define a set of formulas and a logic:

$$W(n) := \big\{\varphi \in Fm : Var(\varphi) \subsetneq \{x_0, \dots, x_n\} \text{ or } Var(\varphi) = \{x_0, \dots, x_{n+1}\}\big\}$$

$$\mathcal{W}_n := \mathcal{L}_{\mathsf{SL}} + \big[W(n) \vdash x_0 \cdot (x_1 \cdot \dots \cdot (x_{n-1} \cdot x_n) \dots)\big]$$

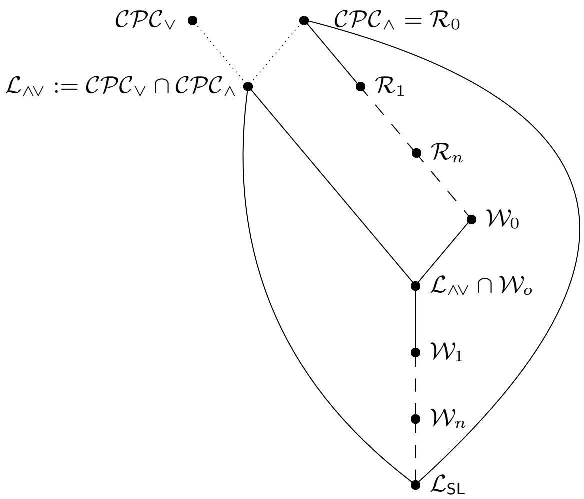
Since the logics $\mathcal{W}_n$ lie between $\mathcal{L}_{\mathsf{SL}}$ and $\mathcal{CPC}_\wedge$, from Theorem 6 it follows that they belong to $\boldsymbol{\mathcal{L}og}(\mathsf{SL})$. Moreover the set of premises $W(n)$ of the additional axiom of $\mathcal{W}_n$ can be finitised by selecting its formulas in which no variable occurs more than once. In particular, $\mathcal{W}_0$ is the weakest logic extending $\mathcal{L}_{\mathsf{SL}}$ in which the deduction $x_0 \cdot x_1 \vdash x_0$ holds; this implies that $\mathcal{W}_0$ is strictly weaker than $\mathcal{CPC}_\wedge$, but not weaker than $\mathcal{CPC}_\vee$.

THEOREM 8.

1. *If $n < m$, then $\mathcal{W}_m < \mathcal{W}_n$.*
2. *If $\mathcal{L} \in \boldsymbol{\mathcal{L}og}(\mathsf{SL})$ and $\mathcal{L}_{\mathsf{SL}} < \mathcal{L}$, then $\mathcal{W}_n < \mathcal{L}$ for some n.*
3. *$\boldsymbol{\mathcal{L}og}(\mathsf{SL})$ is atomless.*

Theorem 7 implies that each logic of semilattices that is strictly weaker than $\mathcal{CPC}_\vee$ is strictly weaker than $\mathcal{CPC}_\wedge$ too. We show that the behaviour of $\mathcal{CPC}_\vee$ and $\mathcal{CPC}_\wedge$ is not symmetrical, by constructing several logics of semilattices strictly weaker than $\mathcal{CPC}_\wedge$ which are not weaker than $\mathcal{CPC}_\vee$. A first example of such logics is $\mathcal{W}_0$; therefore, each logic of semilattices that extends $\mathcal{W}_0$ will be weaker than $\mathcal{CPC}_\wedge$ and not weaker than $\mathcal{CPC}_\vee$. We define an infinite descending chain $\langle \mathcal{R}_n : n \in \mathbb{N} \rangle$ of logics (of semilattices) between $\mathcal{CPC}_\wedge$ and $\mathcal{W}_0$.

Below you can see *a partial picture* of the poset of logics of semilattices $\boldsymbol{\mathcal{L}og}(\mathsf{SL})$. The dotted lines indicate that there is no logic of semilattices strictly between their edges, while dashed and solid lines admit the presence of other logics. The dashed lines indicate the location of the two infinitely descending families of logics mentioned before.



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▸ NIKOLAOS GALATOS AND ROSTISLAV HORČÍK, *Densification via polynomial extensions.*

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The uninorm logic UL is the semilinear extension of full Lambek calculus with exchange FL_e, i.e., it is a logic complete with respect to the class of all FL_e-chains (see [3]). Moreover, the logic UL is known to be standard complete, i.e., it is complete with respect to the class of all FL_e-chains whose universe is the real unit interval $[0,1]$. Nevertheless, there is no algebraic proof of the above fact. The only proofs we have so far are based on a proof-theoretical elimination of the density rule [3, 1].

Interestingly, the proof-theoretical idea from [1] can be translated via residuated frames [4] into an algebraic construction showing that UL is standard complete. This is possible since the residuated frames (introduced in [2] as a relational semantics for substructural logics) are tightly connected with the Gentzen sequent calculus. Inspired by the construction via residuated frames we present a proof that is completely algebraic and resembles standard constructions of ring extensions from classical algebra. We present our proof by viewing FL_e-chains as idempotent semirings and semimodules and by passing to their polynomial extensions.

The crucial step in order to achieve the above-mentioned result is to show that every countable FL_e-chain is embedabble into a dense one. Recall that FL_e-chains are just commutative residuated chains endowed with an extra constant 0 and possibly with a lower and an upper bound. Since these constants are not important for the construction itself, we will work here only with the signature of residuated lattices. Namely, we will show below that given any commutative residuated chain $\mathbf{A}$ that is not dense with a gap $g < h$ (i.e., g, h are elements of the algebra and no element of the algebra is between them) can be embedded into a commutative residuated chain $\bar{\mathbf{A}}$ in which $g < h$ is not a gap.

Having established the above, and assuming we have a countable commutative residuated chain $\mathbf{A}$, we consider all countable commutative residuated chains that have $\mathbf{A}$ as a subalgebra and we order them by the subalgebra relation. By an easy application of Zorn's Lemma to this poset we obtain the existence of a maximal algebra $\mathbf{B}$, which has to be dense by maximality (otherwise, if it has a gap, an isomorphic copy of $\bar{\mathbf{B}}$ has $\mathbf{B}$ as a proper subalgebra).

As $\mathbf{B}$ is countable and dense its order reduct is isomorphic to the rationals. It is well known that the Dedekind-MacNeille completion of $\mathbf{B}$ is also a commutative residuated chain and also that its order reduct is isomorphic to the interval $[0,1]$.

§1. Residuated lattices as semirings and semimodules. Let $\mathbf{A}$ be a commutative residuated lattice, namely a structure $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a commutative monoid and
- the residuation condition holds: $x \cdot y \leq z \Leftrightarrow y \leq x \to z$, for all $x, y, z \in A$.

For every commutative residuated lattice $\mathbf{A}$ we have that $a(b \vee c) = ab \vee ac$, for all $a, b, c \in A$, which together with the fact that $(A, \cdot, 1)$ is a commutative monoid and $(A, \vee)$ is an commutative idempotent semigroup show that the $(\cdot, \vee, 1)$ reduct of $\mathbf{A}$ is an idempotent semiring. (If A has a bottom element B, then we also have that $(A, \vee, B)$ is a monoid and $a \cdot B = B$, for all $a \in A$.)

Moreover, we have the conditions

- $1 \to x = x$,
- $(ab) \to x = a \to (b \to x)$,
- $a \to (x \wedge y) = (a \to x) \wedge (a \to y)$ and
- $(a \vee b) \to x = (a \to x) \wedge (b \to x)$

Which together with the fact that $(A, \wedge)$ is a commutative semigroup show that $(A, \wedge)$ is a semimodule (whose elements we think of as *vectors*) over the semiring $(A, \vee, 1)$ (whose elements we think of as *scalars*) where the action (or *scalar multiplication*) is $\to$, the implication of $\mathbf{A}$. (If $\mathbf{A}$ has a top element T, then we also have that $(A, \wedge, T)$ is a monoid.)

For elements $\bot, \top \notin A$, it is well known that there is a unique commutative residuated chain $\mathbf{A}_\bot$ on the set $A_\bot = A \cup \{\bot, \top\}$ specified by the following conditions

- $\mathbf{A}$ is a subalgebra of $\mathbf{A}_\bot$,
- $\bot < a < \top$ for all $a \in A$,
- $a \cdot \top = \top = \top \cdot a$, for all $a \in A \cup \{\top\}$,
- $b \cdot \bot = \bot = \bot \cdot b$, for all $b \in A \cup \{\bot, \top\}$
- $c \to \top = \top$, for all $c \in A \cup \{\bot, \top\}$
- $\bot \to \bot = \top$ and $b \to \bot = \bot$, for $b \in A \cup \{\top\}$.

§2. Residuated lattice of polynomials. Thinking of $\mathbf{A}_\bot$ as a semiring, we denote by $\mathbf{A}_\bot[X]$ the set of all polynomials with coefficients from $\mathbf{A}_\bot$ over the indeterminate X. We use the notation $p = p_0 \vee p_1 X \vee p_2 X^2 \vee \ldots \vee p_n X^n = \bigvee_{i=0}^n p_i X^i$ for polynomials of degree at most n. As usual, every polynomial of degree at most n can be considered as a polynomial of degree at most m, for $m \geq n$, by appending terms of the form $\bot X^i$.

Clearly $\mathbf{A}_\bot[X]$ is a semiring with operations given by

$$\left(\bigvee_{i=0}^n p_i X^i\right) \vee \left(\bigvee_{i=0}^n q_i X^i\right) = \bigvee_{i=0}^n (p_i \vee q_i) X^i$$

$$\left(\bigvee_{i=0}^n p_i X^i\right) \cdot \left(\bigvee_{i=j}^m q_j X^j\right) = \bigvee_{i=0}^{n+m} \left(\bigvee_{i+j=k} p_i q_j\right) X^k$$

However, since $\mathbf{A}_\bot$ is a residuated lattice, we can further define

$$\left(\bigvee_{i=0}^n p_i X^i\right) \wedge \left(\bigvee_{i=0}^n q_i X^i\right) = \bigvee_{i=0}^n (p_i \wedge q_i) X^i$$

and we also define

$$\left(\bigvee_{i=0}^n p_i X^i\right) \to \left(\bigvee_{j=0}^m q_i X^i\right) = \bigvee_{k=0}^m \left(\bigwedge_{i=0}^n (p_i \to q_{i+k})\right) X^k$$

Lemma 1. *$\mathbf{A}_\bot[X]$ forms a commutative residuated lattice under the above operations.*

We note that for $X \to p = \bigvee_{i=0}^{n-1} p_{i+1} X^i$, which we call the *shift* or *derivative* of p and we denote it by p'.

§3. Linear polynomials. Let $\mathbf{A}(X)$ be the set of all linear polynomials. Given an element $h \in A$, we define a semiring quotient on $\mathbf{A}_\bot[X]$ by the semiring congruence generated by the relation $X^2 = hX$. This congruence is the kernel of the map $\phi_h : \mathbf{A}_\perp[X] \to \mathbf{A}(X)$ given by $\phi_h(p) = p_0 \vee p'(h)X$. This map induces operations on $\mathbf{A}(X)$ so that the map becomes a homomorphism: $p \vee_h q = \phi_h(p \vee q)$, $p \cdot_h q = \phi_h(pq)$, $\phi_h(1)$. More explicitly, for linear polynomials $p = p_0 \vee p_1 X$ and $q = q_0 \vee q_1 X$ we define their join as usual and their product as:

$$(p_0 \vee p_1 X) \cdot_h (q_0 \vee q_1 X) = p_0 q_0 \vee (p_0 q_1 \vee p_1 q_0 \vee p_1 q_1 h)X.$$

Thus, under these operations $\mathbf{A}(X)$ becomes a semiring, being the homomorphic image of $\mathbf{A}_\perp[X]$ under ϕ_h. We further define meet on $\mathbf{A}(X)$ as in $\mathbf{A}_\perp[X]$ and implication $\to_h$ by

$$(p_0 \vee p_1 X) \to_h (q_0 \vee q_1 X) = ((p_0 \to q_0) \wedge (p_1 \to q_1)) \vee ((p_0 \to q_1) \wedge (p_1 h \to q_1))X.$$

Lemma 2. *$\mathbf{A}(X)$ forms a commutative residuated lattice under the above operations.*

§4. The plan. Now we assume that $\mathbf{A}$ is a *commutative residuated chain*, namely a totally ordered commutative residuated lattice. (If $\mathbf{A}$ is bounded it is a UL-algebra.) We also assume that there are elements $g, h \in A$ such that $g < h$ and there is no element of A between g and h, namely $g < h$ forms a *gap*. Our goal is construct a new commutative residuated chain $\bar{\mathbf{A}}$ into which $\mathbf{A}$ will embed, say under an embedding that we call $^\triangleleft : \mathbf{A} \to \bar{\mathbf{A}}$, such that $\bar{\mathbf{A}}$ will contain at least one more element between $g^\triangleleft$ and $h^\triangleleft$.

This can be alternatively rephrased as saying that we actually want to embed into $\bar{\mathbf{A}}$ the partial algebra $\mathbf{A} \cup \{X\}$, namely $^\triangleleft : \mathbf{A} \cup \{X\} \to \bar{\mathbf{A}}$, where X is a new element such that $g < X < h$, in order to have $g^\triangleleft < X^\triangleleft < h^\triangleleft$ in $\bar{\mathbf{A}}$. (In viewing $\mathbf{A} \cup \{X\}$ as a partial algebra we consider the set $A \cup \{X\}$ under the partially defined operations that extend the operations on $\mathbf{A}$ with $a \wedge X = a$ and $a \vee X = X$, for $a \leq g$, and with $a \wedge X = X$ and $a \vee X = a$ for $a \leq h$.)

It will turn out that $\bar{A}$ will be a subset of $\mathbf{A}(X)$ so we actually need an injective function $^\triangleleft : \mathbf{A} \cup \{X\} \to \mathbf{A}(X)$, whose image $[\mathbf{A} \cup \{X\}]^\triangleleft$ will be contained in a chain $\bar{A}$ on which we can define a structure $\bar{\mathbf{A}}$ of a commutative residuated chain, even if $\bar{\mathbf{A}}$ might not be a subalgebra of $\mathbf{A}(X)$.

Since $\mathbf{A}(X)$ is a residuated lattice, its $\wedge$-reduct $\mathbf{A}(X)^\wedge$ is a module over its $(\vee, \cdot, 1)$ reduct $\mathbf{A}(X)^\vee$. From general properties of residuated lattices, it turns out that every $\mathbf{A}(X)^\vee$-submodule of $\mathbf{A}(X)^\wedge$ which forms a closure system is itself a residuated lattice, so we just need to be able to view $\bar{\mathbf{A}}$ as a $\mathbf{A}(X)^\vee$-submodule of $\mathbf{A}(X)^\wedge$.

We will achieve all these by defining a suitable totally ordered $\mathbf{A}(X)^\vee$-module $\mathbf{A}_{\perp,X}$ that contains $A \cup \{X\}$ and define a module morphism $^\triangleleft : \mathbf{A}_{\perp,X} \to \mathbf{A}(X)^\wedge$ that is injective on $A \cup \{X\}$. Then the image $\bar{\mathbf{A}} = [\mathbf{A}_{\perp,X}]^\triangleleft$ will be a residuated lattice (being a $\mathbf{A}(X)^\vee$-submodule of $\mathbf{A}(X)^\wedge$), it will be totally ordered (since it will be the image of the totally ordered module $\mathbf{A}_{\perp,X}$), while $\mathbf{A}$ will embed in it and it will not have $g^\triangleleft < h^\triangleleft$ as a gap (since $^\triangleleft$ will be injective on $A \cup \{X\}$ and provided we show that the image is a closure system such that the closure of a is $a^\triangleleft$, for all $a \in A_\perp \cup \{X\}$).

§5. The module. To motivate the definition of $\mathbf{A}_{\perp,X}$ we note that we want it to contain the set $A_\perp \cup X$, to be totally ordered and also to admit a module action from $\mathbf{A}(X)^\vee$, which we denote by $\mathbin{\backslash\!\backslash}$. Note that for every module action from $\mathbf{A}(X)^\vee$ we will have

$$(p_0 \vee p_1 X) \mathbin{\backslash\!\backslash} b = p_0 \mathbin{\backslash\!\backslash} b \wedge (p_1 X) \mathbin{\backslash\!\backslash} b = p_0 \mathbin{\backslash\!\backslash} b \wedge p_1 \mathbin{\backslash\!\backslash} (X \mathbin{\backslash\!\backslash} b).$$

So it is enough to define the action by elements of $A_\perp$ and by X.

We define the action from $A(X)$ and $A_\perp \cup \{X\}$ by extending the action of $A_\perp$ on itself, since the latter it is a common subset of both and also a residuated lattice (namely it acts on itself by $\to$); for $a, b \in A_\perp$ we simply define $a \bbslash b = a \to b$. For the elements $X \bbslash a$ we note that $b \leq X \bbslash a$ iff $bX \leq a$ iff $X \leq b \to a$ iff $h \leq b \to a$ iff $b \leq h \to a$. In an attempt of being economical and define $A_{\perp,X}$ to be as small as possible, we explore the possibility of defining $X \bbslash a = h \to a$, namely it does not produce a new element of $A_{\perp,X}$ but an old element of $A_\perp$. We also have that $b \leq a \bbslash X$ iff $ba \leq X$ iff $ba \leq g$ iff $b \leq a \to g$. Unlike the above case, we cannot hope to define $a \bbslash X = a \to g$, as for example for $a = 1$ this would collapse X with g, but also because it turns out that this will not yield an action. This means that the elements of the form $a \bbslash X$ are new elements of $A_{\perp,X}$; formally we could define them as pairs (a, X), but we use the more suggestive notation $a \bbslash X$. So we actually define $A_{\perp,X} := A \cup (A \bbslash X)$, and note that it contains $A \cup \{X\}$. This means that with our method it is impossible to define a commutative residuated chain structure on just the set $A \cup \{X\}$, but we need to add new elements that will materialize the implications $X \to a$. Finally, we define $X \bbslash X = 1$ with no reason other than the fact that we want $1 \leq X \to X$ in the resulting chain (which follow by residuation from $1 \cdot X \leq X$). In summary we have motivated the definition

- $b \bbslash a := b \to a$,
- $X \bbslash a = h \to a$
- $b \bbslash (a \bbslash X) = (ab) \bbslash X$
- $X \bbslash X = 1$

which uniquely extends to the action of $A(X)$ to $A_{\perp,X} = A \cup A \bbslash X$ given by:

$$(p_0 \vee p_1 X) \bbslash a = (p_0 \to a) \wedge (p_1 h \to a) \qquad (p_0 \vee p_1 X) \bbslash (a \bbslash X) = (p_0 a \bbslash X) \wedge (p_1 a \to 1)$$

The final step is to define the order structure on $A_{\perp,X}$, which is also used in the computation of the above meets. This order extends the order on $A_\perp$ and sets $a \bbslash X \leq b \bbslash X$ iff $b \leq a$. Finally we set $a \leq b \bbslash X$ if $ab \leq g$, and $a > b \bbslash X$ if $ab > g$. We observe that for this ordering the set of positive elements (elements greater or equal to 1) of $A_{\perp,X}$ is $(A_{\perp,X})^+ = A^+ \cup (\downarrow g) \bbslash X$, where A^+ is the set of positive elements of A.

Concluding we define ${}^\lhd : \mathbf{A}_{\perp,X} \to \mathbf{A}(X)^\wedge$ by setting $z^\lhd$ as the maximum of the set $\{p \in \mathbf{A}(X) : 1 \leq p \bbslash z\}$.

Theorem 3. *Assume that $\mathbf{A}$ is a commutative residuated chain with a gap $g < h$.*

- *The map ${}^\lhd : \mathbf{A}_{\perp,X} \to \mathbf{A}(X)^\wedge$ is a $\mathbf{A}(X)^\vee$-module morphism that is injective on $A \cup \{X\}$.*
- *For all $a \in A$, we have $a^\lhd = a \vee (h \to a)X$ and $(a \bbslash X)^\lhd = (a \to g) \vee (a \to 1)X$.*
- *The image $\bar{\mathbf{A}} = [\mathbf{A}_{\perp,X}]^\lhd$ forms a closure system and the least closed element above $p \in A(X)$ is $p(h)^\lhd \wedge ((p_0 \to g) \wedge (p_1 \to 1) \bbslash X)^\lhd$.*
- *$\bar{\mathbf{A}}$ is a commutative residuated chain and $g^\lhd < X^\lhd < h^\lhd$.*
- *The restriction of ${}^\lhd$ to $\mathbf{A}$ is a residuated lattice embedding, which preserves any existing bounds of $\mathbf{A}$.*

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▸ JOAN GISPERT, *Quasivarieties of MV-algebras and structurally complete Łukasiewicz logics*.
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This paper is a contribution to the study of the lattice of all quasivarieties of MV-algebras.

An *MV-algebra* is an algebra $\langle A; \oplus, \neg, 0\rangle$ satisfying the following equations:

MV1 $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$
MV2 $x \oplus y \approx y \oplus x$
MV3 $x \oplus 0 \approx x$
MV4 $\neg(\neg x) \approx x$
MV5 $x \oplus \neg 0 \approx \neg 0$
MV6 $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

A *lattice-ordered abelian group* (for short, ℓ-group) is an algebra $\langle G, \wedge, \vee, +, -, 0\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, +, -, 0\rangle$ is an abelian group and satisfies the following equation:

$$(x \vee y) + z \approx (x + z) \vee (y + z)$$

For any ℓ-group G and element $0 < u \in G$, let $\Gamma(G, u) = \langle [0, u]; \oplus, \neg, 0\rangle$ be defined by

$$[0, u] = \{a \in G \,|\, 0 \leq a \leq u\}, \quad a \oplus b = u \wedge (a + b), \quad \neg a = u - a.$$

Then, $\langle [0, u]; \oplus, \neg, 0\rangle$ is an MV-algebra. Further, for any ℓ-groups G and H with elements $0 < u \in G$ and $0 < v \in H$, and any ℓ-group homomorphism $f : G \to H$ such that $f(u) = v$, let $\Gamma(f)$ be the restriction of f to $[0, u]$. An element $0 < u \in G$ is called a *strong unit* iff for each $x \in G$ there is an integer $n \geq 1$ such that $x \leq nu$. Then, as proved in [13], (see also [5]) Γ is a categorical equivalence from the category of ℓ-groups with strong unit, with ℓ-homomorphisms that preserve strong units, onto the category of MV-algebras with MV-homomorphisms. Moreover the functor Γ preserves embeddings and epimorphisms.

The following MV-algebras play an important role in the paper.

- $[0, 1] = \Gamma(\mathbb{R}, 1)$, where $\mathbb{R}$ is the totally ordered group of the reals.
- $[0, 1] \cap \mathbb{Q} = \Gamma(\mathbb{Q}, 1) = \langle\{\dfrac{k}{m} : k \leq m < \omega\}; \oplus, \neg, 0\rangle$, where $\mathbb{Q}$ is the totally ordered abelian group of the rationals.

For every $0 < n < \omega$

- $\text{Ł}_n = \Gamma(\mathbb{Z}, n) = \langle\{0, 1, \dots, n\}; \oplus, \neg, 0\rangle$, where $\mathbb{Z}$ is the totally ordered group of all integers. Notice that Ł_n is isomorphic to the subalgebra of $[0, 1]$ given by $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$.
- $\text{Ł}_n^\omega = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, 0)) = \langle\{(k, i) : (0, 0) \leq (k, i) \leq (n, 0)\}; \oplus, \neg, 0\rangle$, where $\times_{lex}$ denotes the lexicographic product.
- $\text{Ł}_n^s = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, s)) = \langle\{(k, i) : (0, 0) \leq (k, i) \leq (n, s)\}; \oplus, \neg, 0\rangle$, where $s \in \mathbb{Z}$ such that $0 \leq s < n$. Notice that $\text{Ł}_n^\omega = \text{Ł}_n^0$.
- $\mathrm{S_n} = \Gamma(T, n)$ where T is the totally ordered dense subgroup of $\mathbb{R}$ generated by $\sqrt{2} \in \mathbb{R}$ and $1 \in \mathbb{R}$. Notice that $T \cap \mathbb{Q} = \mathbb{Z}$.

Since the class of all MV-algebras is definable by a set of equations, it is a variety that we denote by $\mathbf{MV}$. By Chang's Completeness Theorem [4] (see also [5]), $\mathbf{MV}$ is the variety generated by the MV-algebra $[0, 1]$ (or $[0, 1] \cap \mathbb{Q}$), in symbols,

$$\mathbf{MV} = \mathcal{V}([0, 1]) = \mathcal{V}([0, 1] \cap \mathbb{Q}).$$

Proper subvarieties of **MV** are well known. Komori proves in [12](see also [5]) that *two MV-chains generate the same variety if and only if they have same order and same rank.* Using this result he gives the following characterization of all proper subvarieties of **MV**.

THEOREM 1. [12, Theorem 4.11] **V** *is a proper subvariety of MV if and only if there exist two disjoint finite subsets I, J of positive integers, not both empty such that*

$$\mathbf{V} = \mathcal{V}(\{\mathrm{L}_i : i \in I\} \cup \{\mathrm{L}_j^\omega : j \in J\}). \qquad \square$$

A pair (I, J) of finite subsets of positive integers, not both empty is said to be *reduced* iff for every $n \in I$, there is no $k \in (I \setminus \{n\}) \cup J$ such that $k|n$ and for every $m \in J$, there is no $k \in J \setminus \{m\}$ such that $k|m$. In [14] the authors show that there is a 1-1 correspondence between proper subvarieties of **MV** and reduced pairs of finite subsets of positive integers not both empty. Given a reduced pair (I, J), we denote by $\mathcal{V}_{I,J}$ its associated subvariety.

The class of all quasivarieties of MV-algebras is much larger than the class of all varieties. Some special quasivarieties of MV-algebras have been studied [6, 10, 7, 8, 1]. For instance quasivarieties generated by chains. The class of quasivarieties generated by chains contains the class of all varieties and moreover it is a bounded distributive sublattice of the lattice of all quasivarieties of **MV**. An analogous characterization as in the case of varieties is accomplished in [7, Theorem 4.4] where the author proves that *two MV-chains generate the same quasivariety if and only if they have the same order, the same rank, and both contain the same rational elements.* Using this result he gives the following characterization of all quasivarieties generated by MV-chains.

THEOREM 2. **K** *is a quasivariety generated by MV-chains if and only if there are Δ, Γ, Λ subsets of positive integers, not all of them empty, and for every $i \in \Gamma$, a nonempty subset $\gamma(i) \subseteq \mathrm{Div}(i)$ such that*

$$\mathbf{K} = \mathcal{Q}(\{\mathrm{L}_n : n \in \Delta\} \cup \{\mathrm{L}_i^{d_i} : i \in \Gamma \quad d_i \in \gamma(i)\} \cup \{\mathrm{S}_k : k \in \Lambda\}). \qquad \square$$

From the above characterization it follows, that if S is a simple infinite MV-algebra such that $S \cap \mathbb{Q} = \{0, 1\}$, then $\mathcal{Q}(S) = \mathcal{Q}(\mathrm{S}_1)$. Moreover

$\mathcal{Q}(\mathrm{S}_1)$ *is the least MV-quasivariety generated by chains.*

Moreover for every reduced pair (I, J),

$\mathcal{Q}(\{\mathrm{L_i} : i \in I\} \cup \{\mathrm{L_j^1} : j \in J\})$ *is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.*

However not all quasivarieties are generated by chains. Let **V** a variety of any type of algebras, a quasivariety **K** of same type is a **V**-*quasivariety* provided that $\mathcal{V}(\mathbf{K}) = \mathbf{V}$. The purpose of this paper is to study for every variety **V** of MV-algebras the least **V**-quasivariety. The existence is assured by the following general result of Universal Algebra (see for instance [2, 11]).

THEOREM 3. *Let* **V** *a variety of algebras (not necessarily MV-algebras) and let $F_{\mathbf{V}}(X)$ denote the free* **V**-*algebra over the set X of free generators. If X is infinite then, $\mathcal{Q}(F_{\mathbf{V}}(X))$ is the least* **V**-*quasivariety.* $\square$

In the case of MV-algebras, since any subvariety of MV-algebras can be distinguished by an axiom in just one variable [14], the quasivariety generated by the free **V**-algebra over a one free generator is also the least **V**-quasivariety.

COROLLARY 4. *If* $\mathbf{V}$ *is a variety of MV-algebras, then* $\mathcal{Q}(F_{\mathbf{V}}(\{x\}))$ *is the least* $\mathbf{V}$-*quasivariety.* $\square$

We recall that for the case of $\mathbf{V} = \mathbf{MV}$, $F_{\mathbf{MV}}(\{x\})$ is the MV-algebra $\mathcal{M}([0,1])$ of all McNaughton functions in one variable equipped with pointwise MV-operations [5].

Although the previous corollary already characterizes all least $\mathbf{V}$-quasivarieties, we would like to have a nicer or simpler characterization, similar, if possible, to characterizations of subvarieties of $\mathbf{MV}$. Our purpose is to obtain for each subvariety $\mathcal{V}_{I,J}$ a simpler algebra (or a finite set of simpler algebras) than $F_{\mathcal{V}_{I,J}}(\{x\})$, whose generated quasivariety is the least $\mathcal{V}_{I,J}$-quasivariety. We accomplish this by the following result.

THEOREM 5. *Let* $\mathcal{V}_{I,J}$ *be a proper subvariety of* $\mathbf{MV}$*, where* (I,J) *is a reduced pair. Then* $\mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_i : i \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_j^1 : j \in J\})$ *is the least* $\mathcal{V}_{I,J}$-*quasivariety.* $\square$

COROLLARY 6. $\mathcal{V}_{\emptyset,\{1\}}$ *is the least* $\mathcal{V}_{\emptyset,\{1\}}$-*quasivariety.* $\square$

In the case of proper subvarieties we have that for every reduced pair (I,J),
$\mathcal{V}_{I,J} = \mathcal{Q}(\{\mathrm{L}_i : i \in I\} \cup \{\mathrm{L}_j^\omega : j \in J\}$;
$\mathcal{Q}(\{\mathrm{L}_i : i \in I\} \cup \{\mathrm{L}_j^1 : j \in J\}$ is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains;
$\mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_i : i \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_j^1 : j \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety.

Analogously since $\mathbf{MV} = \mathcal{Q}(\mathrm{S})$ for every infinite simple chain S such that $S \cap \mathbb{Q} = [0,1] \cap \mathbb{Q}$ and $\mathcal{Q}(\mathrm{S}_1)$ is the least $\mathbf{MV}$-quasivariety generated by chains, then we may expect $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{S}_1)$ to be the least $\mathbf{MV}$-quasivariety. However this does not hold.

THEOREM 7.
$$\mathcal{Q}(\mathcal{M}([0,1])) \subsetneqq \mathcal{Q}(\mathrm{L}_1 \times \mathrm{S}_1) \subsetneqq \mathcal{Q}(\mathrm{S}_1).$$
Moreover there exists an infinite chain $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n, \dots$ *of MV-quasivarieties such that*
$$\mathcal{Q}(\mathcal{M}([0,1])) \subsetneqq \mathbf{K}_1 \subsetneqq \mathbf{K}_2 \subsetneqq \cdots \mathbf{K}_n \subsetneqq \cdots \subsetneqq \mathcal{Q}(\mathrm{L}_1 \times \mathrm{S}_1). \qquad \square$$

Moreover we give a description of the poset of least $\mathbf{V}$-quasivarieries, Since varieties of MV-algebras are obviously in 1-1 correspondence with all least $\mathbf{V}$-quasivarieties, the reader may expect that both share the same structure of the poset ordered by the inclusion. However, while the poset of the varieties of MV-algebras is a bounded distributive lattice, as shown by the following results, the poset of all least $\mathbf{V}$-quasivarieties is neither a join-, nor a meet-semilattice. these results allow us to find all minimal quasivarieties in the class of all quasivarieties of $\mathbf{MV}$ different from the class of all boolean algebras.

Finally, we apply those results to obtain some results on admissibility theory of finitary extensions of the infinite valued Łukasiewicz calculus.

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§1. Introduction. During the last decade a connection emerged between proof theory and algebra via which cut-elimination, one of the cornerstones of structural proof theory, can be proved by using completions, in particular, the MacNeille completion. This technique, introduced in [1, 4] and further extended and applied in [2], is developed in these papers for a wide range of logics, including substructural ones. The completion used in these papers is far from trivial, and our modest aim in this paper is to establish what the technique of completions boils down to for "strong" logics such as full intuitionistic propositional logic IPC, and what the connection is with other semantical proofs of cut-elimination.
For example, Takeuti in [6] proves the completeness of intuitionistic predicate logic with respect to Kripke models in a way that also provides a proof of cut-elimination. Namely, the method constructs, for a given formula A, a reduction tree based on a cut-free Gentzen calculus for the logic, such that either the tree is a proof of A or one can build a Kripke countermodel from it. This is called the *Schütte method* after Schütte who used a similar technique for type systems in [5]. We wish to compare the Schütte method for cut-elimination to the one introduced in [1, 4]. Also, in these papers on algebraic completions, an important role is played by semantical structures that are closely related to sequent calculi. Here we try to make this relation more precise.
None of the theorems or proofs presented here are deep or original. But we feel that the interesting connection between algebraic completions and cut-elimination could be further explored, and we try to take a step in that direction. We are working on the subject right now and submit this incomplete abstract so that we can hopefully present genuine results during the meeting.

§2. Schütte's method. For the following multi-conclusion Gentzen calculus for IPC we develop, in the style of Schütte and Takeuti [5, 6], a way to obtain countermodels

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for underivable sequents from the calculus. In our setting, a *sequent* is a pair of finite sets written as $\Gamma \Rightarrow \Delta$.

$$\Gamma, p \Rightarrow p, \Delta \;\; \mathrm{A}x \qquad\qquad \Gamma, \bot \Rightarrow \Delta \;\; \mathrm{L}\bot$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \mathrm{L}\wedge \qquad\qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ \mathrm{R}\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \mathrm{L}\vee \qquad\qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \mathrm{R}\vee$$

$$\frac{\Gamma, A \to B \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ \mathrm{L}\to \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}\ \mathrm{R}\to$$

When we say that *from below a rule cannot be applied* to a sequent S, we mean that an application of the rule with conclusion S contains at least one premise equal to S. For example, L$\vee$ can be applied from below to $A \vee B \Rightarrow \Delta$ but not to $A, A \vee B \Rightarrow \Delta$. Given a sequent S a *tableau for* S is a tree labelled with sequents that ends in S (the root has label S), that satisfies the above rules except possibly at the leafs, is *strict*, which means that from below applications of R $\to$ are allowed only when no other rules can be applied, and is *full*, which means that there are no rules except possibly R $\to$ of which one of the leafs is a conclusion. A tableau is *closed* if its leafs are axioms. It is *open* otherwise. It is straightforward to prove the following two lemmas.

Lemma 1. *If a tableau for S is closed, then S holds in* IPC.

Lemma 2. *If all tableaux for S are open, then there is a Kripke countermodel to S based on the tableaux for S.*

Thus in this way one can obtain a Kripke model from the calculus. As every finite Kripke model corresponds to a finite, and thus complete, Heyting algebra, this provides a connection between completions and the calculus as well. What exactly the relation is between these two ways of obtaining completions still needs to be explored.

§3. Gentzen structures. In the papers on algebraic completions discussed above, the completions are based on structures that are closely related to sequent calculi. Here we study such structures that correspond to the single-conclusion variant GC of the cut-free Gentzen calculus above. Thus the rules of GC are obtained from the calculus above by requiring that succedents consist of at most one formula and replacing R$\vee$ by

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

For any set A, we denote by $A^\star$ the free meet-semilattice over A. Recall that $A^\star$ can be realized concretely as the collection of finite subsets of A, and that the mapping $a \mapsto \{a\}$ allows us to view A as a subset of $A^\star$. The binary operation of $A^\star$ will be denoted by the comma symbol and the neutral element by ϵ. By definition, the operation $(\_,\_)$ is associative, commutative and idempotent.
If $G \subseteq X \times Y$ is a relation between two sets X and Y, we use the notation $x \Rightarrow_G y$ to express that the pair (x, y) is in the set G.
By the word *algebra* we always mean: algebra in the signature $(0, \wedge, \vee, \to)$ (not required to satisfy any axioms). Thus, an algebra is simply a tuple $(A, 0, \wedge, \vee, \to)$, where A is a set, $0 \in A$, and $\wedge, \vee, \to$ are binary operations on A. Recall that the *n-generated term algebra*, $T(n)$, consists of the set of $(0, \wedge, \vee, \to)$-terms in the variables $p_1, \ldots, p_n$, equipped with the obvious syntactic operations. We will fix an n throughout and write T for $T(n)$. The elements of $T^\star \times T$ are called *sequents*.

Definition 3. A *Gentzen structure* (or *semi-Heyting structure* [4]) on an algebra $(A, 0, \wedge, \vee, \rightarrow)$ is a relation $G \subseteq A^\star \times A$ which satisfies the following conditions, for any $a, b, c \in A$ and $x \in A^\star$.

1 $x, a \Rightarrow_G a$
2 $x, 0 \Rightarrow_G c$
3 if $x, a, b \Rightarrow_G c$, then $x, a \wedge b \Rightarrow_G c$
4 if $x \Rightarrow_G a$ and $x \Rightarrow_G b$ then $x \Rightarrow_G a \wedge b$
5 if $x, a \Rightarrow_G c$ and $x, b \Rightarrow_G c$ then $x, a \vee b \Rightarrow_G c$
6 if $x \Rightarrow_G a$ or $x \Rightarrow_G b$ then $x \Rightarrow_G a \vee b$
7 if $x \Rightarrow_G a$ and $x, b \Rightarrow_G c$ then $x, a \rightarrow b \Rightarrow_G c$
8 if $x, a \Rightarrow_G b$ then $x \Rightarrow_G a \rightarrow b$.

Lemma 4. *Let A be an algebra. There exists a minimum Gentzen structure on A.*

Proof. Note that, if $(G_i)_{i \in I}$ is a collection of Gentzen structures on an algebra A, then the intersection $\bigcap_{i \in I} G_i$ is again a Gentzen structure. Now take the intersection of all Gentzen structures on A. $\dashv$

We denote the minimum Gentzen structure on A by $G_0(A)$, or, if no confusion can arise, simply by G_0.

Lemma 5. *The minimum Gentzen structure G_0 on the term algebra T consists exactly of the sequents which are provable in* GC.

Proof. Let $G_\vdash \subseteq T^\star \times T$ denote the set of sequents which are provable in GC.
Note that $G_\vdash$ is a Gentzen structure: if one applies any of the rules (1)–(8) in Definition 3 to sequents provable in GC, then one obtains again a sequent which is provable in GC. Therefore, $G_0 \subseteq G_\vdash$, since G_0 is the minimum Gentzen structure.
Conversely, by an induction on the length of proofs one easily shows that, if G' is any Gentzen structure on T, then any sequent provable in GC must be in G'. In particular, $G_\vdash \subseteq G_0$. $\dashv$

This lemma is of course not very surprising: it merely makes the idea explicit that, in the absence of cut, the minimum Gentzen structure on the term algebra plays a role similar to that of the free algebra (which consists of terms modulo provable equivalence in a system with cut). The above definitions can be used to study the admissibility of certain rules, including cut, in an "algebraic" way (where, for now, we leave it a bit vague what we mean by "algebraic").

§4. Questions. We close with some concrete questions concerning the topic discussed above. We have described two ways in which one can obtain algebraic structures that refute a certain underivable sequent. With the *Schütte algebra method* we mean the method where the algebra is the Heyting algebra corresponding to a Kripke counter model obtained via the Schütte method for the underivable sequent, as described in Section 2.3. With the *BJO method* we mean the method in [1] to obtain a Heyting algebra from a Gentzen structure that refutes the sequent (in the case of [1] it is not a Heyting algebra but a weaker algebraic structure, as the logic considered there is substructural).

- What is the relation between the Heyting algebra obtained via the Schütte algebra method and the algebra obtained via the BJO method?
- The Schütte algebra method is easily extendable to predicate logic. What about the BJO method?
- Can the BJO method be adapted to multi-conclusion sequent calculi?

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The admissible rules of a logic are exactly those rules under which its set of theorems is closed. Let us focus on propositional logics, in particular on the consistent extensions of propositional intuitionistic logic (IPC), commonly known as intermediate logics. There is a natural way to interpret a rule as a formula, simply by seeing it as the implication from the conjunction of its assumptions to its conclusion. In classical propositional logic all admissible rules are theorems under this interpretation; in other words, all admissible rules are derivable. For many other intermediate logics the situation is much more interesting.

Consider for instance the weakest intermediate logic, that is to say, IPC itself. Harrop (1960) showed that the rule

$$\neg r \to (p \lor q) / (\neg r \to p) \lor (\neg r \to q)$$

is admissible yet not derivable. Prucnal (1979) later proved that this rule is, in fact, admissible for all intermediate logics. The set of all admissible rules of IPC must contain the set of all derivable rules, hence it could never be finite. Rybakov proved that, although this set is decidable [5], it can not be finitely axiomatised [6].

The above should convince the reader that the notion of admissibility can be quite intricate, even for well-behaved logics such as IPC. In this talk we will consider a class of intermediate logics, known as the subframe logics, and explore some of the structure of their admissible rules. We will show a particular scheme of admissible rules to be admissible in all subframe logics. Using this scheme, we provide a complete description of the admissible rules of the intermediate logic BD_2. Let us spend the rest of this abstract on putting these two results in their proper context.

When one allows a rule to have multiple conclusions, the familiar disjunction property can be formulated as the admissibility of the rule below.

$$p \vee q \big/ \{p, q\}$$

To be precise, a multi-conclusion rule Γ/Δ is said to be *admissible* whenever, for all substitutions σ, we have that if $\vdash \sigma(\phi)$ holds for all $\phi \in \Gamma$ then $\vdash \sigma(\phi)$ for some $\phi \in \Delta$. Spelling out the definitions, to say that the above rule is admissible in a logic is to say that if $\phi \vee \psi$ is derivable, then at least one among ϕ or ψ is derivable, too. This, indeed, means precisely that the logic at hand enjoys the disjunction property. Intermediate logics with the disjunction property abound (there are uncountably many), and IPC is known to be amongst them due to Gödel (1932).

Particular schemes of admissible rules, akin to Harrop's rule, have been studied throughout the years. The following rule scheme describes the so-called Visser rules, variants of which arose independently in the work of Citkin, Skura, de Jongh, Rozière and Visser.

$$\text{(1)} \qquad \bigwedge_{i=1}^{n} (r_i \to p_i) \to r_{n+1} \vee r_{n+2} \Bigg/ \left\{ \bigwedge_{i=1}^{n} (r_i \to p_i) \to r_j \;\middle|\; j = 1, \dots, n+2 \right\}$$

Paraphrasing a result obtained independently by Iemhoff (2001) and Rozière (1992), all admissible rules of IPC can be derived from this particular set of rules. Skura [7] and Iemhoff [4], again independently, showed that IPC is the sole intermediate logic which admits a particular variant of all the above rules. Note that the disjunction property is a direct consequence of the above rule when taking $n = 0$.

Some intermediate logics can be axiomatised by a finite set of formulae from the conjunction-implication fragment of IPC. Zakharyaschev [10] showed that these are precisely the subframe logics, that is, the logics for which any subframe of a frame of the logic is a frame of the logic. An algebraic description of this class was later given by Bezhanishvili and Ghilardi [2]. Due to their most convenient axiomatisation, subframe logics enjoy the finite model property.

We will present some machinery for describing admissibility of certain rules in logics with the finite model property. This machinery depends on two main ingredients. First, we use the universal or characterising model, developed by, among others, Bellissima [1] and Šhekhtman [9]. This model contains copies of all finite models of the logic at hand on a particular set of variables, and as such is complete with respect to all formulae built using those variables. Secondly, we employ the technique developed by Jankov (1963) and de Jongh (1968) to describe finite Kripke models of IPC by means of propositional formulae. This allows us to, intuitively speaking, describe the order in this model through the validity of propositional formulae. Combining these, we can prove the following.

THEOREM 1. *Let L be a logic with the finite model property. The rule below is admissible precisely if for every rooted finite model K of L and each subset $W \subseteq K$ such that W is a model of L there is an extension of W which is again a model of L.*

$$\text{(2)} \qquad \left(\bigvee_{i=1}^{n} r_i \to p\right) \to \bigvee_{j=1}^{n} r_j \Bigg/ \bigvee \left\{ \left(\bigvee_{i=1}^{n} r_i \to p\right) \to r_j \;\middle|\; j = 1, \dots, n \right\}$$

COROLLARY 2. *The rules of (2) are admissible in every subframe logic.*

For the remainder of this abstract, we will focus on the intermediate logic BD_2. This is one of the earliest intermediate logics of interest, introduced by Jankov (1963). Throughout the years it has known several guises, in Hasoi (1967) for instance it appeared as the weakest logic of the second slice. More recently, Chagrov and Zakharyaschev (1997) called this logic BD_2, the intermediate logic of Kripke frames of height at most two. This logic is one of the seven intermediate logics with interpolation, and it is one of the three pre-tabular intermediate logics. On top of all of this, it is, quite clearly, a subframe logic. We can formally define it as

$$\mathsf{BD}_2 := \mathsf{IPC} + q \vee \big(q \to (p \vee \neg p)\big).$$

By virtue of BD_2 being a subframe logic we know that it admits the rules of (2). Moreover, as BD_2 is not equal to IPC it can not possibly admit the rules of (1). Skura [8] observed that a modification of the rule (1), however, is admissible. In this talk we indicate how one can prove the following theorem. This work is based on the paper [3].

THEOREM 3. *All multi-conclusion admissible rules of* BD_2 *follow from the following rules for* $n = 2, 3, \dots$.

$$\left(\bigvee_{i=1}^{n} r_i \to p\right) \to \bigvee_{j=1}^{n} r_j \Big/ \left\{ \neg\neg \left(\bigvee_{i=1}^{n} r_i \to p\right) \to r_j \;\middle|\; j = 1, \dots , n \right\}$$

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Results and considerations presented in this contribution pertain to satisfiability of terms in the class of $\mathrm{FL}_{\mathrm{ew}}$-algebras—the equivalent algebraic semantics of the logic $\mathrm{FL}_{\mathrm{ew}}$, i.e., Full Lambek calculus with extension and weakening (see, e.g., [1, 7]). This algebraic framework is quite general, yet still it preserves many intuitions about satisfiability and provides a clear connection to satisfiablity of terms in classical logic.

The logic $\mathrm{FL}_{\mathrm{ew}}$ can be extended to many well-known non-classical logics, such as the intuitionistic and superintuitionistic logics or Łukasiewicz logic and its extensions; all of these logics can be considered in the language of $\mathrm{FL}_{\mathrm{ew}}$, consisting of $\cdot$ (multiplication),

$\to$ (residuum of $\cdot$), $\wedge$ and $\vee$ (lattice operations), 0 (bottom) and 1 (top); terms are defined in the usual way. A $\mathrm{FL_{ew}}$-algebra carries a bounded lattice order, where the bottom and the top element, endowed with restrictions of the operations considered, form the two-element Boolean algebra $\{0,1\}_{\mathrm{B}}$.

Previous work on many-valued satisfiability includes NP-completeness of satisfiability in the standard MV-algebra ([6]) or NP-completeness of satisfiability in some standard algebras pertaining to Hájek's basic logic and its extensions ([2, 3]). It is also well known that satisfiability in a non-trivial Heyting algebra coincides with the classical satisfiability.

For a $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$, one defines

$$\mathrm{SAT} = \{\varphi \mid \exists e(e(\varphi) = 1)\} \tag{1}$$

$$\mathrm{SATPOS} = \{\varphi \mid \exists e(e(\varphi) > 0)\} \tag{2}$$

where the range of e is the set of evaluations in $\mathcal{A}$. One will notice that SAT and SATPOS are considered as *operators* on the class of $\mathrm{FL_{ew}}$-algebras: given an algebra, each of the two operators yields a particular set of terms. This definition extends to classes of algebras: $\mathrm{SAT}(\mathbb{K}) = \cup_{\mathcal{A}\in\mathbb{K}}\mathrm{SAT}(\mathcal{A})$, and analogously for SATPOS.

If $\mathcal{A}$ is non-trivial, then

$$\mathrm{SAT}(\{0,1\}_{\mathrm{B}}) \subseteq \mathrm{SAT}(\mathcal{A}) \subseteq \mathrm{SATPOS}(\mathcal{A}). \tag{3}$$

It is not obvious whether any of the inclusions in (3) are strict. Our first question about satisfiability can be rendered as follows: for a given $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$, are there any *classically unsatisfiable* terms that are (positively) satisfiable in $\mathcal{A}$?

One can come up with examples of algebras where the answer to this question is "no" (as already mentioned, a non-trivial Heyting algebra can be taken as an example). Within the class of $\mathrm{FL_{ew}}$ algebras, we offer the following characterization, using a subvariety of the variety of $\mathrm{FL_{ew}}$-algebras that is known under the name *weakly contractive algebras* and denoted WCon (see [7]) and which, within the variety of $\mathrm{FL_{ew}}$-algebras, is delimited by the identity $x \wedge \neg x = 0$. In particular, weakly contractive algebras subsume Heyting algebras.

THEOREM 1. *Let $\mathcal{A}$ be a non-trivial $\mathrm{FL_{ew}}$-algebra. The following are equivalent:*
(1) $\mathcal{A}$ is a WCon-*algebra*
(2) $\mathrm{SATPOS}(\mathcal{A}) = \mathrm{SAT}(\{0,1\}_{\mathrm{B}})$
(3) $\mathrm{SATPOS}(\mathcal{A}) = \mathrm{SAT}(\mathcal{A})$

To demonstrate this, first consider that if $\mathcal{A}$ is not a WCon-algebra, i.e., $x \wedge \neg x = 0$ does not hold in $\mathcal{A}$, there is an $a \in \mathcal{A}$ such that $a \wedge \neg a > 0$; hence the term $x \wedge \neg x$ is in $\mathrm{SATPOS}(\mathcal{A})$. At the same time, $x \wedge \neg x$ is never in $\mathrm{SAT}(\mathcal{A})$ for a non-trivial $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$. Thus either of (2) or (3) implies (1). To show (1) implies (2) and (3), it suffices to show that the logic WCon is Glivenko equivalent to classical logic (this is implicit in [1]). The above implies that, within the lattice of logics extending $\mathrm{FL_{ew}}$, WCon is the weakest logic with this property.

Next, we discuss decidability and computational complexity of some satisfiability problems. It follows from [4] that for any non-empty class of non-trivial $\mathrm{FL_{ew}}$-algebras, tautologousness is coNP-hard; one can give a similar argument for satisfiability being NP-hard. It however has to be said that for many (classes of) $\mathrm{FL_{ew}}$-algebras it is not known whether satisfiability is decidable. It has been shown over the timespan of the last few decades that, for any non-empty class $\mathbb{K}$ of algebras given by continuous t-norms, both $\mathrm{SAT}(\mathbb{K})$ and $\mathrm{SATPOS}(\mathbb{K})$ are NP-complete ([6, 2, 3]). NP-completeness has also been proved for satisfiability in the standard NM- and WNM-algebras [5].

On the other hand, one can present a simple cardinality argument, showing that a majority of SAT problems are undecidable, even if one restricts one's attention to rational subalgebras of the standard MV-algebra $[0,1]_{Ł}$. Let $[0,1]_{Ł} \cap \mathrm{Q}$ be the subalgebra of the standard MV-algebra $[0,1]_{Ł}$ on all of its rationals.

THEOREM 2. *There are uncountably many subalgebras of* $[0,1]_{Ł} \cap \mathrm{Q}$. *If* $\mathcal{S}$, $\mathcal{S}'$ *are two such subalgebras, then* $\mathrm{SAT}(\mathcal{S}) = \mathrm{SAT}(\mathcal{S}')$ *iff* $\mathcal{S} = \mathcal{S}'$.

Next, we investigate the difference $\mathrm{SATPOS}(\mathcal{A}) \setminus \mathrm{SAT}(\mathcal{A})$ for a $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$ where this set is non-empty; by Theorem 1, the class of such algebras is exactly the complement of WCon-algebras within the class of $\mathrm{FL_{ew}}$-algebras.

Assume for the moment that both $\mathrm{SAT}(\mathcal{A})$ and $\mathrm{SATPOS}(\mathcal{A})$ are NP-sets; then it follows from its definition that the set $\mathrm{SATPOS}(\mathcal{A}) \setminus \mathrm{SAT}(\mathcal{A})$ is a Δ_2 set within the polynomial hierarchy. Still, one can give a tighter classification. A decision problem P is in the class DP if $P = P_1 \setminus P_2$ for some decision problems $P_1, P_2 \in \mathrm{NP}$. This class has been shown to have complete problems under polynomial-time reducibility (see [8]).

THEOREM 3. *Let* $\mathcal{A}$ *be a non-trivial* $\mathrm{FL_{ew}}$*-algebra. Assume* $\mathrm{SATPOS}(\mathcal{A}) \setminus \mathrm{SAT}(\mathcal{A})$ *is a non-empty set; then it is* DP*-hard.*

If, moreover, both the satisfiability and the positive satisfiability problems for $\mathcal{A}$ are in NP, then the set is DP-complete.

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▸ RAMON JANSANA, *Congruential deductive systems associated with equationally orderable varieties.*
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Every deductive system (or logic) $\mathcal{S}$ has an associated canonical class of algebras[9], denoted $\mathbf{Alg}\mathcal{S}$, and deductive systems are classified in abstract algebraic logic according

[9] The definitions of the concepts used, but not defined, in this abstract can be found in [4].

to the relations they have with the algebras in $\mathbf{Alg}\mathcal{S}$. A deductive system $\mathcal{S}$ has the *congruence property* if the interderivability relation $\dashv_{\mathcal{S}}\vdash$ is a congruence of the algebra of formulas. Two formulas φ and ψ are related by $\dashv_{\mathcal{S}}\vdash$ if and only if they belong to the same theories of $\mathcal{S}$. When this property lifts to every algebra, then $\mathcal{S}$ is said to be *congruential*[10], that is, when for every algebra $\mathbf{A}$ in the language of $\mathcal{S}$ the binary relation $\Lambda^{\mathbf{A}}_{\mathcal{S}}$ on A defined by $\langle a,b\rangle \in \Lambda^{\mathbf{A}}_{\mathcal{S}}$ if and only if a,b belong to the same $\mathcal{S}$-filters[11], is a congruence of $\mathbf{A}$. This is known to be equivalent to saying that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, $\Lambda^{\mathbf{A}}_{\mathcal{S}}$ is the identity relation.

The results in next theorem were first proved in [3] and discussed and proved with different methods in [5].

THEOREM 1. *Every finitary deductive system $\mathcal{S}$ with the congruence property and the property of conjunction*[12] *is congruential and its canonical class of algebras $\mathbf{Alg}\mathcal{S}$ is a variety.*

The proof of the theorem given in [5] consist of three steps, that can be summarized as follows. First it is shown that if $\mathcal{S}$ is a finitary deductive system with the congruence property and the property of conjunction witnessed by a binary term $x \wedge y$, then every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ has an equationally definable order $\leq^{\mathbf{A}}_{\wedge}$, defined by the equation $x \wedge y \approx x$, and the deductive system $\mathcal{S}$ satisfies then that $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if

$$(1) \quad \forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}\ \forall v \in \mathrm{Hom}(\mathbf{Fm},\mathbf{A})\ \forall a \in A((\forall \psi \in \Gamma,\ a \leq^{\mathbf{A}}_{\wedge} v(\psi)) \Longrightarrow a \leq^{\mathbf{A}}_{\wedge} v(\varphi)),$$

for every set of formulas Γ and every formula φ. Secondly it is proved that $\mathbf{Alg}\mathcal{S}$ is a variety, and finally that $\mathcal{S}$ is congruential.

We show that the third step holds under more general assumptions. Condition (1) can be used to associate a finitary deductive system with every equationally orderable quasivariety (by a finite set of equations). Let L be an algebraic language, $\mu(x,y)$ a finite set of L-equations in two variables and Q a quasivariety of L-algebras. We say that Q is *μ-equationally orderable*, cf. [1], if for every algebra $\mathbf{A} \in \mathsf{Q}$ the relation defined on A by the set of equations μ, that is,

$$\leq^{\mathbf{A}}_{\mu} := \{\langle a,b\rangle \in A^2 : \mathbf{A} \models \mu(x,y)[a,b]\},$$

is a partial order of A.

Let Q be a μ-equationally orderable quasivariety of L-algebras. The deductive system $\mathcal{S}^{\leq_\mu}_{\mathsf{Q}}$ is then defined as follows:

$$\Gamma \vdash_{\mathcal{S}^{\leq_\mu}_{\mathsf{Q}}} \varphi \ \text{ iff } \ \forall \mathbf{A} \in \mathsf{Q}\ \forall v \in \mathrm{Hom}(\mathbf{Fm},\mathbf{A})\ \forall a \in A((\forall \psi \in \Gamma)\ a \leq^{\mathbf{A}}_{\mu} v(\psi)) \Longrightarrow a \leq^{\mathbf{A}}_{\mu} v(\varphi)).$$

We refer to $\mathcal{S}^{\leq_\mu}_{\mathsf{Q}}$ as *the deductive system of the μ-order* for Q.

The following facts about $\mathcal{S}^{\leq_\mu}_{\mathsf{Q}}$ are not difficult to prove.

- The deductive system $\mathcal{S}^{\leq_\mu}_{\mathsf{Q}}$ is finitary. This follows from the fact that Q is a quasivariety and μ is a finite set.
- The deductive system $\mathcal{S}^{\leq_\mu}_{\mathsf{Q}}$ has the congruence property.

In general the relation between Q and $\mathbf{Alg}\mathcal{S}^{\leq}_{\mathsf{Q}}$ is the following one.

[10] In [3] these deductive systems are called strongly selfextensional and in [4] fully selfextensional.

[11] A set $F \subseteq A$, is an $\mathcal{S}$-filter of $\mathbf{A}$ if for every set of formulas Γ, every formula φ and every homomorphism v from the algebra of formulas to $\mathbf{A}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $v[\Gamma] \subseteq F$, then $v(\varphi) \in F$.

[12] A deductive system $\mathcal{S}$ has the *property of conjunction* if there is a binary term $x \wedge y$ (not necessarily primitive) such that for all formulas φ, ψ, δ,

$$\varphi, \psi \vdash_{\mathcal{S}} \delta \quad \text{iff} \quad \varphi \wedge \psi \vdash_{\mathcal{S}} \delta.$$

Proposition 1. *Let* Q *be a* μ*-equationally orderable quasivariety. Then*

$$\mathsf{Q} \subseteq \mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}.$$

Moreover, Q *and* $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *generate the same variety.*

When Q is a quasivariety of algebras with a binary term $x \wedge y$ such that in every algebra in Q its interpretation gives a meet-semilattice, then Q is $\{x \wedge y \approx x\}$-equationally orderable and the deductive system $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ (with $\mu(x, y) = \{x \wedge y \approx x\}$) has the property of conjunction. Thus, by Theorem 1, $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ is congruential and from [5] follows that $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ is the variety generated by Q. In particular, when Q is a variety of residuated lattices, the deductive system $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ is the deductive system that for example in [2] is called the logic of Q that preserves degrees of truth, and in this case we have $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu} = \mathsf{Q}$.

If Q is a μ-equationally orderable variety, we do not need to assume that $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ has the property of conjunction to conclude that it is congruential and with $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu} = \mathsf{Q}$.

In the talk we will discuss the following general result.

Theorem 2. *If* Q *is a* μ*-equationally orderable quasivariety and* $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu} = \mathsf{Q}$*, then* $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is congruential.*

A consequence of which is next theorem.

Theorem 3. *Let* Q *be a* μ*-equationally orderable variety. The deductive system* $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is congruential and* $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu} = \mathsf{Q}$.

Next two theorems follow as a consequence of the above discussion and results in [5] and [6].

Theorem 4. *Let* Q *be a* μ*-equationally orderable quasivariety. If* $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *has the property of conjunction for a term* $x \wedge y$*, then:*

1. $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is the variety generated by* Q,
2. $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is* μ*-equationally orderable and the order defined by* $\mu(x, y)$ *in any of its members is a meet-semiilattice with meets defined by* $x \wedge y \approx x$,
3. $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is congruential.*

Theorem 5. *Let* Q *be a* μ*-equationally orderable quasivariety. If* $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *has the deduction-detachment property for a term* $x \to y$ [13]*, then:*

1. $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is congruential,*
2. $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is the variety generated by* Q,
3. $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *is* μ*-equationally orderable and the order defined by* $\mu(x, y)$ *in any of its members is defined also by* $x \to y \approx y \to y$.

Moreover:

Proposition 2. *If* Q *is a* μ*-equationally orderable quasivariety and* $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *has the deduction-detachment property for a term* $x \to y$*, then* $\mathcal{S}_{\mathsf{Q}}^{\leq\mu}$ *satisfies that for every set of formulas* Γ *and every formula* φ,

$$\Gamma \vdash_{\mathcal{S}_{\mathsf{Q}}^{\leq\mu}} \varphi \;\text{ iff }\; \exists\varphi_0, \ldots, \varphi_n \in \Gamma\; \forall\mathbf{A} \in \mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu} \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$$

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[13] A deductive system $\mathcal{S}$ has the deduction-detachment property for a term $x \to y$ if for every set of formulas Γ and every formulas φ, ψ,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi \to \psi.$$

$$v(\varphi_0 \to (\varphi_1 \to \dots (\varphi_n \to \varphi) \dots)) = v(\varphi \to \varphi) \text{ or}$$
$$\forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq\mu} \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\ v(\varphi) = v(\varphi \to \varphi).$$

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§1. Abstract. *Residuated lattices* form the algebraic counterpart of substructural logics [5]. The *connected rotation* construction for t-norms has been introduced in [11] and has been generalized to arbitrary residuated posets in [9], where also the *disconnected* rotation construction was presented for arbitrary residuated posets. The precursor of the disconnected rotation construction goes back to Wronski's reflection construction for BCK-algebras [17]. The reader is also referred to [8, 6]. Both the connected and the disconnected rotation constructions have proved fundamental; for example in the structural description of perfect and bipartite IMTL-algebras [16], of free nilpotent minimum algebras [1], of Nelson algebras [2], and of free Glivenko algebras [3], and have proved useful in many other mathematical applications.

Using rotations one can construct a huge set of examples of positive rank involutive FL_e-algebras. However, we lack examples of negative rank ones. To overcome this, dualization seems to be a natural idea. *De Morgan dualization* can be defined in any algebra which has a binary multiplication $\circledast$ and an order-reversing involution $'$ on its universe by $x \cdot y = (x \circledast y)'$. However, de Morgan dualization does not fit well to the class of residuated lattices, since de Morgan dual of the monoidal operation of a residuated lattice, in general, is not residuated with respect to the same ordering relation. As an attempt to overcome this, and to describe the structure of a certain class of residuated chains, skew dualization has been introduced in [10]. However, skew-dualization is a notion hard to work with. Meyer's "relevant enlargement" construction for relevance logic (see [15]) has been generalized for (possibly non-commutative) residuated posets in [6]. Its commutative version will be named *disconnected co-rotation* construction in this talk.

We introduce the *connected co*-rotation construction, and characterize the class of operations which the connected rotation construction results in associative operations with. The fact that co-rotation constructions are not simply de Morgan duals of their respective rotation construction counterparts is also reflected by the fact that the class of algebras which can be co-rotated and the class of algebras which can be rotated are quite different. Just as both rotation constructions of FL_e-algebras result in positive rank involutive FL_e-algebras, the co-rotations of FL_e-algebras result in non-positive (zero or negative) rank involutive FL_e-algebras; thus providing a wide spectrum of examples for the latter algebra. Also, a construction, called *involutive ordinal sums* will be introduced. This construction generalizes the generalized ordinal sums of Galatos [4] in the zero rank (that is, group-like) case, and constructs group-like FL_e-algebras. Finally, we present new constructions which can construct certain subclasses of involutive FL_e-chains, along with their related decompositions (coined pistil-petal decompositions). Elements of these subclasses can be decomposed into a group-like one and an IMTL-chain (or its skew-dual).

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▶ PETER JIPSEN, *The lattice of varieties generated by small residuated lattices.*
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Residuated lattices are the algebraic models of substructural logics, and have been studied since the late 1930's [5]. The varieties of residuated lattices and full Lambek algebras (FL-algebras) [4] include many well-known varieties of logic, such as the varieties of Boolean algebras, Gödel algebras, Heyting algebras, MV-algebras, MTL-algebras, Basic Logic algebras and involutive FL-algebras. These varieties contain many finite algebras and, apart from the variety of Boolean algebras, they all have infinitely many subvarieties. Recall that a *residuated lattice* is an algebra $(A, \wedge, \vee, \cdot, \mathbf{1}, \backslash, /)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, \mathbf{1})$ is a monoid, and for all $x, y, z \in A$ the equivalences $xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$ hold. An *FL-algebra* is a residuated lattice with an additional constant $\mathbf{0}$ that can denote any element of the algebra. For background about residuated lattices, FL-algebras and notation and terminology of universal algebra, we refer the reader to [2]. In particular, we consider residuated lattices as a subvariety of FL-algebras, defined by the identity $\mathbf{0} = \mathbf{1}$. The lattice of subvarieties of FL-algebras is denoted by $\Lambda(\mathsf{FL})$.

A variety $\mathcal{V}$ is *finitely generated* if there is a finite set $\{A_1, A_2, \ldots, A_n\}$ of finite algebras such that $\mathcal{V} = \mathsf{HSP}\{A_1, A_2, \ldots, A_n\}$, and one can assume that the generating algebras $A_1, \ldots, A_n$ are subdirectly irreducible. Furthermore, since FL-algebras have lattice reducts, they form a congruence distributive variety, so it follows from Jónsson's Lemma that a subvariety is generated by a single finite subdirectly irreducible algebra if and only if it is completely join-irreducible in the lattice of subvarieties. In addition, $\Lambda(\mathsf{FL})$ is a distributive lattices, and the finitely generated varieties form an ideal of this lattice. Hence the structure of this ideal is determined by the poset of join-irreducible varieties in it, i.e., there is a one-one correspondence between finitely generated varieties of FL-algebras and finite downsets in this poset of join-irreducible varieties. By another application of Jónsson's Lemma, if A, B are finite subdirectly irreducible FL-algebras, then $\mathsf{HSP}\{A\} \subseteq \mathsf{HSP}\{B\}$ if and only if $A \in \mathsf{HS}\{B\}$. As a result, one obtains a description of the ideal of finitely generated varieties by computing the so-called HS-poset of finite subdirectly irreducible FL-algebras.

We describe a small part of the bottom of this poset by enumerating the subdirectly irreducible residuated lattices of up to 5 elements and computing their subalgebras and homomorphic images (up to isomorphism). This extended abstract only shows the residuated lattices up to size 4 and a diagram of their HS-poset. Tables for algebras of size 5 and further diagrams of HS-posets can be found in [3]. A longer list of finite residuated lattices is available at `www.chapman.edu/~jipsen/gap/rl.html`. An enumeration of commutative integral residuated lattices up to size 12 is at `vychodil.inf.upol.cz/order/`, [1].

Rather than just providing lists of algebras, we give a view of the HS-poset and arrange the algebras in a way that groups similar algebras together. We consider algebras to be similar if they satisfy the same identities that define specific well-known

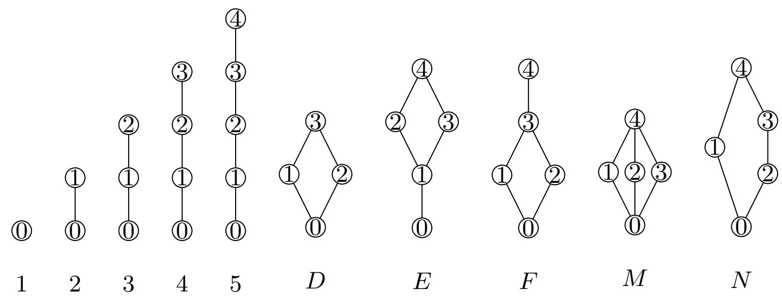

FIGURE 1. Lattices of size up to 5

subvarieties of FL-algebras.

There are 174 residuated lattices with up to 5 elements respectively (see Table 1). The lattice reducts of these algebras are listed in Figure 1. The n-element chain is simply denoted by n, and the remaining lattices are $D = 2 \times 2$, $E = 1 \oplus D$, $F = D \oplus 1$, M and N, where $\oplus$ denotes ordinal sum, and the last two lattices are the 5-element modular lattice and nonmodular lattice respectively, usually referred to as the diamond and the pentagon. Individual residuated lattices are denoted L_n, where $L \in \{1, 2, 3, 4, 5, D, E, F, M, N\}$ and n is an index that enumerates the algebras that have lattice L as reduct. So for example the three 3-element residuated lattices are $3_1, 3_2, 3_3$, and in the lists below they are the 3-element Wasjberg hoop (or MV-algebra if $\mathbf{0} = 0$), the 3-element Brouwerian algebra (or Gödel algebra if $\mathbf{0} = 0$) and the 3-element Sugihara algebra respectively.

n	RL	Chain	D	E	F	M	N	FL	Chain	D	E	F	M	N
1	1	1						1	1					
2	1	1						2	2					
3	3	3						9	9					
4	20	15	5					79	60	19				
5	149	84		20	11	8	26	737	420		97	53	37	130
Total	174	104						828	492					

TABLE 1. Number of residuated lattices and FL-algebras of size ≤ 5

To fully specify a finite residuated lattice, it suffices to give its lattice reduct and a join-preserving monoid operation, since the residuals $\backslash, /$ are uniquely determined by this information, e.g., $z/y = \bigvee\{x \mid xy \leq z\}$. In the tables below, the monoid is presented as a transformation monoid, hence a residuated lattice is given by $\langle L_n, i,$ list of transformations$\rangle$. Here L_n is the lattice reduct with a unique index, i is the element denoted by the identity constant $\mathbf{1}$, and each tuple $t = d_1d_2 \ldots d_m$ of digits is a transformation $t : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, m\}$ where $t(0) = 0$ and $t(k) = d_k$ for $k = 1, \ldots, m$. To construct the operation table for the monoid, simply stack the transformations on top of each other, insert the identity transformation at row i, and add a row and column of zeros. For example, the residuated lattice $\langle D_5, 1, 222, 323\rangle$

RL var	FL var	Description
GBA	BA	(Generalized) Boolean algebras: $xx = x$ and WH, MV
WH	MV	Wajsberg hoops, MV-algebras: involutive BL-algebras
BH	BL	Basic hoops, Basic logic algebras: $x \wedge y = x(x \backslash y)$, CRRL, RFL_{ew}
RBr	GA	Representable Br-algs, Gödel algebras: s.i. are linear Br, HA
Br	HA	Browerian algebras, Heyting algebras: $xy = x \wedge y$
CRL	FL_e	Commutative RL, FL with the exchange rule: $xy = yx$
DRL	DFL	Distributive RL, FL: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
	InFL	Involutive RL, FL: $\mathbf{0}/(x\backslash\mathbf{0}) = x = (\mathbf{0}/x)\backslash\mathbf{0}$
IRL	FL_w	Integral RL: $x \le \mathbf{1}$, FL with weakening: $\mathbf{0} \le x \le \mathbf{1}$
RRL	RFL	Representable RL, FL: s.i. algebras are linear

TABLE 2. Names of subvarieties

has a monoid operation given by

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

An FL-algebra is a residuated lattice with one extra constant $\mathbf{0}$, and there are 828 FL-algebras with up to 5 elements (see Table 1). Note that the constant $\mathbf{0}$ and the least element 0 of each lattice are in general distinct, though they do coincide for FL_o-algebras (defined as FL-algebras where $\mathbf{0}$ is the bottom element). For subvarieties of FL-algebras we use the same names as in [2]. In Table 2 we briefly recall the relevant ones.

From the data in this abstract and in [3] one can make a few observations. Since the bottom element of a residuated lattice (if it exists) must always be a multiplicative zero, the identity element of a nontrivial residuated lattice cannot take the place of the bottom element. However there are other restrictions. The elements that are covered by the identity element need to generate a Boolean algebra under join and meet [5]. Hence there are no integral residuated lattices with the diamond M or pentagon N as lattice reduct. The data in these tables (and longer versions of them) can be used to discover other results of this form, where the noticeable absence of certain configurations in all finite algebras up to a certain size leads to the discovery of results that prove these configurations can never occur in a (finite) residuated lattice. Furthermore, it is currently not known whether the variety of residuated lattices has the amalgamation property. Residuated lattices from Table 3 and [3] are used to test if specific V-formations can be amalgamated.

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RL var	FL var	Name, id, transformations	Sub	Hom
Triv	Triv	$\langle 1_1, 0\rangle$		
GBA	BA	$\langle 2_1, 1\rangle$		
WH	MV	$\langle 3_1, 2, 01\rangle$	2_1	
RBr	GA	$\langle 3_2, 2, 11\rangle$	2_1	2_1
CRRL	RInFL$_e$	$\langle 3_3, 1, 22\rangle$		
WH	MV	$\langle 4_1, 3, 001, 012\rangle$	2_1	
BH	BL	$\langle 4_2, 3, 011, 122\rangle$	3_1 3_2	3_1
		$\langle 4_3, 3, 111, 112\rangle$	3_1 3_2	2_1
RBr	GA	$\langle 4_4, 3, 111, 122\rangle$	3_2	3_2
CIRRL	RInFL$_{ew}$	$\langle 4_5, 3, 001, 022\rangle$	2_1	2_1
CIRRL	RFL$_{ew}$	$\langle 4_6, 3, 001, 002\rangle$	3_1	
IRRL	RFL$_w$	$\langle 4_7, 3, 001, 122\rangle$	2_1	
		$\langle 4_8, 3, 011, 022\rangle$	2_1	
CRRL	RInFL$_e$	$\langle 4_9, 1, 233, 333\rangle$	3_3	
		$\langle 4_{10}, 2, 113, 333\rangle$	2_1 3_3	3_3
CRRL	RFL$_e$	$\langle 4_{11}, 1, 223, 333\rangle$	3_3	
		$\langle 4_{12}, 2, 011, 133\rangle$		
		$\langle 4_{13}, 2, 111, 133\rangle$	3_3	2_1
RRL	RFL	$\langle 4_{14}, 2, 111, 333\rangle$		
		$\langle 4_{15}, 2, 113, 133\rangle$		
GBA	BA	$\langle D_1, 3, 101, 022\rangle$	2_1	2_1
CDRL	DInFL$_e$	$\langle D_{2,1}, 1, 202, 323\rangle$	3_3	
		$\langle D_{3,1}, 1, 213, 333\rangle$	3_3	
		$\langle D_{4,2}, 1, 233, 333\rangle$	3_3	
CDRL	DFL$_e$	$\langle D_5, 1, 222, 323\rangle$	3_3	

TABLE 3. Residuated lattices of size ≤ 4

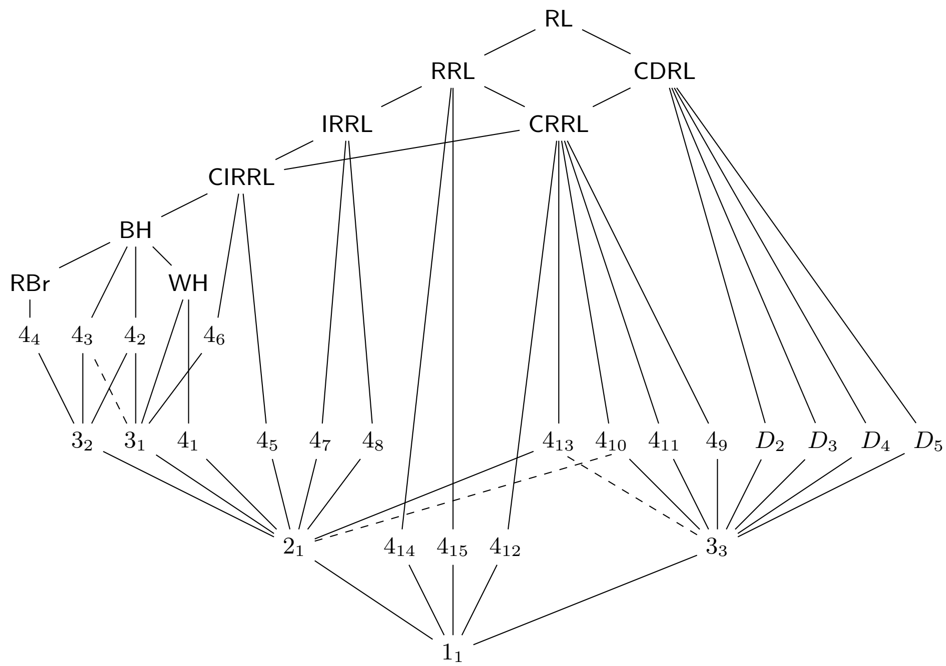

FIGURE 2. The HS-poset of residuated lattices with ≤ 4 elements and some varieties that contain them (remove the dotted lines to get the HS-poset of FL_o-algebras)

▶ TOMÁŠ KROUPA, TOMÁŠ VALLA, *Constructing many-valued logical functions with small influence of their variables.* [15]

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§1. Introduction. The Boolean functions with small influence of their inputs are used in the collective coin flipping algorithms [2]. In this contribution we replace the random bit generator with a random generator over a finite set and we show the existence of finitely-valued Łukasiewicz formulas with small influence of their variables.

§2. Łukasiewicz logic. We repeat basic definitions and results concerning finite-valued Łukasiewicz logic and its Lindenbaum algebra [3, 1]. We consider only finitely-many propositional variables $A_1, \ldots, A_n$. Formulas $\varphi, \psi, \ldots$ are then constructed from these variables and the truth-constant $\overline{0}$ using the following basic connectives: negation $\neg$ and strong disjunction $\oplus$. For any $k \in \mathbb{N}$ the semantics for connectives of $(k+1)$-valued Łukasiewicz logic is given by the corresponding operations of the *finite MV-chain,* which is just the set of rational numbers $Ł_k = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}$ endowed with

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constant zero 0 and the operations of negation $\neg$ and strong disjunction $\oplus$ defined as $\neg a = 1 - a$ and $a \oplus b = \min\{1, a+b\}$, respectively, for each $a, b \in Ł_k$. The structure $\langle Ł_k, \oplus, \neg, 0\rangle$ then becomes an *MV-algebra*. The operations $\odot$, $\wedge$, $\vee$, $\to$ are introduced in the standard way.

In the sequel we consider the expansion of the $(k+1)$-valued Łukasiewicz logic with the truth constants from the chain $Ł_k$. The language of $(k+1)$-*valued Łukasiewicz logic with truth constants* results from the language of $(k+1)$-valued Łukasiewicz logic by adding the truth constant $\bar{r}$ for each $r \in Ł_k$. Every truth constant is a formula. Formulas are built from propositional variables $A_1, \dots, A_n$ and truth constants using the connectives $\oplus$ and $\neg$ as well as other defined connectives of Łukasiewicz logic. Let F_n^k be the Lindenbaum algebra of $(k+1)$-valued Łukasiewicz logic with truth constants over n variables. By [1] we may identify F_n^k with the product $Ł_k^{(Ł_k^n)}$ whose elements are all the functions $f\colon Ł_k^n \to Ł_k$.

§3. Influence of Boolean variables. Boolean functions have a natural interpretation in game theory. A Boolean function is called a *simple game*. Each variable is controlled by a unique *player* and setting this variable to 1 or 0 expresses the yes/no voting scheme. The value of the Boolean function then represents an overall outcome of the voting. Observe that in our notation an n-variable Boolean function $f : Ł_1^n \to Ł_1$ is an element of F_n^1. The problem of measuring influence of a given propositional variable on the values of $f \in F_n^1$ was studied in coalitional game theory [6] and in the field of fault-tolerant computations [2].

The following notations will be used throughout the paper. Let $f \in F_n^k$ and $i \in \{1, \dots, n\}$. For every $y = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) \in Ł_k^{n-1}$, we denote by f_y^{-i} the function $Ł_k \to Ł_k$ defined by $f_y^{-i}(x) = f(y_1, \dots, y_{i-1}, x, y_{i+1}, \dots, y_n)$, where $x \in Ł_k$.

The influence $\beta_i(f)$ of variable x_i on a monotone Boolean function $f \in F_n^1$ is defined as the probability that f_y^{-i} remains non-constant when $y \in \{0,1\}^{n-1}$ is selected at random: $\beta_i(f) = \sum_{y\in\{0,1\}^{n-1}} \frac{f_y^{-i}(1) - f_y^{-i}(0)}{2^{n-1}}$. The number $\beta_i(f)$ is also called the *Banzhaf index* of player i in a coalition game f. We also define the Banzhaf index $\beta(f)$ of f as $\beta(f) = \max\{\beta_1(f), \dots, \beta_n(f)\}$. The Banzhaf index measures the influence of players.

A natural motivation for investigating the players' influence comes from the collective random bit generators. Suppose there are n computers equipped with random generators. The task is to generate one random bit identical for all the machines. Simultaneously, each machine produces a uniform random bit and announces it to other machines. Each of them then has to perform a computation based on these inputs to produce the identical uniform random bit. The question is to which extent is a given random generator resistant towards possible third party attacks and corruption of one machine. The goal of the design of Boolean functions with low variables' influence is to minimize the chance of the attacker to manipulate the result.

Consider the Boolean function $f(x_1, \dots, x_n) = x_k$ called the *dictatorship* of player k. The influence of dictator k is 1 and 0 for other players: $\beta(f) = 1$. The Boolean *majority function* $m \in F_n^1$ is defined as follows. Let n be odd and let $m(x_1, \dots, x_n) = 1$ if there is a set $S \subseteq \{1, \dots. n\}$, $|S| > n/2$, such that $x_i = 1$ for $i \in S$, and 0 otherwise. It follows that $\beta(m) = \Theta(1/\sqrt{n})$. On the other hand, it was shown [5, 4] that for any Boolean function the average influence of a variable is at least $\Omega(1/n)$. Surprisingly, there exists a Boolean function L performing better than the majority functions. In the next theorem we identify the vertices of $\{0,1\}^n$ with the subsets of $\{1, \dots, n\}$.

THEOREM 1 (Ben-Or and Linial [2]). *There exists a construction of the function $L \in F_n^1$ such that $|L^{-1}(0)| = |L^{-1}(1)| = 2^{n-1}$ and $\beta(L) = O\left(\frac{\log n}{n}\right)$.*

The rough idea how the function L is constructed is as follows. Let b be the unique solution of the equation $(2^b-1)^{1/b}=2^{1-1/n}$. Decompose the set $\{1,\dots,n\}$ into n/b blocks of size b and consider the set J of those subsets of $\{1,\dots,n\}$ which contain no block. Let L be defined such that $L(A)=0$ if $A\in J$ and $L(A)=1$ otherwise.

§4. Influence of variables in many-valued logics. We will propose a natural generalization of Boolean Banzhaf index. Let $f\in F_n^k$, $k\geq 1$, and $i\in\{1,\dots,n\}$. The *influence of variable x_i on f* is $\gamma_i(f)=(k+1)^{1-n}\cdot\sum_{y\in Ł_k^{n-1}}\left(\max_{x\in Ł_k} f_y^{-i}(x)-\min_{x\in Ł_k} f_y^{-i}(x)\right)$. It can be shown that $\gamma_i(f)$ is a faithful generalization of the Banzhaf index. A natural next step is to design functions with low variable influence in the setting of the finitely-valued Łukasiewicz logic with truth constants. Our setting is the case of random generators producing a number from a finite set. Note that this setting naturally allows designs that may ask the input generators repeatedly.

4.1. 2^k**-valued logic.** Let us consider the set $Ł_{h-1}$ for $h=2^k$ for some nonnegative integer k. In the sequel, we will naturally identify the elements $S_h=\{0,1,\dots,h-1\}$ with those in $Ł_{h-1}$. Note that we may encode each $x\in\{0,1,\dots,h-1\}$ by a k-element Boolean vector $(x^1,x^2,\dots,x^k)$ representing the binary number x, with x^1 being the highest bit and x^k the lowest. Under the identification of S_h with $Ł_{h-1}$, we may analogously use the Banzhaf index $\gamma_i'(f)=h^{1-n}\sum_{y\in S_h^{n-1}}(\max_{x\in S_h} f_y^{-i}(x)-\min_{x\in S_h} f_y^{-i}(x))$. Let us define the function $f\colon S_h^n\to S_h$ as

$$f(x_1,\dots,x_n)=\left(L(x_1^1,x_2^1,\dots,x_n^1),L(x_1^2,\dots,x_n^2),\dots,L(x_1^k,\dots,x_n^k)\right), \tag{1}$$

where the value $f(x_1,\dots,x_n)$ is the binary representation of a number in S_h.

We shall prove that f has a small variable influence.

Theorem 2. *For $i=1,\dots,n$, $\gamma_i'(f)=O\left(h\frac{\log n}{n}\right)$.*

Proof. The Banzhaf index of the function $L'=L(x_1^1,\dots,x_n^1)$ is $\beta_i(L')=O(\log n/n)$ by [2]. Observe that in the resulting vector of f the value of the highest bit $L(x_1^1,x_2^1,\dots,x_n^1)$ has the same effect on the size of the output value as the sum of all other lower bit orders, and the same holds for the influence of each lower bit. We may thus bound

$$\gamma_i'(f)\leq(2^{k-1}+2^{k-2}+\dots+1)\cdot\beta(L')\leq 2^k\cdot\beta(L')=O\left(h\frac{\log n}{n}\right).$$

⊣

4.2. General many-valued logic. Let us now consider the set S_h with $h>2$ and let ℓ be the smallest integer such that $2^\ell\geq h$. Let us denote by G_2 a random generator producing one uniform random bit. We construct the generator $G_{2\to h}$ that uses G_2 as the input and produces a value from S_h with uniform distribution:

1. Produce a number N by reading random bits $b_1,\dots,b_\ell$ from the G_2.
2. If $N<h$ then return the number N.
3. Otherwise, repeat the whole process.

Lemma 3. *The generator $G_{2\to h}$ produces a result with a uniform distribution over S_h. The expected number of random bits read from G_2 is $\ell 2^\ell/h=\Theta(\log h)$.*

Proof. W.l.o.g., let p denote the probability that 0 is on the output of $G_{2\to h}$. Then

$$p=\frac{1}{2^\ell}+\frac{2^\ell-h}{2^\ell}p$$

since with probability $1/2^\ell$ the result is produced immediately and with probability

$(2^\ell - h)/2^\ell$ the process is repeated independently on the previous round. Solving the equation yields $p = 1/h$.

Denote by E the expected number of random bits needed to produce the result. Similar equation $E = \ell + \frac{2^\ell - h}{2^\ell} E$ holds as ℓ bits are used always and with probability $(2^\ell - h)/2^\ell$ the whole memoryless process repeats. The solution gives $E = \ell 2^\ell / h$. Finally, note that $\ell \approx \log_2 h$ and $h \leq 2^\ell < 2h$. ⊣

Let us denote G_h a random generator producing uniform S_h-valued output. Using analogous technique, we design a generator $G_{h\to 2}$ which reads uniformly distributed random S_h-valued input and produces a uniformly distributed random bit. The process is as follows. If h is even, one random input is read and its parity is returned. If h is odd, the following procedure is used.

1. Read one random number $N \in S_h$ from G_h.
2. If $N \neq h - 1$, return the parity of N.
3. Otherwise, repeat the whole process.

Lemma 4. *The generator $G_{h\to 2}$ produces a uniform random bit. If h is even, $G_{h\to 2}$ reads 1 random input value. If h is odd, the expected number of random values read is $h/(h-1)$.*

Proof. Let p denote the probability that $G_{h\to 2}$ produces, w.l.o.g., 0. Then $p = \frac{(h-1)/2}{h} + \frac{1}{h} p$ as with probability $(h-1)/(2h)$ the result is produced immediately and with probability $1/h$ the process is repeated independently on the previous round. We get $p = 1/2$.

Denote by E the expected number of random S_h-values needed to produce the result. We have that $E = 1 + \frac{1}{h} E$ holds as 1 value is used always and with probability $1/h$ the memoryless process repeats, which gives the solution $E = h/(h-1)$. ⊣

4.3. Function with low influence of variables. We describe the generator G which is given n uniform S_h-valued random generators and produces the S_h-valued output with a low influence of the input generators. Let us denote the input random S_h-value generators by $g_1, \ldots, g_n$.

The generator is constructed as $G = G_{2\to h}\Big(L\big(G_{h\to 2}(g_1), G_{h\to 2}(g_2), \ldots, G_{h\to 2}(g_n)\big)\Big)$, where L is the function from Theorem 1. In another words, the generator $G_{2\to h}$ repeatedly asks for Boolean bits from the function L, which in turn asks the n generators $G_{h\to 2}$ connected to the inputs of L, which in turn ask the input generators $g_1, \ldots, g_n$.

Lemma 5. *The probability that G produces $v \in S_h$ is $1/h$.*

Proof. By Lemma 4, each generator $G_{h\to 2}(g_i)$ produces uniform random bit. By Theorem 1, the function L then produces a uniform random bit. Finally, the generator $G_{2\to h}$ produces uniform S_h-valued output. ⊣

Let ℓ be the smallest integer such that $2^\ell \geq h$.

Lemma 6. *For the generator G, the expectation of the total number of random values produced by the generators $g_1, \ldots, g_n$ in order to obtain one output of G is $n \cdot \ell \cdot 2^\ell/(h-1)$.*

Proof. Recall that by Lemma 4 the expectation of random values needed by the generator $G_{h\to 2}$ to produce single output is $h/(h-1)$. As the generators $g_1, \ldots, g_n$ are independent, linearity of expectation yields that to produce one output bit of the function L, $nh/(h-1)$ input values are needed in the expectation. Since each execution of L is independent on the previous runs, by Lemma 3 we obtain the total expectation $(\ell 2^\ell/h) nh/(h-1) = n\ell 2^\ell/(h-1)$. ⊣

Lemma 7. *For each $i = 1, \ldots, n$, we have $\gamma'_i(G) = O((2^\ell \log n)/n)$.*

PROOF. Observe that the function $L' = L(G_{h\to 2}(g_1), \ldots, G_{h\to 2}(g_n))$ behaves exactly as the function f defined by (1). Theorem 2 yields $\gamma'_i(L') = O((2^\ell \log n)/n$. During the step 1 of the generator $G_{2\to h}$, the function L is called ℓ times, which produces a number N in the range $0, \ldots, 2^\ell - 1$. Less than one half of the possible values of N is rejected and step 1 is repeated independently on the result of the previous iteration. ⊣

We may identify G with a unique function $\widehat{G} : \text{Ł}_{h-1}^n \to \text{Ł}_{h-1}$. Note that $\gamma_i(\widehat{G}) = \gamma'_i(G)/(h-1)$. Since $\ell \approx \log_2 h$, using Lemma 6 and Lemma 7 we may conclude with the following corollary.

COROLLARY 8. *The generator $\widehat{G}$ needs in total $\Theta(n \log h)$ input random values in expectation, and* $\max\limits_{i=1,\ldots,n} \gamma_i(\widehat{G}) = O(\log n/n)$.

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▸ ORI LAHAV AND ARNON AVRON, *Cut-free calculus for second-order Gödel logic.*
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Fuzzy logics form a natural generalization of classical logic, in which truth values consist of some linearly ordered set, usually taken to be the real interval $[0,1]$. They have a wide variety of applications, as they provide a reasonable model of certain very common vagueness phenomena. Both their propositional and first-order versions are well-studied by now (see, e.g., [8]). Clearly, for many interesting applications (see, e.g., [5] and Section 5.5.2 in Chapter I of [6]), propositional and first-order fuzzy logics do not suffice, and one has to use higher-order versions. These are much less developed (see, e.g., [16] and [6]), especially from the proof-theoretic perspective. Evidently, higher-order fuzzy logics deserve a proof-theoretic study, with the aim of providing a basis for automated deduction methods, as well as a complimentary point of view in the investigation of these logics.

The proof theory of propositional fuzzy logics is the main subject of [11]. There, an essential tool to develop well-behaved proof calculi for fuzzy logics is the transition from (Gentzen-style) sequents, to *hypersequents*. The latter, that are usually nothing more than disjunctions of sequents, turn to be an adequate proof-theoretic framework for the fundamental fuzzy logics. In particular, propositional Gödel logic (the logic interpreting conjunction as minimum, and disjunction as maximum) is easily captured by a cut-free hypersequent calculus called HG (introduced in [1]). The derivation rules of HG are the standard hypersequent versions of the sequent rules of Gentzen's LJ for intuitionistic logic, and they are augmented by the *communication rule* that allows "exchange of information" between two hypersequents [2]. In [3], it was shown that HIF, the extension of HG with the natural hypersequent versions of LJ's sequent rules for the first-order quantifiers, is sound and (cut-free) complete for standard first-order Gödel logic.[16] As a corollary, one obtains Herbrand theorem for the prenex fragment of this logic (see [11]).

In this work, we study the extension of HIF with usual rules for *second-order* quantifiers. These consist of the single-conclusion hypersequent version of the rules for introducing second-order quantifiers in the ordinary sequent calculus for classical logic (see, e.g., [7, 15]). We denote by HIF^2 the extension of (the cut-free fragment of) HIF with these rules. Our main result is that HIF^2 is sound and complete for second-order Gödel logic. Since we do not include the cut rule in HIF^2, this automatically implies the admissibility of cut, which makes this calculus a suitable possible basis for automated theorem proving. It should be noted that like in the case of second-order classical logic, the obtained calculus characterizes *Henkin-style* second-order Gödel logic. Thus second-order quantifiers range over a domain that is directly specified in the second-order structure, and it admits full comprehension (this is a domain of *fuzzy sets* in the case of fuzzy logics). This is in contrast to what is called the *standard semantics*, where second-order quantifiers range over *all* subsets of the universe. Hence HIF^2 is practically a system for two-sorted first-order Gödel logic together with the comprehension axioms (see also [4]).

While the soundness of HIF^2 is straightforward, proving its (cut-free) completeness turns out to be relatively involved. This is similar to the case of second-order classical logic, where the completeness of the *cut-free* sequent calculus was open for several years, and known as *Takeuti's conjecture* [14].[17] While usual syntactic arguments for cut-elimination dramatically fail for the rules of second-order quantifiers, Takeuti's conjecture was initially verified by a semantic proof. This was accomplished in two steps. First, the completeness was proved with respect to three-valued non-deterministic semantics (this was done by Schütte in [12]). Then, it was left to show that from every three-valued non-deterministic counter-model, one can extract a usual (two-valued) counter-model, without losing comprehension (this was done first by Tait in [13]). Basically, we take a similar approach. First, we present a non-deterministic semantics for HIF^2 with generalized truth values. Then, we use this semantics to derive completeness with respect to the ordinary semantics. We also note that the main ideas behind the non-deterministic semantics that we use here were laid down in [9], where a proof-theoretic framework for adding non-deterministic connectives to propositional Gödel logic was suggested. In addition, the completeness proof for this semantics is an adaptation of the semantic proof in [10] of cut-admissibility in HIF.

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► SERAFINA LAPENTA, IOANA LEUŞTEAN, *On tensor product in Łukasiewicz logic*.
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MV-algebras are the algebraic counterpart of Łukasiewicz logic. Within several important results, the main achievement in the theory is the categorical equivalence with abelian lattice-ordered groups with strong unit. Several extension of the notion have been defined by endowing an MV-algebra with products: a scalar product leads to the notion of MV-module (and if scalars are taken in $[0, 1]$, we obtain a Riesz MV-algebra) [3, 4]; an internal binary product leads to the notion of PMV-algebra [2]; a combination of both leads to the notion of fMV-algebra [6].

The categorical equivalence for MV-algebras extends naturally to any MV-algebra with

product, and allows us to connect MV-modules, Riesz MV-algebra, PMV-algebra and fMV-algebras with ℓ-modules with strong unit, Riesz Spaces with strong unit, ℓ-rings with strong unit and f-algebras with strong unit respectively.
In lattice ordered structures several tensor products have been defined. We will consider in the following Martinez's tensor product $\otimes_\ell$ of ℓ-groups [9] and Buskes and Van Rooij's tensor product $\overline{\otimes}$ of archimedean ℓ-groups [1].
For the class of MV-algebras, the definition of a tensor product was given by Mundici in both standard (denoted by $\otimes_{mv}$) and semisimple case (denoted by $\otimes_{ss}$) [10] (we recall that semisimple MV-algebras are equivalent to archimedean ℓ-groups). Further property have been investigated by Leuştean and Flondor [5].
Our first result is the following.

THEOREM 1. *Let A, B be MV-algebras and (G, u_G) and (H, u_H) the lattice ordered groups such that $A \simeq \Gamma(G, u_G)$ and $B \simeq \Gamma(H, u_H)$. Then*
1) $\Gamma(G, u_G) \otimes_{mv} \Gamma(H, u_H) \simeq \Gamma(G \otimes_\ell H, u_G \otimes_\ell u_H)$.
If A and B are semisimple and therefore (G, u_G) and (H, u_H) are archimedean,
2) $\Gamma(G, u_G) \otimes_{ss} \Gamma(H, u_H) \simeq \Gamma(G \overline{\otimes} H, u_G \overline{\otimes} u_H)$

The scalar extension property (SEP) is one of the basic properties arising from a tensor product, and while it is straightforward in the non-ordered case, with lattice ordered structures it presents some difficulties. In [5] the property is stated, but it present a wrong argument in its proof, mainly related to the fact that the sum of two homomorphisms of ℓ-groups is not always an homomorphism of ℓ-groups. We correct the result and, in addition, we prove SEP for semisimple MV-algebras.

THEOREM 2. *The following hold.*
1) If A is a unital and semisimple PMV-algebra, M is a semisimple A-MV-module and B is a semisimple MV-algebra, $A \otimes_{ss} B$ is a A-MV-module. As consequence, when A is a Riesz MV-algebra, $A \otimes_{ss} B$ is a Riesz MV-algebra.
2) If A and B are unital and semisimple PMV-algebra, $A \otimes_{ss} B$ is a unital and semisimple PMV-algebra.
3) If A is a unital and semisimple fMV-algebra and P is a unital and semisimple PMV-algebra, $A \otimes_{ss} P$ is a unital and semisimple fMV-algebra.

The notion of tensor PMV-algebra of an MV-algebra has been define in the semisimple framework, following the notion of tensor algebra from the general theory. Again, the results on the tensor PMV-algebra are in partial correction of [8], where the same problem on sums of homomorphisms in lattice ordered structures appears. The first point in the construction was the proof that the semisimple tensor product $\otimes_{ss}$ is associative.

PROPOSITION 1. *Let A, B, C be semisimple MV-algebras and X, Y, Z suitable spaces such that $A \subseteq C(X)$, $B \subseteq C(Y)$, $C \subseteq C(Z)$. Then $A \otimes_{ss} (B \otimes_{ss} C) = (A \otimes_{ss} B) \otimes_{ss} C = \langle a \cdot b \cdot c \mid a \in A, b \in B, c \in C \rangle \subseteq C(X \times Y \times Z)$.*

Then we define the tensor PMV-algebra as the direct limit $\langle T(A), \varepsilon_n \rangle$ of the direct system $\langle T^n(A), \varepsilon_{n,m} \rangle$ where,
-for any $n \in \mathbb{N}$, $T^1(A) = A$ and $T^n(A) = T^{n-1}(A) \otimes_{ss} A$,
-for any $n \leq m$, $\varepsilon_{n,m} : T^n(A) \to T^m(A)$ is defined by $\varepsilon_{n,m}(\mathbf{x}) = \mathbf{x} \otimes_{ss} \underbrace{1 \otimes_{ss} \ldots \otimes_{ss} 1}_{m-n}$ by associativity.

PROPOSITION 2. *$T(A)$ is a unital and semisimple PMV-algebra. Moreover, any ε_n is an embedding.*

Note that any unital and semisimple PMV-algebra is commutative, so $T(A)$ is commutative.

For a PMV-algebra P we denote by $\mathcal{U}(P)$ its MV-algebra reduct. Similarly, for any fMV-algebra M we denote by $\mathcal{U}_{\mathbb{R}}(M)$ its PMV-algebra reduct.

Theorem 3. *Let A be a semisimple MV-algebra. For any semisimple and unital PMV-algebra P and for any homomorphism of MV-algebras $f : A \to \mathcal{U}(P)$ there exists a homomorphism of PMV-algebras $f^{\sharp} : T(A) \to P$ such that $\widetilde{f} \circ \varepsilon_1 = f$.*

Theorem 4. *Let A be a unital and semisimple PMV-algebra. For any unital and semisimple fMV-algebra M and for any homomorphism of PMV-algebras $f : A \to \mathcal{U}_{\mathbb{R}}(M)$ there is a unique homomorphism of fMV-algebras $\widetilde{f} : [0,1] \otimes_{ss} A \to M$ such that $\widetilde{f} \circ \iota_A = f$, where $\iota_A : A \to [0,1] \otimes_{ss} A$ is the embedding in the tensor product.*

Putting together scalar extension property and tensor PMV-algebra, we further prove two adjunctions:

1) from semisimple MV-algebras to unital and semisimple PMV-algebras, where one of the adjoint is the functor $\mathbf{T}$, that maps any MV-algebra in its tensor PMV-algebra;

2) from unital and semisimple PMV-algebras to unital and semisimple fMV-algebras, where where one of the adjoint is the functor $\mathcal{T}$, that maps any unital and semisimple PMV-algebra P in in the fMV-algebra $[0,1] \otimes_{ss} P$.

A further consequence is the amalgamation property for semisimple fMV-algebras.

Theorem 5. *Let A, B, Z unital and semisimple fMV-algebras such that Z embeds in both A and B, with embeddings z_A and z_B. Then there exists a unital and semisimple fMV-algebra E such that both A and B embeds in E, with embeddings f_A and f_B and $f_B \circ z_B = f_A \circ z_A$.*

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▸ SERAFINA LAPENTA, IOANA LEUŞTEAN, *MV-algebras with product and the Pierce-Birkhoff conjecture.*

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An *MV-algebra* [2] is a structure $(A, \oplus, ^*, 0)$ such that $(A, \oplus, 0)$ is an abelian monoid and the following equations are satisfied:

$(x^*)^* = x$, $0^* \oplus x = 0^*$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ for all $x, y \in A$.

MV-algebras are the algebraic structures of Łukasiewicz ∞-valued logic. The variety of MV-algebras is generated by $([0,1], +, ^*, 0)$, where $[0,1]$ is the real unit interval, $x^* = 1 - x$ and $x \oplus y = \min(1, x + y)$ for any $x, y \in [0,1]$.

Since $[0,1]$ is closed to the real product, a fruitful research direction is the study of MV-algebras enriched with a *product operation*, which can be either internal or external. *PMV-algebras* (*product MV-algebras*) are defined in [3] as MV-algebras endowed with an internal product satisfying appropriate axioms. The real structure $([0,1], \oplus, ^*, \cdot, 0)$ does not generate the variety of PMV-algebras but it generates the quasi-variety of PMV+-algebras, which is a proper subclass of PMV-algebras. One can also consider the real product on $[0,1]$ as a scalar multiplication with scalars from $[0,1]$. Our standard model in this case is $([0,1], \oplus, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ where $x \mapsto \alpha x$ is a unary operation for any $\alpha \in [0,1]$. These structures are investigated in [6] under the name of *Riesz MV-algebras*. Note that the variety of Riesz MV-algebras is generated by the standard model $[0,1]$.

A fundamental result in the theory of MV-algebras is their categorical equivalence with the category of abelian lattice-ordered groups with strong unit [13]. Similar results are proved for: PMV-algebras and a subclass of lattice-ordered rings with strong unit; Riesz MV-algebras and Riesz spaces (vector lattices) with strong unit.

Both for PMV-algebras and Riesz MV-algebras logical systems were developed [5, 6] and such systems are conservative extensions of Łukasiewicz logic. One of the main theorems of Łukasiewicz logic states that the term functions corresponding to the formulas of Łukasiewicz logic with n variables ($n \geq 1$) are exactly the continuous $[0,1]$-valued piecewise linear functions with integer coefficients defined on $[0,1]^n$ [10]. This can be seen as a *normal form theorem* for Łukasiewicz logic. A similar result was proved in [6] for the logical system that has Riesz MV-algebras as models; in this case the piecewise linear functions have real coefficients. In [12, Introduction] it is stated that a similar result for PMV-algebras is related to the Pierce-Birkhoff conjecture [1, 4].

Our main issue is to state a variant of the Pierce-Birkhoff conjecture for algebras related to Łukasiewicz logic with product. To do this, we study the class of structures obtained by endowing MV-algebras with both the internal binary product and the scalar product (as a family of unary operations). The following definitions and results are contained in [7].

Definition. An fMV-algebra is a structure $(A, \oplus, \cdot, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ which satisfies the following properties for any $x, y, z \in A$ and $\alpha \in [0,1]$:
(fMV1) $(A, \oplus, \cdot, ^*, 0)$ is a PMVf-algebra,
(fMV2) $(A, \oplus, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ is a Riesz MV-algebra,
(fMV3) $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$.

The fMV-algebras are, obviously, a variety. As in the theory of PMV-algebras, the model $\mathbf{I} = ([0,1], \oplus, \cdot, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ does not generate the variety of fMV-algebras but it generates a proper the quasi-variety. We summarize in the following our main results:

1. the category of fMV-algebras is equivalent with the category of f-algebras [1] with strong unit with unit-preserving morphisms.

2. the variety of fMV-algebras is larger then $HSP(\mathbf{I})$;

3. we characterize $ISP(\mathbf{I})$; we called *FR^+-algebras* the members of $ISP(\mathbf{I})$;

4. for FR$^+$-algebras we prove the subdirect representation w.r.t. totally-ordered structures and a representation as algebras of ${}^*[0,1]$-valued functions, where ${}^*[0,1]$ is an ultrapower of $[0,1]$,

5. the logical system $\mathcal{FMVL}^+$ that has FR$^+$-algebras as models is a conservative extension of Łukasiewicz logic and it is complete w.r.t. $\mathbf{I}$.

Our main focus now is to characterize the term functions associated with the formulas of $\mathcal{FMVL}^+$. If $n \geq 1$ is a natural number then we define the PWP$_u$-*functions* and the ISD$_u$-*functions* as follows:

- a function $f : [0,1]^n \to [0,1]$ is a PWP$_u$-*function* if it is continuous and there is a finite set of polynomials with real coefficients $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ such that for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$; we denote by $PWP(n)_u$ the set of all PWP$_u$-functions;
- a function $f : [0,1]^n \to [0,1]$ is a ISD$_u$-*function* if there is a finite set of polynomials with real coefficients $\{q_{ij} : [0,1]^n \to \mathbb{R} | 1 \leq i \leq m, 1 \leq j \leq k\}$ such that $f = \bigvee_{i=1}^m \bigwedge_{j=1}^k ((q_{ij} \vee \mathbf{0}) \wedge \mathbf{1})$; we denote by $ISD(n)_u$ the set of all ISD$_u$-functions.

Assume FR_n is the free fMV-algebra with n free generators.

Theorem. *The following properties hold:*
(1) $ISD(n)_u \subseteq FR_n \subseteq PWP(n)_u$ for any $n \in \mathbb{N}$,
(2) $ISD(n)_u = FR_n = PWP(n)_u$, for $n \leq 2$.

Conjecture. *$ISD(n)_u = FR_n = PWP(n)_u$ for any $n \geq 3$.*

The above result can be seen as *a local version of the Pierce-Birkhoff conjecture*. We proved it for $n \leq 2$, due to the fact that the Pierce-Birkhoff conjecture is proved in this case [8]. One can see [9] for an analysis of the Pierce-Birkhoff conjecture for $n = 3$. Note that, in general, our local version does not imply, nor it is implied by the Pierce-Birkhoff conjecture.

We worked in the context of fMV-algebras, so the components of the piecewise polynomial functions have real coefficients. A similar approach can be done in the context of PMV-algebras, but in this case the components of the piecewise polynomial functions will have integer coefficients.

We survey the theory of MV-algebras with product (PMV-algebras, Riesz MV-algebras, fMV-algebras) with a special focus on the normal form theorems and their connection with the Pierce-Birkhoff conjecture.

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▸ GARY MAR, *Chaotic, fuzzy and imaginary liars: degrees of truth, truth vectors, and the fractal geometry of paradox.*
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This was sometime a paradox, but now the time gives it proof.
— Shakespeare, *Hamlet*, Act 3, Scene 1

In mathematics the art of proposing a question must be held of higher value than solving it.
— Georg Cantor, doctoral thesis (1867)

During the meta-mathematical period of logic flourishing in the 1930s, the paradox of Liar gave way to proofs of classical limitative theorems — e.g., Gödel's Incompleteness Theorems [9], Church's proof of the Unsolvability of the *Entscheidungsproblem* [5], and Tarski's proof of the Undefinability of Truth [28]. Ways of overcoming these limitations were initially explored by Kleene [15] using partial recursive functions. The semantic equivalent of Kleene's approach uses truth-value gaps to overcome Tarski's Undefinability Theorem. Formal languages with truth-representing truth predicates were constructed by van Fraassen [8], Woodruff and Martin [23], and Kripke [16]. By weakening the assumption of bivalence, these formal solutions exploited meta-language reasoning to prove paradoxical sentences are safely quarantined by forced assignment to truth-value gaps. Skepticism as to whether these truth-value gap theories actually "solve" the paradoxes is supported by strengthened versions of the Liar formalizing the semantic concepts used to block the paradoxes and showing that fundamental semantic principles cannot be expressed without reintroducing paradox (Mar [17]).

An alternative to the Tarskian approach of pathologizing the paradoxes (and seeking to solve them using truth-value gaps and other technical curatives) is to actively seek and geometrically cultivate richer semantic patterns of paradox using many-valued and infinite-valued fuzzy logics with degrees of truth (Mar and Grim [20], Grim, Mar and St. Denis [10], Mar [19], and Hájek [12]).

This approach can be seen as diverging from Tarski's classic analysis of the Liar by

1. generalizing bivalent logical connectives to an infinite-valued Łukasiewiczian logic

with degrees of truth

$$
\begin{aligned}
/\sim p/ &= 1 - /p/ \\
/(p \wedge q)/ &= \textsc{Min}\{/p/, /q/\} \\
/(p \vee q)/ &= \textsc{Max}\{/p/, /q/\} \\
/(p \rightarrow q)/ &= \textsc{Min}\{1, 1 - /p/ + /q/\} \\
/(p \leftrightarrow q)/ &= 1 - \textsc{Abs}(/p/ - /q/)
\end{aligned}
$$

2. replacing Tarski's bivalent (**T**) schema with Rescher's $\mathbf{V}\boldsymbol{v}p$ schema (Rescher [26], p. 81) for many-valued logics

$$
\begin{aligned}
/\mathbf{T}p/ &= 1 - \textsc{Abs}(\boldsymbol{t} - /p/) \\
/\mathbf{V}\boldsymbol{v}p/ &= 1 - \textsc{Abs}(\boldsymbol{v} - /p/)
\end{aligned}
$$

3. modeling *self-reference* as *semantic feedback* thus allowing us to embed the semantics in the mathematics and geometry of dynamical systems theory. This is done by replacing the constant truth-value $\boldsymbol{v}$ in the $\mathbf{V}\boldsymbol{v}$p schema with expressions $S(x_n)$ representing the value the sentence attributes to itself as a function of a previously estimated value x_n.

Experimenting with various substitutions for S(x_n) yields a menagerie of new infinite-valued Liar-like, and quasi-paradoxical, sentences.

- *Continuous-Valued Liars* ("*I am as true as the truth-value* $\boldsymbol{v}$") with $S(x_n) = \boldsymbol{v}$, yielding the *Classical Liar* for $\boldsymbol{v} = 0$, Rescher's fixed-point "solution" to the Liar for $\boldsymbol{v} = 1/2$, and Kripke's *Truth-Teller* for $\boldsymbol{v} = 1$.
- The *Cautious Truth-Teller* ("*I am half as true as I am estimated to be true*") with $S(x_n) = x_n/2$.
- The *Contradictory Liar* ("*I am as true as the conjunction of my estimated value and the estimated value of my negation*") with $S(x_n) = \textsc{Min}\{x_n, 1 - x_n\}$.

The semantic differences among these sentences can be made visually perspicuous using a *web diagram.*[18] The web diagrams for the Continuous-Valued Liars appear as nested series of simple squares ranging from the Classical Liar to the Truth-Teller with a singular fixed-point at $\frac{1}{2}$. The web diagram for the Cautious Truth-Teller is a *fixed-point attractor*, no matter what initial value with which we begin, other than precisely the fixed-point $\frac{2}{3}$: the successively revised estimated values are inevitably drawn toward that fixed-point. The web diagram for the Contradictory Liar, in contrast, is a *fixed-point repellor*: for any values other than the fixed-point $\frac{2}{3}$, the successively revised values are repelled away from $\frac{2}{3}$ until the values settle on the oscillation between 1 and 0, characteristic of the Classical Liar. In short, the Cautious Truth Teller and the Contradictory Liar, while *identical* to the Classical Liar on the values 0 and 1, exhibit *diametrically opposed* semantic behavior in the interval $(0, 1)$. This example provides a justification for degrees of truth in an infinite-valued logic approach: bivalence masks intriguing semantic diversity.

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[18]A web diagram maps the iterations of a linear progression in the $[0, 1]$ interval by reflection through $x = y$ diagonal onto the unit square of the Cartesian coordinate plane: plotting a line vertically from $(x_0, 0)$ to (x_n, x_{n+1}), the web diagram continues the line horizontally from (x_n, x_{n+1}) to (x_{n+1}, x_{n+1}) and then iterates the process by using x_{n+1} for x_n.

Semantics	Algorithm	Geometry
Continuous Liars	$x_{n+1} = 1 - \text{ABS}(\boldsymbol{v} - x_n)$	Nested boxes with period 2
Cautious Truth-Teller	$x_{n+1} = 1 - \text{ABS}(\frac{x_n}{2} - x_n)$	Fixed-Point Attractor at $\frac{2}{3}$
Contradictory Liar	$x_{n+1} = 1 - \text{ABS}(\text{MIN}\{x_n, 1 - x_n\} - x_n)$	Fixed-Point Repellor at $\frac{2}{3}$
Chaotic Liar	$x_{n+1} = 1 - \text{ABS}[(1 - x_n) - x_n]$	Chaotic Tent Function
Fuzzy Logistic Liar	$x_{n+1} = 1 - \text{ABS}[(1 - x_n) - x_n]^2$	Logistic Function

Instead of excluding patterns of semantic paradox by taming semantic *cycles* [3] or seeking semantic *stability* ([14]), this approach seeks to include semantic *complexity* and chaotic *instability*. The simplest generalizations of the classical bivalent Liar in the context of an infinite-valued Łukasiewiczian logic representing self-reference as algorithmic iteration generate semantic chaos. The *Chaotic Liar* ("*I am as true as I am estimated to be false*") is geometrically represented by the chaotic tent function. Using the squaring function for the modifier '*very*' [29], we obtain the *Logistic Liar* ("*I am very true to the extent that I am estimated to be false*") represented by another paradigmatically chaotic function. These semantic generalizations of the paradox of the Liar are *chaotic* in a precise mathematical sense.[19]

Following the lead of the limitative theorems of Gödel [9], Tarski [28], and Church [5] what is initially a paradox of semantic chaos can be turned into proof. Using a *Strengthened Chaotic Liar*, we can use well-known methods to prove the *Incalculability of Chaos*, the index set of partially recursive function defined on the real interval $[0, 1]$ is not effectively calculable (Mar and Grim [20], Mar [19]).

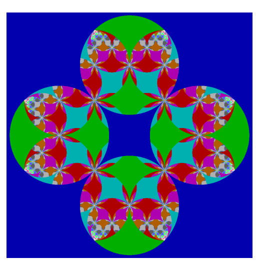
Consider a pair of paradoxical statements known as the *Dualist Liar*:

Aristotle	"What Epimenides says is true."
Epimenides	"What Aristotle says is false."

We can model the Dualist Liar as a pair of dynamical systems:

$$x_{n+1} = 1 - \text{ABS}\ y_n$$
$$y_{n+1} = 1 - \text{ABS}[(1 - x_n) - y_n]$$

Counting the number of iterations required for the ordered pairs (x_n, y_n) to exceed a threshold of the unit circle centered at $(0, 0)$, we obtain an *escape-time diagram*. Self-symmetry on descending scales characteristic of Zeno's paradoxes (Mar and St. Denis [21] and discussed in Stewart [27]) yields *fractal images* of semantic chaos.



A more direct route to fractal images is through the complex numbers. In this paper,

[19]Devaney [7, p. 50], notes that there are stronger and weaker definitions of chaos. Devaney's definition is as follows. A function $f : I \to I$ is *chaotic* on a set I if all three of the following hold:

1. f has *sensitive dependence* on initial conditions: there exists points arbitrarily close to x which eventually separate from x by at least δ under iterations of f (here '$f^n(x)$' represents the nth iteration of the function f) i.e., $\exists\delta > 0\ \forall x \in I\ \forall$neighborhood of N of $x\ \exists y \in N\ \exists n \geq 0 \mid f^n(x) - f^n(y) \mid > \delta$;
2. f is *topologically transitive*: f has points which eventually move under iteration from one arbitrarily small neighborhood to any other, i.e., $\forall$*open sets* $U, V \subset I\ \exists k > 0\ f^k(U) \cap V \neq \varphi$;
3. the *periodic points are dense* on I: there is a periodic point between any two periodic points in the interval I, where a point x is periodic if $\exists n\ f^n(x) = x$.

we propose generalizing logic from degrees of truth in terms of *one-dimensional truth-values* in the [0, 1] real-valued interval to *two-dimensional truth-vectors* in terms of the complex plane using vertical and horizontal axes on the interval [-1, 1] represent *verdicality* and *skew*. A philosophical and mathematical justification of *imaginary truth-vectors* is given and illustrated with some practical applications.

Here, for the first time, we construct the *Imaginary Liar* using truth vectors with an *imaginary* component. The Imaginary Liar is related to the Mandelbrot set and connected by a 1-1 correspondence to the period doubling bifurcation diagram for the Fuzzy Logistic Liar. The paper concludes with some open questions and suggestions for using chaotic dynamical semantics to render some of the paradoxical features of the two-slit experiment in quantum mechanics intuitively understandable.

Paradox is not illogicality, but it has been a trap for logicians: the semantic paradoxes look just a little simpler and more predictable than they actually are. The intricate, but natural, enrichments of formal logic with degrees of truth, fuzzy hedges, mathematical chaos, and imaginary truth-values are not accidental but complex in the same way a intricately notched key precisely fits a lock that opens the door to new possibilities. Our goal here is to offer glimpses into the infinitely complex and fractal patterns of semantic *instability* and *chaos* that have gone virtually unexplored.

Keywords: Liar Paradox, semantic paradoxes, truth-value gaps, fuzzy logic, fuzzy hedges, degrees of truth, dynamical systems, chaos theory, fractals, Chaotic Liar, Logistic Liar, Fuzzy Liar, Imaginary Liar, Quantum Liar, Zeno's paradoxes, truth vectors, Mandelbrot set, quantum mechanics, two-slit experiment, limitative theorems.

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▸ VINCENZO MARRA, DANIEL MCNEILL, AND ANDREA PEDRINI, *Interpreting Łukasiewicz logic into Intuitionistic logic.*

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Łukasiewicz (infinite-valued propositional) logic – denoted L – and Intuitionistic (propositional) logic – denoted Int – are two of the oldest and most well studied systems of non-classical logic. See [2] and [3, 1], respectively, for background. The two logics were conceived with entirely different motivations, and have very different formal properties. Nonetheless, we prove the following.

NOTATION. We fix countable sets of propositional variables X and Y, and write FORM_X for the set of formulæ of L and FORM_Y for the set of formulæ of Int. We use $\neg$ and $\to$ for negation and implication in Int, respectively. We write $\top$ and $\bot$ for the logical constants *verum* and *falsum*, respectively, both in Int and L. We further denote by $\vdash_{\mathsf{L}}$ and $\vdash_{\mathsf{Int}}$ the syntactic consequence relations of L and Int, respectively.

THEOREM. *There exists a deductively closed theory Θ_{L} in Int, and a function $\mathrm{T} : \mathrm{FORM}_X \to \mathrm{FORM}_Y$ satisfying $\mathrm{T}(\top) = \bot$, such that, for each $\alpha, \beta \in \mathrm{FORM}_X$, the following holds.*

$$\beta \vdash_{\mathsf{L}} \alpha \qquad \text{if, and only if,} \qquad \Theta_{\mathsf{L}} \cup \{\mathrm{T}(\alpha)\} \vdash_{\mathsf{Int}} \mathrm{T}(\beta). \tag{1}$$

A corollary about interpreting provability in L into Int follows:

COROLLARY. *With Θ_{L} and $\mathrm{T} : \mathrm{FORM}_X \to \mathrm{FORM}_Y$ as in the theorem, the following holds for each $\alpha \in \mathrm{FORM}_X$.*

$$\vdash_{\mathsf{L}} \alpha \qquad \text{if, and only if,} \qquad \Theta_{\mathsf{L}} \vdash_{\mathsf{Int}} \neg\mathrm{T}(\alpha). \tag{2}$$

SKETCH OF PROOF OF COROLLARY. Taking $\beta = \top$ and using the fact that $T(\top) = \bot$, from the theorem we deduce

$$\vdash_{\mathsf{L}} \alpha \qquad \text{if, and only if,} \qquad \Theta_{\mathsf{L}} \cup \{\mathrm{T}(\alpha)\} \vdash_{\mathsf{Int}} \bot.$$

From the Deduction Theorem for Int, together with the fact that $\neg\varphi$ is logically equivalent to $\varphi \to \bot$ in Int, we obtain:

$$\vdash_{\mathsf{L}} \alpha \qquad \text{if, and only if,} \qquad \Theta_{\mathsf{L}} \vdash_{\mathsf{Int}} \mathrm{T}(\alpha) \to \bot \qquad \text{if, and only if,} \qquad \Theta_{\mathsf{L}} \vdash_{\mathsf{Int}} \neg\mathrm{T}(\alpha).$$

$\dashv$

The proof of the theorem rests on a remarkable property of the lattice L_{fa} of finitely axiomatisable theories in L:

LEMMA 1. *The (distributive) lattice L_{fa} is a (countable)Heyting algebra.*

The set of maximally consistent theories in L carries a natural topology that makes it homeomorphic to $[0,1]^\omega$. The lattice L_{fa} is anti-isomorphic to the lattice of (cylindrified) rational polyhedra in $[0,1]^\omega$. This is proved by passing to Lindenbaum-Tarski algebras, and applying the geometric duality theory of Chang's MV-algebras, the algebraic counterparts of L. Algebraically, the lemma asserts the remarkable fact that *the lattice of principal ideals of F_{L}, the free MV-algebra on ω generators, is a countable Heyting algebra.* This result is part of a more general investigation of the topology of prime spectral spaces of MV-algebras and related structures; see Andrea Pedrini's abstract for further details. It follows that there is an onto homomorphism of Heyting algebras

$$q \colon \mathrm{F}_{\mathsf{Int}} \twoheadrightarrow \mathrm{L}_{\mathsf{fa}},$$

where $\mathrm{F}_{\mathsf{Int}}$ is the free Heyting algebra on ω generators. Now, if we write $\{\bot\}^{\vdash_{\mathsf{L}}}$ for the deductive closure in L of $\bot$, then $\{\bot\}^{\vdash_{\mathsf{L}}}$ is the top element of L_{fa}, the filter $q^{-1}(\{\bot\}^{\vdash_{\mathsf{L}}})$ corresponds to a theory Θ_{L} in Int, and the map q can be used to define the translation map T, leading to a proof of the theorem.

REMARK. From the definition of the map T, as sketched in the preceding paragraph, it is clear why T has the peculiar property of reversing the roles of α and β in passing from L to Int. Explicitly, we have $\{\beta\}^{\vdash_{\mathsf{L}}} \leq \{\alpha\}^{\vdash_{\mathsf{L}}}$ in the (inclusion) order of the Heyting algebra L_{fa} if, and only if, $\alpha \vdash_{\mathsf{L}} \beta$.

At the time of writing, the theorem above is a purely existential result. In further work, we plan to investigate the properties of T and Θ_{L} more closely. Some obvious questions to be addressed include axiomatisability of Θ_{L}, and computability of T.

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▶ VINCENZO MARRA, DANIEL MCNEILL, AND ANDREA PEDRINI, *The space of directions of a polyhedron.*

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We study the Stone-Priestley dual space of the lattice of subpolyhedra of a compact polyhedron, with motivations coming from geometry, topology, ordered-algebra, and non-classical logic. From the perspective of algebraic logic, our contribution is a geometric investigation of lattices of prime theories in Łukasiewicz logic, possibly extended with real constants. Due to space constraints, in this abstract we assume some familiarity with Stone-Priestley duality and polyhedral geometry.

Recall that a *polytope* in $\mathbb{R}^n$ is the convex hull of a finite subset of $\mathbb{R}^n$. Polytopes are thus compact and convex. A *polyhedron* in $\mathbb{R}^n$ is any subset that can be written as the union of finitely many polytopes. Polyhedra are thus compact, but not necessarily convex. Given a polyhedron $P \subseteq \mathbb{R}^n$, let $\operatorname{Sub} P$ denote the collection of all polyhedra contained in P. Observe that $\operatorname{Sub} P$ is a distributive lattice under intersections and unions, with top element P and bottom element $\emptyset$.

Let $\operatorname{Spec}\operatorname{Sub} P$ be the spectral space of prime filters of $\operatorname{Sub} P$, equipped with the dual Stone topology. The main result we announce here is that $\operatorname{Spec}\operatorname{Sub} P$ has a concrete description in terms of a non-Hausdorff completion of the space P which holds great geometric interest. The lattice $\operatorname{Sub} P$ is an instance of a *Wallman basis* of the topological space P. This leads to the natural (Wallman) embedding $P \hookrightarrow \operatorname{Spec}\operatorname{Sub} P$ that extends P from a space consisting of points to one consisting of *directions*. Informally, a first-order direction in a polyhedron P is a point $p \in P$ together with the germ of a half line springing from p, an initial segment of which is contained in P. Higher order directions replace segments with simplices. We now give a precise statement of our main result.

We denote by $\operatorname{conv}\{w_1,\dots,w_k\}$ the convex hull of the points $w_1,\dots,w_k \in \mathbb{R}^n$, by $\mathcal{H}(w_1,\dots,w_k)$ the hyperplane of $\mathbb{R}^n$ orthogonal to each w_i, and by $\mathbb{S}^{n-1}$ the unit $(n-1)$-sphere in $\mathbb{R}^n$.

Definition 1. The *space of directions* of P is the set $\Delta(P) := \bigcup_{k=0}^{n} \Delta_k(P)$, with each layer $\Delta_k(P)$ inductively defined as

$$\begin{aligned}
&\Delta_0(P) := \{p \in \mathbb{R}^n \mid p \in P\} = P,\\
&\Delta_1(P) := \{(p, v_1) \mid p \in \Delta_0(P),\ v_1 \in \mathbb{S}^{n-1} \text{ and } \exists \varepsilon_1 > 0 \text{ s.t. } \operatorname{conv}\{p, p+\varepsilon_1 v_1\} \subseteq P\}\\
&\Delta_k(P) := \{(p, v_1,\dots,v_k) \mid (p, v_1,\dots,v_{k-1}) \in \Delta_{k-1},\ v_k \in \mathbb{S}^{n-1} \cap \mathcal{H}(v_1,\dots,v_{k-1}) \text{ and}\\
&\qquad\qquad \exists \varepsilon_1,\dots,\varepsilon_k > 0 \text{ s.t. } \operatorname{conv}\{p, p+\varepsilon_1 v_1,\dots, p+\varepsilon_1 v_1+\cdots+\varepsilon_k v_k\} \subseteq P\}.
\end{aligned}$$

The topology of $\Delta(P)$ is generated by the basis of closed sets $\{\Delta(Q) \mid Q \in \operatorname{Sub} P\}$.

There is a map

$$\mathrm{I}\colon\ \Delta(P) \longrightarrow \operatorname{Spec}\operatorname{Sub} P$$

that takes a direction $\delta \in \Delta(P)$ to the collection of subpolyhedra Q of P containing it, by which we mean that $\delta \in \Delta(Q)$. Main result:

Theorem 2. *The map* I *is a homeomorphism.*

Remark. In [1], Panti classified by geometric means the prime ℓ-ideals of free finitely generated vector lattices and lattice-ordered Abelian groups, using the notion of direction above. His main tool is the use of directional derivatives of piecewise-linear functions. While we cannot offer a full comparison of the two results here, we direct the reader's attention to the following key points. (1) Our result is independent of the theory of vector lattices and lattice-groups. (2) We do not use piecewise linear maps, nor their derivatives. Everything is encoded by filters of closed polyhedral sets. (3) We remove the algebraic restriction of freeness, which geometrically corresponds to assuming that P is homeomorphic to a sphere. (4) Our motivations are different; our result is a part of long-term project of understanding the PL topology of polyhedra in terms of their non-Hausdorff completion $\Delta(P)$.

We prove this result by direct geometric arguments of some length. If time allows, we discuss selected consequences of the main result, including compactness of the subspace of minimal primes of $\operatorname{Spec}\operatorname{Sub} P$, and the following fundamental property of $\operatorname{Sub} P$.

Lemma 3. *The lattice* $\operatorname{Sub} P$ *is a co-Heyting algebra. Equivalently, its order-dual* $\operatorname{Sub} P^{\mathrm{op}}$ *— isomorphic to the lattice of open subpolyhedra of* P *— is a Heyting algebra.*

By extending the proof of the lemma above somewhat, and applying the geometric duality theory of Chang's MV-algebras, we are able to obtain the following:

Corollary 4. *The lattice of principal MV-ideals of the free MV-algebra on* ω *generators is a Heyting algebra.*

Please see Dan McNeill's abstract for an application of the preceding corollary to an interpretation of Łukasiewicz logic into Intuitionistic logic.

[1] Giovanni Panti, *Prime ideals in free ℓ-groups and free vector lattices*, ***Journal name spelled out, no abbreviations***, vol. 219 (1999), no. 1, pp. 173–200.

▸ ROSSELLA MARRANO, *Ordinal foundation for Łukasiewicz semantics.*
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Hájek's Basic Logic [3] is meant to model a graded notion of truth. In this logic and its extensions — notably Łukasiewicz, Gödel and product logic — the real unit interval is taken as the set of truth values which are then interpreted as degrees of truth. It has been argued (see e.g. [2]) that this formalism suffers from a central difficulty: each sentence is evaluated to a unique real number. This involves both arbitrariness of the choice (how can we justify the choice of the truth value 0.24 over 0.23?) and implausibility of the interpretation (what does it mean for a sentence to be $1/\pi$ true?). This motivates an ordinal perspective on modelling degrees of truth. The key shift in focus is from the point-wise evaluation of sentences to the binary comparison of their truth values.

In this work we restrict attention to Łukasiewicz propositional infinite-valued logic Ł [1]. Let $\mathcal{L}$ be a propositional language and $\mathcal{SL}$ the set of sentences built as usual. We use $\neg$ for negation, $\rightarrow$ for implication, $\oplus$ for strong disjunction, $\bot$ for *falsum* and $\top$ for *verum*. We denote by $\vdash$ the deducibility relation of Ł. The semantics of Ł is given by the notion of *Łukasiewicz valuation*, namely a map $v\colon \mathcal{SL} \rightarrow [0,1]$ such that for any sentences θ, φ:

1. $v(\bot) = 0$,
2. $v(\neg\theta) = 1 - v(\theta)$,
3. $v(\theta \oplus \varphi) = \min\{1, v(\theta) + v(\varphi)\}$.

As an alternative approach, consider a binary relation on the set of sentences $\preceq\, \subseteq \mathcal{SL} \times \mathcal{SL}$ interpreted as "no more true than". The main contribution of this paper is to lay down sufficient conditions for such a relation to be represented by a real-valued function satisfying the conditions 1.–3. above.

More precisely, let $\theta \sim \varphi \Leftrightarrow_{def} \theta \preceq \varphi$ and $\varphi \preceq \theta$. We write $\theta \nsim \varphi$ to mean that $\theta \sim \varphi$ does not hold. We require $\preceq$ to be a preorder:

(A.1): $\preceq$ is a preorder
- **(A.1a):** $\theta \preceq \theta$
- **(A.1b):** $\theta \preceq \varphi, \varphi \preceq \chi \Longrightarrow \theta \preceq \chi$

It follows immediately that $\sim$ is an equivalence relation. Moreover, we need conditions that force the ordering to be compatible with the underlying logic:

(A.2): $\vdash \theta \rightarrow \varphi \Longrightarrow \theta \preceq \varphi$
(A.3): $\theta \preceq \top$

The behaviour of the ordering with respect to the connectives is regulated by the following:

(A.4): $\theta_1 \preceq \theta_2,\ \varphi_1 \preceq \varphi_2 \Longrightarrow \theta_1 \oplus \varphi_1 \preceq \theta_2 \oplus \varphi_2$
(A.5): $\theta \preceq \varphi \Longrightarrow \neg\varphi \preceq \neg\theta$

Notice that by using (A.6), (A.5) can be equivalently formulated in terms of $\rightarrow$. Therefore, the choice of primitive connectives does not affect the result. Lastly, a non-triviality constraint is formulated as follows:

(A.6): $\top \nsim \bot$

Given this, we prove the following:

THEOREM 1. *If* $\preceq\, \subseteq \mathcal{SL}^2$ *satisfies axioms (A.1)–(A.6) then there exists a Łukasiewicz valuation* $v\colon \mathcal{SL} \rightarrow [0,1]$ *such that for all* $\theta, \varphi \in \mathcal{SL}$*:*

$$\theta \preceq \varphi \Longrightarrow v(\theta) \leq v(\varphi).$$

The proof of the theorem rests on a central MV-algebraic result ([1] Corollary 1.2.15). Write [0,1] as usual for the standard MV-algebra $\langle [0,1], \neg, \oplus, 0\rangle$ with $\neg x = 1 - x$ and $x \oplus y = \min\{1, x+y\}$.

Lemma 2. *If M is a non-trivial MV-algebra then there exists at least one homomorphism:*

$$m\colon M \to [0,1].$$

Accordingly, the key-step is to construct an algebra on $(\mathcal{SL}, \preceq)$. To this aim, we take as universe the quotient set $\mathcal{SL}/{\sim} = \{[\theta]_\sim | \theta \in \mathcal{SL}\}$ and we define the following operations:

$$\widetilde{\perp} := [\perp]_\sim := \perp$$

$$\widetilde{\neg}[\theta]_\sim := [\neg\theta]_\sim$$

$$[\theta]_\sim \mathbin{\widetilde{\oplus}} [\varphi]_\sim := [\theta \oplus \varphi]_\sim.$$

It can be proved that $\widetilde{\neg}$ and $\widetilde{\oplus}$ are well-defined and moreover, by virtue of the axioms, the algebra $(\mathcal{SL}/{\sim}, \widetilde{\neg}, \widetilde{\oplus}, \widetilde{\perp})$ turns out to be a non-trivial MV-algebra. By the Lemma there is at least one homomorphism m from $(\mathcal{SL}/{\sim}, \widetilde{\neg}, \widetilde{\oplus}, \widetilde{\perp})$ to $[0,1]$. Define a function $V_\preceq\colon \mathcal{SL} \to [0,1]$ as $m \circ q_\sim$, where $q_\sim$ is the canonical map from $\mathcal{SL}$ to $\mathcal{SL}/{\sim}$. The function $V_\preceq$ is a Łukasiewicz valuation on $\mathcal{SL}$ and, furthermore, it preserves the ordering $\preceq$, that is: $\theta \preceq \varphi \Rightarrow V_\preceq(\theta) \leq V_\preceq(\varphi)$ for all $\theta, \varphi \in \mathcal{SL}$. This is precisely the statement of the Theorem.

We emphasize that this function is in general not unique. However, by using further MV-algebraic result we obtain

Corollary 3. *If $\preceq$ is a total order then $V_\preceq$ is unique.*

Interestingly enough, the method of evaluation of sentences based on comparative judgments induces in a natural way a semantics for Łukasiewicz logic which enjoys both completeness and strong completeness. In order to have a better grasp on this, define the semantical notion of tautology and logical consequence as follows:

$$\models_\preceq \varphi \Leftrightarrow_{def} \forall \preceq \subseteq \mathcal{SL}^2 \text{ satisfying (A.1)–(A.6) } \varphi \sim \top,$$

$$\Theta \models_\preceq \varphi \Leftrightarrow_{def} \forall \preceq \subseteq \mathcal{SL}^2 \text{ satisfying (A.1)–(A.6) if } \forall\theta \in \Theta\ \theta \sim \top \text{ then } \varphi \sim \top.$$

Given this, we prove the following:

Theorem 4. $\forall \varphi \in \mathcal{SL}\ \models_\preceq \varphi \Longleftrightarrow \vdash \varphi$.

Theorem 5. $\forall \varphi \in \mathcal{SL}\ \forall \Theta \subseteq \mathcal{SL}\ \Theta \models_\preceq \varphi \Longleftrightarrow \Theta \vdash \varphi$.

Theorem 1 sets the conditions under which a quantitative evaluation arises from qualitative comparisons. In addition, we argue in favour of the plausibility of the axioms given the interpretation of $\preceq$. This assures that if the sentences can be compared 'well enough' with respect of their truth value then it is as if we attach them a numerical evaluation. Furthermore, Theorems 4 and 5 guarantee that $\preceq$ supplies Ł with an adequate alternative semantics. This goes some way towards providing an ordinal foundation for the notion of degrees of truth in the case of Łukasiewicz logic.

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§1. Introduction. Łukasiewicz many-valued logic is frequently given a real-valued semantics in the interval $[0, 1]$. Numbers are interpreted as "fuzzy" truth values, representing degrees of certainty, with 1 as "certainly true", 0 as "certainly false", and with other values representing intermediate levels of certainty. Traditionally, such degrees of certainty are thought of as conceptually distinct from probabilities, and fuzzy logic is promoted as a formalism for reasoning about non-probabilistic forms of uncertainty.

In [8, 10, 11], building on [6], the current authors have shown that modal extensions of Łukasiewicz many-valued logic are well suited to expressing properties of probabilistic concurrent systems in computer science. The usefulness of Łukasiewicz logic, for this purpose, contrasts with its popular image as a primarily non-probabilistic formalism.

The present work forms part of an attempt to give a retrospective justification for the relevance of Łukasiewicz logic in such probabilistic scenarios. As our main contribution, we give a game interpretation of Łukasiewicz logic, in which the connectives have a purely probabilistic interpretation, and in which the value of a formula is interpreted as a genuine probability. The games we consider are different in spirit from other game-theoretic interpretations of Łukasiewicz logic of which we are aware [4, 12, 2, 8].

The basic idea we work with is that a logical formula represents an event in a probability space, and the value assigned to the formula is simply the probability that the event holds. For example, one might have a formula representing the proposition that it will rain today in Vienna, the probability of which (for a day in July) is 0.29. Similarly, one might have a formula representing that it will rain today in Salzburg, which has probability 0.46. Now consider the conjunction:

It will rain today in both Vienna and Salzburg.

This event has its own probability, but this cannot, of course, be calculated from the probabilities of the individual events, since they are not independent. Instead, there is some correlation between the events. If they were *maximally correlated*, it would rain in Salzburg on every day in which it rains in Vienna, and the probability of the conjunction holding would simply be the probability, 0.29, of it raining in Vienna. If the events were *maximally anticorrelated*, it would never rain in Vienna and Salzburg on the same day, and the probability of the conjunction would be 0. In reality, we (the authors) don't know the actual probability, but it must lie in somewhere between 0 and 0.29.

Now consider the general case of any two events A and B in some probability space. Then we always have the bounds:

$$\begin{array}{rcccl} \max(0, \mathbf{P}(A) + \mathbf{P}(B) - 1) & \leq & \mathbf{P}(A \wedge B) & \leq & \min(\mathbf{P}(A), \mathbf{P}(B)) \\ \max(\mathbf{P}(A), \mathbf{P}(B)) & \leq & \mathbf{P}(A \vee B) & \leq & \min(1, \mathbf{P}(A) + \mathbf{P}(B)) \end{array}$$

In this table, the top-left and bottom-right entries compute the strong Łukasiewicz conjunction and disjunction of the individual probabilities, and the top-right and bottom-left entries compute the weak conjunction and disjunction. Thus the Łukasiewicz connectives give bounds on probabilities computed in joint probability distributions.

The above observations are simple, and by no means original (e.g., see Section 3 of [1]). The contribution of the present work is to turn them into a probabilistic interpretation of Łukasiewicz logic. Our interpretation is built around the idea that the Łukasiewicz connectives express conjunctions and disjunctions in joint probability distributions. Whereas the connectives themselves calculate their values as if events are maximally correlated or anticorrelated, in our interpretation we allow events to be governed by any joint probability distribution. This reflects the fact that, in experience, events are rarely maximally correated or maximally anticorelated. Nonetheless, the Łukasiewicz connectives are pertinent because they establish bounds. Moreover, the combination of upper and lower bounds is accounted for naturally by a game-theoretic interpretation, with one player trying to maximize the probability of a specified outcome and the other trying to minimize it.

§2. Coupling games. We consider the following syntax for Łukasiewicz logic.

$$\varphi ::= A \mid \underline{1} \mid \underline{0} \mid \varphi \sqcap \varphi \mid \varphi \sqcup \varphi \mid \varphi \odot \varphi \mid \varphi \oplus \varphi \mid \neg\varphi$$

Here, A ranges over a set of atomic formulas. Given a valuation function $\mathcal{V}$ from atomic formulas to $[0, 1]$, the function is extended to all formulas by:

$$\begin{aligned} \mathcal{V}(\underline{1}) &= 1 & \mathcal{V}(\underline{0}) &= 0 \\ \mathcal{V}(\varphi \sqcap \psi) &= \min(\mathcal{V}(\varphi), \mathcal{V}(\psi)) & \mathcal{V}(\varphi \sqcup \psi) &= \max(\mathcal{V}(\varphi), \mathcal{V}(\psi)) \\ \mathcal{V}(\varphi \odot \psi) &= \max(0,\, \mathcal{V}(\varphi) + \mathcal{V}(\psi) - 1) & \mathcal{V}(\varphi \oplus \psi) &= \min(1,\, \mathcal{V}(\varphi) + \mathcal{V}(\psi)) \\ \mathcal{V}(\neg\varphi) &= 1 - \mathcal{V}(\varphi) \end{aligned}$$

Our game interpretation involves a number of ingredients. To every atomic formula A we assign an event in a probability space, specified via three pieces of information:

X_A	a measurable space
μ_A	a probability measure on X_A
E_A	an event in X_A.

As usual, by a *measurable space*, we mean a set X_A together with a σ-algebra of subsets of X_A called *events*. We avoid clutter by leaving the σ-algebra implicit in our notation. Although we use σ-algebra-based probability spaces for maximum generality, nothing essential is lost by restricting to finite probability spaces.

The game is played between two players, *Maximizer* and *Minimizer*, where Maximizer is trying to maximize the probability that the property expressed by the formula is true, and Minimizer is trying to minimize this. This is achieved by players using their turns to choose "couplings" of probability measures. Recall (see, e.g., [7]), that a *coupling* of two probability measures, μ_1, μ_2 on X_1, X_2 respectively, is a probability measure μ, on $X_1 \times X_2$, that has μ_1 and μ_2 as marginals.

The game for a formula φ is played on subformulas of φ. Once play of the game for φ has ended, we end up with an event E_φ in a probability space (X_φ, μ_φ). The rules of play depend on the outermost connective of φ:

- If φ is a propositional atom A then there is nothing to do, since we already have X_A, μ_A, E_A.
- If φ has a binary main connective, $\varphi_1 \star \varphi_2$, then proceed as follows:
    1. Recursively play the game for φ_1 and the game for φ_2 to produce $X_{\varphi_1}, \mu_{\varphi_1}, E_{\varphi_1}$ and $X_{\varphi_2}, \mu_{\varphi_2}, E_{\varphi_2}$.
    2. Define:

$$X_{\varphi_1 \star \varphi_2} = X_{\varphi_1} \times X_{\varphi_2}\,, \text{ the product measurable space}$$

$$E_{\varphi_1 \star \varphi_2} = \begin{cases} \{(x_1, x_2) \mid x_1 \in E_1 \,\wedge\, x_2 \in E_2\} & \text{if } \star \in \{\sqcap, \odot\} \\ \{(x_1, x_2) \mid x_1 \in E_1 \,\vee\, x_2 \in E_2\} & \text{if } \star \in \{\sqcup, \oplus\} \end{cases}$$

3. Player P chooses a coupling $\mu_{\varphi_1 \star \varphi_2}$ of μ_{φ_1} and μ_{φ_2}, where P is Maximizer if $\star \in \{\sqcap, \oplus\}$, and P is Minimizer if $\star \in \{\sqcup, \odot\}$.

- If φ is a negation $\neg\varphi'$ then proceed as follows.
  1. Play the game for φ', with the roles of Maximizer and Minimizer reversed, to produce $X_{\varphi'}, \mu_{\varphi'}, E_{\varphi'}$.
  2. Define:

$$
\begin{aligned}
X_{\neg\varphi'} &= X_{\varphi'} \\
\mu_{\neg\varphi'} &= \mu_{\varphi'} \\
E_{\neg\varphi'} &= X_{\varphi'} - E_{\varphi'}
\end{aligned}
$$

We make a few comments on this game. Note that, in contrast to the usual style of logical games, the direction of play runs from the leaves of the syntax tree for φ (the propositional atoms) towards the root (φ itself). In the case of a binary connective $\varphi_1 \star \varphi_2$, it is possible that φ_1 and φ_2 are the same formula. In that case, a separate game is played for each of them, and the probability measures μ_{φ_1} and μ_{φ_2} may be different. In the case of a negation, we have, for the sake of brevity, been somewhat informal about how the role reversal between the players works. This is, however, standard. One could equally well negate the formula using de Morgan duals, pushing all negations to propositional atoms, which would have the same effect.

Note that the four binary connectives are distinguished according to whether they are conjunctions or disjunctions (the definition of $E_{\varphi_1 \star \varphi_2}$), and which player makes the choice of $\mu_{\varphi_1 \star \varphi_2}$.

	Minimizer chooses	Maximizer chooses
Conjunction	$\odot$	$\sqcap$
Disjunction	$\sqcup$	$\oplus$

A *play* of the game for φ results in an event $E_{\varphi'}$ in a probability space $(X_{\varphi'}, \mu_{\varphi'})$ being assigned to every (occurrence of a) subformula φ' in φ. The *result* of the play is the probability $\mu_\varphi(E_\varphi)$ that E_φ holds in (X_φ, μ_φ).

We write $\mathcal{M}_1(X)$ for the set of probability measures on a measurable space X. A *strategy* for Maximizer on φ is a function assigning to every (occurrence of a) subformula $\varphi_1 \star \varphi_2$ in φ, where $\star \in \{\sqcap, \oplus\}$, a function:

$$\sigma^{\mathrm{Max}}_{\varphi_1 \star \varphi_2} : \mathcal{M}_1(X_{\varphi_1}) \times \mathcal{M}_1(X_{\varphi_2}) \to \mathcal{M}_1(X_{\varphi_1} \times X_{\varphi_2})$$

which enjoys the property that $\sigma^{\mathrm{Max}}_{\varphi_1 \star \varphi_2}(\mu_1, \mu_2)$ is a coupling of μ_1 and μ_2. We call such a function a *coupling function*. Note that the notion of strategy is determined, independently of any play of the game, because the measurable spaces X_{φ_1} and X_{φ_2} are determined by the formulas φ_1 and φ_2 alone.

Similarly, a *strategy* for Minimizer on φ is a function assigning to every (occurrence of a) subformula $\mu_{\varphi_1 \star \varphi_2}$ in φ, where $\star \in \{\sqcup, \odot\}$, a coupling function:

$$\sigma^{\mathrm{Min}}_{\varphi_1 \star \varphi_2} : \mathcal{M}_1(X_{\varphi_1}) \times \mathcal{M}_1(X_{\varphi_2}) \to \mathcal{M}_1(X_{\varphi_1} \times X_{\varphi_2})$$

A pair of strategies σ^{Max} and σ^{Min} for φ together determine a play. We write $\mathrm{Res}(\sigma^{\mathrm{Max}}, \sigma^{\mathrm{Min}})$ for the probability given as the result of the play. For each propositional atom A, define $\mathcal{V}(A) = \mu_A(E_A)$. The main result of the paper is a determinacy result for the coupling game for φ. Furthermore, the thereby determined *value* of the game coincides with $\mathcal{V}(\varphi)$.

THEOREM 1.

$$\sup_{\sigma^{\mathrm{Max}}} \inf_{\sigma^{\mathrm{Min}}} \mathrm{Res}(\sigma^{\mathrm{Max}}, \sigma^{\mathrm{Min}}) \;=\; \mathcal{V}(\varphi) \;=\; \inf_{\sigma^{\mathrm{Min}}} \sup_{\sigma^{\mathrm{Max}}} \mathrm{Res}(\sigma^{\mathrm{Max}}, \sigma^{\mathrm{Min}})$$

The proof constructs an optimum strategy for Maximizer, based on building couplings that maximally correlate (or anticorrelate) the events associated with subformulas, according to connective at hand.

We remark that it is easy to extend our game interpretation to cover certain extensions of Łukasiewicz logic. For example, one can easily incorporate connectives for scalar multiplication [3, 13], and/or binary multiplication (and its dual) [5]. Extending the interpretation to include least and greatest fixed points [8, 10] presents more of a challenge, and is a topic of ongoing research. Our hope is to obtain a more natural game interpretation for *Łukasiewicz μ-calculus* [10] than the one given in [8], which is based on a technical reduction to *meta-parity games*.

Finally, we discuss the relevance of coupling games to the use of Łukasiewicz logic as a formalism for the specification and verification of probabilistic systems [8, 9, 10, 11]. In this context, Minimizer represents the environment in which the system is embedded. Since we know nothing about how different probabilistic choices under the control of the environment are correlated, it is appropriate to model them under the assumption that their correlations may be as unhelpful as possible. Maximizer, in contrast, models the possibility of using favourable couplings between probabilistic choices as a mechanism for reasoning about probabilistic behaviour. This approach derives from the "coupling method" in probability theory [7]. We believe it has a vital role to play in the development of general methods for reasoning about probabilisitic computational systems, cf. [9].

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§1. Introduction. One of the main topics of Abstract Algebraic Logic [4, 5] is the study of the ***Leibniz operator***, i.e., a particular map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{P}(A) \to \mathrm{Co}\boldsymbol{A}$, which can be defined for every algebra $\boldsymbol{A}$ and associates a specific congruence with any subset of the universe of $\boldsymbol{A}$. One of its major applications has been to exploit the order-theoretic and set-theoretic behaviour of the operator over the deductive filters $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ of a logic $\mathcal{L}$, for arbitrary algebras $\boldsymbol{A}$, in order to capture interesting facts about its own definability and that of the truth predicate, in models of $\mathcal{L}$. This was essentially discovered by Blok and Pigozzi [1] for algebraizable logics, and their work and that of other scholars (Czelakowski, Herrmann, Jansana, Raftery) on this topic has given rise to a whole hierarchy, called the ***Leibniz hierarchy*** [3, 6], in which logics are classified by means of properties of the Leibniz operator which determine how nicely the Leibniz congruences and the truth predicates can be described in models of the logic.

Until now the two classes of logics which lie at the bottom of the Leibniz hierarchy were that of the ***protoalgebraic logics***, i.e., logics whose Leibniz congruence can be defined by means of a set of formulae $\Delta(x, y, \overline{z})$ in two variables with parameters, and that of the ***truth-equational logics***, i.e., logics whose truth predicates can be defined through sets of equations $\boldsymbol{\tau}(x)$ in one variable. This highlights an asymmetry between the abstract treatment of the definability of the Leibniz congruence and that of the truth predicates: while protoalgebraicity allows the presence of parameters in the definition of the Leibniz congruence, truth-equationality does not admit parameters in the definition of truth predicates.

The starting point of this talk will be to eliminate this asymmetry by introducing a new class of logics, whose truth predicates are defined by means of equations with parameters. Then we will go through the consideration of weaker conditions on the truth predicates of a logic. This will give rise to a small hierarchy, in which logics are classified according to the way their truth predicates are defined; this new hierarchy can be thought of as an extension of the Leibniz hierarchy, since almost all the conditions we take into account turn out to be characterised by a property of the Leibniz operator.

§2. Universal definability In order to consider logics whose truth predicates can be defined by means of equations with parameters, we need to introduce some notation. First, $\mathcal{L}$ denotes an arbitrary propositional logic. Moreover, we say that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ ***almost*** enjoys a certain property over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ when its restriction to $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ enjoys it. In this research one has to take special care of the empty filter, because ***purely inferential logics***, i.e., logics without theorems, play an important role in the study of logics whose truth predicates are defined by equations with parameters. An analogous convention applies to classes of matrices, denoted by $\mathcal{M}$, as follows: we say that a class of matrices $\mathcal{M}$ ***almost*** enjoys a certain property, when every $\langle \boldsymbol{A}, F\rangle \in \mathcal{M}$ such that $F \neq \emptyset$ enjoys it. Finally let us recall that, given a logic $\mathcal{L}$, its lattice of ***theories*** $\mathcal{T}h\mathcal{L}$ is just the lattice of deductive filters over the term algebra built up with countably many variables, denoted by $\boldsymbol{Fm}$. The Leibniz operator over term algebras is denoted simply by $\boldsymbol{\Omega}$.

Definition 1. A ***universal translation*** is a set $\boldsymbol{\tau}(x, \overline{y})$ of equations in a distinguished variable x with parameters $\overline{y}$. An ***equational translation*** is a universal translation without parameters.

Universal and equational translations witness the definability of truth predicates by bounding parameters (if any) by a universal quantifier and considering the *solutions* of the resulting universally quantified equations. More precisely, given a universal translation $\boldsymbol{\tau}(x, \overline{y})$ and an algebra $\boldsymbol{A}$, we put

$$S^{\boldsymbol{A}}(\boldsymbol{\tau}) = \{a \in A : \boldsymbol{A} \models \boldsymbol{\tau}(a, \overline{c}) \text{ for every } \overline{c} \in A\}.$$

Definition 2. Let $\mathcal{M}$ be a class of matrices. A universal (resp. equational) translation $\boldsymbol{\tau}$ ***defines truth*** in $\mathcal{M}$ when $S^{\boldsymbol{A}}(\boldsymbol{\tau}) = F$ for every $\langle \boldsymbol{A}, F\rangle \in \mathcal{M}$. ***Truth is universally (resp. equationally) definable*** in $\mathcal{M}$ when there is a universal (equational) translation which defines truth in $\mathcal{M}$.

A logic $\mathcal{L}$ whose truth predicates are equationally definable in the class $\mathsf{Mod}^*\mathcal{L}$ of its Leibniz-reduced models is called ***truth-equational***; these logics were introduced and characterised by Raftery in [6]. When looking for a generalisation of truth-equational logics which admits parameters in the definition of truth, one is tempted to consider logics whose truth predicates are universally definable in $\mathsf{Mod}^*\mathcal{L}$. However, Corollary 4 will tell us that this is not a good idea, since such logics turn out to coincide with the truth-equational ones. Quite surprisingly it turns out that a suitable generalisation can be achieved by considering the notion of ***almost universal definability***, which can be characterised as follows:

Theorem 3. *The following conditions are equivalent:*

(i) *Truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$.
(ii) $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is almost completely order reflecting over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$, *for every* $\boldsymbol{A}$.
(iii) $\boldsymbol{\Omega}$ *is almost completely order reflecting over* $\mathcal{T}h\mathcal{L}$.

From this Raftery's characterisation of truth-equational logics in terms of the Leibniz operator being completely order-reflecting follows directly. Moreover:

Corollary 4. *The following conditions are equivalent:*

(i) *Truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}$.
(ii) *Truth is universally definable in* $\mathsf{Mod}^*\mathcal{L}$.
(iii) *Truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$ *and* $\mathcal{L}$ *has theorems.*

In particular, Corollary 4 tells us that examples of logics whose truth predicates are almost universally definable in $\mathsf{Mod}^*\mathcal{L}$, but not equationally definable, are forced to be purely inferential. On may wonder whether there really are examples of such logics. Now we provide a general method to construct logics of this kind.

Definition 5 (Cintula and Noguera [2]). A logic $\mathcal{L}$ has a ***protodisjunction*** when there is a term-definable binary connective $\vee$ such that $x \vdash_{\mathcal{L}} x \vee y$ and $y \vdash_{\mathcal{L}} x \vee y$.

Theorem 6. *If* $\mathcal{L}$ *is Fregean and has a protodisjunction, then the universal translation* $\boldsymbol{\tau}(x, \overline{y}) = \{x \vee y \approx x\}$ *almost defines truth in* $\mathsf{Mod}^*\mathcal{L}$.

A result dual to Theorem 6 can be stated for a very weak kind of conjunction. In particular, we obtain that every purely inferential fragment of a Fregean logic with a protodisjunction has its truth predicate almost universally definable in $\mathsf{Mod}^*\mathcal{L}$, but not equationally definable. Some examples of this kind are the $\{\vee\}$- and the $\{\wedge, \vee\}$-fragments of classical propositional logic. Moreover, it is worth remarking that Theorem 6, together with the fact that every logic with theorems has a protodisjunction, implies:

Corollary 7. *Let $\mathcal{L}$ be Fregean. $\mathcal{L}$ has theorems if and only if truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}$.

§3. A hierarchy for truth predicates. We now consider some weaker conditions on the truth predicates of the reduced matrix semantics $\mathsf{Mod}^*\mathcal{L}$ of a logic $\mathcal{L}$, which fit well in the framework of the Leibniz hierarchy. Forgetting about syntactic objects as equations, things become a little more subtle. In particular, our transfer results from $\mathcal{T}h\mathcal{L}$ to filters over arbitrary algebras seem to depend on some cardinality assumptions. This makes useful to introduce the following notation: all along this section L will be the fixed but arbitrary algebraic language we are working in. Moreover, given a cardinal λ, we will denote by $\boldsymbol{Fm}_\lambda$ the term algebra built up with λ variables.

The first condition we will take into account arises from the following considerations. In Theorem 3 we showed that the fact that truth predicates are almost universally definable in $\mathsf{Mod}^*\mathcal{L}$ is equivalent to the fact that the Leibniz operator $\boldsymbol{\Omega}$ is almost completely order reflecting over the theories of the logic $\mathcal{T}h\mathcal{L}$. Therefore it is natural to wonder whether there is any meaningful property of truth predicates which corresponds to the fact that $\boldsymbol{\Omega}$ is simply (almost) order reflecting over deductive filters. In order to answer this question, let us introduce a new concept.

Definition 8. Let $\mathcal{M}$ be a class of matrices and $\mathcal{L}$ the logic it defines. Truth is ***minimal*** in $\mathcal{M}$ when for every $\langle \boldsymbol{A}, F\rangle \in \mathcal{M}$, $F = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\})$.

The first thing it is worth remarking is that there are logics for which truth is minimal but not almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. Nevertheless it seems hard to find a natural example of such logics: for the moment we were able only to construct an *ad hoc* one. Next result tells us that logics whose truth predicates are (almost) minimal are the solution to our original question, i.e., are characterised by the fact that the Leibniz operator is (almost) order reflecting over deductive filters. Even if it is still an open problem whether this condition transfers from the theories of the logics to filters over arbitrary algebras, Corollary 10 provides a positive solution for countable languages.

Theorem 9. *The following conditions are equivalent:*

(i) *Truth is (almost) minimal in* $\mathsf{Mod}^*\mathcal{L}$.

(ii) $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is (almost) order reflecting over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$*, for every* $\boldsymbol{A}$.

(iii) $\boldsymbol{\Omega}$ *is (almost) order reflecting over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}_\lambda$*, where* $\lambda = \max\{\aleph_0, |L|\}$.

Corollary 10. *If* $|L| \leq \aleph_0$*, then* $\boldsymbol{\Omega}$ *is (almost) order reflecting over* $\mathcal{T}h\mathcal{L}$ *if and only if* $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is (almost) almost order reflecting over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every* $\boldsymbol{A}$.

In particular, from Theorem 9 it follows that the fact that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$ implies that it is almost minimal too. However, it is possible to show that this is not true for arbitrary classes of matrices.

The next condition we will consider is that of implicit definability.

Definition 11. Truth is ***implicitly definable*** in $\mathcal{M}$ when matrices in $\mathcal{M}$ are determined by their algebraic reducts, i.e., when $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathcal{M}$ implies that $F = G$.

Obviously, if truth is (almost) minimal in a class of matrices $\mathcal{M}$, then it is also (almost) implicitly definable. Moreover, it is possible to construct examples of logics whose truth predicats are implicitly definable, but not minimal, in $\mathsf{Mod}^*\mathcal{L}$: one of these is the $\{\Box, \top\}$-fragment of the modal system $\mathcal{S}4$. When referred to $\mathsf{Mod}^*\mathcal{L}$, implicit definability can be characterised by the injectivity of the Leibniz operator:

Theorem 12. *The following conditions are equivalent:*

(i) *Truth is (almost) implicitly definable in* $\mathsf{Mod}^*\mathcal{L}$.
(ii) $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is (almost) injective over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$, *for every* $\boldsymbol{A}$.
(iii) $\boldsymbol{\Omega}$ *is (almost) injective over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}_{\lambda}$, *where* $\lambda = \max\{\aleph_0, |L|\}$.

Raftery in [6] posed the following problem: is it true that if the Leibniz operator is injective over the theories of a logic, then it is injective over the deductive filters of arbitrary algebras? In the same paper he provided a positive answer to this problem under the assumption that the language L is mono-unary, i.e., contains only one function symbol, which is unary. Now, Theorem 12, allows us to generalise this result and obtain a positive answer for all countable languages.

Corollary 13. *If* $|L| \leq \aleph_0$, *then* $\boldsymbol{\Omega}$ *is (almost) injective over* $\mathcal{T}h\mathcal{L}$ *if and only if* $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is (almost) injective over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every* $\boldsymbol{A}$.

The last condition on truth predicates we shall consider is the following:

Definition 14. Truth is ***indiscernible*** in $\mathcal{M}$ when matrices in $\mathcal{M}$ are determined up to isomorphism by their algebraic reducts, i.e., when $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathcal{M}$ implies that $\langle \boldsymbol{A}, F\rangle \cong \langle \boldsymbol{A}, G\rangle$.

This notion is still an experimental one. We were able to prove that there are logics whose truth predicates are (almost) indiscernible in $\mathsf{Mod}^*\mathcal{L}$, but not (almost) implicitly definable; among them there are the $\{\neg\}$-fragments of classical and intuitionistic logic and other more exotic creatures. However, for the moment we were not able to characterise logics whose truth predicates are (almost) indiscernible in $\mathsf{Mod}^*\mathcal{L}$ by means of a property of the Leibniz operator.

§3. Conclusion. Along this journey we introduced four classes of logics (eight if we take into account their *almost* versions) and proved that at least three (six) of them fit naturally in (an extended version of) the Leibniz hierarchy. We also proved that in general these classes are all different; however, one can prove (using Corollary 7) that the truth predicates of a Fregean logic $\mathcal{L}$ are indiscernible in $\mathsf{Mod}^*\mathcal{L}$ if and only if they are equationally definable, so that the new hierarchy collapses for Fregean logics with theorems. It is possible to prove that the same happens in the cases of protoalgebraic logics and of mono-unary logics with theorems.

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► M. ANDREW MOSHIER, *Bitopological Duality and Three-valued Logic*.
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Duality theory provides well-understood and parametrizable machinery for relating logics (more generally algebraic structures) of various kinds and their topological semantics [1]. Stone and Priestley duality are of course the best known examples, but many others, including some versions of multi-valued logic have been profitably sent through this machinery. The results, however, are inherently two-valued in that the resulting topological structures are always Stone spaces. Any "multi-valuedness" is carried by the structure of a discrete dualizing (truth value) object. So the topological structure of the semantics is still essentially Boolean.

In this work, we develop techniques to deal directly in three-valued bitopological semantics [3]. One topology provides the possibilities for affirming propositions and another topology provides the possibilities for denying them. The two topologies need not be identical (when negation is not present in the langauge), and it need not be the case that any two distinct models can be separated by a single classical proposition. So the underlying joint topology is not necessarily Boolean. Among other consequences of this, the judgement "p entails p" is not tautological.

The work reported here is closely related to bilattice-based semantics (see for example [2]). However, bilattice-based semantics typically falls within the purview of natural duality and is therefore still essentially Boolean, in so far as the spaces of models are compact zero-dimensional. This work generalizes to settings in which zero-dimensionality is not required. In [4], some of the the frame-theoretic machinery needed to relate the three-valued semantics discussed here with a four-valued bilattice semantics is worked out.

On the side of the object languages we take a very general view in which a "logic" may be nothing more that a partially ordered set (actually, a pre-ordered set would do) of "propositions" ordered by a consequence relation together with a suitable notion of *strong* entailment. This entailment can be motivated by considering how a skeptic might think in a three-valued semantical setting.

Suppose a skeptic considers models of propositions in which he may affirm or deny individual propositions. We only ask him not to be crazy, simultaneously affirming and denying a particular proposition. He is free, as a skeptic should be, to do neither. Suppose in some model under his consideration he can not deny p nor can he affirm q. Then the skeptic should not accept that p entails q. After all, in one of the models he considers p might be true and q false. He can only accept, semantically, that p entails q if it is the case that in every model in which he cannot deny p, he must accept q.

By this interpretation of entailment, p does not necessarily entail p. Indeed, only a "classical" proposition, i.e., one that is denied or affirmed in every model will entail itself.

Suppose our skeptic accepts that p entails r. Being skeptics ourselves, we might ask why. Our friend should be able to produce another proposition that interpolates between p and r. That is, he should be able to elaborate by saying "p entails q and q entails r." This added condition is satisfied trivially if "p entails p", so this merely generalizes the classical setting.

The simplest framework for this is to consider an abstract "logic" merely to be a poset $(L, \sqsubseteq)$ where elements of L are "propositions" and $p \sqsubseteq q$ indicates that p is formally weaker than q. Then a *strong* entailment is a relation $\prec$ on L satisfying:

- $\prec$ is stronger than $\sqsubseteq$ (if $p \prec q$ then $p \sqsubseteq q$);
- $\prec$ is closed under weakening (if $p' \sqsubseteq p \preceq q \sqsubseteq p'$ then $p' \prec q'$);
- $\prec$ is strongly interpolative (if $p_0 \prec r, \ldots, p_{n-1} \prec r$ then there is some q so that $p_0 \prec q, \ldots, p_{n-1} \prec q$ and $q \prec r$, and dually).

We can also ask, e.g., for L to support various logical connectives, and that $\prec$ is compatible with them. Our question is this: When is there a three-valued semantics of L so that $\prec$ is the above sketched skeptic's interpretation of entailment?

In this paper, we show that the answer is "always, when there is already a corresponding representation for the classical two-valued version."

§1. Bitopology. For a topology τ on set X, the *specialization pre-order* is the relation $\sqsubseteq_\tau$ defined by $x \sqsubseteq_\tau y$ if every τ-neighborhood of x is a τ-neighborhood of y. Evidently, $\sqsubseteq_\tau$ is a partial order if and only if τ is T_0, and $\sqsubseteq_\tau$ is trivial if and only if τ is T_1.

A *bitopological space* $X = (X, \tau_-, \tau_+)$ is a set equipped with two topologies, hardly an interesting idea unless the two topologies are somehow related. We will refer to a bitopological space is a *bispace* if the specialization order for τ_+ is opposite the specialization order for τ_-. In a bispace, we think of the specialization order for τ_+ as *the* order on the space and write it as $\sqsubseteq$ when X is clear. A map between bispaces is *bicontinuous* if it is continuous with respect to the two topologies separately.

A bispace is *ordered* if either topology is T_0 (equivalently, if both are). A bispace is *flat* if $\tau_- = \tau_+$. Clearly, a flat ordered space is essentially just a T_1 space. We look for analogues of other topological properties useful for topological representation.

An ordered bispace X is *Hausdorff* if for every $x \not\sqsubseteq_+ y$, there is a neighborhood $u_+ \in \tau_+$ of x and $u_- \in \tau_-$ of y so that $u_+ \cap u_- = \emptyset$. A Hausdorff bispace is 0-*dimensional* if the witnessing opens u_+ and u_- can always be chosen so that $u_+ \cup u_- = X$. A bispace is *compact* if for every $u_+ \in \tau_+$, the set $X \setminus u_+$ is compact with respect to τ_- and vice versa.

In an ordered bispace X, write $\Uparrow x$ for the τ_+ interior of $\uparrow x$, and $\Downarrow y$ for the τ_- interior of $\downarrow y$. Then X is *disjunctive* if every $u \in \tau_+$ is the union of sets $\Uparrow x$ for $x \in u$, likewise for τ_-, and furthermore $x \in \Downarrow y$ if and only if $y \in \Uparrow x$.

A disjunctive bispace is *discrete* if every $\uparrow x$ is τ_+ open (so $\uparrow x = \Uparrow x$), and likewise every $\downarrow x$ is τ_- open. The distinction between disjunctivity and discreteness does not have a topological analogue because a flat disjunctive bispace is a flat discrete bispace, and both of these are essentially just discrete spaces.

We denote some revelant full subcategories of bispaces as follows:

- Pos: discrete;
- Disj: disjunctive;
- bStone: compact 0-dimensional;
- bKHaus: compact Hausdorff.

The full subcategories of each of these determined by flat bispaces are denoted by $\mathsf{f\text{-}Pos}$, and so on. As mentioned, $\mathsf{f\text{-}Pos} = \mathsf{f\text{-}Disj}$. It is clear that these are isomorphic to *Set*. Also $\mathsf{f\text{-}bStone}$ is isomorphic to the category of Stone spaces, $\mathsf{f\text{-}bKHaus}$ to the category of compact Hausdorff spaces.

For a bispace, (X, τ_-, τ_+), let X^∂ denote the bispace with τ_+ exchanged with τ_-. Tychonoff products of bispaces are defined in the obvious way (and are easily seen to be the categorical products). We also need a workable notion of a "closed" sub-object.

For bispace X, say that X_0 is *biclosed* if it is the case that $\downarrow X_0$ is closed with respect to τ_+ and $\uparrow X_0$ is closed with respect to τ_-. A biclosed subset gets a relative bitopology in the obvious way.

LEMMA 1. *Order dualization ($X \mapsto X^\partial$), formation of products and formation of biclosed sub-bispaces all preserve the properties of being ordered, Hausdorff,* 0*-dimensional, compact and disjunctive and discrete.*

A full subcategory of bispaces that is closed under dualization, formation of products and of biclosed sub-bispaces is said to be a *DSP* class. So the categories listed above, and their flattened counterparts, are all DSP classes.

§2. Bitopological Algebras. An *ordered signature* Σ is a collection of symbols Σ with each symbol assigned a string $t(\sigma)$ over the alphabet $\{1, \partial\}$. A *bitopological* Σ*-algebra* is a bispace X with an interpretation of each symbol σ and an operation as follows. If $t(\sigma) = d_0 \dots d_{n-1}$ then σ is interpreted as a bicontinuous operation from $X^{d_0} \times \cdots \times X^{d_{n-1}}$ to X, where X^1 denotes X. In the special case that X is discrete, this just means that σ is interpreted as being monotonic in arguments marked with 1 and antitonic in arguments marked with ∂. A bicontinuous function between bitopological Σ-algebras that preserves operations is, as usual, a Σ*-homomorphism.* This definition makes sense because a map from X to Y is bicontinuous if and only if it is also bicontinuous from X^∂ to Y^∂.

In light of Lemma 1, the notion of an $\mathbb{ISP}$ class in an amiable category makes sense so long as we interpret the $\mathbb{S}$ as meaning biclosed sub-algebra. In the full paper, *ordered Horn theories* are defined, and proved to determine $\mathbb{ISP}$ classes in the bitopological setting. Furthermore, semilattices, distributive lattices, Heyting algebras, Boolean algabras are all determined by such theories, denoted by SL, DL, HA and BA, respectively.

For an ordered Horn theory T and an amiable category of bispaces $\mathcal{C}$, we let $\mathcal{C}(T)$ denote the category of bitopological algebras modelling T in $\mathcal{C}$. In particular, $\mathcal{C}(\emptyset)$ is just $\mathcal{C}$. Notice that T might not be "amiable" itself because the order dual of a T algebra need not be a T algebra. This is the situation for meet semilattices and Heyting algebras, for example.

An ordered Horn theory T has an order dual, denoted by T^∂, in which all inequations are reversed.

§3. Bitopological Duality. Priestley duality (in bitopological form) concerns the bitopological space $2 = \{0, 1\}$ where $\{0\}$ is open in τ_- and $\{1\}$ is open in τ_+. For a (discrete) distributive lattice L, let $\mathrm{spec}(L)$ be the set of all lattice homomorphisms $L \to 2$. This is endowed with a bitopology by insisting that each evaluation map $e_a\colon \mathrm{spec}(L) \to 2$ defined by $e_a(x) = x(a)$ should be bicontinuous. It is routine now to check that $\mathrm{spec}(L)$ is compact and 0-dimensional. Moreover, for a homomorphism $h\colon L \to M$, the function $\mathrm{spec}(h)\colon \mathrm{spec}(M) \to \mathrm{spec}(L)$ defined by $f \mapsto fh$ is bicontinuous.

For a compact 0-dimensional bispace (X, τ_-, τ_+) let $K\Omega(X)$ be the set of all bicontinuous functions $X \to 2$. Then $K\Omega(X)$ is endowed with a bounded distributive lattice structure by insisting that each evaluation map be a bounded lattice homomorphism. Priestley duality can be read as saying that spec and $K\Omega$ are dual equivalences between the categories $\mathsf{Pos}(\mathsf{DL})$ (discrete bounded distributive lattices and lattice homomorphisms) and bStone.

Discrete bounded distributive lattices are dual to compact 0-dimensional bispaces. Discrete bispaces are dual to compact 0-dimensional bounded distributive lattices. Disjunctive bounded distributive lattices are dual to compact Hausdorff bispaces. Disjunctive bispaces are dual to compact Hausdorff bounded distributive lattices.

The mirror image duality, in its more familiar topological form, is due to Banaschewski. It says, roughly, that the category of posets is dually equivalent to the category of Stone distributive lattices. In the bitopological setting, this is a duality between discrete posets and compact 0-dimensional distributive lattices.

Although a lot of fact checking is needed, this is an outline of familiar dualities in a bitopological setting. We summarize a few relevant examples.

THEOREM 2. *The following are dual equivalences:*

- $\mathsf{Pos}(\mathsf{DL}) \equiv_\partial \mathsf{bStone}(\emptyset)$ *[Priestley duality];*
- $\mathsf{Pos}(\mathsf{SL}) \equiv_\partial \mathsf{bStone}(\mathsf{SL})$ *[Hofmann-Mislove-Stralka duality];*
- $\mathsf{Pos}(\emptyset) \equiv_\partial \mathsf{bStone}(\mathsf{DL})$ *[Banaschewski duality];*
- $\mathsf{Pos}(\mathsf{BA}) \equiv_\partial \mathsf{f\text{-}bStone}(\emptyset)$ *[Stone duality].*

§4. Weakening relations. A *weakening relation* between bispaces $X = (X, \sigma_-, \sigma_+)$ and $Y = (Y, \tau_-, \tau_+)$ is a binary relation R that is closed in the topology $\sigma_+ \otimes \tau_-$. In Pos, this amounts to requiring that $x \leq_X x'$, $x\ R\ y$ and $y \leq_Y y'$ implies $x'\ R\ y'$.

Weakening relations compose and $\leq_P$ is identity for composition of weakening relations. So we may define Pos^* denote the category of partially ordered sets and weakening relations. Define bStone^* and bKHaus^* analogously. We will designate a weakening relation with a looped arrow as in $\rho\colon P \looparrowright Q$, and write composition in the "operational" order: $R; S$ rather than $S \circ R$.

For Σ-bitopological algebras X and Y, a Σ-weakening relation is a weakening relation that is also a sub-algebra of $X \times Y$.

For an ordered Horn theory T and amiable $\mathcal{C}$, we let $\mathcal{C}(T)^*$ denote the category in which objects are the same as in $\mathcal{C}(T)$ and the morphisms are Σ-weakening relations.

THEOREM 3. *For any ordered Horn theory T and any amiable category $\mathcal{C}$ of bispaces, $\mathsf{Map}(\mathcal{C}(T)^*)$ is equivalent to $\mathcal{C}(T)$.*

Thus the bitopological dualities we already know are dualities between Map subcategories. These all lift to the general weakening relations.

THEOREM 4. *For ordered Horn theories S and T, if $\mathsf{Pos}(S) \equiv_\partial \mathsf{bStone}(T)$, then $\mathsf{Pos}(S^\partial)^* \equiv \mathsf{bStone}(T^\partial)^*$.*

The corollaries that interest us are, for example, that ordinary posets with weakening relations are equivalent to compact 0-dimensional distributive lattice bispaces with distributive lattice weakening relations.

§5. Strong entailment and three-valued semantics. Bispaces in bStone are essentially semantical spaces in which two-values reign. This is because the relevant open subsets in τ_+ are complemented in τ_-. This is witnessed further by the fact that the joint topology is actually a Stone topology. To get at a semantics that is authentically three valued one needs to remove the 0-dimensionality requirement and consider bKHaus. On the discrete side of this (on the "logical" side), disjunctiveness is the key idea.

THEOREM 5. *For ordered Horn theories T and T', if $\mathsf{Pos}(T)^* \equiv \mathsf{bStone}(T')^*$ then $\mathsf{Disj}(T)^* \equiv \mathsf{bKHaus}(T')^*$.*

In the proof of this result, strong entailment is interpreted in a three-valued setting exactly as a skeptic's entailment. That is, $p \prec q$ holds exactly when every model that does not deny p affirms q.

In the full paper, the main theorem is used to characterize the disjunctive algebras in which $\wedge$ has a "strong" residual. That is, a binary operation $\to$ so that $x \wedge y \prec z$ if and only if $x \prec y \to z$.

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▸ KRYSTYNA MRUCZEK-NASIENIEWSKA, *Semantical and syntactical characterisation of some extensions of the class of* MV*-algebras*.
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Abstract. In the present paper we will ask for the lattice $L(\mathrm{MV}_{\mathrm{Ex}})$ of subvarieties of the variety defined by the set Ex(MV) of all externally compatible identities valid in the variety MV of all MV-algebras. In particular, we will find all subdirectly irreducible algebras from the classes in the lattice $L(\mathrm{MV}_{\mathrm{Ex}})$ and give syntactical and semantical characterization of the class of algebras defined by P-compatible identities of MV-algebras.

Keywords: MV-algebra, variety, identity, P-compatible identity, equational base, subdirectly irreducible algebras

We will consider MV-algebras as systems $\langle A, +, \cdot, ^{-}, 0, 1\rangle$, where A is a nonempty set of elements, 0 and 1 are distinct constant elements of A, $+$ and $\cdot$ are binary operations on the elements of A, and $^{-}$ is a unary operation on elements of A. The class of all MV-algebras will be denoted be MV.

It is known that the set Id(MV) of all identities fulfilled in the class MV determines a variety (i.e., nonempty class of algebras closed under subalgebras, homomorphic images and direct products) MV.

Let $\mathrm{Id}(\tau)$ denote the set of all identities of type τ. For a set $\Sigma \subseteq \mathrm{Id}(\tau)$ we denote by $\mathsf{Cn}(\Sigma)$ the deductive closure of Σ, i.e. $\mathsf{Cn}(\Sigma)$ is the smallest subset of $\mathrm{Id}(\tau)$ containing Σ such that:

1. $x = x \in \mathsf{Cn}(\Sigma)$ for every variable x;
2. $p = q \in \mathsf{Cn}(\Sigma) \Rightarrow q = p \in \mathsf{Cn}(\Sigma)$;
3. $p = q, q = r \in \mathsf{Cn}(\Sigma) \Rightarrow p = r \in \mathsf{Cn}(\Sigma)$;
4. $\mathsf{Cn}(\Sigma)$ is closed under replacement;
5. $\mathsf{Cn}(\Sigma)$ is closed under substitution.

If $\Sigma = \mathsf{Cn}(\Sigma)$ then Σ is called an equational theory.

We will choose from the set Id(MV) a subset E. If the set E is a proper subset of the set Id(MV) and it is an equational theory, then its corresponding variety MV_E is bigger than MV with respect to inclusion.

A natural question to describe a lattice of subvarieties of some bigger variety containing MV arises. A partial answer to this problem is presented in the paper. The research of subvarieties of the variety MV_E is a partial solution of the most general problem in this area: for a fixed language find all equational theories contained between the theory generated by the single identity $x = x$ and the full theory determined by the single identity $x = y$. Logicians inquire about lattices of intermediate logics (for example between intuitionistic logic and classical logic), algebraists search for lattices of subvarieties.

In our case the set E is related to the special structure of terms occurring in the identity. We consider a given type of algebras $\tau : F \to \mathbb{N}$, where F is a set of fundamental operation symbols and $\mathbb{N}$ is the set of non-negative integers.

Let Π_F be the set of all partitions of F and let $P \in \Pi_F$. For any terms p and q of the type τ, the identity $p = q$ is P-compatible iff it is of the form $x = x$ or both p and q are not variables and the outermost operation symbols in p and q belong to the same blok of the partition P.

The notion of P-compatible identity was introduced by J. Płonka [10] and it is a generalization of an externally compatible identity introduced by W. Chromik in [2] and normal identity defined independently by J. Płonka [9] and I. I. Mel'nik [7].

An identity $p = q$ of type τ is externally compatible if it is of the form $x = x$ or the most external fundamental operation symbols in p and q are identical (in other words, P-compatible identity is an externally compatible identity if P consists of singletons only). An identity $p = q$ of type τ is normal if it is of the form $x = x$ or neither p nor q is a variable (i.e., P-compatible identity is a normal identity if $P = \{F\}$).

Classes of algebras that are connected with P-compatible identities were considered for abelian groups , Boolean algebras and others (see [4, 5, 8]).

For a given variety V we will use the following notations:

- $P(V)$—the set of all P-compatible identities satisfied in V,
- $\mathrm{Ex}(V)$—the set of all externally compatible identities satisfied in V,
- $\mathrm{N}(V)$—the set of all normal identities satisfied in V.
- $\mathrm{Id}(V)$—the set of all identities satisfied in V.

The following inclusions are obvious:

$$\mathrm{Ex}(V) \subseteq P(V) \subseteq \mathrm{N}(V) \subseteq \mathrm{Id}(V).$$

One can prove that $P(V)$ is an equational theory. It is known that every equational theory corresponds to a variety of algebras.

Let

- V_P denotes the variety defined by $P(V)$,
- V_{Ex} denotes the variety defined by $\mathrm{Ex}(V)$,
- V_{N} denotes the variety defined by $\mathrm{N}(V)$.

It is a well known fact that the lattice of all equational theories extending the theory $\mathrm{Id}(V)$ is dually isomorphic to the lattice of all subvarieties of the variety V. Thus, for any partition P we have:

$$V \subseteq V_{\mathrm{N}} \subseteq V_P \subseteq V_{\mathrm{Ex}}.$$

Subvarieties of the variety MV have been studied by R. Grigolia, Y. Komori, A. Di Nola, and A. Lettieri. A. Lettieri and A. Di Nola gave equational bases for classes of MV-algebras. Y. Komori determined the lattice of subvarieties of the variety of MV.

In the present paper we will ask for the lattice $L(\mathrm{MV}_{\mathrm{Ex}})$ of subvarieties of the variety defined by the set $\mathrm{Ex}(\mathrm{MV})$. The full description of the lattice $L(\mathrm{MV}_{\mathrm{Ex}})$ is complicated and falls outside the scope of the talk. Of course, every subvariety of the class MV is also a proper subvariety of the variety determined the set $\mathrm{Ex}(\mathrm{MV})$.

Aside from giving a description of some elements of the lattice $L(\mathrm{MV}_{\mathrm{Ex}})$, we will find all subdirectly irreducible algebras from the classes in the lattice $L(\mathrm{MV}_{\mathrm{Ex}})$ and we will give syntactical and semantical characterization of the class of algebras defined by P-compatible identities of MV-algebras.

We use standard algebraic notions (see for example [1]).

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§1. Semantic information. Although the concept and definition of information are subject of continuing debate, (cf. [9]) many would agree that *semantic information*, as contrasted with *data information* such as, e.g, in Shannon's theory of communication, should be such subject matter that admits *semanticization*. According to Carnap and Bar-Hillel, [6] who coined the term "semantic information," such information is to be conveyed by "a certain language system," to which semantic concepts can be applied. Following this line of thought, we use a formal language, $\mathfrak{L}$, the formulae of which are built up from an enumerable set *Var* of propositional variables with use of finitary connectives. For the sake of simplicity, we limit ourselves with the customary assertoric connectives $\wedge$, $\vee$, and $\neg$, denoting formulae of this language by $A, B, \ldots$. The formula algebra is denoted by $\mathfrak{F}$.

Our starting point is an artificial agent (the computer) placed in information flaw. We assume that the computer receives information discretely in the form of reports. On the first glance, it may seem that a report can be identified with a statement or, when a formal language is at hand, with a formula of the language. However, we assume that there is a finite set, $\mathfrak{T}$, of the *degrees of reliability* being used by the reporter to show how much she can trust the piece of information she sends to the computer. In our consideration, $\mathfrak{T}$ will always be nonempty and finite. Thus we define a report to be an expression of the form $A\!:\!\tau$ where A is a statement of a given language $\mathfrak{L}$ and $\tau \in \mathfrak{T}$.

We expect that the computer not only stores the information that enters it, but also can answer questions about constituents of this information and their combinations which can be formed in $\mathfrak{L}$. Also, we expect that some rules can be put into effect, according to which computer's state of knowledge can be modified.

One of the main characteristics of semantic information is that its pieces can be semanticized. The idea of semanticization related to information, though expressed implicitly, can be traced as early as in [6, 3]. In this publications Carnap and Bar-Hillel proposed "as an expicatum for the ordinary concept 'the information conveyed by the statement"', say A, to use the set of all conjuncts obtained from the full conjunctive normal form of A, calling it the content of A.

Our approach differs from that of Carnap and Ber-Hillel's and adapts Belnap's idea of semanticization of a statement. Belnap's approach [4, 5] has two parts, which are not independent. If the computer receives the report $p \vee q : true$, the semanticization of this report can consist in the search for all valuations which make the statement $p \vee q$ true. (This is the *part 1* of Belnap's idea.) Let us denote *true* by $\mathbf{t}$ and assume that $\mathbf{t} \in \mathfrak{T}$. The part 1 certainly presupposes that the connective $\vee$ is interpreted as an operation on the set $\mathfrak{T}$. The *part 2* of Belnap's approach proposes to arrange all pieces of discrete information by an approximation relation and operate with pieces of information in a continuous way relative to this relation.

In accordance with this proposal, we, first, arrange the elements of $\mathfrak{T}$ by $\sqsubseteq$ as a complete semilattice [10, 7]. The relation $\sqsubseteq$, then, induces other relations among the states of computer's knowledge. The $\mathfrak{L}$-connectives are treated as Scott-continuous functions on $\mathfrak{T}$ in the sense of [10]. The modifiers implemented in the computer for updating computer's knowledge are also defined as Scott-continuous operations in a space of possible states of this knowledge. The purpose of this research is to show consequences of this proposal.

§2. Epistemic structure and epistemic state. Semanticization of reports is based on the notion of epistemic structure, each element of which is supposed to represent the degree of reliability or that of informativeness.

Definition 1. *An epistemic structure* (ES) $\mathfrak{T}$ *is an algebraic system* $\langle \mathfrak{T}, \wedge, \vee, \neg, \bot, \sqsubseteq \rangle$ *such that the following conditions are satisfied*:

a) $\langle \mathfrak{T}, \sqsubseteq \rangle$ *is a finite complete semilattice* [10, 7] *with a least element,* $\bot$;
b) *operations* $\wedge$, $\vee$ *and* $\neg$ *are monotone* [10, 7] *with respect to* $\sqsubseteq$.

An ES *is called* ***expanded*** *if, in addition, it has two designated constants* (*or* 0*-ary operations*), $\mathbf{t}$ *and* $\mathbf{f}$, *each different from* $\bot$ *and such that*

$$c) \quad \neg\mathbf{t} = \mathbf{f} \text{ and } \neg\mathbf{f} = \mathbf{t}.$$

Given a poset $\mathcal{P}$ and nonempty $\mathcal{D} \subseteq \mathcal{P}$, $\sqcup\mathcal{D}$ and $\sqcap\mathcal{D}$ denote the least upper bound and the greatest lower bound of $\mathcal{D}$ respectively, if they exist.

Main examples (of expanded ES). We will be illustrating our approach throughout by the following examples.

- ES $\mathfrak{T}_3$: $\mathfrak{T}_3 = \{\bot, \mathbf{t}, \mathbf{f}\}$ and $\bot \sqsubseteq \mathbf{t}, \mathbf{f}$; the operations $\wedge$, $\vee$, and $\neg$ are defined as in Kleene's strong logic [13], §64:

x	$\neg x$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$

$x \wedge y$	$\mathbf{t}$	$\mathbf{f}$	$\bot$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\bot$	$\mathbf{f}$	$\bot$

$x \vee y$	$\mathbf{t}$	$\mathbf{f}$	$\bot$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\bot$
$\bot$	$\mathbf{t}$	$\bot$	$\bot$

- ES $\mathfrak{T}_4$: $\mathfrak{T}_4 = \{\bot, \mathbf{t}, \mathbf{f}, \top\}$ and $\bot \sqsubseteq \mathbf{t}, \mathbf{f} \sqsubseteq \top$; the operations $\wedge$, $\vee$, and $\neg$ are defined as in Belnap-Dunn's logic [5, 8]:

x	$\neg x$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$
$\top$	$\top$

$x \wedge y$	$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$

$x \vee y$	$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$
$\bot$	$\mathbf{t}$	$\bot$	$\bot$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\top$

Definition 2 (valuation, set $\mathrm{V}(s)$, relation $\leq$ on valuations). *Given an* ES $\mathfrak{T}$, *a valuation on* $\mathfrak{T}$ *is a homomorphism of the formula algebra* $\mathfrak{F}$ *to* $\mathfrak{T}$. *A valuation* s *is* ***finite*** *if the set* $\mathrm{V}(\mathrm{s}) = \{p : p \in \mathit{Var} \text{ and } s(p) \neq \bot\}$ *is finite. Given two valuations* s_1 *and* s_2, *we define:*

$$s_1 \leq s_2 \overset{\text{def}}{\Longleftrightarrow} s_1(p) \sqsubseteq s_2(p) \text{ for any } p \in \mathit{Var}. \tag{1}$$

The set of all valuations is denoted by $\mathbf{S}$ *and the finite ones by* $F(\mathbf{S})$. *Accordingly, we define:*

$$\mathbf{S}^* = \langle \mathbf{S}, \leq \rangle.$$

Defining $s^*(p) = \bot$, for any $p \in \mathit{Var}$, we see that s^* is the least element of $\mathbf{S}^*$.

Proposition 1. $\mathbf{S}^*$ *is a domain* [7] *with* $F(\mathbf{S})$ *as the finite elements. Moreover, for any variable* $p \in \mathit{Var}$ *and* $\{s_i\}_{i \in I} \subseteq \mathbf{S}$,

$$\sqcap \{s_i\}_{i \in I}(p) = \sqcap \{s_i(p)\}_{i \in I} \tag{2}$$

and if $\{s_i\}_{i \in I}$ *is directed, then*

$$\sqcup \{s_i\}_{i \in I}(p) = \sqcup \{s_i(p)\}_{i \in I} \tag{3}$$

Next we turn to semanticization of a report which enters the computer. We fix an ES $\mathfrak{T}$ so that our consideration will be relative to $\mathfrak{T}$. We start with the following definition.

Definition 3 (epistemic state). *An epistemic state* (*or simply a* ***state***) *is a nonempty set of valuations. A state is* ***finite*** *if it a finite set of finite valuations. The set of all minimal valuations of a finite state* E *is denoted by* $\mathbf{m}(E)$. *A finite state* E *is* ***minimal*** *if* $\mathbf{m}(E) = E$.

Having received two reports $A{:}\tau_1$ and $A{:}\tau_2$, on the question about A the computer could probably answer that trustfulness of A is not less than $\tau_1 \sqcap \tau_2$ (in the sense of $\sqsubseteq$ on $\mathfrak{T}$). The following definition intends to implement this idea.

Definition 4 (value $E(A)$ of A at E). *The value of a formula* A *at a state* E *is*

$$E(A) = \sqcap \{s(A) : s \in E\}.$$

The last definition induces the following arrangement. For any states E_1 and E_2,

$$E_1 \leq E_2 \overset{\text{def}}{\Longleftrightarrow} \text{for any } s_2 \in E_2, \text{ there is a } s_1 \in E_1 \text{ such that } s_1 \leq s_2. \tag{4}$$

(The intuition behind (4) is well explained in terms of *partial information* in [11], pp. 653–654.)

In the next section we will define a space in which computer's knowledge modifiers operate. The main problem here comes from the intuitive observation that the continuity (in the sense of Scott topology and convergence [10]) of such modifiers will require a larger space than they can actually use.

§3. The space of epistemic states. We think of a finite state, which may be a current state of computer's knowledge, as a point of a domain in the sense of [10]. More than that, the finite states constitute an effective basis [11] of this domain. Besides other things, it means that each point of this domain is the sup of the finite states approximating this point in the sense of (4).

Let us fix an ES and denote by $\mathbf{F}$ and $\mathbf{M}$ the sets of all finite and minimal states, respectively. Accordingly, we then define:

$$\mathbf{F}^* = \langle\{\downarrow E : E \in \mathbf{F}\}, \subseteq\rangle \text{ and } \mathbf{M}^* = \langle\mathbf{M}, \leq\rangle,$$

where $\downarrow E$ is the principle ideal generated by E with respect to $\leq$ defined by (4).

PROPOSITION 2. *$\mathbf{M}^*$ is isomorphic to $\mathbf{F}^*$ and is a lattice. Moreover, denoting the lattice operations of $\mathbf{M}^*$ by $\wedge$ (meet) and $\vee$ (join), for any $E_1, E_2 \in \mathbf{M}$,*

$$E_1 \wedge E_2 = \mathbf{m}(E_1 \cup E_2),$$
$$E_1 \vee E_2 = \mathbf{m}(\{s_1 \sqcup s_2 : s_1 \in E_1, s_2 \in E_2\}),$$

providing that $\{s_1 \sqcup s_2 : s_1 \in E_1, s_2 \in E_2\} \neq \emptyset$.

COROLLARY 2.1. If $\mathfrak{T}$ has a top, then $\boldsymbol{M}^*_{\mathfrak{T}}$ is a lattice. In particular, $\boldsymbol{M}^*_4$ is a lattice.

Let $\mathbf{P}$ be the set of ideals over the pre-ordered set $\langle\mathbf{F}, \leq\rangle$. Then the poset $\mathbf{P}^* = \langle\mathbf{P}, \subseteq\rangle$ is known as the ***upper powerdomain*** (or the ***Smith powerdomain***) over $\langle\mathbf{F}, \leq\rangle$; cf. [11, 10]. In case $\mathfrak{T}_3$ and $\mathfrak{T}_4$, we write $\mathbf{P}^*_3$ and $\mathbf{P}^*_4$, respectively.

PROPOSITION 3. *$\mathbf{P}^*$ is effectively presented domain with the sublattice of compact elements isomorphic to $\mathbf{M}^*$.*

As we will see below, in some cases $\mathbf{P}^*$ allows a transparent description. We proceed with the following definition.

An ES, as well as all structures related to it, is called ***refined*** if for any $E \in \mathbf{F}$ and any state E',

$$E \leq E' \Longleftrightarrow E(A) \sqsubseteq E'(A) \text{ for any formula } A.$$

Next we define: Fore any states E_1 and E_2,

$$E_1 \equiv E_2 \overset{\text{def}}{\Longleftrightarrow} E_1(A) = E_2(A) \text{ for any formula } A.$$

Then, we define:

$$|E| = \{E' : E' \equiv E\},$$

and, next,

$$|E_1| \preceq |E_2| \overset{\text{def}}{\Longleftrightarrow} E_1(A) \sqsubseteq E_2(A) \text{ for any formula } A,$$

and, finally,

$$\mathbf{G}^*_f = \langle\{|E| : E \in \mathbf{F}\}, \preceq\rangle \text{ and } \mathbf{G}^* = \langle\{|E| : E \text{ is a state}\}, \preceq\rangle.$$

PROPOSITION 4. *Let $\mathfrak{T}$ be refined. Then $\mathbf{M}^*$ and $\mathbf{G}^*_f$ associated with this $\mathfrak{T}$ are isomorphic.*

PROPOSITION 5. *Let $\mathfrak{T}$ be refined and have a top element. Then $\mathbf{G}^*$ associated with this $\mathfrak{T}$ is a domain with a basis $\mathbf{G}^*_f$ in the sense of* [10].

Let $\mathbf{P}^*_4$ and $\mathbf{G}^*_4$ be the corresponding structures based on $\mathfrak{T}_4$.

PROPOSITION 6. *Let an* ES *$\mathfrak{T}$ be refined with a top. Then $\mathbf{P}^*_{\mathfrak{T}}$ and $\mathbf{G}^*_{\mathfrak{T}}$ are isomorphic.*

COROLLARY 5.1 (comp. [12]). The domains $\boldsymbol{P}^*_4$ and $\boldsymbol{G}^*_4$ are isomorphic.

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▸ ANTONIO DI NOLA, REVAZ GRIGOLIA, AND GIACOMO LENZI, *On the Logic of Perfect MV-algebras.*

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MV-algebras are algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. There are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals. Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical [1].

As it is well known, MV-algebras form a category which is equivalent to the category of abelian lattice ordered groups (ℓ-groups, for short) with strong unit [6]. Let us denote by Γ the functor implementing this equivalence. In particular each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit. Moreover, the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups, see [5].

Perfect MV-algebras do not form a variety and contain non-simple subdirectly irreducible MV-algebras. It is worth stressing that the variety generated by all perfect MV-algebras is also generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [2]. Algebras from the variety generated by C will be called by $MV(C)$*-algebras*.

The MV algebra C is the subdirectly irreducible MV algebra with infinitesimals. It is generated by an atom c, which we can interpret as a quasi false truth value. The negation of c is a quasi true value. Now quasi truth or quasi falsehood are vague concepts. About quasi truth in an MV algebra, it is reasonable to accept the following propositions:

- there are quasi true values which are not 1;
- 0 is not quasi true;
- if x is quasi true, then x^2 is quasi true (where x^2 denotes the MV algebraic product of x with itself).

In C, to satisfy these axioms it is enough to say that the quasi true values are the co-infinitesimals.

By way of contrast, note that there is no notion of quasi truth in $[0, 1]$ satisfying the previous axioms (there are if we replace the MV product with other suitable t-norms, e.g. the product t-norm or the minimum t-norm).

Let L_P be the logic of perfect algebras which coincides with the set of all Łukasiewicz formulas that are valid in all perfect MV-chains, or equivalently that are valid in the MV-algebra C. Actually, L_P is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom: $(x \oplus x) \odot (x \oplus x) \leftrightarrow (x \odot x) \oplus (x \odot x)$, see [1]. Notice that the Lindenbaum algebra of L_P is an $MV(C)$-algebra.

Let $\mathbf{V}$ be a variety. Recall that an algebra $A \in \mathbf{V}$ is said to be a *free algebra* over $\mathbf{V}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbf{V}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators is finite, then A is said to be a *free algebra of finitely many generators*. We denote a free algebra A with $m \in (\omega + 1)$ free generators by $F_{\mathbf{V}}(m)$.

An algebra A is called *finitely presented* if A is finitely generated, with the generators $a_1, ..., a_m \in A$, and there exist a finite number of equations $P_1(x_1, ..., x_m) = Q_1(x_1, ..., x_m), ..., P_n(x_1, ..., x_m) = Q_n(x_1, ..., x_m)$ holding in A on the generators $a_1, ..., a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, ..., b_m \in B$, such that the equations $P_1(x_1, ..., x_m) = Q_1(x_1, ..., x_m), ..., P_n(x_1, ..., x_m) = Q_n(x_1, ..., x_m)$ hold in B on the generators $b_1, ..., b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

An algebra $A \in \mathbf{V}$ is called *projective*, if for any $B, C \in \mathbf{V}$, any onto homomorphism $\gamma : B \to C$ and any homomorphism $\beta : A \to C$, there exists a homomorphism $\alpha : A \to B$ such that $\gamma\alpha = \beta$. Notice that in varieties, projective algebras are characterized as retracts of free algebras.

A subalgebra A of $F_{\mathbf{V}}(m)$ is said to be *projective subalgebra* if there exists an endomorphism $h : F_{\mathbf{V}}(m) \to F_{\mathbf{V}}(m)$ such that $h(F_{\mathbf{V}}(m)) = A$ and $h(x) = x$ for every $x \in A$.

Let α be a formula of the logic L_P and consider a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ and extend it to all of Φ_m by $\sigma(\alpha(x_1, ..., x_m)) = \alpha(\sigma(x_1), ..., \sigma(x_m))$, where $\Phi_m = \{x_1, ..., x_m\}$ and Φ_m is the set of all formulas containing variables from Φ_m. We can consider the substitution as an endomorphism $\sigma : \Phi_m \to \Phi_m$ of the free algebra Φ_m.

A formula $\alpha \in \Phi_m$ is called *projective* if there exists a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$ [3].

A logic L is *structurally complete* if every rule that is admissible (preserves the set of theorems) should also be derivable.

It holds the following

Theorem 1. *(1) For m-generated $MV(C)$-algebras to be finitely presented is equivalent to be projective;*

(2) There exists a one-to-one correspondence between projective formulas of L_P with m-variables and the m-generated projective subalgebras of the m-generated free algebras of the variety generated by perfect MV-algebras;

(3) L_P is structurally complete.

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▸ TOMASZ POŁACIK, *A semantic approach to conservativity.*
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Every first-order set of sentences T, viewed as a set of axioms, gives rise to a classical theory T^{c} when T is closed under consequences of classical logic, or an intuitionistic one, T^{i}, when T is closed under consequences of intuitionistic logic. While it is clear that every sentence which is intuitionistically provable is also classically provable, the question when the converse holds leads us to the so-called conservativity problem. More precisely, given a class Γ of formulas, we say that the theory T^{c} *is Γ-conservative over its intuitionistic counterpart* T^{i} iff for all $A \in \Gamma$ we have $\mathsf{T}^{\mathsf{i}} \vdash A$ whenever $\mathsf{T}^{\mathsf{c}} \vdash A$.

A typical example is that of classical Peano Arithmetic PA and its intuitionistic counterpart, Heyting Arithmetic HA. The well-known result concerning these two theories states that PA is Π_2-conservative over HA. This fact is proven syntactically using Dialectica interpretation or by means of the so-called negative and Friedman translations.

The negative translation allows us to embed a classical theory into its intuitionistic counterpart. It can be done by using a simple translation of the formulas that assigns to each formula A the formula A^- in the following way: For any formula A:

- $A^- = \neg\neg A$, for atomic formulas A,
- $(A \vee B)^- = \neg(\neg A^- \wedge \neg B^-)$,
- $(\exists x A)^- = \neg\forall x \neg A^-$,
- $(\cdot)^-$ commutes with $\wedge$, $\to$, $\forall$.

For a set of sentences T, we consider the set T^- of all negative translations of the members of T. Then one proves that if a sentence A is *classically* provable from T then its negative translation A^- is *intuitionistically* provable from T^-.

The Friedman translation is defined as follows: Let us fix a formula F then for any formula A (with some obvious restrictions on the free variables of A), we assign to A the formula A^F according to the following rules:

- $A^F = A \vee F$, for atomic formulas A,
- $(\cdot)^F$ commutes with $\wedge$, $\vee$, $\to$, $\forall$, and $\exists$.

It is proven that whenever the formula A is *intuitionistically* derivable from T then the F-translation of A, the formula A^F, is also *intuitionistically* derivable from T^F, the set of F-translations of the formulas in T.

We say that a theory is closed under the negative translation if for every axiom A of

T the formula A^- is also provable in T. Similarly for the Friedman translation. One can check that Heyting Arithmetic is closed under both translations in question. It follows then that, for every formula A, if $\mathsf{PA} \vdash A$ then $\mathsf{HA} \vdash A^-$, and that if $\mathsf{HA} \vdash A$ then also $\mathsf{HA} \vdash A^F$ for every F. These properties enable us to prove the well-known result concerning conservativity of PA over HA. The standard proof exploits a smart combination of the negative and Friedman translation, see [4] for details. We rephrase this result in the following generalized way:

Theorem 1. *Let $\mathsf{T}^{\,i}$ be an intuitionistic theory closed under the Friedman and the negative translation and such that all atomic formulas are decidable in $\mathsf{T}^{\,i}$. Then $\mathsf{T}^{\,c}$ is $\forall\exists$-conservative over $\mathsf{T}^{\,i}$.*

In our talk we consider possible generalizations of this result. However, instead of using syntactic methods, we exploit semantic methods and present some new conservativity results proven by means of Kripke models for first-order theories.

The choice of Kripke semantics seems to be very natural since in this case we can observe an interlpay between intuitionistic and classical theories, and deal with classical models within intuitionistic ones.

For the purpose of this presentation we assume that a first-order Kripke model $\mathcal{K}$ is a collection of classical first-order structures M_w for $w \in W$, called the worlds, partially ordered by the usual (weak) substructure relation. More precisely, we assume that the set W of nodes is partially order by the relation $\leq$ in such the way that M_w is a (weak) substructure of M_v whenever $w \leq v$. The intuitionistic forcing relation $\Vdash$ is defined globally in the model $\mathcal{K}$ in terms of the classical satisfaction relation $\models$ considered locally. In particular, an atomic formula $A(\overline{a})$, with parameters $\overline{a}$ from the world M_w, is forced at a node w, i.e., $w \Vdash A(\overline{a})$, if and only if $A(\overline{a})$ is classically satisfied in the structure M_w according to the usual satisfiability relation, i.e., $M_w \models A(\overline{a})$. However, in general the relation between classical and intuitionistic validity cannot be easily described. In general, the classical validity of a formula A at a world M coincides with intuitionistic validity of A at M in the model $\mathcal{K}$ for formulas built up from atoms, conjunction, disjunction and existential quantifier only. Also, there is no straightforward correlation between the (intuitionistic) theory of the Kripke model and the (classical) theories of its worlds.

Let us describe the main idea behind our semantic method of proving conservativity. In order to prove conservativity of a classical theory $\mathsf{T}^{\,\mathsf{c}}$ over the intuitionistic counterpart $\mathsf{T}^{\,\mathsf{i}}$ with respect to a class of formulas Γ, our approach is as follows. We consider a formula A in Γ which is not derivable intuitionistically in $\mathsf{T}^{\,\mathsf{i}}$. Then we find a suitable Kripke model $\mathcal{K}$ of $\mathsf{T}^{\,\mathsf{i}}$ such that $\mathcal{K}$ refutes A. In particular, we find a node w of $\mathcal{K}$ for which we have $w \nVdash A$. Our goal is now to find a world M in the Kripke model $\mathcal{K}$ such that M is a classical first-order structure that is a model of $\mathsf{T}^{\,\mathsf{c}}$ and M refutes the formula A. Unfortunately, we cannot hope that it will happen in the world M_w, i.e., it may not be the case that $M_w \not\models A$. Moreover, although $\mathcal{K} \Vdash \mathsf{T}^{\,\mathsf{i}}$, in general we cannot hope that $M_w \models \mathsf{T}^{\,\mathsf{c}}$. However, we need only to find *some* node u with $M_u \not\models A$ and $M_u \models \mathsf{T}^{\,\mathsf{c}}$. We show that under suitable assumptions this can be done.

Let T be a set of first-order sentences. Recall that a Kripke model $\mathcal{K}$ is called T-*normal* if all the worlds of $\mathcal{K}$, viewed as classical first-order structures, are models of the theory $\mathsf{T}^{\,\mathsf{c}}$. See [1] for details. Note that not every T-normal Kripke model is a model of $\mathsf{T}^{\,\mathsf{i}}$ and not every Kripke model of $\mathsf{T}^{\,\mathsf{i}}$ is a T-normal one. Our interest will be in intuitionistic theories T that are complete with respect to a class of T-normal models, but not necessarily sound with respect to it.

Let us introduce a class of formulas.

Definition 2. A formula A is *stable over a theory* $\mathsf{T}^{\,i}$ iff for every Kripke model $\mathcal{K}$

of T^{i} and every world M_w in $\mathcal{K}$ we have

$$\text{if } w \Vdash A \text{ then } M_w \models A.$$

A class Γ of formulas is stable over T^{i} if every formula in Γ is. Let us denote the class of all stable formulas by $\mathcal{S}$.

We say that a formula A is *semi-positive* if all antecedents of implications in A are atomic. One can check that every semi-positive formula is stable over every intuitionistic theory. Also, the class of prenex formulas is stable over every intuitionistic theory in which all atomic formulas are decidable. Let us also observe that over classical predicate logic, every formula is equivalent to a semi-positive and a prenex formula.

Finally, let us introduce a class $\mathcal{S} \to \forall\exists$ of formulas defined as follows:

$$\mathcal{S} {\to} \forall\exists = \{C \to D : C \in \mathcal{S} \text{ and } D \in \forall\exists\}.$$

We can now state the following theorem.

Theorem 3. *Assume that* T^{i} *is a theory which is complete with respect to a class of* T*-normal Kripke models. Then* T^{c} *is* $(\mathcal{S}{\to}\forall\exists)$*-conservative over* T^{i}*.*

Let us note that, up to equivalence in classical predicate logic, every formula is equivalent to some $(\mathcal{S}{\to}\forall\exists)$-formula where $\mathcal{S}$ is the class of prenex formulas or semi-positive formulas. Thus, in some sense, Theorem 3 refers to a wide class of formulas.

We will apply Theorem 3 to prove some conservativity results concerning arithmetic. First, note that HA *is not* sound with respect to the class of PA-normal Kripke models. An example of a PA-normal Kripke model that refutes an instance of Σ_2-induction was given by A. Visser and D. Zambella. However, we can prove the following fact.

Theorem 4. *Heyting Arithmetic is complete with respect to* PA*-normal Kripke models.*

So, we can prove our generalization of Π_2-conservativity theorem for arithmetical theories.

Theorem 5. PA *is conservative over* HA *with respect to the class* $\mathcal{P} \to \Pi_2$*-formulas where* $\mathcal{P}$ *is a class of all prenex formulas.*

Our semantic method can be applied also to subsystems of Heyting Arithemetic such as $\mathsf{i}\Delta_0$ and $\mathsf{i}\Sigma_1$. Here the standard syntactic method cannot be applied since the theories in question are not closed under the Friedman translation.

Theorem 6. *The theories* $\mathsf{i}\Delta_0$ *and* $\mathsf{i}\Sigma_1$ *are complete with respect to the classes of* $\mathsf{I}\Delta_0$*-normal and* $\mathsf{I}\Sigma_1$*-normal Kripke models respectively.*

And hence we obtain

Theorem 7. $I\Delta_0$ *is* $(\mathcal{S}{\to}\forall\exists)$*-conservative over* $\mathsf{i}\Delta_0$ *and* $I\Sigma_1$ *is* $(\mathcal{S}{\to}\forall\exists)$*-conservative over* $\mathsf{i}\Sigma_1$*.*

The method presented above can be applied also to theories that are complete with respect to Kripke models with constant domain. In this case we exploit the technique of pruning introduced in [2].

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In this contribution, we investigate logics which expand the four-valued Belnap-Dunn logic by a notion of inconsistency. In order of increasing expressive power, this can take the form of an expansion by an inconsistency *predicate*, an inconsistency *constant*, or a *reductio ad contradictionem* negation. After briefly motivating the study of inconsistency and reviewing some related work, we introduce these logics as the quasiequational logics of a certain class of algebras and then discuss their relational semantics. This leads us to a logic which is a natural conservative extension of both classical and intuitionistic logic.

We shall employ the following notational conventions: de Morgan negation will be denoted by $-$, *reductio ad contradictionem* negation by $\sim$, and intuitionistic implication by $\rightarrow$. The symbol $\neg$ can denote any kind of negation.

In settings where we may be faced with premises containing implicit or even explicit contradictions, it is reasonable to avoid the classical principle of explosion in order to make good use of the premises. In a logic which lacks this principle, contradictions are not *absurd*: they do not entail everything. However, contradiction (inconsistency) is still something we would like to *avoid*. Dually, completeness – knowing whether to accept or reject a proposition – is something we would like to *achieve*. Both of these goals are needed if an epistemological enterprise is to have any interest at all: the goal of avoiding contradiction is attained easily by believing nothing, while the goal of achieving completeness is attained easily by believing everything.

Despite the obvious logical interest of the notions of inconsistency and completeness, connectives which non-trivially represent inconsistency and completeness are seldom considered in non-classical logics. Either the role of the inconsistency constant is played by the absurdity constant, or if a logic contains a separate constant interpreted as representing inconsistency, there is usually nothing to explain what exactly makes it an inconsistency constant.

For instance, Johansson's minimal logic [8] replaces the intuitionistic *reductio ad absurdum* negation $\varphi \rightarrow \bot$ by a *reductio ad contradictionem* negation $\varphi \rightarrow f$, where nothing is assumed about the constant f. Likewise, **FL**-algebras, which feature in the algebraic study of substructural logics, are just residuated lattices expanded by an arbitrary constant 0. One exception to this trend is Odintsov [9], who considers "absurdity as a unary operator", which he in effect defines as $\varphi \rightarrow (\varphi \wedge \neg\varphi)$.

The so-called Logics of Formal Inconsistency [5] (LFIs) are in fact related to the object of our study in name only. Although they have the same goal of "internalizing the very notions of consistency and inconsistency at the object-language level", this notion of inconsistency is entirely different from the one considered here. LFIs treat consistency as "what might be lacking to a contradiction for it to become explosive", whereas our view is simply that inconsistency means entailing a contradiction. What LFIs study is a unary operator $\bigcirc$ such that $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ holds, that is, a unary operator whose presence among the premises makes a given contradiction explosive. By contrast, we are studying a constant 0 such that $\varphi, -\varphi \vdash 0$ holds, that is, an operator whose presence in the conclusion detects a contradiction in the premises.

The present approach to internalizing a notion of inconsistency in the object language is in fact in direct opposition to the main focus of research on LFIs, which is on "boldly" paraconsistent logics. This condition implies that the language does not contain any constant (apart from possibly the trivial truth constant) which is entailed by each contradiction. Since we are not pursuing paraconsistency for its own sake, we do not view the presence of such a constant as a defect but rather as an expressive resource which will enable us to formalize *reductio ad contradictionem* reasoning.

The object of our study will be suitable expansions of the four-valued Belnap-Dunn logic, which was introduced by Belnap [2, 3] as a suggestion for "how a computer should think" and which can naturally be interpreted as the logic of assertion and denial. For a brief history of this logic, see Dunn [6].

The language of the Belnap-Dunn logic contains the distributive lattice connectives $\wedge$, $\vee$, the de Morgan negation $-$, and possibly also the top and bottom constants $\top$ and $\bot$. An algebraic interpretation of this logic is provided by *de Morgan lattices (DMLs)* and *de Morgan algebras (DMAs)*, that is, distributive lattices with an order-inverting involution denoted $-$ and their expansions by $\top$ and $\bot$.

The formulas of the Belnap-Dunn logic are terms in the language of DMLs (DMAs). A sequent $\Gamma \vdash \Delta$ is then valid if and only if the inequality $\bigwedge \Gamma \leq \bigvee \Delta$ is valid in the variety of all DMLs (DMAs). Since we view logic as concerned with deriving sequents from other sequents, we shall concentrate on the *quasi*equational logic of DMLs (DMAs) and say that a sequent follows from a finite set of sequents in case the corresponding quasiequation is valid in the quasivariety of all DMLs (DMAs).

DMLs in fact provide a natural algebraic setting for the study of inconsistency and completeness: each de Morgan lattice comes equipped with a canonical inconsistency predicate, namely the ideal generated by all elements of the form $a \wedge -a$, and a canonical completeness predicate, namely the filter generated by all elements of the form $a \vee -a$.

It will in fact be convenient to admit more general inconsistency predicates. We shall consider the following three kinds of expansions of DMLs by some notion of inconsistency. Since the completeness predicate is just the image of the inconsistency predicate under de Morgan negation, it need not be considered explicitly.

Definition 1. A *de Morgan lattice (algebra) with an inconsistency predicate* is a de Morgan lattice (algebra) equipped with an ideal $\mathcal{I}_0$ such that $a \wedge -a \in \mathcal{I}_0$.

A *de Morgan lattice (algebra) with an inconsistency constant* is a de Morgan lattice (algebra) equipped with a constant 0 such that $a \wedge -a \leq 0$.

A *de Morgan reductio algebra* is a de Morgan algebra with an inconsistency constant equipped with a unary operation $\sim$ such that $a \leq {\sim} b$ if and only if $a \wedge b \leq 0$.

An expansion of a DML with an inconsistency predicate is *standard* in case $\mathcal{I}_0$ is precisely the ideal generated by all elements of the form $a \wedge -a$. A (quasi)variety of expansions of DMLs with an inconsistency predicate is *standard* if it is generated by its standard elements as a (quasi)variety.

The standard (quasi)varieties of expansions of DMLs with an inconsistency predicate form a lattice. We will describe the standard quasivariety lattice of DMLs with an inconsistency predicate and the standard variety lattice of the variety of de Morgan reductio algebras. We also show that the standard quasivariety lattice of DMLs with an inconsistency constant is infinite.

An element of a DML with an inconsistency predicate is called *inconsistent* in case it belongs to $\mathcal{I}_0$, otherwise it is consistent. It is *complete* in case its de Morgan negation is inconsistent. We say that a DML with an inconsistency predicate is *Boolean* if the only inconsistent element is the bottom element, *Kleene* if each inconsistent element is below each complete element, *non-idempotent* if no element is both complete and inconsistent, *totally inconsistent* if each element is inconsistent, and *regular* if for each complete a and consistent b, the meet $a \wedge b$ is consistent.

We now extend the description of the quasivariety lattice of DMLs due to Pynko [10] to the standard quasivariety lattice of DMLs with an inconsistency predicate.

THEOREM 2. *The non-trivial standard quasivarieties of DMLs with an inconsistency predicate are: the quasivarieties of Boolean, Kleene, non-idempotent, regular, non-idempotent Kleene, regular Kleene, non-idempotent regular, non-idempotent or Kleene, non-idempotent or regular, totally inconsistent, and of all DMLs with an inconsistency predicate.*

Since each DML with an inconsistency constant can be viewed as a DML with an inconsistency predicate such that $a \in \mathcal{I}_0$ if and only if $a \leq 0$, the above definitions extend to DMLs with an inconsistency constant.

THEOREM 3. *There are infinitely many standard quasivarieties of (non-idempotent regular) de Morgan lattices with an inconsistency constant.*

To prove this proposition, we use a sequence of quasiequations analogical to those employed by Gaitán and Perea [7] which also shows that infinite meets in the lattice of quasivarieties need not coincide with infinite meets in the lattice of standard quasivarieties. We do not know whether this holds for finite meets. Although there are uncountably many quasivarieties of DMLs with an inconsistency constant by [1], we do not know whether there are uncountably many *standard* ones.

Let us now turn to de Morgan reductio algebras. If $\mathbf{A}$ is a de Morgan reductio algebra, let $\mathbf{A}_0$ be the distributive lattice with pseudocomplementation $([0, \top]_{\mathbf{A}}, \wedge, \vee, \top, 0, \sim)$ and let $\mathbf{A}_0^1$ be the de Morgan algebra $([0 \wedge 1, 0]_{\mathbf{A}}, \wedge, \vee, 0, 0 \wedge 1, \neg)$ such that $\neg a = -a \wedge 0$.

THEOREM 4. $\mathbf{Con}(\mathbf{A}) = \mathbf{Con}(\mathbf{A}_0) \times \mathbf{Con}(\mathbf{A}^0_{0\wedge 1})$.

The algebra $\mathbf{A}$ is therefore subdirectly irreducible if and only if either $\mathbf{A}$ is Kleene and $\mathbf{A}_0$ is a subdirectly irreducible distributive lattice with pseudocomplementation or $\mathbf{A}$ is totally inconsistent and subdirectly irreducible as a DMA. This yields a description of the variety lattice of de Morgan reductio algebras.

THEOREM 5. *The variety of de Morgan reductio algebras is locally finite and has equationally definable principal congruences.*

The algebraic structures introduced above can be given a *relational* interpretation by slightly extending the representation of DMAs due to Białynicki-Birula and Rasiowa [4]. A *frame* in this semantics is a poset equipped with an order-inverting involution δ. The *complex algebra* of a frame is a DMA with an inconsistency constant which suitably expands the bounded distributive lattice of all upsets. The de Morgan negation is interpreted as quasicomplementation, that is, $u \in -a$ if and only if $\delta(u) \notin a$ for each upset a. The inconsistency constant is interpreted by the set of all worlds u such that $u \nleq \delta(u)$. The complex algebra of a frame can be equipped with a Heyting implication, making it a de Morgan *Heyting* algebra (DMHA) with an inconsistency constant.

The *canonical frame* of a DMA with an inconsistency predicate $\mathbf{A}$ is defined as the poset of prime filters on $\mathbf{A}$ equipped with the order-inverting involution $\delta(\mathcal{U}) = \mathbf{A} \setminus -\mathcal{U}$. An embedding of $\mathbf{A}$ into the complex algebra $\mathbf{B}$ of some frame is an embedding of DMAs such that the join of the inconsistent ideal of $\mathbf{A}$ is the inconsistency constant of $\mathbf{B}$.

The complex algebra of the canonical frame of $\mathbf{A}$ is called the *canonical extension* of $\mathbf{A}$, and a standard quasivariety K is called *canonical* if the canonical extension of each standard member of K belongs to K. If a quasivariety is canonical, its quasiequational logic is complete with respect to a suitable class of frames.

THEOREM 6. *Each standard DMA with an inconsistency predicate can be embedded in the complex algebra of its canonical frame.*

THEOREM 7. *All the quasivarieties of DMLs with an inconsistency predicate are standard and canonical as quasivarieties of DMAs (DMHAs) with an inconsistency predicate or an inconsistency constant.*

In particular, the quasiequational logic of all de Morgan Heyting algebras with an inconsistency constant coincides with the semantically defined intuitionistic logic of all frames expanded by de Morgan negation and by the inconsistency constant.

Composing the correspondence betwen sequents and inequalities on DMAs with the representation of DMAs as the complex algebras of frames yields a relational semantics for the expansion of the Belnap-Dunn logic introduced above. The sequent $\Gamma \vdash \Delta$ then holds in this relational semantics if $u \in \bigwedge \Gamma$ implies $u \in \bigvee \Delta$ for each world u under each valuation of propositional atoms.

We say that a world u is *complete* if $\delta(u) \leq u$ and *consistent* if $u \leq \delta(u)$. It is then easy to see that the logic of all complete consistent worlds is precisely classical logic. Unlike the standard semantics for classical logic, this semantics can easily be expanded by an intuitionistic implication without collapsing into material implication.

THEOREM 8. *If Γ and Δ are finite sets of formulas in the language $\wedge, \vee, \top, \bot, -$, then $1, \Gamma \vdash \Delta, 0$ holds if and only if $\Gamma \vdash \Delta$ holds in classical logic. If Γ and Δ are finite sets of formulas in the language $\wedge, \vee, \top, \bot, \to$, then $1, \Gamma \vdash \Delta, 0$ holds if and only if $\Gamma \vdash \Delta$ holds in intuitionistic logic.*

We submit that the logic of complete consistent worlds in this semantics provides a natural way of combining classical negation with intuitionistic implication.

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MOTIVATION. The following theorem was proved in [12].

THEOREM 1. *Let K be a quasivariety that algebraizes (in the sense of [1]) a finitary sentential logic $\vdash$. Then $\vdash$ has a classical inconsistency lemma iff K is relatively semisimple, with equationally definable principal relative congruences (EDPRC) and for every $\boldsymbol{A} \in \mathsf{K}$, the total congruence A^2 is compact in the lattice of all K–congruences of $\boldsymbol{A}$.*

Here, a *classical inconsistency lemma* (CIL) is an abstraction of a familiar feature of classical propositional logic (**CPL**). It requires the existence, for each positive integer n, of a set Ψ_n of formulas in the variables $x_1, \dots, x_n$, such that, for any set $\Gamma \cup \{\alpha_1, \dots, \alpha_n\}$ of formulas,

$$\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \text{ is inconsistent in } \vdash \text{ iff } \Gamma \vdash \Psi_n(\alpha_1, \dots, \alpha_n); \tag{1}$$

$$\Gamma \cup \Psi_n(\alpha_1, \dots, \alpha_n) \text{ is inconsistent in } \vdash \text{ iff } \Gamma \vdash \{\alpha_1, \dots, \alpha_n\}. \tag{2}$$

The realizations of (1) and (2) in **CPL** are

$$\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \text{ is inconsistent iff } \Gamma \vdash \neg(\alpha_1 \wedge \cdots \wedge \alpha_n);$$
$$\Gamma \cup \{\neg(\alpha_1 \wedge \cdots \wedge \alpha_n)\} \text{ is inconsistent in } \vdash \text{ iff } \Gamma \vdash \{\alpha_1, \dots, \alpha_n\}.$$

The case $n = 1$ is of course sufficient for **CPL**, but the remaining cases are not generally redundant.

The equivalent algebraic semantics of an algebraizable logic is normally a quasivariety, and there are quasivarieties, *other* than varieties, that meet all the demands in Theorem 1. Some of them algebraize natural logics (such as the $\rightarrow, \bot$ fragment of $\mathbf{IUML}_3$), see [12, p. 15].

Now it is well known that a *variety* is semisimple with EDPC iff it is *filtral* [6, 7], and the notion of filtrality generalizes naturally to quasivarieties (see below). One might therefore hope to reformulate Theorem 1 as a connection between the CIL and *relatively filtral* quasivarieties—based on an expected equivalence:

$$\text{relatively filtral} = \text{relatively semisimple with EDPRC.}$$

This would square well with the general relationship between deduction-detachment theorems (DDTs) and EDPRC [2], as the existence of a CIL implies that of a DDT.

It seems, however, that relatively filtral quasivarieties have not been studied before. (A literature search for filtral *varieties* reveals possible reasons for this, which cannot be gone into here.) In the present paper, we extend the theory of uniform congruence schemes, ideal varieties and filtrality to quasivarieties. In particular, we prove the equivalence displayed in the previous paragraph—yielding the desired improvement of Theorem 1.

DETAILS. From now on, K denotes a given quasivariety of algebras. Let $\boldsymbol{A}$ be an algebra of the same type. The K–*congruences* of $\boldsymbol{A}$ are the congruences θ such that $\boldsymbol{A}/\theta \in \mathsf{K}$. They always form an algebraic lattice $\mathbf{Con}_{\mathsf{K}}\,\boldsymbol{A}$ when ordered by inclusion. We call them the *relative congruences* of $\boldsymbol{A}$ when K is understood. Expressions like K–*subdirectly irreducible*, K–*simple* and K–*congruence distributive* are defined like their absolute counterparts, but with K–congruences in the role of congruences. For any $X \subseteq A^2$, we use $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}} X$ to denote the smallest K–congruence of $\boldsymbol{A}$ containing X.

Our starting point is the following quasivarietal analogue of Maltsev's Lemma, which corrects a mis-statement in [11]. Unlike a more complicated characterization of K–congruence generation in [5] (and in [4, p. 249]), it leads smoothly to the uniform *relative* congruence schemes defined below it. In applications, X will often be a singleton $\{\langle a, b\rangle\}$; we shall then write $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}} X$ as $\mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, b)$. We use $\boldsymbol{Tm}$ to denote the absolutely free algebra of K's type generated by denumerably many variables $x_1, x_2, x_3, \ldots$.

Lemma 2. ([3, Lem. 4.2]) *For any $c, d \in A$, we have $\langle c, d\rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}} X$ iff there exist a quasi-equation $(\&_{\,i<n}\ \gamma_i \approx \delta_i) \Longrightarrow \gamma \approx \delta$, satisfied by K, and a homomorphism $h\colon \boldsymbol{Tm} \to \boldsymbol{A}$ such that $h(\gamma) = c$ and $h(\delta) = d$ and, for all $i < n$, $\langle h(\gamma_i), h(\delta_i)\rangle \in X$ or $h(\gamma_i) = h(\delta_i)$.*

Definition 3. A triple $\langle \Sigma, \Phi, \langle \gamma, \delta\rangle\rangle$ is called a *uniform relative congruence scheme* (URCS) if $\Sigma \cup \Phi \cup \{\langle \gamma, \delta\rangle\}$ is a finite subset of Tm^2. We call it a URCS *for* K if K satisfies

$$\big(\big(\&_{\,\langle \rho,\sigma\rangle \in \Sigma}\ \rho \approx \sigma\big)\ \&\ \big(\&_{\,\langle \alpha,\beta\rangle \in \Phi}\ \alpha \approx \beta\big)\big) \Longrightarrow \gamma \approx \delta$$

and, for any $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$, the following is true: $\langle c, d\rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, b)$ iff there is a homomorphism $h\colon \boldsymbol{Tm} \to \boldsymbol{A}$ such that

$$h(\rho) = h(\sigma) \text{ for all } \langle \rho, \sigma\rangle \in \Sigma \tag{3}$$

$$\langle h(\alpha), h(\beta)\rangle = \langle a, b\rangle \text{ for all } \langle \alpha, \beta\rangle \in \Phi \tag{4}$$

$$\langle h(\gamma), h(\delta)\rangle = \langle c, d\rangle. \tag{5}$$

We call $\langle \Sigma, \Phi, \langle \gamma, \delta\rangle\rangle$ a *restricted* URCS *for* K if, in addition, its terms involve at most the variables $x_1, \ldots, x_4$ and the homomorphism h in the definition can always be chosen so that it sends $x_1, \ldots, x_4$ to a, b, c, d, respectively—whence (3), (4) and (5) become

$$\rho^{\boldsymbol{A}}(a, b, c, d) = \sigma^{\boldsymbol{A}}(a, b, c, d) \text{ for all } \langle \rho, \sigma\rangle \in \Sigma$$
$$\langle \alpha^{\boldsymbol{A}}(a, b, c, d), \beta^{\boldsymbol{A}}(a, b, c, d)\rangle = \langle a, b\rangle \text{ for all } \langle \alpha, \beta\rangle \in \Phi$$
$$\langle \gamma^{\boldsymbol{A}}(a, b, c, d), \delta^{\boldsymbol{A}}(a, b, c, d)\rangle = \langle c, d\rangle.$$

Definition 4. (cf. [4]) We say that K has *parameterized EDPRC* if there is a set Δ of pairs of terms such that, for any $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$, the following is true: $\langle c, d\rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, b)$ iff there exists a homomorphism $h\colon \boldsymbol{Tm} \to \boldsymbol{A}$, sending $x_1, \ldots, x_4$ to a, b, c, d, respectively, such that

$$h(\alpha) = h(\beta) \text{ for all } \langle \alpha, \beta\rangle \in \Delta.$$

We say that K has *EDPRC* if, moreover, the terms in Δ can be chosen to involve no variable other than $x_1, \ldots, x_4$, i.e., for all $\boldsymbol{A}, a, b, c, d$ as above, we have

$$\langle c, d\rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(a, b) \text{ iff } \Big(\alpha^{\boldsymbol{A}}(a, b, c, d) = \beta^{\boldsymbol{A}}(a, b, c, d) \text{ for all } \langle \alpha, \beta\rangle \in \Delta\Big).$$

Generalizing results in [6], we obtain:

Theorem 5. *A quasivariety has a URCS iff it has parameterized EDPRC. It has a restricted URCS iff it has EDPRC. The set Δ in the definition of EDPRC can therefore be chosen finite.*

In this connection, recall [2] that quasivarieties with EDPRC are relatively congruence distributive and have the relative congruence extension property.

The join semilattice of compact (i.e., finitely generated) K–congruences of an algebra $\boldsymbol{A}$ shall be denoted as $\mathbf{Comp}_{\mathsf{K}}\,\boldsymbol{A}$.

A K–*representation* is an embedding $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i$, where I is a set and each $\boldsymbol{A}_i \in \mathsf{K}$. In this case, we usually identify $\boldsymbol{A}$ with its image $f[\boldsymbol{A}]$ and denote the representation by $\boldsymbol{A} \subseteq \prod_{i\in I} \boldsymbol{A}_i$. Given a filter F over I and an ideal J of the join semilattice $\prod_{i\in I} \mathbf{Comp}_{\mathsf{K}} \boldsymbol{A}_i$, we define relations θ_F and θ_J on $\boldsymbol{A}$ as follows (where $a = \langle a_i : i \in I\rangle$, $b = \langle b_i : i \in I\rangle$ belong to A):

$$\langle a, b\rangle \in \theta_F \text{ iff } \{i \in I : a_i = b_i\} \in F;$$
$$\langle a, b\rangle \in \theta_J \text{ iff } \exists\langle \theta_i : i \in I\rangle \in J \text{ with } \langle a_i, b_i\rangle \in \theta_i \text{ for all } i.$$

Both are K–congruences, as quasivarieties are closed under I, S and P_R (reduced products), and because $\boldsymbol{A}/\theta_J$ clearly satisfies any quasi-equation valid in K (ideals being closed under finite joins). A K–congruence θ of $\boldsymbol{A}$ is then said to be *filtral* [resp. *ideal*] (w.r.t. the representation) if it has the form θ_F [resp. θ_J] for some such F [resp. J]. If this is true of all K–congruences of $\boldsymbol{A}$, we say that the representation $\boldsymbol{A} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ *admits only filtral* [resp. *ideal*] K–congruences. If every K–representation admits only ideal K–congruences, we call K a *relatively ideal* quasivariety.

A *product congruence* of $\boldsymbol{A} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ is a relation of the form $\theta = A^2 \cap \prod_{i\in I} \theta_i$, where each $\theta_i \in \mathbf{Con}_{\mathsf{K}} \boldsymbol{A}_i$ (this means that for all $a, b \in A$, we have $\langle a, b\rangle \in \theta$ iff $\langle a_i, b_i\rangle \in \theta_i$ for every i). Observe that $\boldsymbol{A}/\theta$ embeds into $\prod_{i\in I} \boldsymbol{A}_i/\theta_i$, so θ is a K–congruence of $\boldsymbol{A}$.

Generalizing an observation of Kiss [8], we note:

Lemma 6. *A K–representation $\boldsymbol{A} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ admits only ideal K–congruences iff every compact K–congruence of $\boldsymbol{A}$ is a product congruence.*

This helps to prove the following generalization of a result in [6]:

Theorem 7. *A quasivariety is relatively ideal iff it has EDPRC.*

Lemma 6 also makes it easy to see that

Lemma 8. *If K–representations $\boldsymbol{A} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ admit only ideal K–congruences in those cases where every $\boldsymbol{A}_i$ is K–subdirectly irreducible, then K is a relatively ideal quasivariety.*

Moreover, the next lemma adapts a clever result of Magari [10].

Lemma 9. *Let $\mathsf{X} \subseteq \mathsf{K}$ and suppose that every representation of an algebra as a* subdirect *product of algebras in X admits only ideal* [*resp. filtral*] *K–congruences. Then* any *K–representation $\boldsymbol{A} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ (not assumed subdirect), with every $\boldsymbol{A}_i \in \mathsf{X}$,* admits *only ideal* [*resp. filtral*] *K–congruences.*

Definition 10. We say that K is *relatively filtral* provided that every representation of an algebra as a *subdirect* product of K–*subdirectly irreducible* members of K admits only filtral K–congruences.

It follows at once that every relatively filtral quasivariety is K–semisimple (as there are just two filters over a singleton). In fact, by Lemma 9, even nontrivial *subalgebras* of K–subdirectly irreducible members of K must be K–simple. Moreover, a K–congruence on a subalgebra of a direct product of K–*simple* algebras is filtral iff it is ideal. Therefore, a K–semisimple quasivariety will be relatively filtral if it is relatively ideal. And conversely, relatively filtral quasivarieties are relatively ideal, in view of Lemmas 8 and 9. Thus, by Theorem 7, we obtain:

Theorem 11. *A quasivariety is relatively filtral iff it is relatively semisimple and has EDPRC.*

Corollary 12. *Relatively filtral quasivarieties are relatively congruence distributive and have the relative congruence extension property.*

Corollary 13. *Let* K *be a quasivariety algebraizing a finitary sentential logic* $\vdash$. *Then* $\vdash$ *has a classical inconsistency lemma iff* K *is relatively filtral and the subalgebras of its nontrivial members are nontrivial.*

Here, the condition on subalgebras is equivalent to the compactness demand in Theorem 1; this fact generalizes a result of Kollár [9].

Example. By Corollary 13, the quasivariety generated by the $\to, \bot$ reduct of the 3–element Sugihara monoid is relatively filtral. (It is not a variety.)

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▸ UMBERTO RIVIECCIO, ANDREW CRAIG, *Many-valued modal logic over residuated lattices via duality.*

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One of the latest and most challenging trends of research in non-classical logic is the attempt of enriching many-valued systems with modal operators. This allows us to formalize reasoning about vague or graded properties in those contexts (e.g., epistemic, normative, computational) that require the additional expressive power of modalities.

This enterprise is thus potentially relevant not only to mathematical logic, but also to philosophical logic and computer science.

A very general method for introducing the (least) many-valued modal logic over a given finite residuated lattice is described in [1]. The logic is defined semantically by means of Kripke models that are many-valued in two different ways: the valuations as well as the accessibility relation among possible worlds are both many-valued. The work in [1] also shows that providing complete axiomatizations for such logics, even if we enrich the propositional language with a truth-constant for every element of the given lattice, is a non-trivial problem, which has been only partially solved to date.

In this presentation we report on ongoing research in this direction, focusing on the contribution that the theory of natural dualites [2] can give to this enterprise. We show in particular that duality allows us to extend the method used in [1] to prove completeness with respect to local modal consequence, obtaining completeness for global consequence, too. Besides this, our study is also a contribution towards a better general understanding of quasivarieties of (modal) residuated lattices from a topological perspective.

Following [1], we consider $\mathbf{A}$-valued Kripke models $M = \langle W, R, v\rangle$, where $\mathbf{A}$ is a finite (not necessarily integral, but commutative) residuated lattice. The propositional language is $\langle \wedge, \vee, *, \to, c_0, \ldots, c_n, \Box\rangle$, where $\langle \wedge, \vee, *, \to\rangle$ is the usual signature of residuated lattices (meet, join, fusion and implication); in addition we have a propositional constant c_i $(0 \leq i \leq n)$ for each element of $\mathbf{A}$ and a (necessity) modal operator $\Box$ that will be defined below. A possibility operator (which need not be inter-definable with $\Box$) can also be introduced using a formula similar to (1) below, but here (again, following [1]) we shall focus on a language having only the $\Box$ operator. The *valuation* $v\colon Fm \times W \to A$ assigning to each pair $\langle \varphi, w\rangle$ a value in $\mathbf{A}$ is required to be a $\langle \wedge, \vee, *, \to, c_0, \ldots, c_n\rangle$-homomorphism in the first coordinate. The *accessibility relation* $R\colon W \times W \to A$, viewed as a characteristic function, maps each pair $\langle w, w'\rangle$ to an element of $\mathbf{A}$ which can be seen as the "degree of accessibility" of w' from w. The semantics of the modal operator is given by

$$v(\Box\varphi, w) := \bigwedge\{R(w, w') \to v(\varphi, w') : w' \in W\}. \tag{1}$$

This is readily seen to be a generalization of the classical definition, if we view the two-element Boolean algebra as a particular example of a residuated lattice. Note also that arbitrary meets always exist as we assumed $\mathbf{A}$ to be finite.

The notion of satisfaction is defined as usual: we set $M, w \vDash \varphi$ iff $v(\varphi, w) \geq 1$, where 1 is the neutral element of the monoid $\langle A, *\rangle$ and $\leq$ is the lattice order of $\mathbf{A}$. We can then define a *local modal consequence* relation by setting $\Gamma \vDash_l \varphi$ iff, for every model M and every $w \in W$, $M, w \vDash \gamma$ for all $\gamma \in \Gamma$ implies $M, w \vDash \varphi$. *Global modal consequence* is defined by setting $\Gamma \vDash_g \varphi$ iff it holds that ($M, w \vDash \varphi$ for every $w \in W$) whenever ($M, w \vDash \gamma$ for all $\gamma \in \Gamma$ and for every $w \in W$).

Using the above-defined semantics, it is easy to check that $\vDash_l$ and $\vDash_g$ share the same valid formulas, including:

(i) $\Box\top \leftrightarrow \top$

(ii) $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$

(iii) $\Box(c_i \to \varphi) \leftrightarrow (c_i \to \Box\varphi) \quad (0 \leq i \leq n)$.

where as usual $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\psi \to \varphi)$. It is also easy to show that the *normality* axiom, in the following form, is in general not valid.

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

Failure of normality constitutes one of the main technical difficulties for completeness proofs, for it prevents one from applying the standard canonical model construction of modal logic. However, [1] shows that axiom schemata (i)-(iii), together with the monotonicity rule below, are enough to prove completeness for local consequence:

$$\text{if} \;\; \emptyset \vdash \varphi \to \psi, \;\; \text{then} \quad \emptyset \vdash \Box\varphi \to \Box\psi.$$

By this we mean that it is sufficient to take any (Hilbert-style) axiomatization that is complete with respect to the logic determined (in the $\Box$-free language) by the logical matrix $\langle \mathbf{A}, \uparrow 1 \rangle$, where $\uparrow 1 := \{a \in A : 1 \leq a\}$, and expand it with the above axioms and rule to axiomatize the least local many-valued modal logic over $\mathbf{A}$ (i.e., the logic of *all* $\mathbf{A}$-valued Kripke models). The expressive power gained by including a propositional constant in the language for each element of $\mathbf{A}$, which allows one to employ axiom (iii), is crucial to the completeness proof. We also note that the logic of the matrix $\langle \mathbf{A}, \uparrow 1 \rangle$ is algebraizable, and so in many cases it is easy to obtain a neat axiom system for the non-modal fragment of the logic. The technique of [1] is thus very general as it can be applied to any finite residuated lattice as long as we have a complete axiomatization of its (non-modal) logic. The problem of axiomatizing the global consequence relation in full generality is however left open in [1].

Building on the above results, we employ algebraic logic and natural duality to obtain a more general strategy for proving completeness. This is done through the following steps.

1. Having fixed a finite residuated lattice $\mathbf{A}$, we consider a syntactic calculus that is algebraizable with respect to the (non-modal) logic of the matrix $\langle \mathbf{A}, \uparrow 1 \rangle$. We then expand this calculus as indicated in [1], i.e., adding axioms (i)-(iii) together with the monotonicity rule, either in its weak form shown above (applicable only to valid formulas) or in its strong form, namely: from $\varphi \to \psi$ derive $\Box\varphi \to \Box\psi$. The former defines the calculus $\vdash_l$ for local consequence, the latter $\vdash_g$ for global consequence.

2. We determine the (reduced) algebraic models of $\vdash_l$ and $\vdash_g$ according to the general theory of algebraization of logics (see, e.g., [3]). The calculus $\vdash_g$ is easily shown to be algebraizable w.r.t. a quasivariety $\mathsf{MRL}_{\mathbf{A}}$ of *modal residuated lattices*, the members thereof are algebras $\mathbf{B}$ in the language $\langle \wedge, \vee, *, \to, c_0, \ldots, c_n, \Box \rangle$ such that the $\Box$-free reduct of $\mathbf{B}$ is in $\mathsf{Q}(\mathbf{A})$ and the modal operator satisfies equations corresponding to axioms (i)-(iii) shown above. Algebraizability implies that reduced models of $\vdash_g$ are matrices of the form $\langle \mathbf{B}, \uparrow 1 \rangle$, where $\mathbf{B} \in \mathsf{MRL}_{\mathbf{A}}$. We also show that reduced models of $\vdash_l$ are matrices of the form $\langle \mathbf{B}, F \rangle$ where $\mathbf{B} \in \mathsf{V}(\mathsf{MRL}_{\mathbf{A}})$ and F is a logical filter of the non-modal logic determined by the matrix $\langle \mathbf{A}, \uparrow 1 \rangle$.

3. We study the class $\mathsf{MRL}_{\mathbf{A}}$ using natural duality. This is itself done in two steps.

3.1 We develop a natural duality for $\mathsf{Q}(\mathbf{A})$. This is relatively straightforward, as we are dealing with a finitely-generated quasivariety of algebras having a lattice reduct. We can thus use powerful results from the natural duality toolkit such as the near-unanimity duality theorem [2, Theorem 2.3.4]. The following algebraic insight plays a crucial role in the characterization of the spaces that are dual to our algebras. We know from the near-unanimity theorem that the dual of $\mathsf{Q}(\mathbf{A})$ is a category of structured topological spaces generated by the dualizing object $\langle A; \mathbb{S}(\mathbf{A}^2), \tau \rangle$, where $\langle A, \tau \rangle$ is the Stone space obtained by endowing the universe of $\mathbf{A}$ with the discrete topology, and $\mathbb{S}(\mathbf{A}^2)$ is the family of all subalgebras of the direct power $\mathbf{A}^2$. Thanks to the presence of the constants c_i in the algebraic language, it can be checked that every subalgebra of $\mathbf{A}^2$ is in fact a congruence of $\mathbf{A}$. Moreover, congruences of a residuated lattice $\mathbf{B}$ are in one-to-one correspondence with the *negative idempotents* of $\mathbf{B}$, defined as those elements $a \in B$ such that $a * a = a \leq 1$. The negative idempotents of any residuated lattice $\mathbf{B}$ form a distributive lattice (where the join coincides with that of $\mathbf{B}$) that is dually isomorphic to the lattice of congruences of $\mathbf{B}$. This allows us to take as dualizing structure $\langle A; \{\, \theta_j : j \in \mathsf{JNi}(\mathbf{A}) \backslash \{1\} \,\}, \tau \rangle$, where $\mathsf{JNi}(\mathbf{A})$ is the set of join-irreducible negative idempotents of $\mathbf{A}$ and θ_j is the congruence associated to each $j \in \mathsf{JNi}(\mathbf{A})$. The choice of $\mathsf{JNi}(\mathbf{A})$ is justified by the entailment relations [2, Section 2.4.5] that allow us to restrict ourselves to any set of elements of $\mathbf{A}$ that join-generates the whole lattice of negative idempotents; similarly the neutral element 1 can be dispensed with because $\theta_1 = Id_A$ is a trivial relation. We thus have that $\langle A; \{\, \theta_j : j \in \mathsf{JNi}(\mathbf{A})\backslash\{1\} \,\}, \tau\} \rangle$ yields a (strong) duality on $\mathsf{Q}(\mathbf{A})$. Objects of our dual category are structured topological spaces $\langle X; \{\, R_j : j \in \mathsf{JNi}(\mathbf{A})\backslash\{1\} \,\}, \tau\rangle$ such that $\langle X, \tau\rangle$ is a Stone space and:

i. R_j is an equivalence relation that is clopen-separating. That is, for all $x, y \in X$, if $\langle x, y\rangle \notin R_j$ then there is a clopen set U such that R_j is compatible with U (i.e., $s \in U$ if and only if $t \in U$ for all $\langle s, t\rangle \in R_j$) and $x \in U$ but $y \notin U$.
ii. the collection of sets $\{\, R_j : j \in \mathsf{JNi}(\mathbf{A})\backslash\{1\} \,\}$, ordered by inclusion, forms a poset that is dually order isomorphic to $(\mathsf{JNi}(\mathbf{A})\backslash\{1\}, \leq_{\mathbf{A}})$.

3.2 We try to extend the duality sketched above for residuated lattices in $\mathsf{Q}(\mathbf{A})$ to a duality for modal residuated lattices. In this respect we generalize the work of [5] on dualities for MV-algebras with modal operators. Given a modal residuated lattice $\mathbf{B}$, we consider the space $X(\mathbf{B})$ that is dual to the $\Box$-free reduct of $\mathbf{B}$, which belongs to $\mathsf{Q}(\mathbf{A})$. Elements of $X(\mathbf{B})$ are thus non-modal homomorphisms from $\mathbf{B}$ to $\mathbf{A}$. We define an $\mathbf{A}$-valued relation $R_{\mathbf{B}} : X(\mathbf{B}) \times X(\mathbf{B}) \to A$ as follows: for all $f, g \colon B \to A$,

$$R_{\mathbf{B}}(f, g) := \bigwedge\{\, f(\Box a) \to g(a) : a \in B \,\}.$$

Conversely, consider a pair $\langle X, R\rangle$ such that X is a structured topological space of the type defined above and R is an $\mathbf{A}$-valued relation. According to natural duality, the dual algebra $B(X)$ is given by the structure-preserving continuous functions from X to $\langle A; \{\, \theta_j : j \in \mathsf{JNi}(\mathbf{A})\backslash\{1\} \,\}, \tau\rangle$. Given $h \colon X \to A$, we define, for all $x \in X$,

$$(\Box h)(x) := \bigwedge\{R(x, y) \to h(y) : y \in X\}$$

thus obtaining a modal operator on $B(X)$. We prove that every modal residuated lattice $\mathbf{B}$ is isomorphic to $B(X(\mathbf{B}))$. This in particular involves showing that, for every $f \in X(\mathbf{B})$ and $b \in B$,

$$f(\Box b) = \bigwedge\{R_{\mathbf{B}}(f, g) \to g(b) : g \in X(\mathbf{B})\}. \tag{2}$$

In order to achieve a full duality, one would need to characterize the structured topological spaces $\langle X, R\rangle$ that are of the form $\langle X(\mathbf{B}), R_{\mathbf{B}}\rangle$ for some $\mathbf{B} \in \mathsf{MRL}_{\mathbf{A}}$. This we have not achieved yet, and will be the object of future work. However, the results stated so far are enough to conclude the completeness proof.

4. We prove completeness following a strategy similar to that of [4]. Assuming, e.g., $\Gamma \nvdash_g \varphi$, we use completeness of $\vdash_g$ with respect to its reduced algebraic models to find a counterexample in the form of a reduced matrix $\langle \mathbf{B}, F\rangle$, where $\mathbf{B} \in \mathsf{MRL}_{\mathbf{A}}$ and $F = {\uparrow}1$, together with a homomorphism $k \colon Fm \to B$ such that $k[\Gamma] \subseteq F$ but $k(\varphi) \notin F$. We also know that the $\Box$-free reduct of $\mathbf{B}$ belongs to $\mathsf{Q}(\mathbf{A})$, which means that the matrix $\langle \mathbf{B}, F\rangle$ is (if we ignore the modal operator) also a reduced model of the non-modal logic of $\mathbf{A}$. We consider the dual space $\langle X(\mathbf{B}), R_{\mathbf{B}}\rangle$, which is an $\mathbf{A}$-valued Kripke frame. On this we define a valuation $v(\psi, f) := (f \circ k)(\psi)$ for all formulas $\psi \in Fm$ and all $f \in X(\mathbf{B})$. That v is indeed a modal valuation is ensured by (2) above, which plays the role of the Truth Lemma in usual modal logic. Then $\langle X(\mathbf{B}), R_{\mathbf{B}}, v\rangle$ is an $\mathbf{A}$-valued Kripke model. Since $f({\uparrow}1_{\mathbf{B}}) \subseteq {\uparrow}1_{\mathbf{A}}$ for any (non-modal) homomorphism $f \colon B \to A$, we have $v(\gamma, f) \geq 1_{\mathbf{A}}$, i.e., $v, f \vDash \gamma$ for all $\gamma \in \Gamma$ and all $f \in X(\mathbf{B})$. However, since $k(\varphi) \ngeq 1_{\mathbf{B}}$ (i.e., $k(\varphi) \wedge 1_{\mathbf{B}} \neq 1_{\mathbf{B}}$), we can use the algebraic separation theorem [2, Theorem 1.3.1] to find a homomorphism $g \in X(\mathbf{B})$ such that $g(k(\varphi) \wedge 1_{\mathbf{B}}) = (g \circ k)(\varphi) \wedge 1_{\mathbf{A}} \neq g(1_{\mathbf{B}}) = 1_{\mathbf{A}}$. Hence, $v, g \nvDash \varphi$. The proof of completeness for local consequence is slightly more involved but analogous. One invokes algebraic completeness to obtain a countermodel

in the form of a reduced matrix $\langle \mathbf{B}, F \rangle$ where $\mathbf{B} \in \mathsf{V}(\mathsf{MRL}_{\mathbf{A}})$. Since $\mathbf{B}$ is a homomorphic image of some $\mathbf{B}' \in \mathsf{MRL}_{\mathbf{A}}$ via a homomorphism $\pi \colon B' \to B$, we consider the matrix $\langle \mathbf{B}', \pi^{-1}[F] \rangle$, which defines the same logic as $\langle \mathbf{B}, F \rangle$. If we ignore the modal operator, the matrix $\langle \mathbf{B}', \pi^{-1}[F] \rangle$ is a model of the non-modal logic of $\mathbf{A}$, although it need not be reduced. Then, if we consider its (non-modal) reduction $\langle \mathbf{B}'/\Omega F', F'/\Omega F' \rangle$, we have $\mathbf{B}'/\Omega F \in \mathsf{Q}(\mathbf{A})$ and $F'/\Omega F' = \uparrow 1_{\mathbf{B}'/\Omega F'}$. From this point on we can reason as in the case of global consequence.

As mentioned above, this is ongoing research so there are plenty of open problems and potential further developments. The most obvious one is whether it is possible, and if so how, to upgrade the representation of modal residuated lattices via topological structures to a duality. Another interesting issue, already considered (and partially solved) in [1], is that of axiomatizing extensions of the least modal logic over a residuated lattice that correspond to restrictions on the range of the accessibility relation. One can for example consider the class of *idempotent frames*, where, for all $w, w' \in W$, the value $R(w, w')$ is an idempotent element of $\mathbf{A}$. Finally, it is worth mentioning the (more fundamental) problem of obtaining a complete axiomatization of the least modal logic over $\mathbf{A}$ in a language that avoids using (some of) the constants c_i.

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▸ JONAS ROGGER, *Decidability of order-based modal logics*.
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§1. Introduction. In this work, we consider many-valued modal logics, which are defined in the classical Kripke frame setting with a many-valued semantics at each world. The accessibility relation can be either crisp (as in the classical setting) or many-valued. These logics can be used to model concepts such as necessity, belief (see e.g. [9, 11]), and spatio-temporal relations (see [6, 13]) in the presence of uncertainty, possibility, or vagueness. Furthermore, many-valued description logics may be understood as many-valued multi-modal logics (see e.g. [1, 10]).

In [2, 7, 8], many-valued modal logics are described in quite general settings, restricting mostly to finitely many truth values. Here, we define "order-based" modal logics based more generally on any complete linearly ordered set with operations depending only on the given order. Paradigmatic examples of such logics are the modal logics based on the semantics of infinite-valued Gödel logics (see [3, 4, 12]).

As our main result, we establish decidability of the validity problem for order-based modal logics with certain natural sets of truth values (including Gödel modal logics based on the real unit interval $[0, 1]$). In the generalised version of the semantics used in [3, 4, 12], the finite model property fails for many of these logics, even for their box or diamond fragments (see e.g. [3]). Using analytic calculi, decidability results have been obtained for some of these fragments in [12]. However, decidability for the full order-based modal logics with both modal operators (including Gödel modal logics) has remained open.

As our main tool, we present an alternative Kripke semantics for these logics (having the same valid formulas) that admits the finite model property. As the key ingredient to obtain finite models, we restrict modal formulas at a given world to a particular finite set of truth values.

The material presented here is based on joint work with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez [5].

§2. Order-based modal logics. Let $\mathcal{L}$ be a finite algebraic language that includes the binary operation symbols $\wedge$ and $\vee$ and constant symbols $\bot$ and $\top$, and denote the set of constants of this language by $\mathrm{C}_{\mathcal{L}}$. An algebra $\mathbf{A}$ for $\mathcal{L}$ will be called *order-based* if it satisfies the following conditions:

(1) $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a *complete chain*: i.e., a bounded lattice with order $a \leq^{\mathbf{A}} b$ defined by $a \wedge^{\mathbf{A}} b = a$ that satisfies (i) $a \leq^{\mathbf{A}} b$ or $b \leq^{\mathbf{A}} a$ for all $a, b \in A$, and (ii) $\bigwedge^{\mathbf{A}} B$ and $\bigvee^{\mathbf{A}} B$ exist in A for all $B \subseteq A$ (in particular, $\bot^{\mathbf{A}} = \bigvee^{\mathbf{A}} \emptyset$ and $\top^{\mathbf{A}} = \bigwedge^{\mathbf{A}} \emptyset$).

(2) For each operation symbol $\star$ of $\mathcal{L}$, the operation $\star^{\mathbf{A}}$ is definable in $\mathbf{A}$ by a quantifier-free formula in the first-order language with only $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathcal{L}}$.

Note that, because $\mathbf{A}$ is a complete chain, an implication operation $\to^{\mathbf{A}}$ may always be introduced as the residual of $\wedge^{\mathbf{A}}$:

$$a \to^{\mathbf{A}} b \;=\; \bigvee\nolimits^{\mathbf{A}} \{c \in A : c \wedge^{\mathbf{A}} a \leq^{\mathbf{A}} b\} \;=\; \begin{cases} \top^{\mathbf{A}} & \text{if } a \leq^{\mathbf{A}} b \\ b & \text{otherwise.} \end{cases}$$

This operation is definable in $\mathbf{A}$ by the quantifier-free formula

$$F^{\to}(x, y, z) \;=\; ((x \leq y) \Rightarrow (z \approx \top)) \;\&\; ((y < x) \Rightarrow (z \approx y)),$$

letting $\varphi \leq \psi$ stand for $\varphi \wedge \psi \approx \varphi$ and $\varphi < \psi$ for $(\varphi \leq \psi)$ & $(\varphi \not\approx \psi)$, where & and $\Rightarrow$ are classical conjunction and implication, respectively. Obviously, the following holds:

$$\mathbf{A} \models F^{\to}(a, b, c) \qquad \text{iff} \qquad a \to^{\mathbf{A}} b = c.$$

A negation connective is defined by $\neg\varphi = \varphi \to \bot$ and interpreted by the unary operation

$$\neg^{\mathbf{A}} a \;=\; \begin{cases} \top^{\mathbf{A}} & \text{if } a = \bot^{\mathbf{A}} \\ \bot^{\mathbf{A}} & \text{otherwise.} \end{cases}$$

Furthermore, the following operations are covered by the order-based approach:

$$\Delta^{\mathbf{A}} a \;=\; \begin{cases} \top^{\mathbf{A}} & \text{if } a = \top^{\mathbf{A}} \\ \bot^{\mathbf{A}} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \nabla^{\mathbf{A}} a \;=\; \begin{cases} \bot^{\mathbf{A}} & \text{if } a = \bot^{\mathbf{A}} \\ \top^{\mathbf{A}} & \text{otherwise.} \end{cases}$$

The first operation is the globalization or Baaz Delta operator, and the second is the Nabla operator (definable also as $\nabla^{\mathbf{A}} a = \neg^{\mathbf{A}} \neg^{\mathbf{A}} a$). Another example of an operation definable by a suitable quantifier-free formulas is the dual-implication

$$a \leftarrow^{\mathbf{A}} b \;=\; \bigwedge\nolimits^{\mathbf{A}} \{c \in A : b \leq^{\mathbf{A}} a \vee^{\mathbf{A}} c\} \;=\; \begin{cases} \bot^{\mathbf{A}} & \text{if } b \leq^{\mathbf{A}} a \\ b & \text{otherwise.} \end{cases}$$

We define $\mathcal{L}_{\Box\Diamond}$ as the language $\mathcal{L}$ with additional unary operation symbols $\Box$ and $\Diamond$.

The set of *formulas* $\mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ over $\mathcal{L}_{\Box\Diamond}$, denoted $\varphi, \psi, \ldots$ is defined inductively over a countably infinite set Var of propositional variables, denoted $p, q, \ldots$.

For a fixed order-based algebra $\mathbf{A}$, we define an $\mathbf{A}$-*frame* as a pair $\mathfrak{F} = \langle W, R\rangle$ such that W is a non-empty set of *worlds* and $R\colon W \times W \to A$ is a binary $\mathbf{A}$-*accessibility relation* on W. If $Rxy \in \{\bot^{\mathbf{A}}, \top^{\mathbf{A}}\}$ for all $x, y \in W$, $\mathfrak{F}$ is called a *crisp* $\mathbf{A}$-*frame*.

A $\mathsf{K}(\mathbf{A})$-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$ such that $\langle W, R\rangle$ is an $\mathbf{A}$-frame and $V\colon \mathrm{Var} \times W \to A$ is a mapping, called a *valuation*. This valuation is extended to the mapping $V\colon \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond} \times W \to A$ by $V(\star(\varphi_1, \ldots, \varphi_n), x) = \star^{\mathbf{A}}(V(\varphi_1, x), \ldots, V(\varphi_n, x))$, for each n-ary operation symbol $\star$ of $\mathcal{L}$, and

$$V(\Box\varphi, x) = \bigwedge\nolimits^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\varphi, y) : y \in W\}$$

$$V(\Diamond\varphi, x) = \bigvee\nolimits^{\mathbf{A}}\{Rxy \wedge^{\mathbf{A}} V(\varphi, y) : y \in W\}.$$

A $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame. We will call a formula $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ *valid* in a $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, x) = \top^{\mathbf{A}}$ for all $x \in W$. If φ is valid in all L-models for some logic L, then φ is said to be L-*valid*, written $\models_{\mathsf{L}} \varphi$.

As our main example of an order-based algebra, we consider the standard infinite-valued Gödel algebra

$$\mathbf{G} = \langle [0,1], \wedge, \vee, \to, \bot, \top\rangle.$$

The logics $\mathsf{K}(\mathbf{G})$ and $\mathsf{K}(\mathbf{G})^{\mathsf{C}}$ are the "Gödel modal logics" GK and GK^{C} studied in [2, 3, 4, 12] (in some cases with different names). A more general perspective on Gödel modal logics is obtained by considering the family of Gödel logics defined by algebras $\mathbf{A} = \langle A, \wedge, \vee, \to, \bot, \top\rangle$ satisfying $\{0, 1\} \subseteq A \subseteq [0, 1]$. Three natural examples of universes are the real unit interval $[0, 1]$,

$$G_{\downarrow} = \{0\} \cup \{\tfrac{1}{n} : n \in \mathbb{Z}^+\}, \quad \text{and} \quad G_{\uparrow} = \{1 - \tfrac{1}{n} : n \in \mathbb{Z}^+\} \cup \{1\},$$

where $\mathbf{G}_{\downarrow} = \langle G_{\downarrow}, \wedge, \vee, \to, \bot, \top\rangle$ and $\mathbf{G}_{\uparrow} = \langle G_{\uparrow}, \wedge, \vee, \to, \bot, \top\rangle$.

Let us mention that the logics $\mathsf{K}(\mathbf{G})$, $\mathsf{K}(\mathbf{G}_{\uparrow})$, $\mathsf{K}(\mathbf{G}_{\downarrow})$ and their crisp counterparts are all distinct and that $\mathsf{K}(\mathbf{G}_{\uparrow})$ and $\mathsf{K}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ do have the finite model property (a model is called *finite* if its set of worlds W is finite). $\mathsf{K}(\mathbf{G})$, $\mathsf{K}(\mathbf{G}_{\downarrow})$, $\mathsf{K}(\mathbf{G})^{\mathsf{C}}$, and $\mathsf{K}(\mathbf{G}_{\downarrow})^{\mathsf{C}}$ do not have the finite model property, however, and neither do $\mathsf{K}(\mathbf{G}_{\uparrow})$ and $\mathsf{K}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ if Δ is added to the language. For instance, for $\mathsf{K}(\mathbf{G})$, it is shown in [3] that the following formula provides a counterexample to the finite model property:

$$\varphi = \Box\neg\neg p \to \neg\neg\Box p.$$

Just observe that φ is valid in all finite $\mathsf{K}(\mathbf{G})$-models, but not in the infinite $\mathsf{K}(\mathbf{G})$-model $\mathfrak{M} = \langle \mathbb{Z}^+, R, V\rangle$ where $Rxy = 1$ and $V(p, x) = \frac{1}{x}$ for all $x, y \in \mathbb{Z}^+$.

§3. A new semantics for the modal operators. Consider again the failure of the finite model property for $\mathsf{K}(\mathbf{G})$ described in the last section. The key ingredient is that in $\mathfrak{M}$, for all $x \in W$,

$$Rxy \to^{\mathbf{A}} V(p, y) \overset{y\to\infty}{\longrightarrow} 0, \quad \text{but} \quad Rxy \to^{\mathbf{A}} V(p, y) \neq 0, \text{ for all } y \in \mathbb{Z}^+.$$

This means that there is no "witness-world" y, such that $Rxy \to^{\mathbf{A}} V(p, y) = V(\Box p, x)$, and thus φ requires an infinite countermodel. To remedy this shortcoming, our idea is to restrict modal formulas to only a finite number of possible truth values. In this case, y can act as a "witness-world" even if $Rxy \to^{\mathbf{A}} V(p, y)$ is just "sufficiently close" to $V(\Box p, x)$. For this, we redefine models in the following fashion.

Let us assume that $\mathbf{A}$ is any order-based algebra with universe $[0, 1]$, $G_{\downarrow}$, or $G_{\uparrow}$. We define a $\mathsf{FK}(\mathbf{A})$-*model* as a five-tuple $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$ such that $\langle W, R, V\rangle$ is a

$\mathsf{K}(\mathbf{A})$-model and $T_\Box : W \to \mathcal{P}(A)$ and $T_\Diamond : W \to \mathcal{P}(A)$ are functions satisfying for each $x \in W$, both $T_\Box(x)$ and $T_\Diamond(x)$ contain the values of all constants in $\mathcal{L}$ and

(i) if A is $[0,1]$, then $T_\Box(x) = T_\Diamond(x)$ is finite,

(ii) if A is $G_\downarrow$, then for some $m \in \mathbb{Z}^+$, $T_\Box(x) = \{0, \frac{1}{m}, \frac{1}{m-1}, \dots, \frac{1}{2}, 1\}$ and $T_\Diamond(x) = G_\downarrow$,

(iii) if A is $G_\uparrow$, then for some $m \in \mathbb{Z}^+$, $T_\Box(x) = G_\uparrow$ and $T_\Diamond(x) = \{0, \frac{1}{2}, \dots, \frac{m-1}{m}, 1\}$.

The valuation V is extended to the mapping $V : \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond} \times W$ as for $\mathsf{K}(\mathbf{A})$-models, except

$$V(\Box\varphi, x) = \bigvee\nolimits^{\mathbf{A}} \{r \in T_\Box(x) : r \leq^{\mathbf{A}} \bigwedge\nolimits^{\mathbf{A}} \{Rxy \to^{\mathbf{A}} V(\varphi, y) : y \in W\}\}$$

$$V(\Diamond\varphi, x) = \bigwedge\nolimits^{\mathbf{A}} \{r \in T_\Diamond(x) : r \geq^{\mathbf{A}} \bigvee\nolimits^{\mathbf{A}} \{Rxy \wedge^{\mathbf{A}} V(\varphi, y) : y \in W\}\}.$$

As before, an $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-*model* $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ satisfies the extra condition that $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame, and a formula $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ is *valid* in $\mathfrak{M}$ if $V(\varphi, x) = \top^{\mathbf{A}}$ for all $x \in W$. We then obtain the following results for these new semantics.

Lemma 1. *$\mathsf{FK}(\mathbf{A})$ and $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ have the finite model property.*

In fact, for a $\mathsf{FK}(\mathbf{A})$-counter-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ of a formula φ, $|W|$ will be bounded by an exponential function of the length of φ and thus decidability follows by restricting appropriately to only the truth values that are needed.

Lemma 2. *The validity problems of $\mathsf{FK}(\mathbf{A})$ and $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ are decidable.*

Of course, these results are only fruitful, if there is a tight connection between the new and the original semantics. This connection is given by the following lemma.

Lemma 3. *For all $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$:*

$$\models_{\mathsf{K}(\mathbf{A})} \varphi \quad \textit{iff} \quad \models_{\mathsf{FK}(\mathbf{A})} \varphi \qquad \textit{and} \qquad \models_{\mathsf{K}(\mathbf{A})^{\mathsf{C}}} \varphi \quad \textit{iff} \quad \models_{\mathsf{FK}(\mathbf{A})^{\mathsf{C}}} \varphi.$$

We then immediately obtain decidability of the logics $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ for any order-based algebra $\mathbf{A}$ with universe $[0,1]$, $G_\downarrow$, or $G_\uparrow$.

Theorem 4. *The validity problems of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ are decidable.*

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§1. Introduction. In this work, we provide bases for admissible rules for certain fragments of RM^t, the logic R-mingle extended with a constant t. Admissibility has been studied extensively in the context of transitive modal logics in [3, 4, 5], intermediate logics in [6, 7, 8, 9, 5] and certain many-valued logics in [10, 11, 12], but little is known regarding admissibility of rules in substructural logics. Structural completeness results have been obtained for various fragments of substructural logics, including R-mingle, in [13, 14, 15, 16], and bases have been obtained for all proper fragments of R-mingle in [1]. We extend the results of [1] here to the $\{\to,\neg,t\}$-, $\{\to,\cdot,t\}$-, and $\{\to,t\}$-fragments of RM^t.

First, we give some definitions. We denote by $\mathbf{Fm}_{\mathcal{L}}$ the formula algebra over countably many variables for a language $\mathcal{L}$. A rule is an ordered pair Γ/φ, where $\Gamma\cup\{\varphi\}\subseteq \mathrm{Fm}_{\mathcal{L}}$, and a logic L is understood as a finitary structural consequence relation $\vdash_{\mathsf{L}}$ over $\mathbf{Fm}_{\mathcal{L}}$. A rule Γ/φ is called *derivable* in L if $\Gamma\vdash_{\mathsf{L}}\varphi$, and *admissible* in L if for every substitution (homomorphism) $\sigma\colon \mathbf{Fm}_{\mathcal{L}}\to\mathbf{Fm}_{\mathcal{L}}$:

$$\vdash_{\mathsf{L}} \sigma(\psi) \text{ for all } \psi\in\Gamma \quad\Rightarrow\quad \vdash_{\mathsf{L}}\sigma(\varphi).$$

A logic L is called *structurally complete* if for $\Gamma\cup\{\varphi\}\subseteq\mathrm{Fm}_{\mathcal{L}}$: Γ/φ is L-derivable if and only if (henceforth iff) it is L-admissible. Define

$$\Gamma\mid\!\sim_{\mathsf{L}}\varphi :\Leftrightarrow \Gamma'\mid\!\sim_{\mathsf{L}}\varphi \text{ is } \mathsf{L}\text{-admissible for some finite } \Gamma'\subseteq\Gamma.$$

Then $\mid\!\sim_{\mathsf{L}}=\{(\Gamma,\varphi)\in\mathcal{P}(\mathrm{Fm}_{\mathcal{L}})\times\mathrm{Fm}_{\mathcal{L}}\mid \Gamma\mid\!\sim_{\mathsf{L}}\varphi\}$ is also a logic. For a set of rules $\mathcal{R}$, let $\mathsf{L}+\mathcal{R}$ denote the smallest logic containing $\mathsf{L}\cup\mathcal{R}$. Then $\mathcal{R}$ is called a *basis* for the admissible rules of L if the logics $\mathsf{L}+\mathcal{R}$ and $\mid\!\sim_{\mathsf{L}}$ coincide.

There is an algebraic notion of admissiblity corresponding to the one just given. For a class of $\mathcal{L}$-algebras K, a quasi-identity $\Sigma\Rightarrow\varphi\approx\psi$ is *admissible* in K, if for every homomorphism $\sigma\colon\mathbf{Fm}_{\mathcal{L}}\to\mathbf{Fm}_{\mathcal{L}}$:

$$\vDash_{\mathsf{K}}\sigma(\varphi')\approx\sigma(\psi') \text{ for all } \varphi'\approx\psi'\in\Sigma \quad\Rightarrow\quad \vDash_{\mathsf{K}}\sigma(\varphi)\approx\sigma(\psi).$$

Moreover, a quasi-identity is admissible in K iff it is valid in the free algebra of K on countably infinite many generators, $\mathbf{F}_{\mathsf{K}}(\omega)$ (see, e.g., [5]).

§2. RM^t. The logic RM^t is formulated in the language $\mathcal{L}_t=\{\wedge,\vee,\to,\cdot,\neg,t\}$ with binary operations $\wedge,\vee,\to$ and $\cdot$, a unary operation $\neg$, and a constant t.

Let $\mathbf{Z}^\circ$ denote the algebra $\langle \mathbb{Z} \setminus \{0\}, \wedge, \vee, \to, \cdot, \neg, 1\rangle$, where $\wedge$ and $\vee$ act as min and max, respectively, $\neg$ is the usual minus operation, and the constant t is interpreted as 1. The binary operations $\to$ and $\cdot$ are defined as follows:

$$x \to y = \begin{cases} \neg x \vee y & \text{if } x \leq y \\ \neg x \wedge y & \text{if } x > y \end{cases} \qquad\qquad x \cdot y = \begin{cases} x \wedge y & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |x| > |y|. \end{cases}$$

Using the same interpretations for the operation symbols, we also define for $n \in \mathbb{N}\setminus\{0\}$,

$$\mathbf{Z}_{2n} = \langle \{-n, \dots, -1, 1, \dots, n\}, \wedge, \vee, \to, \cdot, \neg, 1\rangle,$$
$$\mathbf{Z}_{2n+1} = \langle \{-n, \dots, -1, 0, 1, \dots, n\}, \wedge, \vee, \to, \cdot, \neg, 1\rangle,$$

and denote by $\mathbf{Z}_k^m$ the multiplicative or $\{\to, \neg, t\}$-reduct of $\mathbf{Z}_k$.

Let us denote by SM the variety $\mathbb{V}(\mathbf{Z}^\circ)$ of *Sugihara monoids* generated by $\mathbf{Z}^\circ$. For convenience, let us write $|\varphi|$ to mean $\varphi \to \varphi$. For a rule Γ/φ define

$$\Gamma \models_{\mathsf{SM}} \varphi \quad \Leftrightarrow_{def} \quad \{\psi \approx |\psi| \mid \psi \in \Gamma\} \models_{\mathsf{SM}} \varphi \approx |\varphi|.$$

Then, for each rule Γ/φ:

$$\Gamma \models_{\mathsf{SM}} \varphi \quad \Leftrightarrow \quad \Gamma \vdash_{\mathsf{RM}^t} \varphi,$$

that is, SM provides an equivalent algebraic semantics for the logic RM^t.

Let L be a logic for a language $\mathcal{L}$, and $\mathcal{L}' \subseteq \mathcal{L}$. The $\mathcal{L}'$*-fragment* of L is

$$\mathsf{L}' =_{\text{def}} \{\Gamma/\varphi \mid \Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}'}, \Gamma \vdash_{\mathsf{L}} \varphi\}.$$

We recall that if a quasivariety Q provides an equivalent algebraic semantics for L with translations in $\mathcal{L}'$, then the class $\mathsf{Q}{\restriction}\mathcal{L}'$ of $\mathcal{L}'$-subreducts of algebras from Q provides an equivalent algebraic semantics for L' (see [17] for further details). In particular, $\mathsf{SM}{\restriction}\mathcal{L}'$ is an equivalent algebraic semantics for the $\mathcal{L}'$-fragment of RM if $\{\to\} \subseteq \mathcal{L}'$.

It is proved in [1] that the logic R-mingle RM has the following seven distinct fragments containing $\to$:

$$\{\to, \wedge, \neg\}, \{\to, \wedge\}, \{\to, \cdot, \vee\}, \{\to, \vee\}, \{\to, \neg\}, \{\to, \cdot\}, \{\to\}.$$

Therefore, RM^t has at most seven fragments containing $\to$ and t:

$$\{\to, \wedge, \neg, t\}, \{\to, \wedge, t\}, \{\to, \cdot, \vee, t\}, \{\to, \vee, t\}, \{\to, \neg, t\}, \{\to, \cdot, t\}, \{\to, t\}.$$

Note that the $\{\to, \wedge, \neg, t\}$-fragment is just the whole of RM^t and the $\{\to, \wedge, \neg\}$-fragment is RM as $\cdot$ and $\vee$ can be defined using $\to, \wedge$, and $\neg$ (see [1]).

The variety generated by the multiplicative subreducts of Sugihara monoids, $\mathsf{SM}{\restriction}\{\to, \neg, t\}$, coincides with the variety generated by the multiplicative reduct of the four-element subalgebra of $\mathbf{Z}^\circ$, that is, by $\mathbf{Z}_4^m = \langle \{-2, -1, 1, 2\}, \to, \neg, t\rangle$.

Lemma 1. *$\mathbb{V}(\mathsf{SM}{\restriction}\{\to, \neg, t\}) = \mathbb{V}(\mathbf{Z}_4^m)$.*

§3. Bases for admissible rules. The $\{\to, \wedge\}$-, $\{\to, \cdot, \vee\}$-, and $\{\to, \vee\}$-fragments of RM are structurally complete (see [1]). Bases for the admissible rules of the $\{\to, \neg\}$-, $\{\to, \cdot\}$-, and $\{\to\}$-fragments are provided in [1], but no basis has yet been found for the admissible rules of the whole logic RM. In this work, we provide bases for the $\{\to, \neg, t\}$-, $\{\to, \cdot, t\}$-, and $\{\to, t\}$-fragments of RM^t. Surprisingly, the bases for these fragments are the same as for the corresponding fragments without t of RM.

Note that for two classes of algebras K_1 and K_2, $\mathbb{V}(\mathsf{K}_1) = \mathbb{V}(\mathsf{K}_2)$ implies that a rule Γ/φ is admissible in K_1 iff it is admissible in K_2. Hence, Lemma 1 tells us that a rule is admissible in $\mathsf{SM}{\restriction}\{\to, \neg, t\}$ iff it is admissible in $\mathbf{Z}_4^m$.

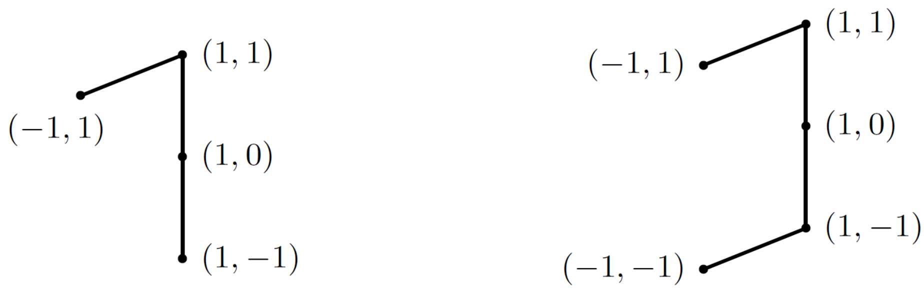

FIGURE 1. The algebras **A** (left) and **B** (right)

LEMMA 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *denote the subalgebras of the* $\{\to, t\}$*- and* $\{\to, \cdot, t\}$*-reduct of* $\mathbf{Z}_2^m \times \mathbf{Z}_3^m$*, respectively, illustrated in Figure 1. Then admissibility in* $\mathsf{SM}\restriction\{\to, t\}$*,* $\mathsf{SM}\restriction\{\to, \cdot, t\}$*, and* $\mathsf{SM}\restriction\{\to, \neg, t\}$ *are equivalent to validity in* $\mathbf{A}$*,* $\mathbf{B}$*, and* $\mathbf{Z}_2^m \times \mathbf{Z}_3^m$*, respectively.*

Let us now define the following implication ([2]):

$$\varphi \Rightarrow \psi =_{def} (\varphi \to |\psi|) \to (\varphi \to \psi).$$

The modus ponens rule for this connective is:

$$\text{(A)} \quad \{p, p \Rightarrow q\}/q.$$

This rule is admissible in all three fragments considered. In fact, we have the following result:

THEOREM 3. $\{\text{(A)}\}$ *is a basis for the* $\{\to, t\}$ *and* $\{\to, \cdot, t\}$ *fragments of* RM^t.

However, the admissible rules for the last fragment, the multiplicative fragment $\{\to, \neg, t\}$, do not have a finite basis. In this case, we use the following rules introduced in [1]: For each $n \in \mathbb{N} \setminus \{0\}$,

$$(\mathrm{R}_n) \quad \{\neg((p_1 \to p_1) \leftrightarrow \ldots \leftrightarrow (p_n \to p_n))\}/q,$$

where the connective $\leftrightarrow$ is defined as

$$\varphi \leftrightarrow \psi =_{\mathrm{def}} (\varphi \to \psi) \cdot (\psi \to \varphi).$$

THEOREM 4. *A basis for the admissible rules of the* $\{\to, \neg, t\}$*-fragment of* RM^t *is*

$$\{\text{(A)}\} \cup \{(\mathrm{R}_n) \mid n \in \mathbb{N} \setminus \{0\}\}.$$

Moreover, the admissible rules of this fragment do not have a finite basis.

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§1. Introduction. The notion of *fuzzy quantifier* as a generalization of the classical 'for all" and 'there exists' was introduced by L.A. Zadeh in 1975 [4]. This provided a semantics for fuzzy modifiers such as *most, many, few, almost all*, etc. and introduced the idea of reasoning with syllogistic arguments along the lines of '*Most* men are vain; Socrates is a man; therefore, it is *likely* that Socrates is vain', where vanity is given as a fuzzy predicate. This and numerous succeeding publications [6, 7, 8, 9, 10, 11] developed well-defined semantics also for *fuzzy probabilities* (e.g., *likely, very likely, uncertain, unlikely*, etc.) and fuzzy *usuality modifiers* (e.g., *usually, often, seldom*, etc.). In addition, Zadeh has argued at numerous conferences over the years that these modifiers offer an appropriate and intuitively correct approach to nonmonotonic reasoning.

The matter of exactly how these various modifiers are interrelated, however, and therefore of a concise semantics for such syllogisms, was not fully explored. Thus while a new reasoning methodology was suggested, it was never developed. The present work has grown out of an effort to realize this goal. The paper [1] defined a formal logic **Q** for 'qualified syllogisms', together with a detailed discussion of how the logic may be used to address some well-known issues in the study of default-style nonmonotonic reasoning. A short summary was recently presented as [2]. That system falls short of the overall goal, however, in that it deals only with crisp predicates. A recent work [3] takes the next step by creating a logic that accommodates fuzzy predicates. The present abstract overviews these efforts.

Both logics are named **Q**, since in their syntax they are identical; their differences reside only in their semantics. Section 2 provides the rationale for a probablisitic interpretation of the fuzzy modifiers and illustrates the types of syllogisms in concern. Section 3 defines the formal languages. Section 4 briefly summarizes the semantics.

§2. Intuitive Motivation. *Qualified syllogisms* are classical Aristotelean syllogisms that have been 'qualified' through the use of fuzzy quantification, usuality, and likelihood. Some examples are

Most Swedes are tall.
Helge is a Swede.
———
It is *likely* that Helge is tall.

Usually, if someone has a child in college, she is middle-aged.
Vera has a child in college.
———
It is *likely* that Vera is middle aged.

Very few wealthy people are frugal.
Robert is wealthy.
———
It is *very unlikely* that Robert is frugal.

Here *Swede* and *has a child in college* are crisp predicates, while *tall*, *middle-aged*, *wealthy*, and *frugal* are fuzzy predicates. From a common-sense perspective, such arguments are certainly intuitively correct. A more detailed analysis is as follows.

First, note that there is a natural connection between fuzzy quantification and fuzzy likelihood. To illustrate, the statement

Most Swedes are tall.

may be regarded as equivalent with

If x is a Swede, then it is *likely* that x is tall.

The implicit connection is provided by the notion of a statistical sampling. In each case one is asserting

Given a person randomly selected from the population of Swedes,
there is a *high probability* that the person will be tall.

Suppose we express this equivalence as

$$(\mathit{Most}\ x)(\text{Swede}(x) \to \text{Tall}(x)) \leftrightarrow (\text{Swede}(x) \to \mathit{Likely}\text{Tall}(x))$$

Then the first of the two syllogisms involving Helge can be reduced to an application of this formula, together with the syllogism

$$\begin{array}{l} \text{Swede}(x) \to \mathit{Likely}\text{Tall}(x) \\ \underline{\text{Swede(Helge)}\qquad\qquad} \\ \mathit{Likely}\text{Tall(Helge)} \end{array}$$

This follows because the left side of the equivalence is the first premise of the original syllogism, and the right side of the equivalence is the first premise of the above syllogism. A key observation to be made here is that the latter syllogism follows by instantiating x with Helge and applying ordinary (classical) Modus Ponens. This suggests that the desired formulation of fuzzy quantification and fuzzy likelihood may be obtained by adjoining classical logic with an appropriate set of modifiers. It also suggests that the modifiers of interest may be introduced in the manner of either quantifiers or modal

Quantification	*Usuality*	*Likelihood*
all	always	certainly
very many/almost all	very often/almost always	very likely/almost certainly
most	usually	likely
many/about half	frequently/often	uncertain/about 50-50
few/some	seldom/occasionally	unlikely
very few/almost no	very seldom/almost never	very unlikely/almost certainly not
no	never	certainly not

TABLE 4. Interrelations across the three kinds of modifiers.

operators, and that the semantics for such a system could be based on some version of probability theory.

A second observation is that there is a similar connection between the foregoing two concepts and the concept of usuality. Based on the same idea of a statistical sampling, one has that

> *Usually*, if someone is a Swede, then the person is tall.

is equivalent with the former two assertions. Thus one should be able to include usuality modifiers along with quantifiers and likelihood modifiers in a similar extension of classical logic.

System **Q** is an outgrowth of these various insights and reflections. In addition to the syllogisms given above, it allows for expression of all similar syllogisms as represented by the lines of Table 1, where the two 'Helge' and 'Vera' examples are given by the first and last entries of the third line, and the 'Robert' example is given by first and last entries of the sixth line.

§3. Formal Syntax. We begin by defining the desired class of formal languages. Let the modifiers in Table 1, in top-down then left-right order, be represented by $\mathbf{Q}_3, \ldots, \mathbf{Q}_{-3}$, $\mathbf{U}_3, \ldots, \mathbf{U}_{-3}$, $\mathbf{L}_3, \ldots, \mathbf{L}_{-3}$. As *symbols* select: an (*individual*) *variable*, denoted by x; countably infinitely many (*individual*) *constants*, denoted generically by $a, b, \ldots$; countably infinitely many unary *predicate symbols*, denoted generically by $p, q, r, \ldots$; seven *logical connectives*, denoted by $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\dot{\rightarrow}$, $\ddot{\neg}$, and $\ddot{\vee}$; the above-mentioned modifiers $\mathbf{Q}_i$, $\mathbf{U}_i$, and $\mathbf{L}_i$; and *parentheses* and *comma*, denoted as usual. As will be seen, the dotted connectives are used to formalize part of the metalanguage. Let the *formulas* be the members of the sets

$$\mathrm{F}_1 = \{p(x) | p \text{ is a predicate symbol}\}$$

$$\mathrm{F}_2 = \mathrm{F}_1 \cup \{\neg P, (P \vee Q), (P \wedge Q) | P, Q \in \mathrm{F}_1 \cup \mathrm{F}_2\}^{20}$$

$$\mathrm{F}_3 = \{(P \rightarrow Q) | P, Q \in \mathrm{F}_2\}$$

$$\begin{aligned}\mathrm{F}_4 = \{&\mathbf{L}_3(P \dot{\rightarrow} \mathbf{L}_i Q), \mathbf{L}_3(P \dot{\rightarrow} \mathbf{Q}_i Q), \mathbf{L}_3(P \dot{\rightarrow} \mathbf{U}_i Q),\\ &\mathbf{Q}_3(P \dot{\rightarrow} \mathbf{L}_i Q), \mathbf{Q}_3(P \dot{\rightarrow} \mathbf{Q}_i Q), \mathbf{Q}_3(P \dot{\rightarrow} \mathbf{U}_i Q),\\ &\mathbf{U}_3(P \dot{\rightarrow} \mathbf{L}_i Q), \mathbf{U}_3(P \dot{\rightarrow} \mathbf{Q}_i Q), \mathbf{U}_3(P \dot{\rightarrow} \mathbf{U}_i Q) |\\ &P, Q \in \mathrm{F}_2 \cup \mathrm{F}_3, i = -3, \ldots, 3\}\end{aligned}$$

$$\mathrm{F}_5 = \{\mathbf{L}_i P, \mathbf{Q}_i P, \mathbf{U}_i P, | P, Q \in \mathrm{F}_2 \cup \mathrm{F}_3, i = -3, \ldots, 3\}$$

$$\mathrm{F}_6 = \mathrm{F}_4 \cup \mathrm{F}_5 \cup \{\ddot{\neg} P, (P \ddot{\vee} Q) | P, Q \in \mathrm{F}_4 \cup \mathrm{F}_5 \cup \mathrm{F}_6\}$$

---

[20] This notation abbreviates the usual inductive definition, in this case the smallest class of formulas containing F_1 together with all formulas that can be built up from formulas in F_1 in the three prescribed ways.

$\mathrm{F}_1' = \{P(a/x) | P \in \mathrm{F}_1$ and a is an individual constant$\}$

$\mathrm{F}_2' = \{P(a/x) | P \in \mathrm{F}_2$ and a is an individual constant$\}$

$\mathrm{F}_3' = \{P(a/x) | P \in \mathrm{F}_3$ and a is an individual constant$\}$

$\mathrm{F}_4' = \{\mathbf{L}_3(P \dot{\rightarrow} \mathbf{L}_i Q)(a/x) | \mathbf{L}_3(P \dot{\rightarrow} \mathbf{L}_i Q) \in \mathrm{F}_4$, a is an individual constant, and $i = -3, \ldots, 3\}$

$\mathrm{F}_5' = \{\mathbf{L}_i P(a/x) | P \in \mathrm{F}_5$, a is an individual constant, and $i = -3, \ldots, 3\}$

$\mathrm{F}_6' = \mathrm{F}_4' \cup \mathrm{F}_5' \cup \{\ddot{\neg} P, (P \ddot{\vee} Q) | P, Q \in \mathrm{F}_5' \cup \mathrm{F}_6'\}$

where $P(a/x)$ denotes the formula obtained from P by replacing every occurrence of the variable x with an occurrence of the constant a. As abbreviations take

$(P \ddot{\wedge} Q)$ for $\ddot{\neg}(\ddot{\neg} P \ddot{\vee} \ddot{\neg} Q)$

$(P \ddot{\rightarrow} Q)$ for $(\ddot{\neg} P \ddot{\vee} Q)$

$(P \ddot{\leftrightarrow} Q)$ for $((P \ddot{\rightarrow} Q) \ddot{\wedge} (Q \ddot{\rightarrow} P))$

Formulas without modifiers are *first-* or *lower-level* formulas, and those with modifiers are *second-* or *upper-level*. The members of the set $\mathrm{F}_1 \cup \mathrm{F}_1'$ are *elementary first-* or *lower-level* formulas, and the members of $\mathrm{F}_4 \cup \mathrm{F}_4' \cup \mathrm{F}_5 \cup \mathrm{F}_5'$ are *elementary second-* or *upper-level* formulas. A lower-level formula is *open* if it contains the variable x, and *closed* if not.

By a *language* L is meant any collection of symbols and formulas as described above. Languages differ from one another essentially only in their choice of individual constants and predicate symbols. As an example, the first of the foregoing syllogisms can be written in a language employing the individual constant a for Tweety and the predicate symbols α and β for Bird and CanFly—and, for clarity, writing these names instead of the symbols—as

$$\begin{array}{l} \mathbf{Q}_1(\mathrm{Swede}(x) \rightarrow \mathrm{Tall}(x)) \\ \mathbf{L}_3\mathrm{Swede}(\mathrm{Helge}) \\ \hline \mathbf{L}_1\mathrm{Tall}(\mathrm{Helge}) \end{array}$$

In words: For *most* x, if x is a Swede then x is tall; it is *certain* that Helge is a Swede; therefore it is *likely* that Helge is tall.

§4. Semantics. The expanded semantics uses Tarski-style interpretations where predicate symbols are assigned as their meanings crisp or fuzzy subsets of the underlying universe of discourse U. A fuzzy-predicate symbol is taken as representing a linguistic term in the term set for a linguistic variable V (cf. [5]). A *belief valuation* β is defined on lower-level formulas so that, for elementary lower-level formulas $p(a)$, if p is a crisp-predicate symbol, $\beta(p(a))$ is the probability that the predicate represented by P is true about the inviidual represented by a, and if p is fuzzy-predicate symbol, $\beta(P(a))$ is the probability that the linguistic term represented by p is the one in the term set for V for which the membership of the individual represented by a is largest. Analogous considerations apply for elementary lower-level formulas of the form $p(x)$. This leads to a mapping defined on lower-level formulas qualifies as a probability measure. In turn, β is used to define a bivalent *truth valuation* v on upper-level formulas.

This semantics validates the types of syllogistic arguments described in Section 2. In addition, at the upper-level, it validates the axioms and inference rules of classical propositional calculus, as well as vesions of the additional axioms and inference rules needed for the classical first-order predicate calculus. The logic can be used to provide intuitively plausible resolutions of some well-known puzzles of default-style nonmonotonic reasoning. See [1] for details.

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▸ ESKO TURUNEN, *Connecting Fuzzy Sets and Pavelka's Fuzzy Logic: An Extended Abstract.*
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Keywords Fuzzy sets, mathematical fuzzy logic, MV–algebras.

§1. Introduction. During the last decades Fuzzy Set Theory has become an important method in dealing with vagueness in engineering, economics and many other applied sciences. Alongside this development, there has been significant segregation: *fuzzy logic in broad sense* include everything that is related to fuzziness and is mostly oriented to real-world applications, while *fuzzy logic in narrow sense*, also called *mathematical fuzzy logic*, develops mathematical methods to model vagueness and fuzziness by well-defined logical tools. These two approaches do not always meet each other. This is unfortunate, since theory should always reflect practice, and practice should draw upon the best theories. In this work, we try to bridge the gap between practical applications of Fuzzy Set Theory and mathematical fuzzy logic. Our guiding principle is to explain in logic terms the fuzzy logic concepts that are used in many real world applications, therefore we are trying to stay as close as possible to practical applications of fuzzy sets. In this respect, our approach is very different from the mainstream approach, where the idea is to generalize classical first order logic concepts to many valued logics.

We demonstrate how continuous $[0, 1]$–valued fuzzy sets can be naturally interpreted as open formulas in a simple first order fuzzy logic of Pavelka style (cf. [3, 2, 5, 1]); a logic whose details we discuss here. The main idea is to understand truth values as continuous functions; for single elements $x_0 \in X$ the truth values are constant functions defined by the membership degree $\mu_\alpha(x_0)$, for open formulas $\alpha(x)$ they are the membership functions $\mu_\alpha : X \curvearrowright [0,1]$, where the base set X is scaled to the unit interval $[0,1]$, for universally closed formulas $\forall x\alpha(x)$ truth values are definite integrals understood as constant functions. We also introduce existential quantifiers $\exists^a$, where $a \in [0,1]$. We show that this logic is complete in Pavelka's sense and generalize all classical tautologies that are definable in the language of this logic. However, all proofs, many details and a deeper description of the starting points of our approach are omitted due to space limitations. These shortcomings will be discussed in detail in a forthcoming full paper.

§2. Language and Semantics: Fuzzy Sets as Open Formulas.

2.1. Language. When a real world knowledge engineer applies Fuzzy Set Theory, one of the first things to do is to introduce fuzzy (sub)sets, say $P, R, S, \cdots, T$. A fuzzy (sub)set A is defined by its *membership function* $\mu_A : X \curvearrowright [0,1]$, where X is (the base) set; therefore we identify fuzzy sets $P, R, S, \cdots, T$ with their membership functions and denote them by $\overline{P}(x), \overline{R}(x), \cdots, \overline{T}(x)$ instead of $\mu_P, \mu_R, \mu_S, \cdots \mu_T$ and assume that they are continuous $[0,1]$-valued functions defined on $[0,1]$. After all, from real life application point of view, this assumption is not very restrictive; in most applications the underlying set X can be mapped (scaled) in one–one way to the real unit interval $[0,1]$. Moreover, we are in the realm of Pavelka style fuzzy logic (cf. [3, 5, 2, 1]), thus we assume that the interval $[0,1]$ is equipped with the standard MV-algebra structure, see e.g. [5]. Using the terminology in [6], the logic we now define would be called *monadic logic of type* $\langle 1 \rangle$. Thus, our approach is very different from the mainstream approach on mathematical first order fuzzy logics; for a complete survey of them and a list of related relevant literature can be found in [1, 2].

We study the simplest first order fuzzy logic; we assume that there is only a finite number of unary predicates, namely the fuzzy sets $P, R, S, \cdots, T$ and only one free variable x in the language under consideration; we use notation $P(x), R(x), S(x), \cdots, T(x)$; they are (elementary) *open formulas.* $P(x_0)$, where $x_0 \in [0,1]$, is a *constant formula* of the language. The logic connectives are as usual: `or`, `and`, `imp`, `not`. Open formulas and constant formulas can be combined with logical connectives and the result is an open formula if at least one of the components is an open formula, otherwise the result is a constant formula. Open and constant formulas are *well defined formulas.* Contrary to the original Pavelka logic [3], there are no truth constants in our approach.

There is also a *universal quantifier* $\forall$ in the language. If $\alpha(x)$ is an open formula, then $\forall x\alpha(x)$ is a *closed formula*; read $\forall x$: 'for an average x' as $\forall x\alpha(x)$ indicates the average characteristics of the whole fuzzy set α; such formulas are well defined formulas of the language under consideration. However, in this interpretation `not` $\forall x\alpha(x)$ does not have any clear and unambiguous meaning, therefore `not` $\forall x\alpha(x)$ does not belong to the set of well formed formulas of the language. Finally, there are *existential quantifiers* $\exists^a$ for each $a \in [0,1]$. If there is an $x_0 \in [0,1]$ such that $\overline{\alpha}(x_0) = a \in [0,1]$, then $\exists^a x\alpha(x)$ is defined and is a *closed formula.* Notice that $\exists^a$ is a constructive existential quantifier; it establishes constructively the existence of a certain element x, while `not` $\exists^a$ might not be constructive it the same sense. Therefore $\exists^a x\alpha(x)$ is a well formed formula while `not` $\exists^a x\alpha(x)$ is not defined. A detailed discussion about these quantifiers will be presented in a forthcoming full paper.

From now on by a formula we mean a well formed formula.

2.2. Semantics. A *truth value* of logical formula α in a usual fuzzy logic approach is a value $v(\alpha)$ on the unit real interval $[0,1]$, which is equipped with a suitable algebraic structure, typically generated by a continuous t-norm. In our approach truth values are continuous functions defined on the unit real real interval $[0,1]$ and equipped with point wise defined standard MV-operations. Thus, we can utilize Pavelka style fuzzy logic framework.

When a real world knowledge engineer defines the membership functions of the fuzzy sets $P, R, \cdots, T$ he fixes their meaning (recall that in our approach we scale the base set X to the real unit interval). In logic terminology, he/she gives the semantics. We follow this line and associate with all formulas α a continuous function $v(\alpha) : [0,1] \curvearrowright [0,1]$, denoted by $\overline{\alpha}$, in the following way. For elementary open formulas A we define $v(A(x)) = \overline{A}(x)$, for constant formulas $A(x_0)$, $x_0 \in [0,1]$, we define $v(A(x_0)) = \overline{a}(x)$, where $\overline{a}$ is the constant function $\overline{a} : [0,1] \curvearrowright [0,1]$; $\overline{a}(x) = a$ and $\overline{A}(x_0) = a$. For formulas closed by the universal quantifier we set

$$v(\forall x \alpha(x)) = \int_0^1 \overline{\alpha}(x)dx = b,$$

where x is free variable in α, thus denoted by $\alpha(x)$, and the value b of the definite integral is understood as a constant function $\overline{b} : [0,1] \curvearrowright [0,1]$. For formulas closed by the existential quantifiers we set

$$v(\exists^a x\alpha(x)) = a \text{ if, for some } x_0 \in [0,1], v(\alpha(x_0)) = \overline{\alpha}(x_0) = a,$$

where the value a is understood as a constant function $\overline{a}(x) = a$ for all $x \in [0,1]$. As discussed above, there are infinitely many existential quantifiers $\exists^a$, one for each $a \in [0,1]$. In more general setting, if there is no such $x_0 \in [0,1]$ that $\overline{\alpha}(x_0) = a$, then $v(\exists^a x\alpha(x))$ is not defined. Moreover, if $v(\alpha) = \overline{\alpha}$ and $v(\beta) = \overline{\beta}$, then we interpret the logical connectives by point wise defined MV–operations, that is,

$$\begin{array}{lllll}
v(\alpha \,\texttt{and}\, \beta) & = & \overline{\alpha} \odot \overline{\beta} & = & \max\{\overline{\alpha} + \overline{\beta} - 1, 0\}, \\
v(\alpha \,\texttt{or}\, \beta) & = & \overline{\alpha} \oplus \overline{\beta} & = & \min\{\overline{\alpha} + \overline{\beta}, 1\}, \\
v(\alpha \,\texttt{imp}\, \beta) & = & \overline{\alpha} \to \overline{\beta} & = & \min\{1 - \overline{\alpha} + \overline{\beta}, 1\}, \\
v(\texttt{not}\, \alpha) & = & \overline{\alpha}^* & = & 1 - \overline{\alpha},
\end{array}$$

whenever the corresponding formulas are defined.

It is clear from the above construction that any well formed formula α has exactly one valuation, in fact, the only valuation it is the membership function $\overline{\alpha} : [0,1] \curvearrowright [0,1]$ of α. By their construction all valuations are continuous functions on the real unit square $[0,1]^2$.

Notice that that, using Pavelka style notation, $\models_a \alpha$ has the same meaning than $v(\alpha) = \overline{a}$, where $\overline{a}$ is the membership function – the only truth value – of α. Here we list tautologies that are later taken as logical axioms. It is a routine task to show that they are 1-tautologies whenever the corresponding formulas are defined

$$\begin{array}{ll}
(T_1) & \models_1 \alpha \,\texttt{imp}\, (\texttt{not}\,\texttt{not}\, \alpha), \\
(T_2) & \models_1 (\texttt{not}\, \alpha \,\texttt{or}\, \texttt{not}\, \beta) \,\texttt{imp}\, \texttt{not}(\alpha \,\texttt{and}\, \beta), \\
(T_3) & \models_1 (\texttt{not}\, \alpha \,\texttt{and}\, \texttt{not}\, \beta) \,\texttt{imp}\, \texttt{not}(\alpha \,\texttt{or}\, \beta), \\
(T_4) & \models_1 (\texttt{not}\, \alpha \,\texttt{or}\, \beta) \,\texttt{imp}\, (\alpha \,\texttt{imp}\, \beta), \\
(T_5) & \models_1 (\alpha \,\texttt{and}\, \texttt{not}\, \beta) \,\texttt{imp}\, \texttt{not}(\alpha \,\texttt{imp}\, \beta), \\
(T_6) & \models_1 (\texttt{not}\, \alpha(x_0) \,\texttt{or}\, \beta) \,\texttt{imp}\, (\exists^a x\alpha(x) \,\texttt{imp}\, \beta), \text{ where } x_0 \text{ justifies } \exists^a \alpha(x), \\
(T_7) & \models_1 (\forall x \,\texttt{not}\, \alpha(x) \,\texttt{or}\, \beta) \,\texttt{imp}\, (\forall x \alpha(x) \,\texttt{imp}\, \beta).
\end{array}$$

In their seminal book [4], pages 297 - 298, Rasiowa and Sikorski list classical tautologies for quantified formulas. All classically valid tautologies, whenever the corresponding formulas are defined, are valid also in our approach.

2.3. Syntax, Axioms and Rules of Inference, Completeness. In Pavelka style logic setting, fuzzy set $P, R, S, \cdots, T$ and their membership functions $\overline{P}, \overline{R}, \overline{S}, \cdots, \overline{T}$ would correspond to special axioms of a fuzzy theory $\mathcal{T}$. This is our starting point.

The logical axioms in our approach correspond to the 1-tautologies (T_1) - (T_7). Special axioms are the open elementary formulas $A(x)$ with truth degree $\overline{A}(x)$, their negations $\texttt{not}\, A(x)$ with truth degree $1 - \overline{A}(x)$ as well as constant elementary formulas $A(x_0)$ and their negations with obvious truth degrees. Rules of inference are the following

Generalized Modus Ponens:

$$\frac{\alpha, \alpha \;\texttt{imp}\; \beta}{\beta} \quad , \quad \frac{\overline{\alpha}, \overline{\gamma}}{\overline{\alpha} \odot \overline{\gamma}}$$

Rule of Bold Conjunction:

$$\frac{\alpha, \beta}{\alpha \;\texttt{and}\; \beta} \quad , \quad \frac{\overline{\alpha}, \overline{\beta}}{\overline{\alpha} \odot \overline{\beta}}$$

Rule of Bold Disjunction:

$$\frac{\alpha, \beta}{\alpha \;\texttt{or}\; \beta} \quad , \quad \frac{\overline{\alpha}, \overline{\beta}}{\overline{\alpha} \oplus \overline{\beta}}$$

Rules for existential quantifiers:

$$\frac{\alpha(x_0)}{\exists^a x \alpha(x)} \quad , \quad \frac{\overline{\alpha}(x_0) = \overline{a} \text{ for some } x_0 \in [0,1]}{\overline{a}}$$

Rule for universal quantifier:

$$\frac{\alpha(x)}{\forall x \alpha(x)} \quad , \quad \frac{\overline{\alpha}(x)}{\int_0^1 \overline{\alpha}(x)dx}$$

It is routine to show that they are fuzzy rules of inference in Pavelka's sense. Then we prove by induction of the length of formulas

Theorem 1. *If the truth value (i.e. the degree of validity, as there is only one valuation) of a formula α is $\overline{\alpha}$, then there is also an $\mathcal{R}$-proof for α whose value is $\overline{\alpha}$ (by Soundness, this value cannot be greater than $\overline{\alpha}$)*

§3. Conclusion and Future Work. In this work we have shown how real life applications of Fuzzy Set Theory are naturally connected to mathematical fuzzy logic, in particular to Pavelka style fuzzy logic. The key idea is to understand the given fuzzy sets and their membership functions as special axioms of a fuzzy theory à la Pavelka. The truth values of such fuzzy sets, interpreted as elementary unary predicates of the logic, are their membership functions. The truth value of a formula containing universal quantifier $\forall$, 'the average x', is obtained via definite integral when viewed as a constant function. We also have existential quantifiers in the language under consideration; they are constructive in the sense that, for $\exists^a x\alpha(x)$ to be a well-formed formula, we must be able to point out an element that has the required property at a degree a. Logic connectives are interpreted by standard MV-operations point wisely; the set of truth values is the MV-algebra of continuous functions on the real unit interval. Then all classical tautologies that are definable in our approach are tautologies also in our setting. This kind of a simple first order fuzzy logic enjoys Pavelka style completeness.

Our approach have several advantages. It combines real-world applications to more theoretical studies, it establishes a link from mathematical fuzzy logic to continuous functions and its formalism is easy to understand and implement e.g. to MatLab and Mable programs. The degree of subset-hood as well as the degree of similarity of two fuzzy sets can be naturally defined by logic terms in our approach, and the formalism can be extended to first order languages with more than one free variable. Our approach also offers a simple way to introduce such generalized quantifiers as 'Almost all but not all', 'Most', 'Many', etc. However, these and many other interesting extensions are left for a future work.

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▸ THOMAS VETTERLEIN, ANNA ZAMANSKY, *An application of distance-based approximate reasoning for diagnostic questionnaires in healthcare.*
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Introduction. The *assessment of symptoms* is an essential aspect of the diagnosis of various disorders in medical science in general and in psychiatry and psychology in particular. As opposed to *signs*, which are objective findings detected by the clinician, *symptoms* are subjective experiences reported by a patient, such as the complaint of feeling depressed or anxious. To find out if a symptom is present, clinicians must rely on the patient's self-report; naturally, no objective tests are available. The need for a reliable measurement of symptoms has led to the development of standardized assessment methods based on *rating scales*. Such questionnaires are frequently used in routine healthcare [2, 1].

Designing and evaluating assessment questionnaires is a challenging task taking years to accomplish and is usually a collaborative effort of several experts. For instance, during the development of the ICD-10 Symptom Rating (ISR) questionnaire [1], which deals with psychological disorders, a panel of experts selected a list of syndromes to be tested and proposed for each of them symptoms deemed suitable for a reliable, honest, and valid self-rating by patients. A pilot study was carried out to check linguistic and contextual comprehensibility of the formulation, viability, etc. On the basis of the pilot study the formulations of the items as well as the scales were refined. The panel of experts also decided—in an intuitive manner—on the method of calculating the total score from the score of the single items: the mean value was taken [1]. It is interesting to note that no formal justification of this choice was provided. The experts furthermore discussed the possibility of adding weights, but decided against it—again on the basis of informal considerations.

We conclude that there is a need for a more formal approach to the analysis of the score computation in assessment questionnaires. In this talk we make some steps in this direction by proposing a degree-based logical framework. Our formalism is taylored to the particular application that we have in mind, and we address the difficulties of applying well-known degree-based logics like in particular fuzzy logics.

The problem of defining a logical framework to capture the way scores of questionnaires are processed in the medical domain is closely related to an issue often discussed in connection with fuzzy logic: vagueness. Let us consider an example. One of the disorders assessed by an ISR questionnaire [1] is the "Depressive Syndrome", and one of the questions to evaluate this disorder is "I feel down and depressed". The patient is asked to answer this question by choosing an element on a four-element linear scale, ranging from "1 - does not apply" to "4 - applies extremely". Apparently, we can understand the question as vague and the answer as a degree to which, from the point of view of the patient, the indicated statement fits to the patient's actual state.

Having to do with vague statements, we might, in a first reaction, call for fuzzy logic. In fact, in certain medical decision support systems, the way how degrees assigned to signs and symptoms are processed resemble methods well-known in fuzzy logic [3]. In our context, however, the method to aggregate degrees is, although very simple, totally different from what is known in fuzzy logics as we know them.

A formal framework for medical questionnaires. Our approach is based on the following considerations. The items appearing in the questionnaire are identified by the experts as symptoms of a disease that can occur in any possible combination. We accordingly consider the scores as elements of a multi-dimensional space; the scores provided by a particular patient are points of this space, also called *worlds*. The disease δ in question is then identified by two sets of worlds: the set of scores that (fully) confirm the disease and the set of scores that (fully) exclude the disease. Interesting, in our case, both these sets are singletons: δ is confirmed only if all answers are clearly positive (i.e., rated "4"), and excluded only if all answers are fully negative (i.e., rated "1").

Furthermore, the degree to which δ holds at an arbitrary world w is determined from its distance from the confirming cases in relation to its distance from the excluding cases. Here, the distance is taken to be the sum of the differences between the scores. On this basis we are naturally led to use the mean value and to provide a justification of the way in which the total score of a patient is computed, e.g., in the ISR questionnaire.

Our ideas suggest the following formal framework, which we shall roughly outline. Clinical entities will be represented by variable symbols $\alpha, \beta, \ldots$. We assume here that we deal with two types of such entities, diseases and symptoms; these types, however, are not formally distinguished and we could in principle also include, e.g., signs. Furthermore, clinical entities are vague and thus subject to gradation. Accordingly, we allow to assign *degrees* to symbols. A *graded variable* is an expression of the form (α, t), where α is a variable and t is an element of the real unit interval $[0, 1]$. The intended meaning is that α applies to a patient to the degree t. If α is not vague, t equals 0 or 1.

Graded variables are treated as boolean. We allow to build from them compound formulas by means of the boolean connectives $\wedge, \vee, \neg$, understood like in classical propositional logic.

The graded variables themselves are interpreted in a particular way, which we explain next. Our model is the following. First of all, we declare certain variables $\varphi_1, \ldots, \varphi_n$ as *basic* and the remaining ones as *dependent*. We assume that the basic variables can be assigned degrees in an arbitrary way, but that the degrees of all other variables are determined by the degrees of the basic variables.

Accordingly, we define $W = [0, 1]^n$ as our set of worlds. Each graded variable (α, t) is interpreted by a particular subset of W, denoted $[(\alpha, t)]$; the sets $[(\alpha, t)]$ are required to form partition of W. We say that α has the degree t at $w \in W$ if $t \in [0, 1]$ is such that $w \in [(\alpha, t)]$.

In case of the basic variables, the interpretation is fixed. In fact, each $w \in W$ is meant to correspond to a certain evaluation of $\varphi_1, \ldots, \varphi_n$ with degrees; accordingly, we put

$$[(\varphi_i, t)] \ = \ \{(a_1, ..., a_n) \in W : a_i = t\}.$$

For the case of the dependent variables, we need some preparations. First, we make W a metric space by putting

$$d((a_1, ..., a_n), (b_1, ..., b_n)) \ = \ |a_1 - b_1| + \ldots + |a_n - b_n|.$$

We additionally define the distance of $w \in W$ from $A \subseteq W$ by $d(w, A) = \min\{d(w, a) : a \in A$.

Consider next a basic variables φ, and a degree $t \in [0, 1]$. Observe how $[(\varphi, t)]$ is determined from $[(\varphi, 1)]$ and $[(\varphi, 0)]$ by means of the metric. Namely, $[(\varphi, t)]$ consists exactly of those $w \in W$ that have the distance t from $[(\varphi, 0)]$ and the distance $1 - t$ from $[(\varphi, 1)]$. Alternatively, we can say that $w \in [(\varphi, t)]$ if and only if

$$\frac{d(w, \varphi^-)}{d(w, \varphi^+) + d(w, \varphi^-)} \ = \ t,$$

where $\varphi^+ = [(\varphi, 1)]$ and $\varphi^- = [(\varphi, 0)]$. In a sense, $[(\varphi, \frac{1}{2})]$ is located half-way between φ^+ and φ^-, and for a general $t \in [0, 1]$, $[(\varphi, t)]$ is a sort-of weighted mean of the sets φ^+ and φ^-.

We shall use this same principle for the dependent variables as well. That is, let δ be a variable that is not basic. Then we require, just as above, that

$$[(\delta, t)] \ = \ \{w \in W : \frac{d(w, \delta^-)}{d(w, \delta^+) + d(w, \delta^-)} \ = \ t\},$$

where $\delta^+ = [(\delta, 1)]$ and $\delta^- = [(\delta, 0)]$. In other words, the truth degree of δ is at any world uniquely determined once we know the clear cases of δ, that is, the set of those worlds at which δ is clearly false and the set of those worlds at which δ is clearly true.

This is basically all we need. The notion of a theory is defined as a set of formulas built up from graded variables and the consequence relation is defined on a semantical basis. It is then easily checked that the way clinician evaluate the scores of questionnaire can be emulated in our framework.

Conclusion. Our proposal of a "questionnaire logic" is a first step towards a formal framework for score calculations in assessment questionnaires. We believe that such a framework can deepen our understanding how syndroms and their symptoms as modeled by the questionnaires are related, why the patient's particular scores are aggregated in a particular way, or how missing scores affect the overall calculation (a question left open in [1]).

From the point of view of logics, the biggest challenge is to define a proof system based on derivation rules in a common style. We are, however, not sure if this is a feasible concern.

From a foundational perspective, we think that the idea to model vague propositions by a pair of prototypes and counterexamples in a metric space and to determine truth degrees from the distances from these sets deserves to be further explored.

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▸ AMANDA VIDAL, FRANCESC ESTEVA AND LLUIS GODO, *Axiomatising a fuzzy modal logic over the standard product algebra.*
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Modal extensions of main systems of mathematical fuzzy logic are a family of logics that are still under research. Several papers have been published on this topic treating different aspects, see for instance [9] for modal extension of Łukasiewicz logic, [3, 4, 2] for modal extensions of Gödel fuzzy logic or [1] for modal logics over finite residuated lattices. However, the study of modal extensions over the product fuzzy logic Π, with semantics based on Kripke structures where both worlds and accessibility relations are evaluated over the standard product algebra, has remained open. We present here some results that partially fill this gap for the case of Kripke semantics with crisp accessibility relations and when the underlying product fuzzy logic is expanded with truth-constants, Δ operator and with two infinitary inference rules. We also explore the algebraic semantics for this modal logic.

§1. Enforcing propositional strong completeness of product logic. Propositional product logic Π is known to be finitely strong complete but not strongly complete with respect to the standard product chain $[0,1]_\Pi$, i.e. the product algebra over the real unit interval with the usual product of reals as monoidal operation, see [8]. For technical reasons, for proving the completeness results refering to the modal expansion of product logic, we need to work over an strongly complete logic, even if we only focused in finitary completeness results. In [11], Montagna defined an expansion of the BL logic with a storage operator $*$ and an infinitary rule (where ψ^k denotes $\psi \& \overset{k}{\ldots} \& \psi$)

$$(R_M)\ \frac{\chi \vee (\varphi \to \psi^k) \text{ for all } k \in \omega}{\chi \vee (\varphi \to \psi^*)},$$

This expansion was proved to be strongly complete (for infinite theories) with respect to the corresponding class of expanded standard BL-chains. In particular, for Product logic $*$ coincides with the Monteiro-Baaz operator Δ in $[0,1]$.

On the other hand, in [12] the addition of rational truth constants to product logic was studied, and it was proven that the extension of product logic with Δ and natural axioms for the constants was finitely strong complete with respect to the *canonical* standard product algebra $[0,1]_{\Pi^c_\Delta}$ (where the rational constants are interpreted by its name). Moreover, in the frame of rational Pavelka-like logics, Cintula in [7] had already proven that the addition of two infinitary inference rules made this logic to be strongly complete.

Let Π^c_Δ be the infinitary logic defined by the following axioms and rules:

- Axioms of Π (propositional product logic) (see for instance [8]);
- Axioms referring to rational constants over product logic [12];
- Axioms of the Δ operator ([8]) plus $\neg\Delta\overline{c}$ for each $c \in (0,1)_\mathbb{Q}$;

- Rules of Modus Ponens and Necessitation for Δ: $\varphi \vdash \Delta\varphi$;
- The infinitary rules

$$(\mathbf{R_1})\ \frac{\overline{c} \to \varphi, \text{ for all } c \in (0,1)_{\mathbb{Q}}}{\varphi} \qquad (\mathbf{R_2})\ \frac{\varphi \to \overline{c}, \text{ for all } c \in (0,1)_{\mathbb{Q}}}{\neg\varphi};$$

It is clear that Π^c_Δ is algebraizable and that its algebraic semantics is given by the class $\mathcal{P}^c_\Delta$ of algebras $\mathbf{A} = \langle A, \odot, \to, \Delta, \{c^{\mathbf{A}}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle$ where

- $\langle A, \odot, \to, \Delta, \mathbf{0}^{\mathbf{A}}\rangle$ is a Π_Δ-algebra.
- The rational constants $\{c^{\mathbf{A}}\}_{c\in[0,1]_{\mathbb{Q}}}$ form a subalgebra isomorphic to $[0,1]_{\mathbb{Q}}$ (as Π_Δ-algebras) such that for each $c, d \in (0,1)_{\mathbb{Q}}$ the following equations and generalised quasi-equations hold:

$$d^{\mathbf{A}} \odot c^{\mathbf{A}} = (d\cdot c)^{\mathbf{A}}, \qquad d^{\mathbf{A}} \to c^{\mathbf{A}} = \min\{1, (c/d)^{\mathbf{A}}\}, \qquad \Delta c^{\mathbf{A}} = \mathbf{0};$$
$$\text{If } x \geq c^{\mathbf{A}} \text{ for all } c \in (0,1)_{\mathbb{Q}} \text{ then } x = 1,$$
$$\text{If } x \leq c^{\mathbf{A}} \text{ for all } c \in (0,1)_{\mathbb{Q}} \text{ then } x = \mathbf{0}.$$

Due to the above two generalised quasi-equations, [11, Lemma 10] yields that any $\mathcal{P}^c_\Delta$-chain is archimedean. Now, following similar arguments from [11], one can prove that any consistent set of formulas can be extended to a complete theory over Π^c_Δ (closed under R_1 and R_2). It is then routine to show that the Lindenbaum sentence algebra of this complete theory is a $\mathcal{P}^c_\Delta$-chain, and hence archimedean. Finally, using results about product algebras from [6], one can also prove that for any countable archimedean chain from $\mathcal{P}^c_\Delta$ there is a complete embedding (i.e. preserving sups and infs) of that chain into the canonical standard product algebra $[0,1]^c_\Delta$. This gives the following completeness results.

Theorem 1 (Strong Completeness of Π^c_Δ). *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then the following conditions are equivalent:*

- $\Gamma \vdash_{\Pi^c_\Delta} \varphi$; - $\Gamma \models_{\mathcal{P}^c_\Delta} \varphi$;
- $\Gamma \models_{C\mathcal{P}^c_\Delta} \varphi$; - $\Gamma \models_{[0,1]_{\Pi^c_\Delta}} \varphi$,

where $C\mathcal{P}^c_\Delta$ is the class the linearly ordered algebras in $\mathcal{P}^c_\Delta$.

§2. Expanding product fuzzy logic with $\Box$ and $\Diamond$. In this section we expand the logic Π^c_Δ with the two usual modalities $\Box$ and $\Diamond$, we define a Kripke semantics for them and show an adequate complete axiomatization.

We start with the semantics. The notion of Kripke frame is as usual: a frame is a pair $\mathfrak{F} = \langle W, R\rangle$ with $W \neq \emptyset$ and $R \subseteq W \times W$. Given a product algebra $\mathbf{A} \in \mathcal{P}^c_\Delta$, an **A-Kripke model** $\mathsf{M} = (W, R, e)$ is just a Kripke frame $\langle W, R\rangle$ endowed with an evaluation of variables in $\mathbf{A}$ for each world $e\colon W \times \mathcal{V} \to A$ This evaluation is extended to non-modal formulas by its corresponding operations in $\mathbf{A}$, i.e. fulfilling $e(w, \varphi\&\psi) = e(w,\varphi) \odot e(w,\psi)$, $e(w, \varphi \to \psi) = e(w,\varphi) \to e(w,\psi)$, $e(w, \Delta\varphi) = \Delta(e(w,\varphi))$ and $e(w, \overline{c}) = c^{\mathbf{A}}$ and to modal formulas by:

$$e(w, \Box\varphi) := \inf\{e(v,\varphi) :\ Rwv = 1\}; \qquad e(w, \Diamond\varphi) := \sup\{e(v,\varphi) :\ Rwv = 1\};$$

A model $\mathsf{M} = (W, R, e)$ where these two values are defined for each $w \in W$ will be called *safe*, and we will denote the class of safe models by PK. For $\mathsf{M} = (W, R, e) \in \mathsf{PK}$ and $w \in W$ we write $\mathsf{M} \models_w \varphi$ whenever $e(w,\varphi) = 1$, and $\mathsf{M} \models \varphi$ whenever $\mathsf{M} \models_w \varphi$ for all $w \in W$.

Then, as usual in modal logics, two notions of logical consequence can be defined, a local and a global one. They are respectively defined as follows:

- $\Gamma \models^l_{\mathsf{PK}} \varphi$ if for any $\mathsf{M} = (W, R, e) \in \mathsf{PK}$ and any $w \in W$, if $\mathsf{M} \models_w \Gamma$ then $\mathsf{M} \models_w \varphi$;
- $\Gamma \models^g_{\mathsf{PK}} \varphi$ if for any $\mathsf{M} \in \mathsf{PK}$, if $\mathsf{M} \models \Gamma$ then $\mathsf{M} \models \varphi$.

A proposed axiomatization for the local consequence $\models^l_{\mathsf{PK}}$ is the following. Let $\mathbf{K}_\Pi$ be the logic defined by the following axioms and rules:

Π^c_Δ: Axioms and rules from Π^c_Δ

$(\mathbf{K})$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$(\mathbf{A}_\Box 1)$: $(\overline{c} \to \Box\varphi) \leftrightarrow \Box(\overline{c} \to \varphi)$

$(\mathbf{A}_\Box 2)$: $\Delta\Box\varphi \leftrightarrow \Box\Delta\varphi$

$(\mathbf{A}_\Diamond 1)$: $\Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$

$(\mathbf{N}_\Box)$: if φ is a theorem, then $\Box\varphi$ is a theorem as well.

The corresponding axiomatization for the global consequence $\models^g_{\mathsf{PK}}$ will be as above just replacing the necessitation rule $(\mathbf{N}_\Box)$ by the more general rule

$(\mathbf{N}^G_\Box)$: from φ derive $\Box\varphi$

We will denote this latter logic by $\mathbf{K}^g_\Pi$. There are two interesting observations about the modal logic $\mathbf{K}_\Pi$. First, it holds that, for an arbitrary theory Γ and any formula φ, $\Gamma \vdash_{\mathbf{K}_\Pi} \varphi$ implies that $\Box\Gamma \vdash_{\mathbf{K}_\Pi} \Box\varphi$, where $\Box\Gamma = \{\Box\psi : \psi \in \Gamma\}$. Second, since the necessitation rule $(\mathbf{N}_\Box)$ only affects theorems, it also holds

$$\Gamma \vdash_{\mathbf{K}_\Pi} \varphi \text{ iff } \Gamma \cup Th_{\mathbf{K}_\Pi} \vdash_{\Pi^c_\Delta} \varphi,$$

where $Th_{\mathbf{K}_\Pi}$ stands for the set of theorems of $\mathbf{K}_\Pi$, and where in the right-hand deduction formulas starting by a modal symbol are understood as new propositional variables.

Then, a natural procedure to check that the logic $\mathbf{K}_\Pi$ indeed axiomatizes the local consequence $\models_{\mathsf{PK}}$ is through the usual canonical model construction. In what follows we denote by $Fm^\star$ the algebra of propositional formulas built from the extended set of variables $\mathcal{V}^\star = \mathcal{V} \cup \{(\Box\varphi)^*, (\Diamond\varphi)^* \mid \varphi \text{ is a modal formula}\}$, that is, we introduce a new propositional variable for each formula starting with a modal operator.

Definition 2. The **canonical model** is the $[0,1]_{\Pi^c_\Delta}$-model $\mathbf{M}_c = (W_c, R_c, e)$ where:
- $W_c := \{w \in Hom(Fm^\star, [0,1]_{\Pi^c_\Delta}) : \ w([Th_{\mathbf{K}_\Pi}]) \subseteq \{1\}\}$;
- $R_c := \{(w,v) \in W_c \times W_c : \text{for any } \varphi \in Fm^\star, \text{if } w((\Box\varphi)^*)) = 1 \text{ then } v(\varphi) = 1\}$;
- $e : W \times \mathcal{V}^* \to [0,1]$ such that $e(w,x) := w(x)$ for all $x \in \mathcal{V}^*$.

Next step is to check that the so-called *Truth Lemma* holds true, i.e. for any φ we have $e(w,\Box\varphi) = w((\Box\varphi)^*)$ and $e(w,\Diamond\varphi) = w((\Diamond\varphi)^*)$. This directly gives the following completeness theorem.

Theorem 3 (Kripke Completeness). *For any set of modal formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{K}_\Pi} \varphi \text{ iff } \Gamma \models_{\mathsf{PK}^l} \varphi.$$

§3. Algebraic semantics. In this section we study the algebraic semantics of the modal systems $\mathbf{K}_\Pi$ and $\mathbf{K}^g_\Pi$. We begin by classifying these logics in the Leibniz hierarchy of Abstract Algebraic Logic. It turns out that $\mathbf{K}^g_\Pi$ is algebraizable and that $\mathbf{K}_\Pi$ is not (even if it is still equivalential). Nevertheless, it turns out that the classes of algebras associated with these two logics coincide, and are given by the generalized quasi-variety $\mathcal{MP}^c_\Delta$ of modal product algebras $\mathbf{A} = \langle A, \odot, \to, \Delta, \Box, \Diamond, \{c\}_{c\in[0,1]_\mathbb{Q}}\rangle$ where
- $\langle A, \odot, \to, \Delta, \{c\}_{c\in[0,1]_\mathbb{Q}}\rangle \in \mathcal{P}^c_\Delta$;
- For every $x, y \in A$, $\Box(x \to y) \leq \Box x \to \Box y$;
- For every $x \in A$, $c \in [0,1]_\mathbb{Q}$ $\Box(x \to c^\mathbf{A}) = \Diamond x \to c^\mathbf{A}$ and $\Box(c^\mathbf{A} \to x) = c^\mathbf{A} \to \Box x$;
- For every $x \in A$, $\Box\Delta x = \Delta\Box x$;
- $\Box 1 = 1$.

One can check that the reduced filters of the global modal logic are just $\{1\}$, and thus we obtain the following **completeness** result for any set of modal formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathbf{K}^g_\Pi} \varphi \text{ iff } \Gamma \models_{\mathcal{MP}^c_\Delta} \varphi.$$

However, the study of the local modal logic is not so neat. It is a general fact that any logic is strongly complete with respect to its class of reduced models, but for non-algebraizable logics these do not need to form a well-behaved class. Nevertheless, gaining inspiration from [10], we can provide a nice characterization of the reduced models of $\mathbf{K}_\Pi$ and thus a more concrete algebraic completeness result.

More precisely, it can be proven that the deductive filters of $\mathbf{K}_\Pi$ over a modal product algebra $\mathbf{A}$, in symbols $\mathcal{F}i_{\mathbf{K}_\Pi}\mathbf{A}$, coincide with those of the non-modal logic Π^c_Δ over the non-modal reduct of $\mathbf{A}$. Then, the reduced filters can be characterized using the concept of *open* filter of $\mathbf{A}$, i.e., the ones closed under the $\Box$ operator.

Theorem 4. *$\langle \mathbf{A}, F\rangle$ is a reduced model of $\mathbf{K}_\Pi$ if and only if $\mathbf{A} \in \mathcal{MP}^c_\Delta$, $F \in \mathcal{F}i_{\mathbf{K}_\Pi}\mathbf{A}$ and $\{1\}$ is the maximum open filter in $\mathcal{F}i_{\mathbf{K}_\Pi}\mathbf{A}$ such that $\{1\} \subseteq F$.*

As we developed two semantics for our modal logics, namely the Kripke and the algebraic ones, it is natural to study their relationship. We describe a way of translating the Kripke semantics into the algebraic one by associating a modal product algebra to each safe $\mathbf{A}$-Kripke model (see for instance [5] for the classical case). More precisely, let $\mathbf{A} \in \mathcal{P}^c_\Delta$ and a safe $\mathbf{A}$-Kripke model $\mathbf{M}$. We say that $\mathbf{M}^+ = \langle \mathbf{A}^W, \odot, \to, \Delta, \Box, \Diamond, \{c\}_{c\in[0,1]_\mathbb{Q}}\rangle$ is the *dual algebra of* $\mathbf{M}$, where

$$f \odot g := [v \mapsto f(v) \odot g(v)];\ \ \Box f := [v \mapsto \inf\{f(w) :\ Rvw\}];\ \ c^{\mathbf{M}^+} := [v \mapsto c^{\mathbf{A}}].$$
$$f \to g := [v \mapsto f(v) \to g(v)];\ \ \Diamond f := [v \mapsto \sup\{f(w) :\ Rvw\}];$$

The *dual evaluation* $e^+ \colon Fm \to \mathbf{M}^+$ is given by $e^+(\varphi) = [v \mapsto e(v, \varphi)]$. It turns out that $\mathbf{M}^+ \in \mathcal{MP}^c_\Delta$, and applying this translation to the Canonical Model it is possible to obtain a second completeness result of $\mathbf{K}_\Pi$ with respect to $\mathcal{MP}^c_\Delta$.

Theorem 5 (Algebraic completeness). *For any set of modal formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbf{K}_\Pi} \varphi \textit{ iff } \Delta\Gamma \models^{\leq}_{\mathcal{MP}^c_\Delta} \varphi,$$

where $\Theta \models^{\leq}_{\mathcal{MP}^c_\Delta} \chi$ means that for any $\mathbf{A} \in \mathcal{MP}^c_\Delta$, h homomorphism from the algebra of modal formulas into $\mathbf{A}$ and $a \in \mathbf{A}$, if $a \leq h(\theta)$ for all $\theta \in \Theta$, then $a \leq h(\chi)$.

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