This document describes the calculus **LK** as used in `proof.dtd`:

1 The Calculus

A *sequent* is a pair of sequences of formulas, inference on the left (right) side of the sequent occurs only at the outermost left (right) formula.

The Axioms:

We allow arbitrary atomic sequents as axioms. The logical axioms are of the form $A \vdash A$ for A atomic. For equality we use the reflexivity axiom scheme $\vdash t = t$ for all terms t.

The Rules:

1. propositional

$$\frac{\Gamma \vdash \Delta, A \quad \Pi \vdash \Lambda, B}{\Gamma, \Pi \vdash \Delta, \Lambda, A \wedge B} \wedge : r$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge : l1 \qquad \frac{A, \Gamma \vdash \Delta}{B \wedge A, \Gamma \vdash \Delta} \wedge : l2$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Pi \vdash \Lambda}{A \vee B, \Gamma, \Pi \vdash \Delta \Lambda} \vee : l$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee : r1 \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A} \vee : r2$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Pi \vdash \Lambda}{A \rightarrow B, \Gamma, \Pi \vdash \Delta, \Lambda} \rightarrow : l \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \rightarrow : r$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \neg : r \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \neg : l$$

2. first-order

$$\frac{\Gamma \vdash \Delta, A\{x \leftarrow \alpha\}}{\Gamma \vdash \Delta, (\forall x)A} \forall : r \qquad \frac{A\{x \leftarrow t\}, \Gamma \vdash \Delta}{(\forall x)A, \Gamma \vdash \Delta} \forall : l$$

$$\frac{\Gamma \vdash \Delta, A\{x \leftarrow t\}}{\Gamma \vdash \Delta, (\exists x)A} \exists : r \qquad \frac{A\{x \leftarrow \alpha\}, \Gamma \vdash \Delta}{(\exists x)A, \Gamma \vdash \Delta} \exists : l$$

For the $\forall : r$ and the $\exists : l$ rules the variable α must not occur in Γ nor in Δ nor in A.

For the $\forall : l$ and the $\exists : r$ rules the term t must not contain a variable that is bound in A.

3. equality

$$\frac{\Gamma \vdash \Delta, s = t \quad \Pi \vdash \Lambda, A[s]_\Xi}{\Gamma, \Pi \vdash \Delta, \Lambda, A[t]_\Xi} = (\Xi) : r1 \qquad \frac{\Gamma \vdash \Delta, s = t \quad A[s]_\Xi, \Pi \vdash \Lambda}{A[t]_\Xi, \Gamma, \Pi \vdash \Delta, \Lambda} = (\Xi) : l1$$

where Ξ is a set of positions in A.

$$\frac{\Gamma \vdash \Delta, t = s \quad \Pi \vdash \Lambda, A[s]_\Xi}{\Gamma, \Pi \vdash \Delta, \Lambda, A[t]_\Xi} = (\Xi) : r2 \qquad \frac{\Gamma \vdash \Delta, t = s \quad A[s]_\Xi, \Pi \vdash \Lambda}{A[t]_\Xi, \Gamma, \Pi \vdash \Delta, \Lambda} = (\Xi) : l2$$

where Ξ is a set of positions in A.

4. structural

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A_1, \ldots, A_n}\ w : r \qquad \frac{\Gamma \vdash \Delta}{A_1, \ldots, A_n, \Gamma \vdash \Delta}\ w : l$$

where $n > 0$

$$\frac{\Gamma \vdash A_1^{(m_1)}, \ldots, A_n^{(m_n)}}{\Gamma \vdash A_1, \ldots, A_n}\ c(m_1, \ldots, m_n) : r \qquad \frac{A_1^{(m_1)}, \ldots, A_n^{(m_n)} \vdash \Delta}{A_1, \ldots, A_n \vdash \Delta}\ c(m_1, \ldots, m_n) : l$$

where $m_i > 0$ for $1 \leq i \leq n$ and $A^{(k)}$ denotes k copies of the formula A.

$$\frac{A_1, \ldots, A_n \vdash \Delta}{A_{\sigma(1)}, \ldots, A_{\sigma(n)} \vdash \Delta}\ \pi(\sigma) : l \qquad \frac{\Gamma \vdash A_1, \ldots, A_n}{\Gamma \vdash A_{\sigma(1)}, \ldots, A_{\sigma(n)}}\ \pi(\sigma) : r$$

where σ is a permutation given as list of cycles

$$\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

5. definitions

$$\frac{F[B]_\Xi, \Gamma \vdash \Delta}{F[D]_\Xi, \Gamma \vdash \Delta}\ d(\Xi, \Pi, \mu) : l \qquad \frac{\Gamma \vdash \Delta, F[B]_\Xi}{\Gamma \vdash \Delta, F[D]_\Xi}\ d(\Xi, \Pi, \mu) : r$$

where Ξ is a set of positions in F, Π is a position translation and μ is a position.

Let P be the set of all positions, i.e. the set of all finite sequences of natural numbers including the empty sequence. A position translation is a function $\Pi : P \to \mathcal{P}(P)$ translating each position to a set of positions in a disjoint way, i.e. it satisfies:

$$\forall p_1, p_2 \in P : p_1 \neq p_2 \Longrightarrow \Pi(p_1) \cap \Pi(p_2) = \emptyset$$

The position translation Π of the definition rules translates positions from B into sets of positions in F.

The position μ in the parameter gives the position of the *defined symbol* relative to the $\xi \in \Xi$. For direct definitions μ is the empty position, for indirect definitions, e.g. a function symbol defined via a predicate, μ is not the empty position.

Example 1: We want to define a symbol $S(x)$ denoting the square $x \cdot x$ of x. An application of this definition rule could be

$$\frac{\Gamma \vdash 2 + 2 = 2 \cdot 2}{\Gamma \vdash 2 + 2 = S(2)} \; d(\Xi, \Pi, \mu) : r$$

with

$$\Xi = \{2\}$$

denoting the position of S in the conclusion and

$$\begin{aligned} \Pi(1\nu) &= \{1\nu, 2\nu\} \text{ for all positions } \nu \\ \Pi(\nu) &= \emptyset \text{ for all positions } \nu \text{ not starting with } 1 \end{aligned}$$

which denotes the fact that the 2 at position 1 in $S(2)$ has two ancestors in the premise: one at position 1 and one at position 2.

The main symbol position μ relative to the $\xi \in \Xi$ is the empty position because S is defined directly.

Example 2: We want to define the intersection of two sets x and y as follows:

$$m \in I(x, y) := m \in x \wedge m \in y$$

An instance of this definition is

$$\frac{\Gamma \vdash m \in S_1 \wedge m \in S_2}{\Gamma \vdash m \in I(S_1, S_2)} \; d(\Xi, \Pi, \mu) : r$$

with

$$\Xi = \{\varepsilon\}$$

and

$$\Pi(1\nu) = \{11\nu, 21\nu\} \text{ for all } \nu \text{ (to describe } m)$$
$$\Pi(21\nu) = \{12\nu\} \text{ for all } \nu \text{ (to describe } x)$$
$$\Pi(22\nu) = \{22\nu\} \text{ for all } \nu \text{ (to describe } y)$$
$$\Pi(\nu) = \emptyset \text{ for all other positions } \nu$$

Note that for this case of an indirect definition (of the function symbol I) the function Π cannot be described by a list of sets of position of the length of the arity of I (nor of $\in$).

The main symbol position μ relative to the $\xi \in \Xi$ is $\mu = 2$ denoting the fact that this definition rule defines the occurrence of I at position 2.

6. replacement

The replace rules are used for replacing formulas by other formulas. These rules are in the calculus for backwards compatibility. The validity of the replacement is the responsibility of the user and is not formal part of the rule.

$$\frac{A, \Gamma \vdash \Delta}{B, \Gamma \vdash \Delta}\; r:l \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B}\; r:r$$

2 XML Code

For the specification of the calculus-independent part, see `proof.dtd`.

2.1 Rule Types

The attribute `type` of the tag `rule` specifies the type of a rule and is defined as follows:

Rule	attribute `type`
$\wedge : r$	`andr`
$\wedge : l1$	`andl1`
$\wedge : l2$	`andl2`
$\vee : l$	`orl`
$\vee : r1$	`orr1`
$\vee : r2$	`orr2`
$\rightarrow : l$	`impll`
$\rightarrow : r$	`implr`
$\neg : r$	`negr`
$\neg : l$	`negl`
$\forall : r$	`forallr`
$\forall : l$	`foralll`
$\exists : r$	`existsr`
$\exists : l$	`existsl`
$=: l1$	`eql1`
$=: r1$	`eqr1`
$=: l2$	`eql2`
$=: r2$	`eqr2`
$w : r$	`weakr`
$w : l$	`weakl`
$c : r$	`contrr`
$c : l$	`contrl`
$\pi : r$	`permr`
$\pi : l$	`perml`
cut	`cut`
$r : r$	`replr`
$r : l$	`repll`
$d : r$	`defr`
$d : l$	`defl`

2.2 Rule Parameters

1. If the type of a rule is `perml` or `permr` then the attribute `param` must contain a string specifying a permutation. The permutation is given in the notation that is common in mathematics: a list of cycles with (and) as delimiters, e.g.:

   ```
   (1 2)(4 5 6)
   ```

   denotes the permutation

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{matrix}$$

assuming that the length of the permuted part of the sequent is 6. If the length of the permuted part of the sequent is longer than the largest position occurring in the permutation then the rest is considered to be padded with the identity permutation.

2. If the type of a rule is one of `eql1`, `eqr1`, `eql2`, `eqr2` then the attribute `param` must contain a string specifying a set of positions. A set of positions is specified as a list of positions where double-occurrences are ignored.

   Each position in the list starts with `(` and ends with `)`. The position is given relative to the auxiliary formula of the paramodulation rule, for a unary formula 1 is expected, for a quantifier formula 2, for a binary formula a 1 denotes the left subformula and a 2 the right subformula, for atom formulas and function symbols (in a term) the first (sub-)term has position 1, the second has position 2, and so on, e.g.

   ```
   (1 1)(2 2 1 1)(2 2 2)
   ```

   denotes the set of occurrences of x in the formula:

   $$P(x) \rightarrow (\exists z)Q(f(x, f(z, z)), x)$$

3. If the type of a rule is one of `defr`, `defl` the the attribute `param` must contain a string specifying a set of positions and a position translation. The set of positions is specified in exactly the same way as in the case of the equality-rules.

   The set of positions is separated by a `;` from the string specifying the position translation. The complete grammar for the parameter string `A` is:

   $$\begin{aligned} \text{position: } \mathtt{P} &= \underline{\mathtt{(}}\, \mathit{num}^* \,\underline{\mathtt{)}} \\ \text{position translation: } \mathtt{T} &= \underline{\mathtt{[}}\, \mathtt{P}\ \mathtt{P}^+ \,\underline{\mathtt{]}} \\ \text{parameter: } \mathtt{A} &= \mathtt{P}^+ \,\underline{\mathtt{;}}\, \mathtt{T}^* \,\underline{\mathtt{;}}\, \mathtt{P} \end{aligned}$$

   A position translation `T` is interpreted as translating all positions having as prefix the given first postiton (`P`) to the respective set of positions where this prefix is replaced by those given in the list of positions ($\mathtt{P}^+$).

   Example 1: The parameter string for our first example is

   `(2);[(1)(1)(2)];()`

   The set Ξ consists of the single element 2 and the position prefix 1 is translated to both 1 and 2. The defined symbol occurrence is at position $2 \cdot \varepsilon = 2$.

Example 2: The parameter string for our second example is

`();[(1)(1 1)(2 1)][(2 1)(1 2)][(2 2)(2 2)];(2)`

The set Ξ consists of the empty position, the position prefix 1 is translated to both 11 and 21, the position prefix 21 is translated to 12 and 22 is translated to 22. The defined symbol occurrence is at position $\varepsilon \cdot \mathtt{2} = \mathtt{2}$

4. If the type of a rule is one of `contrr`, `contrl` then the attribute `param` must contain a comma-separated list of integers describing the number of copies in the premise sequent of each of the formula occurrences in the conclusion sequent. For example, the contraction rule

$$\frac{A, A, B, C, C, C \vdash D}{A, B, C \vdash D}\ c:l$$

would be described by the parameter string

```
2,1,3
```